CSC B36 Additional Notes

simple and complete induction

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⋆ Introduction

Given a predicate $P(n)$, where $n \in \mathbb{N}$, if we want to prove that $P(n)$ is true for all $n \in \mathbb{N}$, then one option is to use induction. It is not the only option and it is not always an appropriate option.

Here are some examples of a predicate $P(n)$.

(E1) The sum of all natural numbers between 0 and n equals $\frac{n(n+1)}{2}$. I.e., $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

(E2) The set of all n-bit strings has exactly 2^n elements.

Instead of wanting to prove $P(n)$ for **all** natural numbers n, sometimes we get variations. Here are some examples.

(V1) Prove $P(n)$ for any $n \geq b$, where b is some constant.

(V2) Prove $P(n)$ for all odd numbers n.

(V3) Prove $P(n)$ for every n which is a power of 2.

Usually such variations can be restated in the form "prove $Q(n)$ for all $n \in \mathbb{N}$" for some related predicate $Q(n)$.

- (V1) is the same as "prove $Q(n)$ for all $n \in \mathbb{N}$" if we define $Q(n)$ to be $P(n+b)$.
- (V2) is the same as "prove $Q(n)$ for all $n \in \mathbb{N}$" if we define $Q(n)$ to be $P(2n+1)$.
- (V3) is the same as "prove $Q(n)$ for all $n \in \mathbb{N}$" if we define $Q(n)$ to be $P(2^n)$.

⋆ Simple Induction

(Simple induction is sometimes also called weak induction.)

In a simple induction proof, we prove two parts.

Part 1 — Basis: $P(0)$.

Part 2 — Induction Step: $\forall i \geq 0, \Big(P(i) \rightarrow P(i+1)\Big)$.

Note 1: Informally, part 2 says,

$P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, $P(2)$ implies $P(3)$, and so on.

Formally, there are other ways to say the same thing. For example,

$$\forall i > 0, \Big(P(i-1) \rightarrow P(i)\Big).$$

Each formal way of saying part 2 can lead to a slightly different proof (if we use a direct proof), which explains why there are many variations of induction proofs.

By induction, we can conclude from proving these two parts that $P(n)$ is true for every $n \in \mathbb{N}$. Let us examine why this is a valid conclusion.

From part 1, we can conclude $P(0)$, which should be obvious.

Part 2 says $\Big(P(i) \rightarrow P(i+1)\Big)$ is true for all $i \geq 0$. So in particular it must be true for $i = 0$. Thus we get $\Big(P(0) \rightarrow P(1)\Big)$. Now if we combine this with $P(0)$, which we just concluded above, we can conclude $P(1)$.

From part 2 again, this time with $i = 1$, we get $\Big(P(1) \rightarrow P(2)\Big)$. Now if we combine this with $P(1)$, which is the conclusion from the previous step, we can conclude $P(2)$.

Applying the same argument over and over, we can conclude $P(3)$, $P(4)$, $P(5)$, and so on. Thus $P(n)$ is true for all $n \in \mathbb{N}$.

Aside: In a sense, using induction is a formal way to convey the idea of "and so on".

⋆ Complete Induction

(Complete induction is sometimes also called strong induction.)

In a complete induction proof, we also prove two parts.

Part 1 — Basis: $P(0)$.

Part 2 — Induction Step: $\forall i \geq 1, \Big((\forall j, [0 \leq j < i \rightarrow P(j)]) \rightarrow P(i)\Big)$.

Note 2: Using "$\cdots$" as a way of saying "and so on", part 2 can be informally written as

$$\forall i \geq 1, \Big([P(0) \wedge \cdots \wedge P(i-1)] \rightarrow P(i)\Big).$$

Note 3: Sometimes there can be more than one case in the basis. If there are b cases in the basis, then here are our two parts.

Part 1 — Base Cases: $P(0), \cdots, P(b-1)$.

Part 2 — Induction Step: $\forall i \geq b, \Big([P(0) \wedge \cdots \wedge P(i-1)] \rightarrow P(i)\Big)$.

By induction, we can conclude from proving these two parts that $P(n)$ is true for every $n \in \mathbb{N}$. Let us examine why this is a valid conclusion.

By proving the bases cases, we can conclude all of $P(0), \cdots, P(b-1)$, which should be obvious.

Part 2 says $[P(0) \wedge \cdots \wedge P(i-1)] \rightarrow P(i)$ is true for all $i \geq b$. So in particular it must be true for $i = b$. Thus we get $[P(0) \wedge \cdots \wedge P(b-1)] \rightarrow P(b)$. Now if we combine this with $P(0) \wedge \cdots \wedge P(b-1)$, which we just concluded above, we can conclude $P(b)$.

From part 2 again, this time with $i = b+1$, we get $[P(0) \wedge \cdots \wedge P(b)] \rightarrow P(b+1)$. The antecedent (part to the left of $\rightarrow$) is exactly what we get from the base cases plus what we get from the previous paragraph. Therefore we can conclude the consequent, namely $P(b+1)$.

