EAST Lectures

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Abstract

These are the lectures notes the EAST workshop in 2022. They are rough and should be used with caution.

Contents

Chapter 1

Spaces of diffeomorphisms and embeddings

The goal of this lecture series is to introduce you to a family of spaces of geometric interest. Simultaneously, much is known and much is unknown about their homotopy types; there are many questions to answer but a solid foundation of knowledge to build on. Moreover, we will see that in their study many themes and tools of modern homotopy theory interact.

In this first lecture, we will introduce the objects of interest, explain the relationship between them, and pose the questions that we will attempt to answer. For background and facts about differential topology, see Wall's recent book [Wal16].

1.1 Diffeomorphism groups

We start with discussing the diffeomorphism groups of smooth manifolds.

1.1.1 Diffeomorphisms

The following notion is likely familiar:

Definition 1.1.1. A *diffeomorphism* is a smooth map $\psi\colon M \to N$ between two closed smooth manifolds which has a smooth inverse.

Of course, the identity map of any manifold is diffeomorphism and the composition of two diffeomorphisms is once more diffeomorphism, so the set of diffeomorphisms $\varphi\colon M \to M$ forms a group. We can topologise this in the smooth topology (using semi-norms that control derivatives on compact subsets) and with respect to this topology it is a topological group as long as M is compact.

Definition 1.1.2. The *diffeomorphism group* $\mathrm{Diff}(M)$ of a closed smooth manifold M, is the topological group of diffeomorphisms $\varphi\colon M \to M$ in the smooth topology.

Remark 1.1.3. For those worried about point-set topology issues, it may be comforting to know that this is homotopy equivalent to a CW-complex.

A useful generalisation is to allow manifolds with boundary. It is then the convention to only consider diffeomorphisms that are the identity near the boundary if M is compact, $\mathrm{Diff}_\partial(M)$ is the topological group of diffeomorphisms $\varphi\colon M \to M$ a neighbourhood ∂M of pointwise, in the smooth topology.

Of course, we can consider any closed manifold as a manifold with boundary (that just happens to be empty) and so we will generally use the notation $\mathrm{Diff}_\partial(M)$ even for closed M.

Example 1.1.4 (Diffeomorphisms of a 1-disc). Let us start with understanding $\mathrm{Diff}_\partial(D^1)$. A diffeomorphism $\psi\colon M \to N$ induces a homeomorphism $\mathrm{Diff}_\partial(M) \to \mathrm{Diff}_\partial(N)$ by the formula $\varphi \mapsto \psi \circ \varphi \circ \psi^{-1}$ so we may replace D^1 by $[0,1]$, with the purpose of writing easier formulas.

To understand $\mathrm{Diff}_\partial([0,1])$, it is helpful to recall a consequence of the inverse function theorem: a smooth map between compact manifolds is a diffeomorphism if and only if it is (i) injective, and (ii) it derivatives is injective at each point. Thus here we can identify

$\mathrm{Diff}_\partial([0,1])$ as the space of smooth functions $f\colon [0,1] \to [0,1]$ that (i) are the identity near 0 and 1, and (ii) have positive derivative everywhere. But this space is contractible by linear interpolation to the identity:

$$f_t := (1-t)\cdot f + t\cdot \mathrm{id} \qquad \text{for } t \in [0,1].$$

The upshot is that $\mathrm{Diff}_\partial(D^1) \simeq *$.

Example 1.1.5 (Diffeomorphisms of a 2-torus). For a non-trivial example, we consider the 2-dimensional torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. This has many interesting diffeomorphisms. Firstly, as $\mathrm{GL}_2(\mathbb{Z})$ are exactly those linear automorphisms of $\mathbb{R}^2$ preserving the lattice $\mathbb{Z}^2$, each such matrix A descends to diffeomorphism $\varphi_A\colon \mathbb{T}^2 \to \mathbb{T}^2$. Moreover, composition of linear maps goes to composition of diffeomorphisms so we get a homomorphism

$$\mathrm{GL}_2(\mathbb{Z}) \longrightarrow \mathrm{Diff}(\mathbb{T}^2).$$

Secondly, $\mathbb{T}^2$ is a Lie group so acts on itself by multiplication , giving rise to a homomorphism

$$\mathbb{T}^2 \longrightarrow \mathrm{Diff}(\mathbb{T}^2)$$

of topological groups. This can be combined with the previous construction to a homomorphism

$$\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \longrightarrow \mathrm{Diff}(\mathbb{T}^2)$$

of the topological groups, out of the semi-direct product for the evident action; it is a consequence of the theory of 2-dimensional manifolds that this is an equivalence [Gra73], (see [Hat11, Appendix B] for an exposition).[1]

Exercise 1. Generalise the previous example to give a homomorphism

$$\mathrm{GL}_d(\mathbb{Z}) \ltimes \mathbb{T}^d \longrightarrow \mathrm{Diff}(\mathbb{T}^d).$$

(This is not an equivalence unless $d \leqslant 3$, but this should not be clear. For $d \geqslant 5$ its group of path components is not even finitely-generated, by a contribution of a $(\mathbb{Z}^2)^{\oplus\infty}$ from an A^{nil}-term.)

1.1.2 Manifold bundles

To any topological group G, we can assign a classifying space BG, well-defined up to homotopy. This classifies principal G-bundles, in the sense that it carries a universal such bundle $EG \to BG$ so that pullback gives a natural bijection

$$[X, BG] := \frac{\{\text{maps } X \to BG\}}{\text{homotopy}} \longrightarrow \frac{\{\text{principal } G\text{-bundles } E \to X\}}{\text{isomorphism}}$$
$$[f] \longmapsto [f^*EG].$$

Strictly speaking, here we need to restrict to paracompact X, or to numerable principal G-bundles; we will forego these details here and later.

This applies to $G = \mathrm{Diff}_\partial(M)$, and says that the classifying space $B\mathrm{Diff}_\partial(M)$ classifies principal $\mathrm{Diff}_\partial(M)$-bundles. We can interpret this more geometrically: for us, a *manifold bundle with fibre M and trivialised boundary* is a locally trivial fibre bundle whose fibres are manifolds diffeomorphic to M and whose transition maps lie in $\mathrm{Diff}_\partial(M)$. The reason for the terminology "trivialised boundary" is that taking fibrewise boundary yields a subbundle $\partial E' \to X$ which is in fact canonically isomorphic to a trivial bundle $\pi_2\colon \partial M \times X \to X$. We conclude that:

[1] Throughout these notes, equivalence of spaces means weak homotopy equivalence.

Theorem 1.1.6. *The following map is a natural bijection:*

$$\frac{\{\textit{principal } G\textit{-bundles } E \to X\}}{\textit{isomorphism}} \longrightarrow \frac{\left\{\begin{matrix}\textit{smooth manifold bundles } E' \to X \textit{ with fibre} \\ M \textit{ and trivialised boundary bundle}\end{matrix}\right\}}{\textit{isomorphism}}$$
$$[E \to X] \longmapsto [E \times_{\mathrm{Diff}_\partial(M)} M \to X].$$

In other words, the homotopy type $B\mathrm{Diff}_\partial(M)$ is that of the moduli space of manifolds diffeomorphic to M with boundary identified with ∂M; such a moduli space should exactly classify the type of bundles given above. This can be more concrete: in Chapter 3 we will construct a space of submanifolds $\mathbb{R}^\infty$ diffeomorphic to M whose resulting homotopy type is that of $B\mathrm{Diff}_\partial(M)$. That is, one can really model $B\mathrm{Diff}_\partial(M)$ as a space of manifolds.

1.1.3 Applications of the classifying property

As a first application of Theorem 1.1.6, we get a geometric description of the homotopy groups of $B\mathrm{Diff}_\partial(M)$:

Corollary 1.1.7. *Pullback of the universal bundle gives a bijection*

$$\pi_i\, B\mathrm{Diff}_\partial(M) \xrightarrow{\cong} \frac{\left\{\begin{matrix}\textit{smooth manifold bundles } E' \to S^i \textit{ with fibre } M \textit{ with} \\ \textit{trivialised boundary bundle and trivialised at } * \in S^i\end{matrix}\right\}}{\textit{isomorphism}}.$$

Example 1.1.8. We know that $\pi_i\, BG \xrightarrow{\cong} \pi_{i-1}\, G$. For $B\mathrm{Diff}_\partial(M)$ we thus get an isomorphism between $\pi_0\, \mathrm{Diff}_\partial(M)$, the group of diffeomorphisms up to isotopy, and the set of isomorphism classes of manifold bundles over S^1 with fibre over $*$ identified with M. In other direction, this sends φ to the mapping torus $M_\varphi \to S^1$. In the other direction, this sends a bundle to its monodromy.

As a second application of Theorem 1.1.6, we get an interpretion of the cohomology groups of $B\mathrm{Diff}_\partial(M)$ in terms of characteristic classes of manifold bundles. Recall that a *characteristic class valued in* $H^i(-;\Bbbk)$ of manifold bundles as above is an assignment

$$\alpha\colon \frac{\left\{\begin{matrix}\text{smooth manifold bundles } E' \to X \text{ with fibre} \\ M \text{ with trivialised boundary bundle}\end{matrix}\right\}}{\text{isomorphism}} \longrightarrow H^i(X;\Bbbk)$$

natural in X, using pullbacks on both sides. This naturality implies that any characteristic uniquely determines and is uniquely determined by a cohomology class $\overline{\alpha} \in H^i(B\mathrm{Diff}_\partial(M);\Bbbk)$:

Corollary 1.1.9. *Pullback of cohomology classes gives a bijection*

$$H^i(B\mathrm{Diff}_\partial(M);\Bbbk) \xrightarrow{\cong} \left\{\begin{matrix}\textit{characteristic classes valued in } H^i(-;\Bbbk) \textit{ of} \\ \textit{smooth manifold bundles } E' \to X \textit{ with fibre } M \\ \textit{and trivialised boundary bundle}\end{matrix}\right\}.$$

1.2 Embedding spaces

A good strategy for studying a space is to map into or out of it, and a better one is to situate it in cofibre or fibre sequences. Applying these strategies to diffeomorphism groups will lead us to study spaces of embeddings.

1.2.1 Isotopy extension

Recall that an embedding $e\colon M \hookrightarrow N$ is a smooth map that is a diffeomorphism onto its image (which necessarily must be a submanifold of N).

Remark 1.2.1. Equivalently an embedding is a smooth map that is injective, whose derivatives are injective everywhere, and which is a homeomorphism onto its image (if M is compact this is implied by injectivity).

Suppose that $e\colon M \hookrightarrow N$ is an embedding of compact manifolds with boundary, mapping M into the interior of N. This gives a decomposition $N = M \cup_{\partial M} P$ with $P := \mathrm{cl}(N \backslash M)$ (this is a manifold with boundary $\partial P = \partial M \sqcup \partial N$). This allows us to identify $\mathrm{Diff}_\partial(M)$ as the closed subgroup of $\mathrm{Diff}_\partial(N)$ fixing P pointwise. In other words, acting by composition with diffeomorphisms of N on the inclusion $\iota\colon P \hookrightarrow N$ gives a map

$$\mathrm{Diff}_\partial(N) \longrightarrow \mathrm{Emb}_{\partial N}(P, N),$$

with $\mathrm{Diff}_\partial(M)$ is the *set-theoretic fibre* over ι. We need to make precise the right terms:

Definition 1.2.2. The *space of embeddings* $\mathrm{Emb}_{\partial N}(P, N)$ is the space of embeddings $P \hookrightarrow N$ that agree with ι near $\partial N \subset \partial P$ in the smooth topology.

Remark 1.2.3. Here the manifolds P and N happens to have the same dimension; this is of course irrelevant for defining a space of embeddings.

Exercise 2. Two smooth manifolds P and P' are isotopy equivalent if there are embeddings $P \hookrightarrow P'$ and $P' \hookrightarrow P$ that are inverse up to isotopy. Prove that in this case $\mathrm{Emb}(P, N) \simeq \mathrm{Emb}(P', N)$ and give a generalisation with boundary.

The *isotopy extension theorem* will tell us $\mathrm{Diff}_\partial(M)$ is not only the set-theoretic fibre but also the *homotopy fibre* [Wal16, Theorem 6.1.1]:

Theorem 1.2.4. *Under the above assumptions, the action map*

$$\mathrm{Diff}_\partial(N) \to \mathrm{Emb}_{\partial N}(P, N)$$

is a Serre fibration.

