CSCB63 – Design and Analysis of Data Structures

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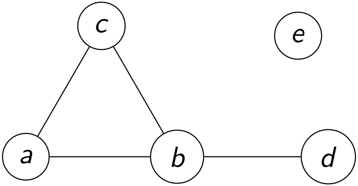
[1] based on notes by Anna Bretscher and Albert Lai

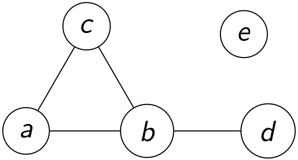
introduction

- cities and highways between them
- computers and network cables between them
- people and relationships
- in a board game: a state and legal moves to other states



a graph

undirected graph



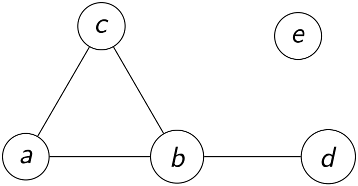
An undirected graph is a pair (V, E) of:

- V: a set of vertices (above: $\{a, b, c, d, e\}$)
- E: a set of edges, where an edge is a pair of vertices
  (above: $\{\{a, c\}, \{a, b\}, \{b, c\}, \{b, d\}\}$)
  (usually, no edge from a vertex to itself)
  undirected graph — no direction specified, bidirectional

graph terminology: incident, endpoint, degree

Edge incident on vertex, vertex is an endpoint of edge: e.g.,
$\{a, c\}$ is incident on a; a is an endpoint of $\{a, c\}$
$\{a, c\}$ is incident on c; c is an endpoint of $\{a, c\}$
$\{a, c\}$ is not incident on b; b is not an endpoint of $\{a, c\}$

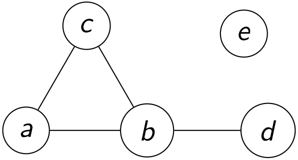
Degree of vertex: how many edges are incident on it.



vertex	a	b	c	d	e
degree	2	3	2	1	0

graph terminology: adjacent

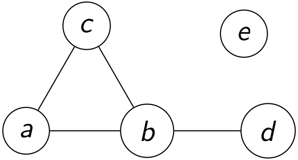
Two vertices are adjacent iff there is an edge between them.



	a	b	c	d	e
a		$\surd$	$\surd$		
b	$\surd$		$\surd$	$\surd$	
c	$\surd$	$\surd$			
d		$\surd$			
e					

	is adjacent to
a	b, c
b	a, c, d
c	a, b
d	b
e	

storing a graph: adjacency matrix

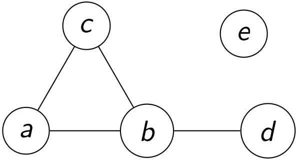


	a	b	c	d	e
a		$\surd$	$\surd$		
b	$\surd$		$\surd$	$\surd$	
c	$\surd$	$\surd$			
d		$\surd$			
e					

Adjacency matrix = store this in a 2D array
Let $n = |V|$ and $m = |E|$. Then in terms of n and m:

- space: $\Theta(n^2)$
- "who are adjacent to v?" time: $\Theta(n)$
- "are v and w adjacent?" time: $\Theta(1)$

storing a graph: adjacency lists



	is adjacent to
a	b, c
b	a, c, d
c	a, b
d	b
e	

Adjacency lists = store this in a 1D array or dictionaryUse a list or a set for each entry on the right.

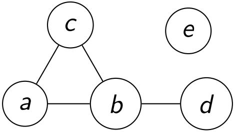
Let $n = |V|$ and $m = |E|$. Then in terms of n, m, and $degree(v)$:

- space: $\Theta(n + m)$
- "who are adjacent to v?" time: $\Theta(deg(v))$
- "are v and w adjacent?" time: $\Theta(deg(v))$
- optimal for graph searches

graph terminology: (simple) path, reachable

A (simple) path is a non-empty sequence of vertices in which

- consecutive vertices are adjacent
- vertices are distinct



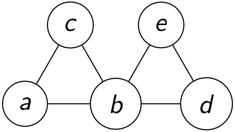
$\langle d \rangle$ is a path, length 0.
$\langle d, b, c \rangle$ is a path, length 2.
$\langle d, b, c, b \rangle$ is a not a (simple) path.
$\langle d, a, b \rangle$ is not a path.

v is reachable from u iff there is a path from u to v.

graph terminology: (simple) cycle

A (simple) cycle is a non-empty sequence of vertices in which

- consecutive vertices are adjacent
- first vertex = last vertex
- vertices are distinct except first=last; edges used are distinct
- $\langle v \rangle$ is not a cycle



