CSC B36 Additional Notes

how NOT to write proofs

⋆ Introduction

Certain mistakes are common in proofs, especially ones written by those who are just learning to do proofs — and that usually includes a significant percentage of CSCB36 students. I call these mistakes "classic" errors, and I have seen them many times. In these notes, we illustrate these mistakes and suggest ways to avoid them.

⋆ Example of a badly written proof

Let function f be defined by $f(x) = x^2 - 2$.

Here is a "proof" for the statement:

For any integer $n \geq 2$, $f(n) = n$.

Note: For the purpose of talking about this proof, the lines are numbered.

PROOF:

1. Let $P(n) = f(n) = n$ for all $n \in \mathbb{N}$.
2. Basis: $n = 2$.
3. $f(2) = 2^2 - 2$
4. $2 = 2$
5. Therefore $f(2)$ holds.
6. Induction Step: Let $n > 2$.
7. Assume $P(n)$ holds. I.e., $n^2 - 2 = n$.
8. WTP: $P(n+1)$ holds.
9. $P(n+1) = (n+1)^2 - 2 = n + 1$
10. $n^2 + 2n + 1 - 2 = n + 1$
11. $2n + 1 = 1$
12. $n = -n$
13. $n^2 = (-n)^2$
14. $n^2 = n^2$
15. Therefore true. $\square$

⋆ Faults in above proof and how to avoid them

(a) [line 1] Equating predicates to function or numeric values.
A predicate is a statement which can be true or false. In our case, $P(n)$ is a statement about n, which we take to be a natural number. When we write "$P(n) = f(n)$", we are saying "the statement $P(n)$ equals the value of $f(n)$", which is not what we mean to say.

Advice: For this course, use the following convention for writing predicates. I.e., use a colon (:). Here is how our predicate should be written.

$P(n)$: $f(n) = n$

(b) [line 1 *again!*] Quantifying a predicate variable.
When we add "for all $n \in \mathbb{N}$", we introduce a new variable n which is different from the n in $P(n)$.

It should be clear that $\forall x\, Q(x)$ and $\forall y\, Q(y)$ have the same meaning, namely "every element (in whatever domain we are talking about) has property Q". Therefore by adding "for all $n \in \mathbb{N}$" to the definition of $P(n)$, we have created a statement that is independent of the n in $P(n)$, which is not what we mean to do.

Advice: Never quantify a variable that appears before the : in the definition of a predicate. A good way to write line 1 is to split it into two lines as follows.

For $n \in \mathbb{N}$, we define a predicate $P(n)$ as follows.

$P(n)$: $f(n) = n$.

(c) [lines 2-4] The infamous STACK!
Lines 2-4 consist of a sequence of equations, one following another with no explanation in between. What is meant by these lines is not clear. Do we mean line 2 implies line 3 and line 3 implies line 4? Or do we mean line 2 iff line 3 and line 3 iff line 4?

Advice: Never write a stack of equalities. This also applies to stacks of inequalities. When trying to prove an equality — $f(n) = n$ in our case, work from the left hand side (LHS) toward the right hand side (RHS), or vice versa. Here is a good way to write lines 2-4.

$$\begin{aligned} \text{For } n = 2,\ f(n) &= f(2) \\ &= 2^2 - 2 && \text{[definition of } f\text{]} \\ &= 2 = n. && \text{as wanted.} \end{aligned}$$

(d) [lines 2-4 *again!*] No justification.
Remember that a good proof is a convincing argument. So whenever some step may be unclear, you should justify it. Notice how we provided the justification on the second last line in (c) above.

Advice: The amount of justifcation depends on the intended reader and purpose of your proof. For this course, the reader is your marker as well as your classmates, and the main purpose is to convince the marker that you understand what you are doing.

(e) [line 5] Mixing up predicates and function/numeric values (*again!*).
"$f(2)$ holds" means "the value of the function f evaluated at 2 is true", which is not what we mean to say. What we mean to say is that $f(2) = 2$, or more to the point, $P(2)$ is true. Here is a good way to write line 5.

Therefore $P(2)$ holds.

(f) [line 6] Off-by-one error (in induction step).
When n is an integer, saying "$n > 2$" is the same as saying "$n \geq 3''$. Try applying the argument we used in class (or the part starting at the last line of page 17 in the course notes) and see how using $n > 2$ would not allow us to conclude $P(3)$. Here is a correct way to start the induction step.

Let $n \geq 2$.

(g) [line 7] Identify the induction hypothesis (IH).
Line 7 is correct, but stylistically poor.

Advice: In an induction proof, we should always identify the IH. Here is a good way to write line 7.

Assume $P(n)$ holds. I.e., $n^2 - 2 = n$. [IH]

(h) [line 9] Mixing up predicates and function/numeric values (*once more!*).
See (a) and (e) above. What we really want to do in line 9 is to say what $P(n+1)$ means. Here is a good way to do this.

$P(n+1)$ means $(n+1)^2 - 2 = (n+1)$.

(i) [lines 9-14] STACK (*again!*).
See (c) above. The added danger of having these dreaded stacks is that they could lead you to faulty conclusions. An observant reader might have seen that what we are trying to prove, namely "$f(n) = n$ for all $n \geq 2$", is actually not true! Indeed within these lines is where our induction proof falls apart. Do you see how? (If not, then read the next point to find out.) There is no good way to write lines 9-14 because what we are trying to prove is false.

(j) [lines 9-14] Starting with the desired conclusion.
In any argument, if we start with the desired conclusion and conclude the same thing, then we really have not proved anything. Say we want to prove that "Nick is smart", and we begin with "Nick is smart", then present some argument resulting in $5 = 5$. Then all we have proved is

> "Nick is smart" imples $5 = 5$, or "Nick is smart" $\rightarrow$ TRUE.

We know from CSCA65/7 that any implication is always true whenever the consequent is true. I.e., $P \rightarrow Q$ is always true whenever Q is true. Proving such a statement does not allow us to draw any conclusions about the antecedent — "Nick is smart" in our example, or P in $(P \rightarrow Q)$.

Advice: Never start by assuming what we want to prove.

(k) [line 15] Meaningless statement.
"Therefore true" is practically meaningless. "Therefore" implies some sort of conclusion is being drawn as a result of some argument. However, "true" is always true! We do not need an argument to conclude "true". What we really mean to say in line 15 is that we have the desired conclusion. Here is a good way to write line 15.

> Therefore $P(n+1)$ holds as wanted.

(l) [line 9-14] Identify where the IH is used.
Somewhere in the induction step, the induction hypothesis should be used to justify something. Wherever the IH is used, it should be stated. In our case, do you see how it is used to go from line 10 to line 11? So we should add something like [IH] to the end of line 11.

Advice: Always indicate where the IH is used.