Time Series

- Collection of R.V.’s \( \{X_t\} \) indexed by time \( t \)
  - Assume \( t \) discrete (\( t=0, \pm 1, \pm 2, \ldots \)), and \( X_t \) real (\( \mathbb{R} \))
  - Define mean function \( \mu_t=E[X_t] \), and variance function \( \sigma^2_t=\text{Var}[X_t] \)

- TS called *strictly stationary* if its joint distributions do not change with time-shifts
  \[
  F_{t_1,\ldots,t_n}(\cdot) = F_{t_1+s,\ldots,t_n+s}(\cdot), \quad \forall \ s \quad \text{and} \quad t_1,\ldots,t_n
  \]

- For strictly stationary TS, both \( \mu_t \) & \( \sigma^2_t \) are *constant*
Autocovariance Function

- Even strictly stationary TS are difficult to work with: need to model $F_{1,\ldots,n}(x_1,\ldots,x_n) \ \forall n \geq 1$
- Easier way to describe dependence structure is to look at covariances at different times
- Autocovariance function of a TS is defined as

$$\gamma(s,t) = \text{Cov}[X_s, X_t] = E[(X_s - \mu_s)(X_t - \mu_t)]$$

$$= E[X_sX_t] - \mu_s\mu_t$$

- Note: $\text{Cov}[X_t, X_t] = \text{Var}[X_t] = \sigma_t^2$
Autocorrelation Function

- Magnitude of covariance is difficult to interpret (depends on measurement scale), so we equivalently use correlation
- Autocorrelation function of a TS is defined as

\[
\rho(s, t) = \frac{\gamma(s, t)}{\sqrt{\gamma(s, s)\gamma(t, t)}} = \frac{\text{Cov}[X_s, X_t]}{\sqrt{\text{Var}[X_s]\text{Var}[X_t]}} = \frac{\gamma(s, t)}{\sqrt{\sigma^2_s\sigma^2_t}}
\]

- Always have: \(-1 \leq \rho(s, t) \leq +1\)
Autocorrelation Function

- Correlation only measures linear dependence, not general dependence, i.e.
  - If $X_s \perp X_t$ (indep.) $\rightarrow \gamma(s,t) = 0$, but
  - If $\gamma(s,t) = 0 \not\rightarrow X_s \perp X_t$
- E.g. If $Y \sim N(0,1)$ and $X_s = Y$, $X_t = Y^2$, then:
  - What is $\gamma(s,t)$?
  - $\gamma(s,t) = \text{Cov}(X_s, X_t) = E[X_s \cdot X_t] - \bar{X}_s \cdot \bar{X}_t$
    - $\gamma(s,t) = E[Y^3] - 0 = 0$
  - Is $X_s \perp X_t$? \(\neg\)
- Will use autocorrelation function to describe (linear) dependence structure of TS
Autocovariance for Strictly Stationary Time Series

- For *strictly stationary* TS, the autocovariance function $\gamma(s,t)$ depends *only on the time difference* between $s$ and $t$, call it $h = |t-s|
- I.e. $\gamma(s,t) = \gamma(h)$, for all $s, t$ such that $h = |t-s|
- Proof: without loss of generality, assume $t = s + h$

\[
\gamma(s,t) = \text{Cov} \left( X_s, X_t \right) = \left( \text{by strict stationarity, joint distr.} \right) \left( \text{of } (X_s, X_t) = \text{joint distr } (X_0, X_h) \right)
\]
\[
= \text{Cov} \left( X_0, X_h \right) = \gamma(0,h) = \gamma(h) \]

[drop first argument, since it will always be 0]
Why Autocovariance?