Proof sketch. Let us indicate the proof of the lifting property in

$$\begin{array}{ccc} D^0 & \longrightarrow & \mathrm{Diff}_\partial(N) \\ \downarrow & & \downarrow \\ D^0 \times [0,1] & \longrightarrow & \mathrm{Emb}_{\partial N}(P, N), \end{array}$$

assuming that the adjoint $E\colon P \times [0,1] \to N$ of the bottom map is smooth.[2] Moreover, by composing all data with a diffeomorphism of N, we may suppose that the top map picks out id_N. Then we must prove that any smooth 1-parameter family of embeddings $e_t\colon P \hookrightarrow N$ starting at the inclusion is induced by a 1-parameter family of diffeomorphism $\varphi_t\colon N \to N$ starting at the identity. To do so, one observes that $\tilde{E} = (E, \pi_2)\colon P \times [0,1] \to N \times [0,1]$ is a smooth embedding and that pushing forward the vector field $\frac{\partial}{\partial t}$ on $P \times [0,1]$ along $d\tilde{E}$, we get a partially defined vector field on $N \times [0,1]$ so that flowing along it starting in $P \times \{0\}$ recovers the map $\tilde{E}$. The family of diffeomorphisms is then obtained by extending this vector field arbitrarily to one whose $[0,1]$-coordinate is $\frac{\partial}{\partial t}$ and flowing along it. □

[2] By smooth approximation techniques we are in fact allowed to make this assumption for the purpose of the homotopy fibre, that is, only prove the lifting property for maps with smooth adjoints.

Thus there is a fibre sequence

$$\mathrm{Diff}_\partial(M) \longrightarrow \mathrm{Diff}_\partial(N) \longrightarrow \mathrm{Emb}_{\partial N}(P, N).$$

The path components of $\mathrm{Emb}_{\partial N}(P, N)$ given by orbit of ι under $\mathrm{Diff}_\partial(N)$ is the quotient $\mathrm{Diff}_\partial(N)/\mathrm{Diff}_\partial(M)$, so there is a map

$$\mathrm{Diff}_\partial(N) /\!\!/ \mathrm{Diff}_\partial(M) \longrightarrow \mathrm{Diff}_\partial(N)/\mathrm{Diff}_\partial(M) = \mathrm{Emb}_{\partial N}(P, N)_{\iota\text{-orbit}},$$

which by previous fibre sequence is an equivalence. We conclude:

Corollary 1.2.5. *There is a fibre sequence*

$$\mathrm{Emb}_{\partial N}(P, N)_{\iota\text{-}orbit} \longrightarrow B\mathrm{Diff}_\partial(M) \longrightarrow B\mathrm{Diff}_\partial(N).$$

Remark 1.2.6. Let me explain how to remove the restriction to the ι-orbit. To remove it, observe that different embeddings $P \hookrightarrow N$ can have non-diffeomorphic complements; if we pick a set of representatives $M_1, M_2, \ldots$ up to diffeomorphism relative to $\partial M \backslash \partial P$ for these, then there is a fibre sequence

$$\mathrm{Emb}_{\partial N}(P, N) \longrightarrow \bigsqcup_i B\mathrm{Diff}_\partial(M_i) \longrightarrow B\mathrm{Diff}_\partial(N).$$

1.2.2 Embeddings of discs

Embedding spaces can be quite complicated, but there is one example of fundamental importance to this subject in general and the following lectures in particular, which is rather straightforward to understand.

Recall that $\mathrm{Fr}(TN)$ denote the frame bundle of a d-dimensional manifold N, the principal $\mathrm{GL}_d(\mathbb{R})$-bundle over N whose associated vector bundle is TN.

Theorem 1.2.7. *Taking the derivative at the origin induces an equivalence*

$$\mathrm{Emb}(\mathbb{R}^d, N) \longrightarrow \mathrm{Fr}(TN).$$

Proof. Recall that $\mathrm{Imm}(M, N)$ is the space of immersions in the smooth topology, and that $\mathrm{Bun}(TM, TN)$ is the space of (fibrewise injective) maps of vector bundles in the compact-open topology. These feature in the Smale–Hirsch theorem, saying that the map

$$\mathrm{Imm}(M, N) \longrightarrow \mathrm{Bun}(TM, TN)$$

taking the derivative is an equivalence if M has no compact components.

Let us now take $M = \mathbb{R}^d$, then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Emb}(\mathbb{R}^d, N) & \xrightarrow{\text{derivative at } 0} & \mathrm{Fr}(TM) \\ \big\downarrow & & \big\uparrow \text{evaluate at } 0 \\ \mathrm{Imm}(\mathbb{R}^d, N) & \xrightarrow[\simeq]{\text{derivative}} & \mathrm{Bun}(T\mathbb{R}^d, TN) \end{array}$$

with bottom map an equivalence. The right vertical map is an equivalence because any immersion is an embedding near the origin and we can shrink the $\mathbb{R}^d$. The left vertical map is an equivalence because $\mathbb{R}^d$ is contractible and the space of bundle maps from $T\mathbb{R}^d|_0$ into TM is exactly the frame bundle. □

Since D^d and $\mathbb{R}^d$ are isotopy equivalent, it follows from this and Exercise 2 that the map

$$\mathrm{Emb}(D^d, M) \longrightarrow \mathrm{Fr}(TM)$$

taking the derivative at the origin, is also an equivalence.

Example 1.2.8 (Diffeomorphisms of spheres). Let us combine Theorem 1.2.4 and Theorem 1.2.7 to relate diffeomorphisms of the sphere S^d and the disc D^d. These theorems yield that there is a fibre sequence

$$\mathrm{Diff}_\partial(D^d) \longrightarrow \mathrm{Diff}(S^d) \longrightarrow \mathrm{Emb}(D^d, TS^d) \simeq \mathrm{Fr}(TS^d).$$

The action of $\mathrm{O}(d+1)$ on S^d by diffemorphisms canonically lifts to an action on the frame bundle $\mathrm{Fr}(TS^d)$. This restricts in turn to an action on the equivalent subspace $\mathrm{Fr}^{\mathrm{O}}(TS^d) \subset \mathrm{Fr}(TS^d)$ of orthogonal frames with respect to the standard Riemannian metric, exhibiting $\mathrm{Fr}^{\mathrm{O}}(TS^d)$ as an $\mathrm{O}(d+1)$-torsor. We conclude that in the above fibre sequence, not only is the right term equivalent to $\mathrm{O}(d+1)$ but the right map admits a section. We thus obtain an equivalence

$$\mathrm{Diff}(S^d) \simeq \mathrm{O}(d+1) \times \mathrm{Diff}_\partial(D^d).$$

Warning: this splitting is not compatible with the topological group structure.

1.2.3 The Weiss fibre sequence

Sometimes the fibre sequence in Corollary 1.2.5 can be delooped further. This may not seem striking at first, but is technically quite convenient. The inclusion $\partial N \hookrightarrow N$ always extends to an embedding $\partial N \times [0,1] \hookrightarrow N$. This is known as a collar and is unique up to isotopy. Using it, we can produce a diffeomorphism

$$N \cup_{\partial N \times \{1\}} \partial(N \times [0,1]) \cong N$$

sending $\partial N \times \{0\}$ to ∂N by the obvious identification. Applying isotopy extension to N on the left side, we get a fibre sequence

$$\mathrm{Diff}_\partial(\partial N \times [0,1]) \longrightarrow \mathrm{Diff}_\partial(N) \longrightarrow \mathrm{Emb}(N, N).$$

Here all terms are topological monoids under composition. The left two are group-like, and the right one is when we restrict to those embeddings isotopic to diffeomorphisms fixed a neighbourhood of the boundary (i.e. those in the image of the right map), indicated by $(-)^{\cong}$. The left map is one of monoids, and since the right map is equivalent to the inclusion $\mathrm{Diff}_\partial(N) \hookrightarrow \mathrm{Emb}(N, N)$ (don't restrict to a collar but just relax the boundary condition), we get maps fitting in a fibre sequence

$$B\mathrm{Diff}_\partial(\partial N \times [0,1]) \longrightarrow B\mathrm{Diff}_\partial(N) \longrightarrow B\mathrm{Emb}^{\cong}(N, N).$$

In fact, one can do even better. Stacking in the $[0,1]$-direction yields an operation

$$\star \colon B\mathrm{Diff}_\partial(\partial N \times [0,1]) \times B\mathrm{Diff}_\partial(\partial N \times [0,1]) \longrightarrow B\mathrm{Diff}_\partial(\partial N \times [0,1])$$

which is associative up to homotopy. Like when studying fundamental groups, the issue is that $x \star (y \star z)$ involves diffeomorphisms inserted on $[0,1/2]$, $[1/2,3/4]$, and $[3/4,1]$ while $(x \star y) \star z$ involves diffeomorphisms insert on $[0,1/4]$, $[1/4,1/2]$, and $[1/2,1]$. This is dealt with by relaxing the notion of associative algebra to E_1-algebra, where operations are indexed by embeddings of intervals in an interval. Such E_1-algebras also can be delooped, and with some care one deduces the existence of the following *Weiss fibre sequence* [Kup19]:

Theorem 1.2.9. *There is a fibre sequence*

$$B\mathrm{Diff}_\partial(N) \longrightarrow B\mathrm{Emb}^{\cong}(N, N) \longrightarrow B^2\mathrm{Diff}_\partial(\partial N \times [0,1]).$$

Here is an interpretation: stacking into a collar gives an E_1-action of the E_1-algebra $B\mathrm{Diff}_\partial(\partial N \times [0,1])$ on $B\mathrm{Diff}_\partial(N)$, and the homotopy quotient of this action is $B\mathrm{Emb}^{\cong}(N, N)$. This is in fact how the proof proceeds.

Remark 1.2.10. In analogy with Remark 1.2.6, we may ask to remove or at least weaken the restrictions imposed on the path components of $\mathrm{Emb}(N, N)$. In particular, we would prefer homotopy-theoretic conditions over geometric ones. This can be done in dimension $d \geqslant 5$, when the s-cobordism theorem applies; if we pick a set of representatives $W_1, W_2, \cdots$ of cobordisms $\partial N \rightsquigarrow \partial N$ with the property that $N \cup_{\partial N} W_i \cong N$ relative to ∂N, then there is a fibre sequence

$$B\mathrm{Diff}_\partial(N) \longrightarrow B\mathrm{Emb}^{\simeq}(N, N) \longrightarrow B\Big(\bigsqcup_i B\mathrm{Diff}_\partial(W_i)\Big)$$

with superscript $(-)^{\simeq}$ indicating that we restrict to embeddings that are homotopy equivalences, and the B on the right is taken with respect to the stacking operation $\star$.

Example 1.2.11 (Diffeomorphisms of annuli). If $N = D^d$, then we get

$$B\mathrm{Diff}_\partial(D^d) \longrightarrow B\mathrm{Emb}^{\cong}(D^d, D^d) \simeq B\mathrm{SO}(d) \longrightarrow B^2\mathrm{Diff}_\partial(S^{d-1} \times [0, 1])$$

describing the homotopy type of diffeomorphisms of an annulus in terms of diffeomorphisms of a disc and $\mathrm{O}(d)$. In the middle, we used Theorem 1.2.7 and that an embedding $D^d \to D^d$ is isotopic to a diffeomorphism fixing a neighbourhood of the boundary if and only if it is orientation-preserving.

1.3 Questions about diffeomorphism groups and embedding spaces

Having introduced diffeomorphism groups, spaces of embeddings, and the relationship between them, we can now ask the questions that these lectures is interested in. The first one is as important as it is vague:

Question 1.3.1. What is the homotopy type of $B\mathrm{Diff}_\partial(M)$?

Since embedding spaces appear as the relative terms we try to compare different diffeomorphism groups, it is inevitable we also ask:

Question 1.3.2. What is the homotopy type of $\mathrm{Emb}(M, N)$?