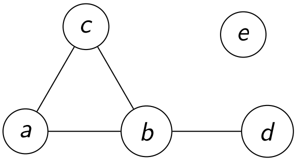
$\langle b, c, a, b \rangle$ is a simple cycle, length 3. ($\langle b, c, a \rangle$ in some books.)
$\langle b, c, a, b, d, e, b \rangle$ is not a (simple) cycle: uses b in the middle
$\langle b, d, b \rangle$ is not a cycle: it uses $\{b, d\}$ twice

graph terminology: (dis)connected, component

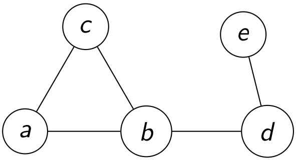
A graph is connected iff between every two distinct vertices there is a path.

A graph is disconnected iff it is not connected.

Disconnected:



Connected:



Component: maximal subset of vertices reachable from each other. (Sometimes also include their edges.)

E.g., the graph on the left has two components: $\{a, b, c, d\}$, $\{e\}$

tree: definition and results

A tree is a graph that is connected and has no cycles.

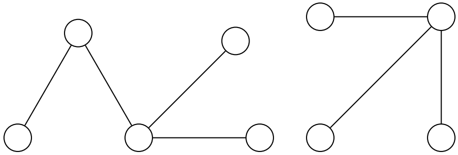
Equivalently:

- between every two vertices, a unique simple path
- connected, but disconnected if any edge removed
- connected, and $|E| = |V| - 1$
- no cycles, but has a cycle if any edge added
- no cycles, and $|E| = |V| - 1$

Exercise: convince yourself that these are equivalent!

graph terminology: forest

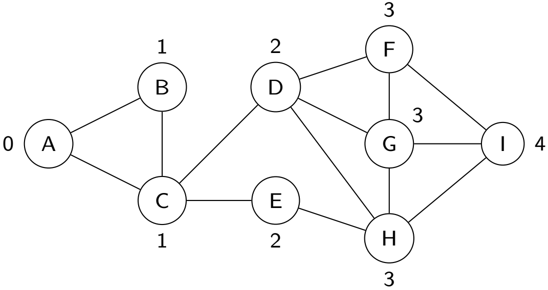
A forest is a collection of trees (may be disconnected). A forest has no cycles.



Breadth-First Search

Specify or arbitrarily pick a start vertex.

0. visit the start vertex
1. visit vertices 1 edge away from the above
2. visit unvisited vertices 1 edge away from the above
3. visit unvisited vertices 1 edge away from the above
4. ...

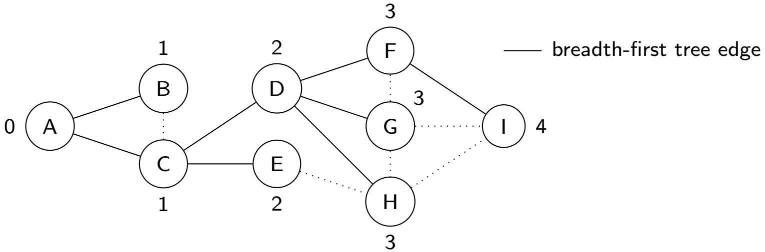


Breadth-First Search

```
0. start := pick a vertex
1. queue := new Queue()
2. queue.enqueue(start)
3. mark start as seen
   // distance(start) = 0

4. while not queue.is_empty():
5.   u := queue.dequeue()
6.   for each v in u's adjacency list:
7.     if v is not seen:
8.       queue.enqueue(v)
9.       mark v as seen
         // edge {u,v} is a "breadth-first tree edge"
         // u is v's "predecessor"
         // distance(v) = distance(u) + 1
```

Breadth-First Search



BFS finds:

- whether a vertex is reachable from *start*
- if yes, a shortest path and distance
- a tree consisting of the reachable vertices from *start*
- the component containing *start*

Shortest paths and the tree are non-unique: sensitive to orders of vertices in adjacency lists.

Breadth-First Search

BFS running time:

1. we enqueue and dequeue each vertex once:
   - only enqueue unseen vertices; mark as seen right after enqueue
2. we consider each edge twice:
   - each edge incident on 2 vertices
3. we find each vertex's adjacency list once:
   - right after dequeue (line 6)
4. check v's "seen" status $deg(v)$ times:
   - once from every node adjacent to it (line 7)

Assume $\Theta(1)$ time for

- marking/checking a vertex's "seen" status
- finding a vertex's adjacency list

Then BFS total time: $\Theta(|V| + |E|)$.

Exercise: What if the assumption doesn't hold?