Want to model dependence structure of TS

- General case is intractable: need joint distr. at all possible times \( F_{t_1,\ldots,t_n}(x_{t_1},\ldots,x_{t_n}), \forall t_1,\ldots,t_n \)
  - Even with strict stationarity, would need to know joint distr. \( F_{1,\ldots,n}(x_1,\ldots,x_n) \) for all \( n \)

- Autocovariance function (or ACF) describes all possible (linear) dependencies of TS
  - In case of strict stationarity, its becomes simple & tractable 1D function \( \gamma(h) \)
Weak Stationarity

- TS with finite variance is called *weakly stationary* (or just *stationary*) if:
  1. $\mu_t$ is constant (indep. of $t$)
  2. $\gamma(s, t) = \gamma(h)$ depends on $s, t$ only through their difference $h = |s - t|

- Strict stationarity $\rightarrow$ (weak) stationarity, but
- (Weak) Stationarity $\not\rightarrow$ Strict stationarity in general, **unless** TS is Normally distributed
  - Dependence structure of Normal distr. is entirely defined by covariances
Weak Stationarity

- Autocovariance function for stationary TS is
  \[ \gamma(h) = \text{Cov}[X_t, X_{t+h}] = E[X_tX_{t+h}] - \mu^2, \quad \forall h \geq 0 \]
  - Note that \( \gamma(0) = \text{Var}[X_t] = \sigma^2 \)

- ACF for stationary TS is \( \rho(h) = \frac{\gamma(h)}{\gamma(0)}, \quad \forall h \geq 0 \)
  - Note that:
    - Always have \( \rho(0) = 1 \)
    - If we know \( \gamma(h) \), we can get \( \rho(h) \)
    - If we know \( \rho(h) \) and \( \gamma(0) \), we can get \( \gamma(h) \)
Example (White Noise)

- Series of uncorrelated R.V.'s \( \{W_t\} \) with zero mean (\( \mu_t = 0 \)) & constant variance (\( \sigma_t^2 = \sigma^2 \)) is called White Noise, denoted by \( W_t \sim \text{WN}(0, \sigma^2) \)

- Find ACF of \( \{W_t\} \)

\[
\gamma(0) = \sigma_t^2 = \sigma^2 \quad \& \quad \gamma(h) = 0 \quad , \quad \forall \ h \neq 0
\]

\[
\Rightarrow \rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 
1, & \text{if } h = 0 \\
0, & \text{if } h \neq 0
\end{cases}
\]
Example (Random Walk)

- Consider series with $X_0 = 0$ and
  
  $$X_t = X_{t-1} + W_t, \ \forall t \geq 1, \text{ for } W_t \sim WN(0, \sigma_w^2)$$

- Find mean & variance functions

  $$\mu_t = \mathbb{E}[X_t] = \mathbb{E}[X_{t-1} + W_t] = \mathbb{E}[X_{t-1}] + \mathbb{E}[W_t] \Rightarrow \mu_t = \mu_{t-1} = \mu_0 = \mathbb{E}[X_0] = 0$$

  $$\sigma_t^2 = \text{Var}[X_t] = \text{Var}[X_{t-1} + W_t] = \text{Var}[X_{t-1}] + \text{Var}[W_t] \Rightarrow \sigma_t^2 = \sigma_{t-1}^2 + \sigma_w^2$$

  $$\Rightarrow \text{Var}[X_t] = \sigma_t^2 = t \cdot \sigma_w^2$$
Example (cont’ed)

- Find the autocovariance function

Let’s assume \( s < t \): \( \gamma(s, t) = \text{Cov}(X_s, X_t) = \text{Cov}(X_s, X_s + W_{s+1} + \ldots + W_t) = \text{Cov}(X_s, X_s) + \text{Cov}(X_s, W_{s+1}) + \ldots + \text{Cov}(X_s, W_t) \)

\[ = \sigma_s^2 = s \cdot \sigma_w^2 \]

Similarly, if \( t \leq s \) \( \Rightarrow \gamma(s, t) = \text{Cov}(X_s, X_t) = \sigma_t^2 = t \cdot \sigma_w^2 \)

\( \Rightarrow \) In general, we have \( \gamma(s, t) = \min(s, t) \cdot \sigma_w^2 \), \( \forall s, t \)