We can interpret these questions in several ways, all of which interact:

	Description	**Example**
computationally	We can ask for the homotopy groups of homology groups of $B\mathrm{Diff}_\partial(M)$ or $\mathrm{Emb}(M,N)$. There is nothing holding one back from computing more refined invariants, such as k-invariants or generalised homology theories, but not much progress as been made.	For d sufficiently large, $\pi_1\, B\mathrm{Diff}_\partial(D^d)$ surjects onto $\mathrm{coker}(J)_{d+1}$, the cokernel of the stable J-homomorphism $\pi_{d+1}\,\mathrm{O} \to \pi_{d+1}\,\mathbb{S}$.
qualitatively	We can ask whether these spaces are finite CW-complexes, of finite type, nilpotent, etc., or about large scale structure of the previously mentioned computations.	If M and N have finite fundamental groups and satisfy $\dim(M) \leqslant \dim(N) - 3$, then each path-component of $\mathrm{Emb}(M,N)$ has degreewise finitely-generated homotopy groups.
structurally	We can ask what additional structures are present on these spaces. These could take the shape of additional higher-algebraic structures, or of non-trivial filtrations.	For all $d \geqslant 0$, the space $B\mathrm{Diff}_\partial(D^d)$ is a d-fold loop space. In fact, it is equivalent to $\Omega^d_0 \frac{\mathrm{Top}(d)}{\mathrm{O}(d)}$ as long as $d \neq 4$.
comparatively	We can ask for the construction of interesting maps into or out of these spaces. The former amounts to finding methods to construct diffeomorphisms or embeddings, and the latter to finding invariants of them	If M is oriented and closed of dimension $2n$ with n odd, then there is an interesting map $B\mathrm{Diff}(M) \to \Omega^\infty \mathrm{KSp}(\mathbb{Z})$.

Chapter 2

Calculus of embeddings

We will start with discussing an approach one can use to get at embeddings. It takes inspiration from the Smale–Hirsch theorem describing immersions and gives a homotopy-theoretic description of certain spaces of embeddings. Our approach is inspired by [BdBW13], and the original references for the theory are [Wei99, Wei11, GW99]. Many useful properties of embedding calculus are established in [KK21, KK22].

2.1 Smale–Hirsch as a sheaf

Throughout this sections all manifolds have dimension d unless mentioned otherwise.

2.1.1 A sheaf property for immersions and bundle maps

Suppose M and N both have dimension d. Recall that the Smale–Hirsch theorem says that the derivative map

$$\mathrm{Imm}(M, N) \longrightarrow \mathrm{Bun}(TM, TN)$$

is an equivalence if M has no closed component. In this case, $\mathrm{Imm}(M, N)$ inherits all the good properties of $\mathrm{Bun}(TM, TN)$. For the property we are interested in, we fix N and think of M as variable. More precisely, let Man be the topological category whose objects are d-dimensional manifolds (without boundary) and whose morphism spaces from M to M' is given by the space of embeddings $\mathrm{Emb}(M, M)$. You can take the coherent nerve and regard this as an ∞-category $\mathcal{M}\mathrm{an}$; at this stage in the development of the theory this is optional but as we do more advanced things it will become mandatory. Some references include [Lur09, Cis19, Lan21].

Every embedding $e\colon M \to M'$ induces a map

$$- \circ de\colon \mathrm{Bun}(TM', TN) \longrightarrow \mathrm{Bun}(TM, TN).$$

This depends continuously on e and satisfies $- \circ \mathrm{id} = \mathrm{id}$ as well as $(- \circ e')(- \circ e) = (- \circ ee')$ so we can think of it as a continuous functor

$$\begin{aligned} \mathrm{Bun}(T-, TN)\colon \mathsf{Man}_d^{\mathrm{op}} &\longrightarrow \mathsf{Top} \\ M &\longmapsto \mathrm{Bun}(TM, TN), \end{aligned}$$

and taking coherent nerves we get one of the following objects:

Definition 2.1.1. A *presheaf on* $\mathcal{M}\mathrm{an}_d$ is a functor $F\colon \mathcal{M}\mathrm{an}^{\mathrm{op}} \to \mathcal{S}$.

Note that given an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M, a map $g\colon TM \to TN$ is uniquely determined by its restrictions $g|_{U_i}\colon TU_i \to TN$; conversely, any collection of $g_i\colon TU_i \to TN$ agreeing on overlaps can be glued a unique map $g\colon TM \to TN$. That is, it is an ordinary *sheaf* of sets on $\mathcal{M}\mathrm{an}$ with respect to the usual notion of open coverings. We could ask whether it is a homotopy sheaf as well, a notion obtained by replacing a limit of sets in the definition of an ordinary sheaf with a limit in the ∞-category of spaces:

Definition 2.1.2. A presheaf $F\colon \mathcal{M}\mathrm{an}^{\mathrm{op}} \to \mathcal{S}$ is a *sheaf* if for any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M the natural map

$$F(M) \longrightarrow \lim_{U_J \in N\mathcal{U}} F(U_J)$$

is an equivalence, with $N\mathcal{U}$ the nerve of the open cover $\mathcal{U}$ (here $J \subset I$ is finite and $U_J = \bigcap_{j \in J} U_j$).

Remark 2.1.3. It is technically and practically convenient to allow hypercovers, but for simplicity we will forego this.

Some of the proofs of Smale–Hirsch establish this along the way. Let us give a different prove that will generalise more cleanly. Note that only depends on the space M with the vector bundle TM, or equivalent $\mathrm{GL}_d(\mathbb{R})$-space $\mathrm{Fr}(TM)$. More generally, the frame bundle construction gives an equivalence of categories $\mathcal{V}\mathrm{ect}_d \to \mathrm{Fun}(\mathrm{GL}_d(\mathbb{R}), \mathcal{S})$ with domain the ∞-category of spaces with d-dimensional vector bundle over it. This yields an identification

$$\mathrm{Bun}(TM, TN) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Fun}(\mathrm{GL}_d(\mathbb{R}),\mathcal{S})}(\mathrm{Fr}(TM), \mathrm{Fr}(TN)).$$

Lemma 2.1.4. *For any open cover $\mathcal{U}$ of M, the natural map*

$$\operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Fr}(TU_J) \longrightarrow \mathrm{Fr}(TM)$$

is an equivalence.

Proof. Since colimits in functor categories are detected objectwise and coherent nerves takes homotopy colimits to colimits in ∞-categories, it suffices to prove that in Top the map

$$\operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Fr}(TU_J) \longrightarrow \mathrm{Fr}(TM)$$

is an equivalence. This follows because $\{\mathrm{Fr}(TU_i)\}_{i \in I}$ is an open cover of $\mathrm{Fr}(TM)$ and the theorem of Dugger–Isaksen [DI04] that for any open cover $\mathcal{V} = \{V_i\}_{i \in I}$ of a topological space X, the map $\mathrm{hocolim}_{V_J \in N\mathcal{V}} V_J \to X$ is an equivalence. ☐

This proves that $M \mapsto \mathrm{Fr}(TM)$ is a sheaf, using the commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{GL}_d(\mathbb{R}),\mathcal{S})}(\mathrm{Fr}(TM), \mathrm{Fr}(TN)) & \xleftarrow{\simeq} & \mathrm{Map}_{\mathrm{Fun}(\mathrm{GL}_d(\mathbb{R}),\mathcal{S})}(\operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Fr}(TU_J), \mathrm{Fr}(TN)) \\ & \searrow \quad \swarrow{\scriptstyle\simeq} & \\ & \lim_{U_J \in N\mathcal{U}} \mathrm{Map}_{\mathrm{Fun}(\mathrm{GL}_d(\mathbb{R}),\mathcal{S})}(\mathrm{Fr}(TU_J), \mathrm{Fr}(TN)) & \end{array}$$

with top map an equivalence by the previous lemma and right map an equivalence by the universal property of a colimit.

2.1.2 Right Kan extension as sheafification

Anticipating the coming sections, we will call an ordinary open cover of a manifold a $\mathcal{J}_1$-cover. Then there is a (homotopy) sheafification functor

$$\tau_{\mathcal{J}_1} : \mathrm{Fun}(\mathcal{M}\mathrm{an}_d^{\mathrm{op}}, \mathcal{S}) =: \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an}_d) \longrightarrow \mathcal{S}\mathrm{h}_{\mathcal{J}_1}(\mathcal{M}\mathrm{an}_d) \subset \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an}_d).$$

This is a localisation, i.e. the left adjoint to the fully faithful functor given by the inclusion of $\mathcal{J}_1$-sheaves into presheaves.

To understand this more concretely, we reinterpret the category $\mathrm{Fun}(\mathrm{GL}_d(\mathbb{R}), \mathcal{S})$. There is a map of group-like monoids

$$\mathrm{GL}_d(\mathbb{R}) \longrightarrow \mathrm{Emb}(\mathbb{R}^d, \mathbb{R}^d)$$

which is an equivalence as an application of the equivalence $\mathrm{Emb}(\mathbb{R}^d, M) \to \mathrm{Fr}(TM)$. Observing that $\mathrm{Emb}(\mathbb{R}^d, \mathbb{R}^d)$ is the endomorphism space of the object $\mathbb{R}^d$ of $\mathcal{M}\mathrm{an}$, we recognise that we can identify

$$\mathrm{Fun}(\mathrm{GL}_d(\mathbb{R}), \mathcal{S}) \simeq \mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{=1})$$

where $\mathcal{D}\mathrm{isc}_{=1} \subset \mathcal{M}\mathrm{an}$ is the full subcategory on the object $\mathbb{R}^d$. Moreover, letting $\mathcal{D}\mathrm{isc}_{\leqslant 1} \subset \mathcal{M}\mathrm{an}$ denote the full subcategory on the objects $\varnothing$ and $\mathbb{R}^d$, we can identify $\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{=1})$ with the full subcategory of $\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{\leqslant 1})$ of those presheaves whose value at $\varnothing$ is contractible.

Under this identification the $\mathrm{GL}_d(\mathbb{R})$-space $\mathrm{Fr}(TM)$ corresponds to the presheaf on $\mathcal{D}\mathrm{isc}_{\leqslant 1}$ given by $U \mapsto \mathrm{Emb}(U, M)$. The upshot is then that we can rewrite bundle maps as

$$\mathrm{Bun}(TM, TN) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{\leqslant 1})}(\mathrm{Emb}(-, M), \mathrm{Emb}(-, N)).$$

The reason for doing so, is that the right side can be recognised as a right Kan extension, using [Lur09, 4.3.2]:

Proposition 2.1.5. *For the inclusion of full subcategory $\mathcal{C}_0 \to \mathcal{C}$, the right Kan extension functor* $\mathrm{RKan}\colon \mathcal{PS}\mathrm{h}(\mathcal{C}_0) \to \mathcal{PS}\mathrm{h}(\mathcal{C})$ *is given by*

$$\mathrm{RKan}(F)(c) = \mathrm{Map}_{\mathcal{PS}\mathrm{h}(\mathcal{C}_0)}(\mathrm{Map}_{\mathcal{C}}(-, c), F).$$

Once more anticipating later sections, we make the following definition:

Definition 2.1.6. The functor $T_1\colon \mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{\leqslant 1}) \to \mathcal{PS}\mathrm{h}(\mathcal{M}\mathrm{an})$ is given by taking the right Kan extension along the inclusion $\mathcal{D}\mathrm{isc}_{\leqslant 1} \to \mathcal{M}\mathrm{an}$.

Notation 2.1.7. We will also write T_1 for the functor $\mathcal{PS}\mathrm{h}(\mathcal{M}\mathrm{an}) \to \mathcal{PS}\mathrm{h}(\mathcal{M}\mathrm{an})$ obtained by first restricting to $\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{\leqslant 1})$ and then right Kan extending back to $\mathcal{PS}\mathrm{h}(\mathcal{M}\mathrm{an})$.

Theorem 2.1.8. *The sheafification functor $\tau_{\mathcal{J}_1}$ agrees with T_1.*

Proof. For any open cover $\mathcal{U}$ of M we have a commutative diagram

$$\begin{array}{ccc}
\operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Emb}(\mathbb{R}^d, U_J) & \longrightarrow & \mathrm{Emb}(\mathbb{R}^d, M) \\
\downarrow \simeq & & \downarrow \simeq \\
\operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Fr}(TU_J) & \xrightarrow{\simeq} & \mathrm{Fr}(TM)
\end{array}$$

where the bottom horizontal map is an equivalence by Lemma 2.1.4 and hence so is the top horizontal map. As colimits of presheaves are detected objectwise, from this and a similar argument with $\varnothing$ in place of $\mathbb{R}^d$, we deduce that

$$\operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Emb}(-, U_J) \longrightarrow \mathrm{Emb}(-, M)$$

is an equivalence in $\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{\leqslant 1})$ and hence that

$$\mathrm{Map}_{\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{\leqslant 1})}(\mathrm{Emb}(-, M), F) \longrightarrow \lim_{U_J \in N\mathcal{U}} \mathrm{Map}_{\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc}_{\leqslant 1})}(\mathrm{Emb}(-, U_J), F),$$

is an equivalence. This says that $\mathrm{RKan}_{\leqslant 1} F$ is a $\mathcal{J}_1$-sheaf.

