Rationalizing the Denominator or Numerator

Sometimes the denominator or numerator of a fraction has two terms and involves square roots, such as $3 - \sqrt{5}$ or $\sqrt{2} + \sqrt{3}$. The denominator or numerator may be rationalized by multiplying by an expression that makes the denominator or numerator a difference of two squares.

**Example:**

$$\frac{1}{1 + \sqrt{2}} = \frac{1}{1 + \sqrt{2}} \cdot \left(\frac{1 - \sqrt{2}}{1 - \sqrt{2}}\right)$$

$$= \frac{1 - \sqrt{2}}{1^2 - (\sqrt{2})^2}$$

$$= \frac{1 - \sqrt{2}}{-1}$$

$$= \sqrt{2} - 1$$

**Exercise:** Rationalize the numerator of $\frac{\sqrt{4 + x} - 2}{x}$.

**Answer:** $\frac{1}{\sqrt{4 + x} + 2}$

**Equations**

- An equation is a statement that two mathematical expressions are equal. For example, $3+5=8$. We can also have equations that contain variables that stand for numbers. For example, $4x + 7 = 19$.
- Solving an equation means finding the values of $x$ which make the equation true.
- The values are called the solutions of the equation and are said to satisfy the equation.

**Solving Linear Equations**

A linear equation is of the form: $ax + b = c$, where $a, b, c$ are real numbers and $a \neq 0$. We can solve these equations as follows:
\[ax + b = c\]
\[ax + b - b = c - b \quad \text{[subtract } b \text{ from both sides of the equation]}\]
\[ax = c - b\]
\[(ax)/a = (c - b)/a \quad \text{[divide both sides by } a]\]
\[x = (c - b)/a\]

Therefore, the solution to the equation is: \(x = (c-b)/a\). In other words, the equation \(ax + b = c\) is true if and only if \(x = (c-b)/a\).

**Example:** Solve \(4x + 7 = 19\).

\[
4x + 7 = 19 \\
4x + 7 - 7 = 19 - 7 \\
4x = 12 \\
(4x)/4 = 12/4 \\
x = 3
\]

Therefore, the solution to \(4x + 7 = 19\) is \(x = 3\).

**Equations involving fractions**

Suppose we have an equation of the following form:

\[
\frac{7}{x+2} = \frac{4}{x-3}
\]

This equation is not in the form of a linear equation, however, we can re-write it in this form and then use the above method to solve it.
\[
\frac{7}{x + 2} = \frac{4}{x - 3}
\]

\[
(x + 2)(x - 3) \frac{7}{x + 2} = (x + 2)(x - 3) \frac{4}{x - 3} \quad \text{[multiply both sides by (}x + 2)(x - 3)\text{]}
\]

\[
7(x - 3) = 4(x + 2) \\
7x - 21 = 4x + 8 \\
3x = 29 \\
x = \frac{29}{3}
\]

- An alternative solution that avoids multiplying both sides by LCD is as follows:

\[
\frac{7}{x + 2} - \frac{4}{x - 3} = 0
\]

Assuming that \(x\) is neither -2 or 3 and combining fractions gives:

\[
\frac{3x - 29}{(x - 3)(x + 2)} = 0
\]

A fraction can be zero only when its numerator is zero, and its denominator is not. Therefore, \(x = \frac{29}{3}\).

**Radical Equations**

A radical equation is one in which an unknown occurs in a radical.

**Example:** Solve \(\sqrt{x^2 + 33} - x = 3\).

**Solution:** We first isolate the radical on one side. Then, we raise both sides to power 2, and solve the equation in the standard techniques.

\[
\sqrt{x^2 + 33} = x + 3 \\
(x + 3)^2 = x^2 + 6x + 9 \\
24 = 6x \\
4 = x
\]

- By raising both sides to power 2, we might obtain some fake solutions. Thus, we must check any resulting solutions by substitution.
Example: Solve $2x = 1 - \sqrt{2 - x}$

\[
\begin{align*}
2x &= 1 - \sqrt{2 - x} \\
2x - 1 &= -\sqrt{2 - x} \\
4x^2 - 4x + 1 &= 2 - x \quad \text{[Square both side]} \\
4x^2 - 3x - 1 &= 0 \\
(4x + 1)(x - 1) &= 0
\end{align*}
\]

Therefore, $x = -1/4, 1$ satisfy the last equation. But, since we squared both sides in solving this equation we may have introduced new solutions. So, we must check whether or not the solutions we found were actually solutions of the original equation.