& ACF is \( \rho(s, t) = \frac{\gamma(s, t)}{\sqrt{\gamma(s, s) \cdot \gamma(t, t)}} = \frac{\min (s, t) \sigma_w^2}{\sqrt{s \cdot \sigma_w^2 \cdot t \cdot \sigma_w^2}} = \frac{\min (s, t)}{\sqrt{s \cdot t}} \)
Example (Random Walk with Drift)

- Consider series with $X_0=0$ and
  
  $$X_t = \delta + X_{t-1} + W_t, \quad \forall t \geq 1, \text{ for } W_t \sim WN(0, \sigma_w^2)$$

- Find mean & variance functions,
  
  $$\mu_t = E[X_t] = E[S + X_{t-1} + W_t] = E[S] + E[X_{t-1}] + E[W_t] = \delta + \mu_{t-1}$$

  $$\Rightarrow \mu_t = \delta + \mu_{t-1} = \delta + (\delta + \mu_{t-2}) = \ldots = \delta + \delta + \ldots + \delta + \mu_0$$

  $$\Rightarrow \mu_t = \delta \cdot t$$

  $$\sigma_t^2 = t \cdot \sigma_w^2$$ (random walk with drift differs from simple random walk only by a constant $\delta$; variance is the same)
Example (cont’ed)

- Define **differenced** series \( Y_t = X_t - X_{t-1} = \nabla X_t, \ \forall t \geq 1 \) (\( \nabla \) is called the difference operator), and show that \( \{Y_t\} \) is stationary

\[
Y_t = X_t - X_{t-1} = (X_{t-1} + \delta + W_t) - X_{t-1} = \delta + W_t \implies
\]

\( \implies \{Y_t\} \) is a white noise series + a constant =

\( \implies \{Y_t\} \sim WN(\delta, \sigma_w^2) \implies \text{Stationary} \)

\[
\begin{align*}
\mu_t & = E(Y_t) = \delta \\
\sigma_t^2 & = \text{Var}(Y_t) = \sigma_w^2 \\
\rho(s,t) & = \begin{cases} 1, & s=t \\ 0, & s \neq t \end{cases}
\end{align*}
\]
Example (Moving Average)

- Consider series \( X_t = \frac{W_t + W_{t-1}}{2} \), for \( W_t \sim WN(0, \sigma_w^2) \)

- Show that \( \{X_t\} \) is stationary

\[
\mu_t = \mathbb{E}[X_t] = \mathbb{E}\left[ \frac{W_t + W_{t-1}}{2} \right] = \frac{1}{2} \left( \mathbb{E}[W_t] + \mathbb{E}[W_{t-1}] \right) = 0, \forall t
\]

\[
\sigma_t^2 = \text{Var}[X_t] = \text{Var}\left[ \frac{W_t + W_{t-1}}{2} \right] = \frac{1}{4} \left( \text{Var}[W_t] + \text{Var}[W_{t-1}] \right) = \frac{1}{4} \left( \sigma_w^2 + \sigma_w^2 \right) = \frac{1}{2} \sigma_w^2
\]
Let's assume \( t = s + 1 \Rightarrow y(s, t) = \text{Cov} (X_s, X_t) = \)

\[ = \text{Cov} (X_s, X_{s+1}) = \text{Cov} (\frac{W_s + W_{s-1}}{2}, \frac{W_{s+1} + W_s}{2}) = \]

\[ = \frac{1}{4} \cdot \left[ \text{Cov} (W_s, W_{s+1}) + \text{Cov} (W_s, W_s) + \text{Cov} (W_{s-1}, W_{s+1}) + \text{Cov} (W_{s-1}, W_s) \right] = \]

\[ = \frac{1}{4} \sigma_w^2 \]

Let's assume \( t = s + 2 \Rightarrow y(s, t) = \text{Cov} (X_s, X_t) = \)

\[ = \text{Cov} (X_s, X_{s+2}) = \text{Cov} (\frac{W_s + W_{s-1}}{2}, \frac{W_{s+2} + W_{s+1}}{2}) = \]

\[ = \frac{1}{4} \cdot \left[ \text{Cov} (W_s, W_{s+2}) + \text{Cov} (W_s, W_{s+1}) + \text{Cov} (W_{s-1}, W_{s+2}) + \text{Cov} (W_{s-1}, W_{s+1}) \right] = 0 \]