By the universal property of $\mathcal{J}_1$-sheaficaiton, this induces the dashed map in the commutative diagram

$$\begin{array}{ccc}
F & \longrightarrow & \tau_{\mathcal{J}_1} F \\
\downarrow & \swarrow & \downarrow \\
T_1 F & \longrightarrow & T_1 \tau_{\mathcal{J}_1} F
\end{array}$$

and evaluating on $\mathbb{R}^d$ yields

$$\begin{array}{ccc} F(\mathbb{R}^d) & \longrightarrow & \tau_{\mathcal{J}_1} F(\mathbb{R}^d) \\ \downarrow \simeq & \swarrow & \downarrow \simeq \\ T_1 F(\mathbb{R}^d) & \longrightarrow & T_1 \tau_{\mathcal{J}_1} F(\mathbb{R}^d) \end{array}$$

showing that the dashed map has a left and right inverse and hence is an equivalence as well. That the vertical maps are equivalences uses that right Kan extension evaluated on an object in the full subcategory is just the identity. By the analogous argument, the map $\tau_{\mathcal{J}_1} F(\varnothing) \to T_1 F(\varnothing)$ is an equivalence.

Every manifold M admits a good open cover $\mathcal{U}$, where each intersection of finitely many elements is either empty or an open disc [BdBW13]: take sufficiently small geodesically convex open balls with respect to some Riemannian metric. Using such a cover we get

$$\begin{array}{ccc} \tau_{\mathcal{J}_1} F(M) & \longrightarrow & T_1 F(M) \\ \downarrow \simeq & & \downarrow \simeq \\ \lim\limits_{U_J \in N\mathcal{U}} \tau_{\mathcal{J}_1} F(U_J) & \xrightarrow{\simeq} & \lim\limits_{U_J \in N\mathcal{U}} T_1 F(U_J) \end{array}$$

with vertical maps equivalences by the sheaf property and bottom horizontal map an equivalence since $\tau_{\mathcal{J}_1} F(\mathbb{R}^d) \to T_1 F(\mathbb{R}^d)$ and $\tau_{\mathcal{J}_1} F(\varnothing) \to T_1 F(\varnothing)$ are. □

This implies that it is easy to detect an $\mathcal{J}_1$-sheafification:

Lemma 2.1.9. *A map $F \to G$ from a presheaf to a $\mathcal{J}_1$-sheaf is a $\mathcal{J}_1$-sheafification if it is an equivalence on $\varnothing$ and $\mathbb{R}^d$.*

Because $\mathrm{Imm}(\varnothing, N) \to \mathrm{Bun}(T\varnothing, TM)$ and $\mathrm{Imm}(\mathbb{R}^d, N) \to \mathrm{Bun}(T\mathbb{R}^d, TN)$ are equivalences, this leads to the following interpretation of the Smale–Hirsch map: the map

$$\mathrm{Imm}(-, N) \longrightarrow \mathrm{Bun}(T-, TN) \tag{2.1}$$

identifies the right side as the $\mathcal{J}_1$-sheafification of the left side. The Smale–Hirsch theorem can be interpreted as a "convergence theorem" giving conditions under which the $\mathcal{J}_1$-sheafification—think, "best approximation satisfying descent for $\mathcal{J}_1$-covers—is an equivalence.

However, notice that it is also true that $\mathrm{Emb}(\varnothing, N) \to \mathrm{Bun}(T\varnothing, TM)$ and $\mathrm{Emb}(\mathbb{R}^d, N) \to \mathrm{Bun}(T\mathbb{R}^d, TN)$ are equivalences so it is also true that the derivative map

$$\mathrm{Emb}(-, N) \longrightarrow \mathrm{Bun}(T-, TN) \tag{2.2}$$

is the $\mathcal{J}_1$-sheafification. This is no contradiction: many presheaves have the same sheafification and the map (2.1) is a good approximation but (2.2) is usually not.

2.2 Embedding calculus as higher-order sheaves

How could we build a better approximation to $\mathrm{Emb}(-, N) \in \mathcal{PSh}(\mathcal{M}\mathrm{an})$? The evident issue with $\mathcal{J}_1$-covers is that they can never capture injectivity but only local injectivity. This is why embeddings and immersions have the same $\mathcal{J}_1$-sheafification: immersions are exactly the local embeddings. In this section we explain one solution to this.

2.2.1 A better type of sheaves

As the introduction to this section suggests, we need to keep track of values near multiple points. This can be done by replacing $\mathcal{D}\mathrm{isc}_{\leqslant 1}$ with the full subcategory on finitely many open discs:

$$\mathcal{D}\mathrm{isc} \subset \mathcal{M}\mathrm{an}.$$

Remark 2.2.1. A presheaf on $\mathcal{D}\mathrm{isc}$ is the same as a right framed E_d-module; if it is symmetric monoidal then it is a framed E_d-algebra.

Let us call an open cover of a manifold a $\mathcal{J}_\infty$*-cover* if any finite subset is contained in an element of the cover. To develop the theory of $\mathcal{J}_\infty$-sheaves we need the following exercise and lemma:

Exercise 3. Let $\mathrm{Conf}_k(M) \coloneqq \mathrm{Emb}(\{1,\dots,k\}, M)$ denote the configuration space of k ordered points and $\mathrm{FrConf}_k(TM) \coloneqq \mathrm{Conf}_k(M) \times_{M^k} \mathrm{Fr}(TM)^k$ the configuration space of k ordered points with frames. Prove that the map

$$\mathrm{Conf}_k(M) \longrightarrow \mathrm{FrConf}_k(TM)$$

taking the derivatives at the origins, is an equivalence.

Lemma 2.2.2. *For any $\mathcal{J}_\infty$-cover $\mathcal{U}$ of M, we have an equivalence*

$$\operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Emb}(-, U_J) \longrightarrow \mathrm{Emb}(-, M)$$

of presheaves on $\mathcal{D}\mathrm{isc}$.

Proof. The previous lemma shows that the vertical maps in the commutative diagram

$$\begin{array}{ccc} \operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{Emb}(\{1,\dots,k\} \times \mathbb{R}^d, U_J) & \longrightarrow & \mathrm{Emb}(\{1,\dots,k\} \times \mathbb{R}^d, M) \\ \downarrow \simeq & & \downarrow \\ \operatorname*{colim}_{U_J \in N\mathcal{U}} \mathrm{FrConf}_k(TU_J) & \xrightarrow{\ \simeq\ } & \mathrm{FrConf}_k(TM) \end{array}$$

are equivalences. Moreover, for a $\mathcal{J}_\infty$-cover $\mathcal{U} = \{U_i\}_{i \in I}$ the subsets $\mathrm{FrConf}_k(TU_i)$ are an open cover of $\mathrm{FrConf}_k(TM)$, so invoking once more [DI04] the bottom horizontal map is an equivalence. The result follows once we recall that colimits of presheaves are detected objectwise. ☐

Arguing as before, this identifies $\mathcal{J}_\infty$-sheafification of $F \in \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{D}\mathrm{isc})$ as a right Kan extension

$$\tau_{\mathcal{J}_\infty} F(M) = \mathrm{Map}_{\mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{D}\mathrm{isc})}(\mathrm{Emb}(-, M), F).$$

Definition 2.2.3. The functor $T_\infty \colon \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{D}\mathrm{isc}) \to \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an})$ is given by taking the right Kan extension along the inclusion $\mathcal{D}\mathrm{isc} \to \mathcal{M}\mathrm{an}$.

Notation 2.2.4. We will also write T_∞ for the functor $\mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an}) \to \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an})$ obtained by first restricting to $\mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{D}\mathrm{isc})$ and then right Kan extending back to $\mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an})$.

In particular, we can applying this $F = \mathrm{Emb}(-, N)$ and obtain an approximation

$$\mathrm{Emb}(M, N) \longrightarrow T_\infty \mathrm{Emb}(M, N). \tag{2.3}$$

2.2.2 Convergence of embedding calculus

The most important result in embedding calculus concerns the quality of the map (2.3). It has a similar condition to the Smale–Hirsch theorem. The following is the main result of [GW99], based on [Goo90, GK08, GK15]. It uses the notion of *handle dimension*: M has handle dimension $\leqslant h$ if it is the interior of a compact manifold with handle decomposition with only handles of index $\leqslant h$.

Theorem 2.2.5. *The map* $\mathrm{Emb}(M,N) \to T_\infty \mathrm{Emb}(M,N)$ *is an equivalence if* $\mathrm{hdim}(M) \leqslant \dim(N) - 3$.

Remark 2.2.6. We say that "embedding calculus converges for embeddings $M \hookrightarrow N$" if the map $\mathrm{Emb}(M,N) \to T_\infty \mathrm{Emb}(M,N)$ is an equivalence. The above gives a sufficient criterion when this holds but not a necessary one. For example, it always converges when $d = 2$ [KK21]. However, usually there is no convergence in handle codimension $\leqslant 2$ [KK22].

To give you a sense of the proof, let us consider the case $M = S^i \times \mathbb{R}^{d-i}$ for $i \leqslant d-3$. It is proven by induction over i of the stronger statement that has finite disjoint unions of such manifolds for $i' \leqslant i$; the initial case $i = -1$ follows because on finite disjoint unions of $\mathbb{R}^d$ the right Kan extension is the identity as these are objects of $\mathcal{D}\mathrm{isc}$.

Let us focus on the induction step for a single manifold $M = S^i \times \mathbb{R}^{d-i}$. It is the interior of a manifold with $\overline{M} = S^i \times D^{d-i}$ with a single 0-handle and a single i-handle: $\overline{M} = D^d \cup_{(S^{i-1}\times D^{d-i})} (D^i \times D^{d-i})$. We may use $\overline{M}$ instead of M since it is isotopy equivalent. By cutting the i-handle $D^i \times D^{d-i}$ along codimension 1 submanifolds Q of the form $S^i_r \times D^{d-i}$ with $0 < r < 1$ we get a $\mathcal{J}_\infty$-cover $\mathcal{U}$ of $\overline{M}$. Its elements and their intersections are disjoint unions of $\mathbb{R}^d$'s and $S^{i-1} \times D^{d-i+1}$'s. Thus we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Emb}(M,N) & \longrightarrow & T_\infty \mathrm{Emb}(M,N) \\ \downarrow & & \downarrow \simeq \\ \lim\limits_{U_J \in N\mathcal{U}} \mathrm{Emb}(U_J, N) & \xrightarrow{\simeq} & \lim\limits_{U_J \in N\mathcal{U}} T_\infty \mathrm{Emb}(U_J, N) \end{array}$$

with right vertical map an equivalence since $T_\infty \mathrm{Emb}(-,N)$ is a $\mathcal{J}_\infty$-sheaf and bottom horizontal map an equivalence by the induction hypothesis. It remains to prove that the left vertical map is an equivalence; this is a consequence of the multiple disjunction results of [GK15] and it is where the hypothesis $i \leqslant d-3$ is used.

The general case is of course just a more complicated handle induction.

Remark 2.2.7. Historically, the use of embedding calculus has been restricted to situations of convergence. However, it can be as powerful as a source of invariants in cases where convergence is unknown.

2.3 Using embedding calculus

Now that we have set up an approximation from the right to spaces of embeddings, which in certain nice situations is an equivalence. This has at least two uses:

1. The space $T_\infty \mathrm{Emb}(M,N)$ only depends on M and N through the presheaves $\mathrm{Emb}(-,M)$ and $\mathrm{Emb}(-,N)$ on $\mathcal{D}\mathrm{isc}$.
2. The filtration on $\mathcal{D}\mathrm{isc}$ by numbers of discs will induce a filtration $T_\infty \mathrm{Emb}(M,N)$. This will turn out to be surprisingly computable.

2.3.1 Knots in smooth 4-manifolds

As an application of the first use of embedding calculus is to spaces of knots in smooth 4-manifolds, i.e. $\mathrm{Emb}(S^1, M)$. Viro asked whether the homotopy type of this space can detect exotic smooth structures. The answer is no, in many cases:

Theorem 2.3.1. *If M, N are 1-connected smooth 4-manifolds that are homeomorphic, then* $\mathrm{Emb}(S^1, M) \simeq \mathrm{Emb}(S^1, N)$.