Applying the same argument over and over, we can conclude $P(b+2)$, then $P(b+3)$, $P(b+4)$, and so on. Thus $P(n)$ is true for all $n \in \mathbb{N}$.

⋆ When to use what?

Here are some commonly asked questions.

- When should we use simple induction?
- When should we use complete induction?
- When do we need more than one base case?
- When should we use "$i \geq 0$", "$i > 0$", $P(i-1)$, $P(i+1)$, etc?

Short answer:

We should use whatever that allows us to conclude the desired result.

Long answer:

Whatever we do, we must be able to apply the same logical arguments like those above to draw our conclusion, that is, $P(n)$ holds for all $n \in \mathbb{N}$. Novice proof writers may frequently make bad choices, but should eventually discover them when they try to argue that $P(n)$ is true for every $n \in \mathbb{N}$. With experience, we can all become so proficient at induction that we rarely make a mistake. *So practice lots!!!*

⋆ Example

Let us use an example to illustrate some of what we discussed above. Consider the following inductively defined function which maps from positive integers to positive integers.

$$f(n) = \begin{cases} n & \text{if } 1 \leq n < 3; \\ f(\lfloor \frac{n}{3} \rfloor) + f(\lceil \frac{2n}{3} \rceil) + 1 & \text{if } n \geq 3. \end{cases}$$

We will use induction to prove that $f(n) \leq 2n - 1$ for all integers $n \geq 1$.

Suppose we start by using simple induction. At the induction step, we would assume $f(n) \leq 2n - 1$ (for some $n \geq 1$) as our induction hypothesis (IH) and try to prove that $f(n+1) \leq 2(n+1) - 1$. From the function definition, we have, for $n + 1 \geq 3$,

$$f(n+1) = f(\left\lfloor \frac{n+1}{3} \right\rfloor) + f(\left\lceil \frac{2(n+1)}{3} \right\rceil) + 1.$$

However, $\lfloor \frac{n+1}{3} \rfloor$ does not equal n. So our IH, which tells us something about $f(n)$, but nothing about $f(\lfloor \frac{n+1}{3} \rfloor)$, is of little use to us.[1] At this point, we should realize that simple induction will not work and we should be using complete induction.

Suppose we now start using complete induction. For the basis, we prove that $f(1) \leq 2(1) - 1$. For the induction step, we consider an arbitrary integer $i \geq 2$, and we assume $f(j) \leq 2j - 1$ whenever $1 \leq j < i$ as our IH, then try to prove that $f(i) \leq 2i - 1$. We run into trouble when we do this for $i = 2$. When $i = 2$, our IH becomes just $f(1) \leq 2(1) - 1$, and we are trying to prove $f(2) \leq 2(2) - 1$. The former inequality is not useful for proving the latter. I.e., knowing $f(1) \leq 1$ does not help us prove $f(2) \leq 3$. At this point, we should realize that we need more than one case in our basis. We need the $n = 2$ case also.

[1] Similar statements can be made about $\left\lceil \frac{2(n+1)}{3} \right\rceil$ and $f(\left\lceil \frac{2(n+1)}{3} \right\rceil)$ when $n + 1 \geq 4$.

Here then is our complete proof.

Proof that $f(n) \le 2n - 1$ for all integers $n \ge 1$:

BASE CASES:

Let $n = 1$.

$$
\begin{aligned}
\text{Then } f(n) &= 1 && \text{[definition of } f\text{]} \\
&\le 2(1) - 1 && \text{[arithmetic]} \\
&= 2n - 1
\end{aligned}
$$

as wanted.

Let $n = 2$.

$$
\begin{aligned}
\text{Then } f(n) &= 2 && \text{[definition of } f\text{]} \\
&\le 2(2) - 1 && \text{[arithmetic]} \\
&= 2n - 1
\end{aligned}
$$

as wanted.

INDUCTION STEP:

Let $n \ge 3$.

Suppose $f(j) \le 2j - 1$ whenever $1 \le j < n$. [IH]

$$
\begin{aligned}
\text{Then } f(n) &= f(\lfloor \tfrac{n}{3} \rfloor) + f(\lceil \tfrac{2n}{3} \rceil) + 1 && \text{[definition of } f\text{; } n \ge 3\text{]} \\
&\le (2\lfloor \tfrac{n}{3} \rfloor - 1) + (2\lceil \tfrac{2n}{3} \rceil - 1) + 1 && \text{[since } n \ge 3 \text{, then } 1 \le \lfloor \tfrac{n}{3} \rfloor < n \text{ and } 1 \le \lceil \tfrac{2n}{3} \rceil < n\text{;} \\
& && \quad \text{so IH applies (twice)]} \\
&= 2(\lfloor \tfrac{n}{3} \rfloor + \lceil \tfrac{2n}{3} \rceil) - 1 && \text{[simple algebra]} \\
&= 2n - 1 && [\lfloor \tfrac{n}{3} \rfloor + \lceil \tfrac{2n}{3} \rceil = n]
\end{aligned}
$$

as wanted.

Therefore, by induction, $f(n) \le 2n - 1$ for all integers $n \ge 1$. $\square$