For $x = -1/4$, the left hand side is: $2(-1/4) = -1/2$, and the right hand side is: $1 - \sqrt{2 - (-1/4)} = 1 - \sqrt{9/4} = -1/2$. So, $x = -1/4$ is a solution of the original equation.

For $x = 1$, the left hand side is: $2(1) = 2$, and the right hand side is: $1 - \sqrt{2 - (1)} = 1 - 1 = 0$. Since the left hand side does not equal the right hand side, $x = 1$ is not a solution of the original equation.

**Quadratic equations**

A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$ where $a, b, c$ are real numbers and $a \neq 0$. These equations can be solved by factoring, completing the square, or by using the quadratic formula. A quadratic equation may have zero, one, or two solutions in the real numbers.

**Quadratic Formula**

The solutions of $ax^2 + bx + c = 0$ are given by the formula:

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

The expression $D = b^2 - 4ac$ is called the **discriminant**.
If $D > 0$, the equation has two solutions.

If $D = 0$, the equation has one solution.

If $D < 0$, the equation has no solutions.

Example: Solve $2x^2 + 3x - 4 = 0$.

Answer:

$$x = \frac{-3 + \sqrt{41}}{4}, \frac{-3 - \sqrt{41}}{4}$$

This means we can factor the equation as follows:

$$2x^2 + 3x - 4 = \left(x - \frac{-3 + \sqrt{41}}{4}\right)\left(x - \frac{-3 - \sqrt{41}}{4}\right)$$

Completing the Square

Given an expression $x^2 + bx$, we can make this a perfect square by adding $\left(\frac{b}{2}\right)^2$. Then we have,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

We can use the method of completing the square to solve quadratic equations.

Example: Solve $x^2 - 8x + 13 = 0$.

$$x^2 - 8x + 13 = 0 \quad \Rightarrow \quad x^2 - 8x = -13 \quad [\text{Complete the square}]$$

$$x^2 - 8x + 16 = -13 + 16 \quad \Rightarrow \quad (x - 4)^2 = 3 \quad [\text{Take the square root}]$$

$$x - 4 = \pm \sqrt{3} \quad \Rightarrow \quad x = 4 \pm \sqrt{3}$$

Therefore, the solutions are:

$$x = 4 + \sqrt{3}, \ 4 - \sqrt{3}$$
Quadratic-Form Equations

A quadratic-form equation is an equation which is not quadratic, but can be transformed into a quadratic.

Example: solve

\[
\frac{1}{x^6} + \frac{9}{x^3} + 8 = 0.
\]

Solution: The equation can be written as:

\[
\left(\frac{1}{x^3}\right)^2 + 9\left(\frac{1}{x^3}\right) + 8 = 0.
\]

Then, we can substitute the variable \( w \) for \( \frac{1}{x^3} \).

Absolute Value

The absolute value of a real number \( a \) is defined as

\[
|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}
\]

Remark: The absolute value of \( x \) is distance from \( x \) to 0.

Properties of Absolute Value:

1. \(|a| \geq 0\)
2. \(|a| = |-a|\)
3. \(|ab| = |a||b|\)
4. \(|\frac{a}{b}| = \frac{|a|}{|b|}\)
5. \(|a^n| = |a|^n\)

Remark: The distance between the points \( x \) and \( y \) on the real line is equal to \(|x - y|\).
Equations Involving Absolute Value

Example: Solve the following equation: $|2x - 5| = x + 3$

By the definition of absolute value we have

$$|2x - 5| = \begin{cases} 2x - 5 & \text{if } 2x - 5 \geq 0 \\ -(2x - 5) & \text{if } 2x - 5 < 0 \end{cases}$$

$$= \begin{cases} 2x - 5 & \text{if } x \geq 5/2 \\ -2x + 5 & \text{if } x < 5/2 \end{cases}$$