Proof. We will input a fact that the presheaves $\mathrm{Emb}(-, M)$ and $\mathrm{Emb}(-, N)$ are equivalent as presheaves on $\mathcal{D}\mathrm{isc}$, via an obstruction-theoretic argument. Then we argue follows:

$$\begin{aligned}
\mathrm{Emb}(S^1, M) &\simeq T_\infty \mathrm{Emb}(S^1, M) && \text{convergence} \\
&= \mathrm{Map}_{\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc})}(\mathrm{Emb}(-, S^1), \mathrm{Emb}(-, M)) && \\
&\simeq \mathrm{Map}_{\mathcal{PS}\mathrm{h}(\mathcal{D}\mathrm{isc})}(\mathrm{Emb}(-, S^1), \mathrm{Emb}(-, N)) && \text{use equivalence of presheaves} \\
&= T_\infty \mathrm{Emb}(S^1, N) && \\
&\simeq \mathrm{Emb}(S^1, n) && \text{convergence.}
\end{aligned}$$

□

2.3.2 The embedding calculus tower

Now we move on to the much more involved task of constructing a filtration on $T_\infty \mathrm{Emb}(M, N)$ and rendering it computable. The inclusion $\mathcal{D}\mathrm{isc}_{\leqslant 1} \hookrightarrow \mathcal{D}\mathrm{isc}$ can be filtered as

$$\mathcal{D}\mathrm{isc}_{\leqslant 1} \hookrightarrow \mathcal{D}\mathrm{isc}_{\leqslant 2} \hookrightarrow \mathcal{D}\mathrm{isc}_{\leqslant 3} \hookrightarrow \cdots \hookrightarrow \mathcal{D}\mathrm{isc} \subset \mathcal{M}\mathrm{an}$$

where $\mathcal{D}\mathrm{isc}_{\leqslant k} \subset \mathcal{M}\mathrm{an}$ is the full subcategory of at most k open discs. The right Kan extension functor T_k along $\mathcal{D}\mathrm{isc}_{\leqslant k} \to \mathcal{M}\mathrm{an}$ is a model for the $\mathcal{J}_k$-sheafification, where a $\mathcal{J}_k$-cover is an open cover where every subset of cardinality at most k is contained in an element of the open cover. Clearly a $\mathcal{J}_k$-sheaf is a $\mathcal{J}_{k+1}$ sheaf, so this tower of categories over $\mathcal{M}\mathrm{an}$ yields a tower of functors $\mathcal{PS}\mathrm{h}(\mathcal{M}\mathrm{an}) \to \mathcal{PS}\mathrm{h}(\mathcal{M}\mathrm{an})$ under the identity

$$T_1 \longleftarrow T_2 \longleftarrow T_3 \longleftarrow \cdots \longleftarrow T_\infty \longleftarrow \mathrm{id}_{\mathcal{M}\mathrm{an}}.$$

Moreover, since $\mathcal{D}\mathrm{isc} = \mathrm{colim}_k \, \mathcal{D}\mathrm{isc}_{\leqslant k}$, we have $T_\infty = \lim_k T_k$.

Applying this to the presheaf $F = \mathrm{Emb}(-, N)$ and evaluating at M, we get a tower

$$T_1 \mathrm{Emb}(M, N) \longleftarrow T_2 \mathrm{Emb}(M, N) \longleftarrow \cdots \longleftarrow T_\infty \mathrm{Emb}(M, N) \longleftarrow \mathrm{Emb}(M, N).$$

On the one hand, we saw before that the right-most term is $\mathrm{Bun}(TM, TN)$. On the other hand, if convergence holds, the left-most map is an equivalence. This yields a strategy: we can attempt to understand

$$\mathrm{Emb}(M, N) \simeq T_\infty \mathrm{Emb}(M, N) \simeq \lim_{k \to \infty} T_k \mathrm{Emb}(M, N)$$

by understanding $\mathrm{Bun}(TM, TN)$ and the fibres of the maps $T_k \mathrm{Emb}(M, N) \to T_{k-1} \mathrm{Emb}(M, N)$. The former is straightforward because bundle maps are essentially continuous maps with some linear data added, and the latter will be the focus of the next subsection.

2.3.3 The layers of the tower

Given an element $x \in T_{k-1} \mathrm{Emb}(M, M)$, we want to understand the fibre

$$\mathrm{fibre}_x(T_k \mathrm{Emb}(M, N) \to T_{k-1} \mathrm{Emb}(M, N)).$$

The answer is described by the slogan:

> It is a relative section space of a bundle over a configuration space of M, of a bundle whose fibre is built from configuration spaces of N.

To make this precise, it is irrelevant that we started with $T_k\mathrm{Emb}(M,N)$; any right Kan extension of $G \in \mathcal{PSh}(\mathcal{D}\mathrm{isc}_{\leqslant k})$ will do and working in this generality is clarifying. The original proof appear in [Wei99, Wei11], but I will explain a streamlined approach that will appear in forthcoming joint work with Krannich.

The first observation is that we can consider the above fibre as the value at M of a functor

$$\begin{aligned} \mathrm{fib}_x G \colon \mathcal{O}(M)^{\mathrm{op}} &\longrightarrow \mathcal{S} \\ U &\longmapsto \mathrm{fib}_{x|_U}(G(U) \to T_{k-1}G(U)) \end{aligned}$$

where $\mathcal{O}(M)$ is the poset of open subsets of M ordered by inclusion, and $x|_U$ is shorthand for the image of x under the map $G(M) \to G(U)$. That, $\mathrm{fib}_x G$ is a presheaf on $\mathcal{O}(M)$. This has two properties which will allow us to recognise it. Firstly, since both G and $T_{k-1}G$ are $\mathcal{J}_k$-sheaves so is $\mathrm{fib}_x G$. Secondly, since $G(U) \to T_{k-1}G(U)$ are equivalences when U consists of at most $k-1$ discs, on each U we have $\mathrm{fib}_x G(U) \simeq *$.

The element $x|_U$ is given by a map $\mathrm{Emb}(-,U) \to T_{k-1}G$, so any map $\mathrm{Emb}(-,U) \to G$ in $\mathcal{PSh}(\mathcal{D}\mathrm{isc}_{\leqslant k})$ that restrict to it canonically lies over under the left Kan extension of $x|_U$ along the inclusion $\mathcal{D}\mathrm{isc}_{\leqslant k-1} \hookrightarrow \mathcal{D}\mathrm{isc}_{\leqslant k}$, and over its right Kan extension. Evaluating these at $\underline{k} \times \mathbb{R}^d$ with $\underline{k} = \{1,\ldots,k\}$, we get a map from $\mathrm{fib}_x G(U)$ to the space of $\Sigma_k \ltimes \mathrm{GL}_d(\mathbb{R})$-equivariant dashed maps

$$\begin{array}{ccc} (\mathrm{LKan}\,\mathrm{Emb}(-,U))(\underline{k}\times\mathbb{R}^d) & \longrightarrow & (\mathrm{LKan}\,T_{k-1}G)(\underline{k}\times\mathbb{R}^d) \\ \downarrow & & \downarrow \\ \mathrm{Emb}(\underline{k}\times\mathbb{R}^d, U) & \dashrightarrow & G(\underline{k}\times\mathbb{R}^d) \\ \downarrow & & \downarrow \\ (\mathrm{RKan}\,\mathrm{Emb}(-,U))(\underline{k}\times\mathbb{R}^d) & \longrightarrow & (\mathrm{RKan}\,T_{k-1}G)(\underline{k}\times\mathbb{R}^d) \end{array}$$

making the diagram commute; all of this is happening in ∞-categories, so making the diagram commute requires the data of homotopies in the top and bottom squares.

We make two observations. Firstly, by definition (and some abuse of notation) we have that $\mathrm{RKan}\,T_{k-1}G = T_{k-1}G$. Secondly, this construction is natural in U, yielding a presheaf that we denote $\mathrm{fib}'_x G \in \mathcal{PSh}(\mathcal{O}(M))$, which by construction comes with a map of such presheaves

$$\mathrm{fib}_x G \longrightarrow \mathrm{fib}'_x G.$$

To understand $\mathrm{fib}'_x G$ and prove this map is an equivalence, we need a computation:

Exercise 4. Prove that the map $(\mathrm{LKan}\,\mathrm{Emb}(-,U))(\underline{k}\times\mathbb{R}^d) \to \mathrm{Emb}(\underline{k}\times\mathbb{R}^d, U)$ is equivalent to the inclusion into $\mathrm{FrConf}_k(U)$ of an open neighbourhood of the fat diagonals (where any two or more points get close).

With some work, this gives a description of the space of dashed maps. Writing

$$C_k(U) \coloneqq \mathrm{Conf}_k(U)/\Sigma_k$$

for the configuration space of k unordered points in U, there is a bundle

$$\begin{array}{c} \mathrm{fib}_x\big(G(\underline{k}\times\mathbb{R}^d) \to T_{k-1}G(\underline{k}\times\mathbb{R}^d))\big) \times_{\Sigma_k \wr \mathrm{GL}_d(\mathbb{R})} \mathrm{FrConf}_k(U) \\ \downarrow \\ C_k(U) \end{array}$$

with vertical map induced by the quotient map $\mathrm{FrConf}_k(U) \to C_k(U)$. This has section on an open neighbourhood $\Delta_k(U) \subset C_k(U)$ of the diagonals, provided by the top part of the diagram. Then the space of dashed maps is the space of sections of this bundle that agree with the given section on $\Delta_k(U)$.

Remark 2.3.2. The space $C_k(U)$ is a non-compact manifold, and it is the interior of a topological manifold $\overline{C}_k(U)$ with boundary $\partial\overline{C}_k(U)$ such that pair $(\overline{C}_k(U), \partial\overline{C}_k(U))$ is equivalent to $(C_k(U), \Delta_k(U))$. If $U = M$ is a closed manifold, or more generally compact, then $\overline{C}_k(M)$ is compact as well.

Lemma 2.3.3. *The map* $\mathrm{fib}_x G \to \mathrm{fib}'_x G$ *is an equivalence.*

Proof. Using an elaboration fo the techniques used before one establishes that $\mathrm{fib}'_x G$ is also a $\mathcal{J}_k$-sheaf. Thus this to verify that this map is an equivalence, it suffices to verify it is so on U diffeomorphic to $S \times \mathbb{R}^d$ with $|S| \leqslant k$. For $|S| < k$ this amounts to observing that $\mathrm{fib}'_x G(S \times \mathbb{R}^d)$ is contractible because $\Delta_k(S \times \mathbb{R}^d) \to C_k(S \times \mathbb{R}^d)$ is an equivalence.

For $|S| = k$ we have that $C_k(S \times \mathbb{R}^d)$ has a contractible path component where each disc in $S \times \mathbb{R}^d$ contains a unique point, and inclusion $\Delta_k(S \times \mathbb{R}^d) \to C_k(S \times \mathbb{R}^d)$ is an equivalence onto the other path components. Thus in this case the section space describing $\mathrm{fib}'_x G(S \times \mathbb{R}^d)$ simplifies to $\mathrm{fib}_x\big(G(\underline{k} \times \mathbb{R}^d \to T_{k-1}G(\underline{k} \times \mathbb{R}^d))\big)$, which is exactly $\mathrm{fib}_x G(S \times \mathbb{R}^d)$. ☐

Example 2.3.4. We end with a description of $\mathrm{fib}_x\big(G(\underline{k} \times \mathbb{R}^d) \to T_{k-1}G(\underline{k} \times \mathbb{R}^d))\big)$ in terms of G. The subsets $S \times \mathbb{R}^d$ for $S \subsetneq \underline{k}$ provide a $\mathcal{J}_{k-1}$-cover of $\underline{k} \times \mathbb{R}^d$, so the map

$$T_{k-1}G(\underline{k} \times \mathbb{R}^d)) \longrightarrow \lim_{S \subsetneq \underline{k}} G(S \times \mathbb{R}^d)$$

is an equivalence, where the limit is over reverse inclusion: there is a map $G(S \times \mathbb{R}^d) \to G(S' \times \mathbb{R}^d)$ only when $S' \subseteq S$. That is, it is the homotopy limit of the punctured cubical diagram $\underline{k} \supsetneq T \mapsto G((\underline{k} \backslash T) \times \mathbb{R}^d)$ and hence $\mathrm{fib}_x\big(G(\underline{k} \times \mathbb{R}^d) \to T_{k-1}G(\underline{k} \times \mathbb{R}^d))\big)$ is the so-called total homotopy fibre of the cubical diagram $\underline{k} \supseteq T \mapsto G((\underline{k} \backslash T) \times \mathbb{R}^d)$. See [MV15] for a detailed discussion of the homotopy theory of cubical diagrams.