Similarly, \( y(s, t) = 0 \), \( \forall |s - t| \geq 3 \Rightarrow \)

\[ y(h) = \begin{cases} \sigma_w^2/2, & \text{if } h = 0 \\ \sigma_w^2/4, & \text{if } h = 1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow p(h) = \frac{y(h)}{\delta(0)} = \begin{cases} 1, & \text{if } h = 0 \\ 1/2, & \text{if } h = 1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow \{X_t\} \text{ is stationary} \]
Example (Autoregression)

- Consider series $X_t = .5X_{t-1} + W_t$, for $W_t \sim WN(0, \sigma_w^2)$
- Show that $\{X_t\}$ is stationary

By iterated substitution, we have:

$X_t = .5X_{t-1} + W_t = .5X_{t-2} + .5W_{t-1} + W_t$

$= .5^2X_{t-2} + .5^2W_{t-2} + .5W_{t-1} + W_t$

$\Rightarrow X_t = \sum_{i=0}^{\infty} (.5)^iW_{t-i}$

$\mu_t = E[X_t] = E\left[ \sum_{i=0}^{\infty} (.5)^iW_{t-i} \right] = \sum_{i=0}^{\infty} (.5)^iE(W_{t-i}) = 0$

$\sigma_t^2 = Var[X_t] = Var\left[ \sum_{i=0}^{\infty} (.5)^iW_{t-i} \right] = \sum_{i=0}^{\infty} (.5)^{2i}Var[W_{t-i}] = \sigma_w^2 \sum_{i=0}^{\infty} (.5)^{2i}$

$\Rightarrow \sigma_t^2 = \sigma_w^2 \left( \frac{\sum_{i=0}^{\infty} (.5)^{2i}}{1 - .25} \right) = \sigma_w^2 \cdot 4 = 4\sigma_w^2$, constant
Let's assume \( t = s+1 \) \( \Rightarrow \gamma(s, t) = \text{Cov}(X_s, X_{s+1}) = \sigma_s^2 \)
\[
= \text{Cov}(X_s, 0.5 \cdot X_s + W_{s+1}) = 0.5 \cdot \text{Cov}(X_s, X_s) + \text{Cov}(X_s, W_{s+1}) = 0.5 \sigma_s^2 + 0.5 \sigma_w^2
\]
\[
= 0.5 \cdot (1.33) \cdot \sigma_w^2
\]

Let's assume \( t = s+2 \) \( \Rightarrow \gamma(s, t) = \text{Cov}(X_s, X_{s+2}) = \)
\[
= \text{Cov}(X_s, 0.5 \cdot X_{s+1} + W_{s+2}) = 0.5 \cdot \text{Cov}(X_s, X_{s+1}) + \text{Cov}(X_s, W_{s+2}) =
\]
\[
= (0.5)^2 \sigma_s^2
\]

In general, \( \gamma(s, s+h) = (0.5)^h \sigma_s^2 = (0.5)^h \cdot (1.333...) \sigma_w^2, \forall h \geq 0 \)
\[
\Rightarrow \rho(h) = \frac{\gamma(h)}{\gamma(0)} = (0.5)^h, \forall h \geq 0
\]
Getting Started with R

- Download & install software (for free) from: [http://cran.r-project.org/](http://cran.r-project.org/)
  - Short intro to R given in: [http://cran.r-project.org/doc/contrib/Paradis-rdebuts_en.pdf](http://cran.r-project.org/doc/contrib/Paradis-rdebuts_en.pdf)

- Webpage [http://www.stat.pitt.edu/stoffer/tsa3/](http://www.stat.pitt.edu/stoffer/tsa3/) has data & code for all examples in textbook
  - Load package “astsa” onto R to get data