We need to consider two cases:

i) $x \geq 5/2$

In this case, the equation $|2x - 5| = x + 3$ becomes $2x - 5 = x + 3$. Solving this equation gives $x = 8$. Since this is larger than $5/2$ this is a valid solution of the equation.

ii) $x < 5/2$

The equation $|2x - 5| = x + 3$ becomes $-2x + 5 = x + 3$. Solving this equation gives $x = 2/3$. Since $2/3$ is less than $5/2$ it is a valid solution of the equation.

Therefore, the solutions of $|2x - 5| = x + 3$ are $x = 2/3$ and $x = 8$.

Exercise: Solve $|3x - 7| = 2x + 5$.

Answer: $x = 2/5$ and $x = 12$

Inequalities

- In a linear inequality all terms are either constants or a multiple of the variable. For example $3x < 9x + 4$ is a linear inequality.

- Solving an inequality means to find all numbers that make the inequality true.
We have the following rules for manipulating inequalities.

1. \( A \leq B \Leftrightarrow A + C \leq B + C \)
2. \( A \leq B \Leftrightarrow A - C \leq B - C \)
3. If \( C > 0 \), then \( A \leq B \Leftrightarrow CA \leq CB \)
4. If \( C < 0 \), then \( A \leq B \Leftrightarrow CA \geq CB \)
5. If \( A > 0 \) and \( B > 0 \), then \( A \leq B \Leftrightarrow 1/A \geq 1/B \)
6. If \( A \leq B \) and \( C \leq D \), then \( A + C \leq B + D \)

**Example:** Solve \( 3x < 9x + 4 \) and express the solution set in interval notation.

We manipulate the inequality to isolate \( x \).

\[
3x < 9x + 4
\]
\[
-6x < 4
\]
\[
x > -2/3
\]

Therefore, the solution set is \((-2/3, \infty)\).

Answer: \((-\infty, 3]\)

- **Nonlinear inequalities** involve squares and other powers of the variable.

- To solve nonlinear inequalities, we use factoring together with the following principle:

  ★ If a product or quotient has an even number of negative factors, then its value is positive.
  ★ If a product or quotient has an odd number of negative factors, then its value is negative.

**Example:** Solve \( x^2 - 2x - 8 > 0 \).

Solution:

\[
x^2 - 2x - 8 = (x - 4)(x + 2)
\]
Therefore, we need to solve the inequality

\[(x - 4)(x + 2) > 0.\]

\((x - 4)(x + 2) = 0\) has two solutions \(x = -2\) and \(x = 4\). These values divide the number line into the following intervals \((-\infty, -2)\), \((-2, 4)\), and \((4, \infty)\). Then we decide whether each factor in the inequality is positive or negative on each of these intervals. We can organize this information in the following table.

<table>
<thead>
<tr>
<th>Interval</th>
<th>(-\infty, -2)</th>
<th>-2</th>
<th>(-2, 4)</th>
<th>4</th>
<th>((4, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of (x + 2)</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>sign of (x - 4)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>sign of ((x - 4)(x + 2))</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

We need to find when \((x - 4)(x + 2) > 0\). We can read this information off from the above table. It occurs when

\[(-\infty, -2) \cup (4, \infty).\]

**Absolute Value Inequalities**

We can also have inequalities which have absolute values. When solving these inequalities it can be easier to work with equivalent forms of these inequalities which do not have absolute values. These equivalent forms are summarized in the chart below.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Equivalent Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>x</td>
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<td>(</td>
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<td>x</td>
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</tbody>
</table>

**Example:** Solve the following inequality.

\[|2x + 1| \leq 4.\]
The given inequity is equivalent to

\[-4 \leq 2x + 1 \leq 4\]
\[-5 \leq 2x \leq 3\]
\[-\frac{5}{2} \leq x \leq \frac{3}{2}\]

In interval notation the solution is

\((-\frac{5}{2}, \frac{3}{2})\).