2.3.4 Summary, questions, and an example

We have explained the construction of a tower

$$\begin{array}{ccc}
 & & T_\infty \mathrm{Emb}(M,N) \simeq \lim\limits_{k \to \infty} T_k \mathrm{Emb}(M,N) \\
 & \overset{①}{\nearrow} & \downarrow \\
 & & \cdots \\
 & & \downarrow \\
 & \nearrow & T_2 \mathrm{Emb}(M,N) \\
 & & \downarrow \\
\mathrm{Emb}(M,N) & \longrightarrow & T_1 \mathrm{Emb}(M,N) \simeq \mathrm{Bun}(TM,TN)
\end{array} \tag{2.4}$$

with ① an equivalence if $\mathrm{hdim}(M) \leqslant \dim(N) - 3$, and the fibres of the vertical maps $\mathrm{fibre}_x(T_k\mathrm{Emb}(M,N) \to T_{k-1}\mathrm{Emb}(M,N))$ equivalent to relative section spaces

$$\mathrm{Sect}_{\Delta_k(M)}\left(\begin{array}{c} \mathrm{tohofib}_x\big(\underline{k} \supsetneq T \mapsto \mathrm{Conf}_{\underline{k}\backslash T}(N)\big) \times_{\Sigma_k} \mathrm{Conf}_k(M) \\ \downarrow \\ C_k(M) \end{array}\right), \tag{2.5}$$

where we have cancelled k copies of $\mathrm{GL}_d(\mathbb{R})$ to get the expression for the total space. This is also referred to as the *cardinality filtration* on $\mathrm{Emb}(M,N)$.

Under the assumption of convergence, this gives an inductive strategy for obtaining computational or qualitative results about the homotopy type of spaces of embeddings. For the computation of homotopy groups, this can be succinctly packaged in a Bousfield–Kan spectral sequence, as in [BK72]:

$$E^1_{st} = \begin{cases} \pi_{t-s}(\mathrm{Bun}(TM, TN)) & \text{if } s = 1 \\ \pi_{t-s}(\text{section space (2.5) for } k = s) & \text{if } s > 1 \end{cases} \Longrightarrow \pi_{t-s}(\mathrm{Emb}(M, N)). \tag{2.6}$$

Remark 2.3.5. This is not your usual spectral sequence, because some entries are groups or even pointed sets: it is *fringed spectral sequence*.

Question 2.3.6. Can one describe the differentials in this spectral sequence?

Question 2.3.7. Experience teaches us that the cardinality filtration is "wasteful", e.g. the vast majority of entries in Bousfield–Kan spectral sequence cancel. Can it modified to make it less so?

Question 2.3.8. One can also apply embedding calculus to the spectral presheaf $M \mapsto \mathrm{Emb}(M, N)_+ \wedge E$ for one's favourite spectrum E (e.g [Wei04]). Are there particular spectra E for which the tower admits a simpler description or has special properties?

Question 2.3.9. Since the proof in [Goo90] only works for smooth manifolds, convergence of embedding calculus in the generality of Theorem 2.2.5 is not known in the world of PL- or topological manifolds. Is it true?

Let us finish with an example. It is slightly circular, since it is a byproduct of the proof in [GW99] of convergence:

Example 2.3.10 (Quantitative convergence). Let us investigate the section spaces (2.5) in more detail. On the one hand, the configuration space $C_k(M)$ is $k\dim(M)$-dimensional. On the other hand, we can estimate the connectivity of $\mathrm{tohofib}_x\big(\underline{k} \supseteq T \mapsto \mathrm{Conf}_{\underline{k}\backslash T}(N)\big)$ using isotopy extension and the Blakers–Massey theorem.

The basic input is that total homotopy fibres of cubes can be computed iteratively. That is, given a $\underline{k}$-cube $\underline{k} \supset T \mapsto X(T)$ and a basepoint x in $X(\varnothing)$ we have

$$\mathrm{tohofib}\Big(\underline{k} \supseteq T \mapsto X(T)\Big) \simeq \mathrm{tohofib}\Big(\underline{k-1} \supseteq T' \mapsto \mathrm{fib}(X(T') \to X(T' \cup \{k\}))\Big).$$

By isotopy extension, the fibres are given by those of of $\mathrm{Conf}_{T' \cup \{k\}}(N) \to \mathrm{Conf}_{T'}(N)$, which by isotopy extension are $N \backslash x(T')$. But computing the total homotopy fibre of the $\underline{k-1}$-cube

$$\underline{k-1} \supset T' \longmapsto N \backslash x(\underline{k-1} \backslash T')$$

is easier. The reason is that the cube is strongly cocartesian with all maps $(\dim(N)-1)$-connected, so by higher Blakers–Massey [MV15, Theorem 6.2.1] is $(1-(k-1)+(k-1)(\dim N - 2)) = k\dim(N-2) + \text{const}$-cartesian and hence the total homotopy fibre has this connectivity.

By obstruction theory, if the base of a section space is b-dimensional and the fibre is f-connected, then the section space is $(f-b)$-connected. Thus this section space is $k(\dim(N) - \dim(M) - 2) + \text{const}$-connected. In particular, as long as the codimension of M is at least 3 the connectivity tends to infinity with k. This gives a hint of why this hypothesis appears in Theorem 2.2.5; in fact, this only known proof of this qualitative convergence result proves a quantitative one along the way.

Exercise 5. Prove that we may replace $\dim(M)$ by $\mathrm{hdim}(M)$ in the previous example.

Chapter 3

Stabilised diffeomorphisms and cobordism categories

Today we describe an approach to get at diffeomorphism groups, almost distinct from the embedding calculus used to get at embedding spaces (we will use the notion of a $\mathcal{J}_1$-sheaf in a proof). This is the approach using cobordism categories and homological stability, due to Galatius and Randal-Williams [GRW10, GRW14, GRW18, GRW17, GRW20].

3.1 Spaces of manifolds

Let us circle back to and expand on a topic discussed in the first lecture.

3.1.1 Classifying spaces via spaces of manifolds

Whitney proved that every closed d-dimensional manifold M admits an embedding into $\mathbb{R}^{2d}$. In other words, the space $\mathrm{Emb}(M, \mathbb{R}^{2d})$ is non-empty. This can be generalised to saying that $\mathrm{Emb}(M, \mathbb{R}^{d+n})$ is approximately $\frac{n-d}{2}$-connected; think of this as a topological space of *parametrised* submanifolds of $\mathbb{R}^{d+n}$ diffeomorphic to M. The homotopy quotient

$$\mathrm{Emb}(M, \mathbb{R}^{d+n}) /\!\!/ \mathrm{Diff}(M) := \mathrm{Emb}(M, \mathbb{R}^{d+n}) \times_{\mathrm{Diff}(M)} E\mathrm{Diff}(M)$$

by the diffeomorphism group $\mathrm{Diff}(M)$ fits into fibre sequence

$$\mathrm{Emb}(M, \mathbb{R}^{d+n}) \longrightarrow \mathrm{Emb}(M, \mathbb{R}^{d+n}) /\!\!/ \mathrm{Diff}(M) \longrightarrow B\mathrm{Diff}(M)$$

where the right map is about $\frac{n-d}{2}$-connected. Moreover, the action of $\mathrm{Diff}(M)$ on $\mathrm{Emb}(M, \mathbb{R}^{d+n})$ is evidently and has slices [Mic80, §13] so the map from the homotopy quotient to the actual quotient

$$\mathrm{Emb}(M, \mathbb{R}^{d+n}) /\!\!/ \mathrm{Diff}(M) \longrightarrow \mathrm{Emb}(M, \mathbb{R}^{2d+n})/\mathrm{Diff}(M)$$

is an equivalence. The latter can be thought of as a topological space of *unparametrised* submanifolds of $\mathbb{R}^{d+n}$ diffeomorphic to M. Letting $n \to \infty$ and using the colimit topology, we get a topological space $\mathrm{Emb}(M, \mathbb{R}^{\infty})/\mathrm{Diff}(M)$ which satisfies:

Proposition 3.1.1. *The quotient* $\mathrm{Emb}(M, \mathbb{R}^{\infty})/\mathrm{Diff}(M)$ *is homotopy equivalent to* $B\mathrm{Diff}(M)$.

3.1.2 Spaces of manifolds that can go to infinity

Switching back to finite-dimensional Euclidean spaces, we can take a disjoint union over diffeomorphism classes of d-dimensional M to get a topological space $\widehat{\Psi}_d(\mathbb{R}^{d+n})$ whose underlying set is the set of closed submanifolds of $\mathbb{R}^{d+n}$. For later use, we create a version of this topological space where pieces of our manifolds can disappear to infinity. For a each compact subset $K \subset \mathbb{R}^{d+n}$ we can take the quotient by the equivalence relation $\sim_K$ where $X, X' \subset \mathbb{R}^{d+n}$ satisfy $X \sim_K X'$ if and only if $X \cap K = X' \cap K$. This gives a quotient space $\widehat{\Psi}_d(\mathbb{R}^{d+n})/{\sim_K}$ and if $K \subset K'$ there is a map

$$\widehat{\Psi}_d(\mathbb{R}^{d+n})/{\sim'_K} \longrightarrow \widehat{\Psi}_d(\mathbb{R}^{d+n})/{\sim_K}.$$

Taking a colimit in the category of topological spaces over the category of compact subsets of $\mathbb{R}^{d+n}$

$$\Psi_d(\mathbb{R}^{d+n}) := \operatorname*{colim}_{K \subset \mathbb{R}^{d+n}} \widehat{\Psi}_d(\mathbb{R}^{d+n})/\sim_K.$$

Its points are submanifolds of $\mathbb{R}^{d+n}$ that are closed as subsets (but not necessarily as submanifolds) and an interesting example of a converging sequence is one whose intersection with any compact subset is eventually constant.

We now compute the homotopy type of $\Psi_d(\mathbb{R}^{d+n})$. This uses the notion of a *Thom space* $\mathrm{Thom}(\xi)$ of a vector bundle $\xi\colon E \to B$. Here are two equivalent descriptions up to based homotopy: (1) one fibrewise one-point compactifies ξ and collapsing the section at ∞ to a point (if B is compact, we can just take the one-point compactification directly), (2) take the homotopy cofibre of the projection map $E\backslash 0\text{-section} \to B$.

We denote the *Grassmannian of d-planes in* $\mathbb{R}^{d+n}$ by $\mathrm{Gr}_d(\mathbb{R}^{d+n})$ and this has a d-dimensional canonical vector bundle $\gamma_{d,n}$ of d-planes over it: the fibres are $(\gamma_{d,n})|_V = v$. It also has an n-dimensional vector bundle $\gamma_{d,n}^\perp$ of their orthogonal complements: the fibres are $(\gamma_{d,n}^\perp)|_V = V^\perp$. Then one can write a map

$$\begin{aligned} \gamma_{d,n}^\perp &\longrightarrow \Psi_d(\mathbb{R}^{d+n}) \\ (W, v) &\longmapsto W + v. \end{aligned}$$

Its image is the subspace of affine planes, and it extends to the one-point compactification by mapping ∞ to $\varnothing$:

$$\mathrm{Th}(\gamma_{d,n}^\perp) \longrightarrow \Psi_d(\mathbb{R}^{d+n})$$

Lemma 3.1.2. *This map gives an equivalence* $\mathrm{Thom}(\gamma_{d,n}^\perp) \xrightarrow{\simeq} \Psi_d(\mathbb{R}^{d+n})$.

Proof sketch. There is a cover $\Psi_d(\mathbb{R}^{d+n})$ by two open subsets[1]: U_0 consists of those manifolds avoiding the origin, and U_1 consists of those manifolds with a unique point closest to the origin. Using [DI04] once more, there is a homotopy pushout square

$$\begin{array}{ccc} U_0 \cap U_1 & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & \Psi_d(\mathbb{R}^{d+n}). \end{array}$$

Its intersection with $\mathrm{Th}(\gamma_{d,n}^\perp)$ is the homotopy pushout square given by

$$\begin{array}{ccc} \gamma_{d,n}^\perp\backslash 0\text{-section} & \longrightarrow & \mathrm{Th}(\gamma_{d,n}^\perp)\backslash 0\text{-section} \\ \downarrow & & \downarrow \\ \gamma_{d,n}^\perp & \longrightarrow & \mathrm{Th}(\gamma_{d,n}^\perp), \end{array}$$

and this square hence maps to the previous one. To prove that the map on the bottom-right corners is an equivalence, it thus suffices to the map on the remaining three corners is so. One argues as follows:

- $U_0 \simeq *$ and $\mathrm{Th}(\gamma_{d,n}^\perp)\backslash 0\text{-section} \simeq *$: zooming in on the origin deformation retracts U_0 onto $\varnothing$ so it is contractible, and this restricts to affine planes.
- $U_1 \simeq \gamma_{d,n}^\perp$: given an element $X \in U_1$, zooming in on the point closest to the origin we obtain an affine d-plane V_X.

