Height of the WBT

Claim:

$$height(T) \leq \log(size(T) + 1)/\log(4/3)$$

for all weight-balanced trees T.

Proof. By induction on size of the tree.

Base. $height(nil) = 0 = \log(size(nil) + 1)/\log(4/3)$

IH. Suppose $\forall k \in \mathbb{N}, 0 \leq k < n, height(T') \leq \log(k + 1)/\log(4/3)$ where $size(T') = k$.

Show. $height(T) \leq \log(n + 1)/\log(4/3)$ where $size(T) = n$.

Height of the WBT

Show. $height(T) \leq \log(n+1)/\log(4/3)$ where $size(T) = n$.

WLOG assume that $height(T.left) \leq height(T.right)$, thus $height(T) = height(T.right) + 1$.

Let $(l, r) = (size(T.left), size(T.right))$.
Then

$$\begin{aligned}
& size(T) + 1 \\
=& n + 1 \\
=& l + r + 1 + 1 \\
\geq& (r+1)/3 + r + 1 && [\text{since } l + 1 \geq (r+1)/3] \\
=& (r+1) * 4/3 \\
\therefore& r + 1 \leq (n+1)/(4/3)
\end{aligned}$$

Height of the WBT

Show. $height(T) \leq \log(n+1)/\log(4/3)$ where $size(T) = n$.

$$
\begin{aligned}
& height(T) \\
= & height(T.right) + 1 && \text{IH} \\
\leq & \log(r+1)/\log(4/3) + 1 && \text{result above} \\
\leq & \log((n+1)/(4/3))/\log(4/3) + 1 \\
= & (\log(n+1) - \log(4/3))/\log(4/3) + 1 \\
= & \log(n+1)/\log(4/3) - 1 + 1 \\
= & \log(n+1)/\log(4/3)
\end{aligned}
$$