[1] Actually U_1 is not open—to see this, show its complement is not closed—which is why this is only a proof sketch; one solution is to require that the distance function $|| - ||$ has a unique non-degenerate minimum. See [GRW10] for the details.

· $U_0 \cap U_1 \simeq \gamma_{d,n}^{\perp} \backslash 0$-section: argue as above.

Thus the map on bottom-right corners is an equivalence and we get that $\mathrm{Th}(\gamma_{d,n}^{\perp}) \to \Psi_d(\mathbb{R}^{d+n})$ is an equivalence. □

We could replace $\mathbb{R}^{d+n}$ with any subset $U \subseteq \mathbb{R}^{d+n}$, by taking a colimit over compact subsets $K \subset U$ of quotients by $\sim_K$, or even any $(d+n)$-dimensional manifold M. Restriction makes this natural and continuous in embeddings. Thus we have a presheaf $\Psi_d(-)$ on $\mathcal{M}\mathrm{an}_{d+n}$, which has the following striking property [RW11] resembling the Smale–Hirsch h-principle for immersions, whose proof combines Gromov's theory of microflexible sheaves [Gro86] with some relatively standard differential topology:

Theorem 3.1.3. *The map $\Psi_d(-) \to T_1\Psi_d(-)$ is an equivalence on those $(d+n)$-dimensional manifolds that have no compact components.*

Our computation of $\Psi_d(\mathbb{R}^d)$ yields a computation of $T_1\Psi_d(M)$ for a $(d+n)$-manifold M; it is equivalent to the space of sections

$$\mathrm{Sect}\left(\begin{matrix}\Psi_d(\mathbb{R}^{d+n}) \times_{\mathrm{GL}_{d+n}(\mathbb{R})} \mathrm{Fr}(TM) \\ \downarrow \\ M\end{matrix}\right).$$

To prove this, construct a "scanning" map from $T_1\Psi_d(-)$ to the presheaf given by these spaces of sections, and observe that both are $\mathcal{J}_1$-sheaves and take the same values on $\mathbb{R}^{d+n}$.

3.2 The homotopy type of the cobordism category

We will use this to compute the homotopy type of the classifying space $B\mathcal{C}\mathrm{ob}_d$ of cobordism category of d-dimensional manifolds. Platonically, $\mathcal{C}\mathrm{ob}_d$ is an ∞-category with the following properties:

- its objects are $(d-1)$-dimensional closed manifolds,
- its morphism spaces
  $$\mathrm{Map}_{\mathcal{C}\mathrm{ob}_d}(M,N) \simeq \bigsqcup_{[W]} B\mathrm{Diff}_\partial(W)$$
  where the disjoint union is over all diffeomorphism classes of d-dimensional compact cobordisms $W\colon M \rightsquigarrow N$, and
- its composition is concatenation of cobordisms.

We will indicate why its homotopy type is given by an infinite loop space of the Thom spectrum $\mathrm{MTO}(d)$ built from the Thom spaces that appeared in the previous section, given by $\mathrm{MTO}(d)_{d+n} = \mathrm{Thom}(\gamma_{d,n}^{\perp})$. More directly, it is the Thom spectrum of the virtual bundle given by the opposite of the canonical bundle over $B\mathrm{O}(d)$. This is a theorem of Galatius–Madsen–Tillmann–Weiss [GTMW09], generalising work of Madsen–Weiss in dimension 2 [MW07].

Theorem 3.2.1. *$B\mathcal{C}\mathrm{ob}_d \simeq \Omega^{\infty-1}\mathrm{MTO}(d)$.*

3.2.1 Outline of proof of Theorem 3.2.1

To construct $\mathcal{C}\mathrm{ob}_d$ and compute the homotopy type of its classifying space, it is helpful to introduce a category Cob_d enriched in topological spaces; $\mathcal{C}\mathrm{ob}_d$ will be coherent nerve of the associated category enriched in simplicial sets.

Definition 3.2.2. We will set

$$\mathsf{Cob}_d := \operatorname*{colim}_{n\to\infty} \mathsf{Cob}_d(\mathbb{R}^{d+n}),$$

where $\mathsf{Cob}_d(\mathbb{R}^{d+n})$ is the category enriched in topological spaces given by:

- its objects are given by closed $(d-1)$-dimensional submanifolds $M \subset [0,1]^{d+n-1}$,
- its space of morphisms $\mathrm{Map}_{\mathsf{Cob}_d(\mathbb{R}^{d+n})}(M,N)$ is given by the space of pairs $(t,X) \in [0,\infty) \times \Psi_d(\mathbb{R}^{d+n})$ satisfying
  (i) $X \subset [0,1]^{d+n-1} \times \mathbb{R}$,
  (ii) $X \cap ([0,1]^{d+n-1} \times (-\infty,\epsilon]) = M \times (-\infty,\epsilon]$ for some small $\epsilon > 0$, and
  (iii) $X \cap ([0,1]^{d+n-1} \times [t-\epsilon,\infty)) = N \times [t-\epsilon,\infty)$,
- its composition is given by sending (t,X) and (t',X') to $(t+t',X'')$ with $X'' \cap ([0,1]^{d+n-1} \times (-\infty,t]) = X \cap ([0,1]^{d+n-1} \times (-\infty,t])$ and $X'' \cap ([0,1]^{d+n-1} \times [t,\infty)) = (t \cdot e_{d+n} + X' \cap ([0,1]^{d+n-1} \times [t,\infty))$.

Informally, the morphisms from M to N are spaces of embedded unparametrised cobordisms $W\colon M \rightsquigarrow N$ and the composition concatenates these after translating one of them. We have that

$$B\mathcal{C}\mathrm{ob}_d \simeq B\mathsf{Cob}_d \simeq \operatorname*{colim}_{n\to\infty} B\mathsf{Cob}_d(\mathbb{R}^{d+n}),$$

so it will suffice to compute $B\mathsf{Cob}_d(\mathbb{R}^{d+n}) = |N_\bullet \mathsf{Cob}_d(\mathbb{R}^{d+n})|$. This will be done by recognising the simplicial space $N_\bullet \mathsf{Cob}_d(\mathbb{R}^{d+n})$ as equivalent to something more geometric. Let us spell out what it is: its p-simplices are disjoint unions over $(p+1)$-simplices $M_0,\ldots,M_p \subset [0,1]^{d+n-1}$ of the space of $(t_0 = 0 \leqslant t_1 \leqslant \cdots \leqslant t_p, X) \in [0,\infty)^{p+1} \times \Psi_d(\mathbb{R}^{d+n})$ satisfying

(i) $X \subset [0,1]^{d+n-1} \times \mathbb{R}$,
(ii) $X \cap ([0,1]^{d+n-1} \times (t_i-\epsilon, t_i+\epsilon)) = M_i \times (t_i-\epsilon,t_i+\epsilon)$ for $i = 0,\ldots,p$,
(iii) $X \cap ([0,1]^{d+n-1} \times (-\infty,\epsilon]) = M_0 \times (-\infty,\epsilon]$, and
(iv) $X \cap ([0,1]^{d+n-1} \times [t_p-\epsilon,\infty)) = M_p \times [t_p-\epsilon,\infty)$.

The face map d_i forgets the pair (t_i, M_i) and for d_0, d_p modifies the cobordism appropriately so that (iii) and (iv) hold again. The degeneracy map s_j duplicate the pair (t_j, M_j).

Let us successively relax more conditions in this definition.

(1) If we allow all $(t_0 \leqslant \cdots \leqslant t_p) \in \mathbb{R}^{p+1}$ we get a larger simplicial space $\widetilde{N}_\bullet \mathsf{Cob}_d(\mathbb{R}^{d+n})$ and an inclusion of simplicial spaces

$$N_\bullet \mathsf{Cob}_d(\mathbb{R}^{d+n}) \longrightarrow \widetilde{N}_\bullet \mathsf{Cob}_d(\mathbb{R}^{d+n})$$

which is a levelwise equivalence and hence induce an equivalence on (thick) geometric realisation.

(2) If also we drop the conditions (iii) as well as (iv) and forego the modifications in d_0, d_p we get a simplicial space $\Psi_d^+([0,1]^{d+n-1} \times \mathbb{R})_\bullet$ (the notation will become clear later) with map of simplicial spaces

$$\Psi_d^+([0,1]^{d+n-1} \times \mathbb{R})_\bullet \longrightarrow \widetilde{N}_\bullet \mathsf{Cob}_d(\mathbb{R}^{d+n})$$

enforcing conditions (iii) and (iv). This is a levelwise equivalence by pushing the non-constant parts in $[0,1]^{d+n-1} \times (-\infty,t_0]$, resp. $[0,1]^{d+n-1} \times [t_p,\infty)$, to $-\infty$, resp. ∞ and hence induces an equivalence on (thick) geometric realisations.

(3) We replace condition (ii) by the condition that $X \pitchfork [0,1]^{d+n-1} \times \{t_i\}$. Equivalent t_i is a regular value of the map $X \to \mathbb{R}$ and since regular values are open $X \to \mathbb{R}$ is a manifold bundle near t_i and can be straightened out using the isotopy extension theorem. This proves that the inclusion map

$$\Psi_d^{+}([0,1]^{d+n-1} \times \mathbb{R})_\bullet \longrightarrow \Psi_d^{\pitchfork}([0,1]^{d+n-1} \times \mathbb{R})_\bullet$$

induces an equivalence on (thick) geometric realisation.

The upshot is:

Lemma 3.2.3. $B\mathsf{Cob}_d(\mathbb{R}^{d+n}) \simeq |\Psi_d^{\pitchfork}([0,1]^{d+n-1} \times \mathbb{R})_\bullet|$.

We now identify the right side. To do so, we write $\Psi_d([0,1]^{d+n-1}) \subset \Psi_d(\mathbb{R}^{d+n})$ for the subspace of X contained in $[0,1]^{d+n-1} \times \mathbb{R}$ and observe that forgetting all t_i's yields an augmentation

$$\Psi_d^{\pitchfork}([0,1]^{d+n-1} \times \mathbb{R})_\bullet \longrightarrow \Psi_d([0,1]^{d+n-1} \times \mathbb{R}),$$

which induces a map of (thick) geometric realisations. This is the map in the following lemma:

Lemma 3.2.4. *The map* $|\Psi_d^{\pitchfork}([0,1]^{d+n-1} \times \mathbb{R})_\bullet| \to \Psi_d([0,1]^{d+n-1} \times \mathbb{R})$ *is an equivalence.*

Proof sketch. It turns out this map is nice enough so that it suffices that its fibres are contractible; it is a microfibration in the sense of [Wei05]. By inspecting, the fibre over X is the geometric realisation of the simplicial space whose space of p-simplices is the space of $(p+1)$-tuples $(t_0 \leqslant \cdots \leqslant t_p) \in \mathbb{R}^{p+1}$ such that all t_i are regular values of $X \to \mathbb{R}$. This has contractible geometric realisation because the set of regular values is non-empty and open, hence infinite. ☐

Proof of Theorem 3.2.1. We have equivalences

$$\begin{aligned} B\mathcal{C}\mathrm{ob}_d &\simeq \operatorname*{colim}_{n\to\infty} B\mathsf{Cob}_d(\mathbb{R}^{d+n}) \\ &\simeq \operatorname*{colim}_{n\to\infty} |\Psi_d^{\pitchfork}([0,1]^{d+n-1} \times \mathbb{R})_\bullet| \\ &\simeq \operatorname*{colim}_{n\to\infty} \Psi_d([0,1]^{d+n-1} \times \mathbb{R}) \\ &\simeq \operatorname*{colim}_{n\to\infty} \Omega^{d+n-1}\Psi_d(\mathbb{R}^{d+n}) \\ &\simeq \operatorname*{colim}_{n\to\infty} \Omega^{d+n-1}\mathrm{Thom}(\gamma_{d,n}^{\perp}) \\ &= \Omega^{\infty-1}\mathrm{MTO}(d), \end{aligned}$$

where the fourth and fifth equivalences follow from Theorem 3.1.3 and the subsequent identification of $T_1\Psi_d(M)$ as a section space. ☐

3.2.2 Adding tangential structures

We will need that a similar result holds for manifolds with additional structure. We define a *tangential structure* as a $\mathrm{GL}_d(\mathbb{R})$-space Θ.

Remark 3.2.5. More homotopy-theoretically, this is equivalent to a space $\theta\colon B \to B\mathrm{O}(d)$. In one direction, we can send Θ to $\Theta /\!\!/ \mathrm{O}(d) \to B\mathrm{O}(d)$. In the other direction, we can θ to $\mathrm{Fr}(\theta^*\gamma)$ with γ the universal bundle over $B\mathrm{O}(d)$.

A Θ-*structure on* M is a map $\mathrm{Fr}(TM) \to \Theta$ of $\mathrm{GL}_d(\mathbb{R})$-spaces, and we want to define a cobordism category $\mathcal{C}\mathrm{ob}_d^{\Theta}$ of manifolds with Θ-structures. Fixing a Θ-structure on ∂W amounts to specifying $\mathrm{GL}_d(\mathbb{R})$-equivariant map $\ell_\partial\colon \mathrm{Fr}(T\partial W \oplus \epsilon) \to \Theta$. Doing so, we let

$$\mathrm{Map}_\partial^{\mathrm{GL}_d(\mathbb{R})}(\mathrm{Fr}(TW), \Theta)$$

denote the space of Θ-structures ℓ on W extending ℓ_∂. Then the homotopy quotient

$$B\mathrm{Diff}^\Theta_\partial(W) := \mathrm{Map}^{\mathrm{GL}_d(\mathbb{R})}_\partial(\mathrm{Fr}(TW),\Theta) /\!\!/ \mathrm{Diff}_\partial(W)$$

classifies manifolds bundles with fibre W, trivialised boundary bundle, and fibrewise Θ-structure extending a given one on the boundary.

Platonically, $\mathcal{C}\mathrm{ob}^\Theta_d$ is an ∞-category with the following properties:

- its objects are $(d-1)$-dimensional manifolds M with Θ-structure on $TM \oplus \epsilon$,
- its morphism spaces
$$\mathrm{Map}_{\mathcal{C}\mathrm{ob}_d}(M,N) \simeq \bigsqcup_{[W]} B\mathrm{Diff}^\Theta_\partial(W)$$
where the disjoint union is over all diffeomorphism classes of d-dimensional cobordisms $W\colon M \rightsquigarrow N$, and
- its composition is concatenation of cobordisms.

It may be defined like $\mathcal{C}\mathrm{ob}_d$ in terms of embedded bordisms with Θ-structures.

Then there is a Thom spectrum $\mathrm{MT}\Theta$ defined analogous to $\mathrm{MTO}(d)$ with Grassmannians of d-planes replaced by Grassmannians of d-planes with Θ-structure: writing $\mathrm{Gr}_d(\mathbb{R}^{d+n}) = V_d(\mathbb{R}^{d+n})/\mathrm{GL}_d(\mathbb{R})$ this is $V_d(\mathbb{R}^{d+n}) \times_{\mathrm{GL}_d(\mathbb{R})} \Theta$. More directly, it is the Thom spectrum of the virtual bundle given by minus the canonical bundle over $B = \Theta /\!\!/ \mathrm{GL}_d(\mathbb{R})$. The analogue of Theorem 3.2.1 is [GTMW09]:

Theorem 3.2.6. $B\mathcal{C}\mathrm{ob}^\Theta_d \simeq \Omega^\infty \mathrm{MT}\Theta$.

Example 3.2.7. An interesting choice is $\Theta = \mathbb{R}^d$, or equivalently $\theta\colon * \to B\mathrm{O}(d)$. The tangential structure encoded by this is that of a framing. In this case, we get $\mathrm{MT}\Theta = \mathbb{S}^{-d}$.

Exercise 6. For Θ_X given by a space X with trivial $\mathrm{GL}_{2n}(\mathbb{R})$-action, what is the Thom spectrum $MT\Theta$?

3.3 Stable homology and homological stability

If W is a closed d-dimensional manifold then we think of it as a cobordism $W\colon \varnothing \rightsquigarrow \varnothing$ so there is a map

$$B\mathrm{Diff}(W) \longrightarrow \Omega_\varnothing B\mathcal{C}\mathrm{ob}_d \simeq \Omega^\infty \mathrm{MTO}(d)$$

where $\Omega_\varnothing$ denotes based loops at the object $\varnothing$. More generally if $\partial W = M$ and W has a Θ-structure extending a fixed one on M, there is a map

$$B\mathrm{Diff}^\Theta_\partial(W) \longrightarrow P_{\varnothing M} B\mathcal{C}\mathrm{ob}^\Theta_d$$

wbere $P_{\varnothing M}$ denotes path from $\varnothing$ to M. Of course, this path space can not be non-empty since the left side is not, and hence the right side is non-canonically equivalent to $\Omega B\mathcal{C}\mathrm{ob}^\Theta_d \simeq \Omega^\infty \mathrm{MT}\Theta(2n)$.

Given the correct choice of Θ for a given W, these turns out to capture a lot of the homology of classifying spaces of diffeomorphism groups. The general statement can be found in [GRW18, GRW17] but would take too long to state (see [GRW20] for an overview). Instead, let us focus on the important example of the $2n$-dimensional manifold

$$W_{g,1} := D^{2n} \#(S^n \times S^n)^{\#g}.$$

For $n = 1$ this is a surface of genus g with one boundary component, so you should think of $W_{g,1}$ has a higher-dimensional generalisation of such a surface.

Since S^{2n-1} is $(2n-2)$-connected, it has a θ-structure for $\theta\colon B\mathrm{O}(2n)\langle n\rangle \to B\mathrm{O}(2n)$ with $B\mathrm{O}(2n)\langle n\rangle$ the n-connective cover (killing π_i for $i \leqslant n$), which is unique up to contractible space of choices. Moreover, since $S^{2n-1} = \partial W_{g,1} \to W_{g,1}$ is $(n-1)$-connected, the space of extensions of this to $W_{g,1}$ is also contractible. That is, for this tangential structure Θ we have that

$$B\mathrm{Diff}^\Theta_\partial(W_{g,1}) \longrightarrow B\mathrm{Diff}_\partial(W_{g,1})$$

is an equivalence and hence the map $B\mathrm{Diff}_\partial(W_{g,1}) \to \Omega^\infty \mathrm{MTO}(2n)$ canonically lift to $\Omega^\infty \mathrm{MT}\Theta(2n)$. It is this lifted map that we can use to the compute the homology of $B\mathrm{Diff}_\partial(W_{g,1})$, at least in high dimensions. The following was proven in [GRW18]:

Theorem 3.3.1. *For $2n \geqslant 6$, the map*

$$H_*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Z}) \longrightarrow H_*(\Omega^\infty_0 MT\Theta(2n); \mathbb{Z})$$

is an isomorphism in degrees $* < \frac{g-3}{2}$. Here the subscript $(-)_0$ denote we restrict to the zero component.

Proof sketch. This combines two results, proven independently:

(1) *homological stability*: the homology is independent of g in range tending to ∞ with g,

(2) *stable homology*: a computation of the homology as $g \to \infty$.

For (1), one follows the usual strategy of proving homological stability. It is proven in [GRW18] (with improvements in [GRW17]) This studies the action of the group in question on the "complex of subobjects used in the stabilisation", which crucially needs to increases in connectivity with g. There is general technology to produce these complexes, and here they amount to a simplicial complex whose p-simplices are roughly $p+1$ disjoint embedded copies of $W_{1,1}$ in $W_{g,1}$, attached to the boundary. Using the Whitney trick, constructing these becomes a question of finding orthogonal hyperbolic summands in the intersection form, and the analogous algebraic complex was previously considered in the context of homological stability for unitary groups.

For (2), one reduces the size of $B\mathcal{C}\mathrm{ob}^\Theta_d$ until the only object is S^{2n-1} and the only morphisms are $W_{g,1}$'s. It proven in [GRW14] using [GRW10]. This is done by performing parametrised surgery in a simplicial space like $\Psi^\pitchfork_d([0,1]^{d+n-1} \times \mathbb{R})_\bullet$. ☐

Remark 3.3.2. Analogous results are true with tangential structure or local coefficients, when stabilising a different manifold than S^{2n}, or with different boundary. One caveat is that the stable homology computation ought to always work, but stability requires that the fundamental group has finite unitary stable rank as far as we know.

Rationally the right side is easy to understand.

Example 3.3.3. For any spectrum E, we have that

$$H^*(\Omega^\infty_0 E; \mathbb{Q}) \cong \mathbb{Q}[\mathrm{Hom}(\pi_{*>0}(E), \mathbb{Q}))].$$

Indeed, we only need that $\Omega^\infty_0 E$ is a path-connected H-space. Moreover, in spectra the map Hurewicz homomorphism $H^*(E; \mathbb{Q}) \to \mathrm{Hom}(\pi_*(E), \mathbb{Q})$ induced by the Hurewicz homomorphism, is an isomorphism. For a Thom spectra of an orientable virtual bundle, like $MT\Theta$, we can compute the latter using the Thom isomorphism.

Applying this in the case of Theorem 3.3.1, this yields

$$H^*(\Omega^\infty_0 \mathrm{MT}\Theta(2n); \mathbb{Q}) = \mathbb{Q}[H^{*-2n>0}(B\mathrm{O}(2n)\langle n \rangle)].$$

Using our knowledge of the cohomology of $B\mathrm{O}(2n)$, we can prove that this polynomial algebra is generated by classes κ_c of degree $|c| - 2n$ where c runs over monomials in the Euler class e and the Pontryagin classes p_i for $i > \frac{n}{4}$, satisfying $|c| - 2n > 0$. We conclude that

$$H^*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Q}) \cong \mathbb{Q}[\kappa_c] \qquad \text{for } * < \tfrac{g-3}{2}.$$

Remark 3.3.4. The classes κ_c are named so because as characteristic classes for $W_{g,1}$-bundles they have a geometric construction: they are *generalised Miller–Morita–Mumford classes.* Given a $W_{g,1}$-bundle $\pi\colon E \to M$ with trivialised boundary bundle, we do the following:

1. We glue in a trivial bundles $B \times D^{2n}$ to get $\overline{\pi}\colon \overline{E} \to B$.
2. We take the vertical tangent bundle $T_{\overline{\pi}}\overline{E}$.
3. We apply $c \in H^*(BO(2n);\mathbb{Q})$ to get $c(T_{\overline{\pi}}\overline{E}) \in H^{|c|}(\overline{E};\mathbb{Q})$.
4. We fibre integrate to get

$$\kappa_c \coloneqq \pi_!(c(T_{\overline{\pi}}\overline{E})) \in H^{|c|-2n}(B;\mathbb{Q}).$$

Here $\pi_!$ is the pushforward, which can be constructed by using the Serre spectral sequence or using the Becker–Gottlieb transfer in stable homotopy theory.

In general, the computation of the homology of the infinite loop spaces $\Omega_0^\infty \mathrm{MT}\Theta(2n)$ is quite hard; H_1 was mostly computed in [GRW16] with a final ambiguity resolved in [BHS19], and H_2 is work-in-progress by one of my students. Let us some of the former as a taste for the techniques involved (see [GRW16] for more details):

Example 3.3.5. We start with observing there is a map $\mathrm{MTO}(d) \to \Sigma^{-d}\mathrm{MO}$: this is explictly given by

$$\begin{aligned} \mathrm{MTO}(d)_{d+n} &= \mathrm{Th}(\gamma_{d,n}^{\perp} \to \mathrm{Gr}_d(\mathbb{R}^{d+n}) \\ &\xrightarrow{\cong} \mathrm{Th}(\gamma_{n,d} \to \mathrm{Gr}^n(\mathbb{R}^{d+n})) \\ &\to \mathrm{Th}(\gamma_{n,\infty} \to \mathrm{Gr}_n(\mathbb{R}^\infty)) = (\Sigma^{-d}\mathrm{MO})_{n+d}. \end{aligned}$$

and for tangential structure $\theta\colon BO(2n)\langle n\rangle \to BO(2n)$ there is a corresponding map $\mathrm{MT}\Theta(d) \to \Sigma^{-2n}\mathrm{MO}\langle n\rangle$ with the latter the Thom spectrum for the map $\mathrm{BO}\langle n\rangle \to \mathrm{BO}$. The pullback square

$$\begin{array}{ccc} \frac{O}{\mathrm{O}(2n)} \simeq \frac{\mathrm{O}\langle n\rangle}{\mathrm{O}(2n)\langle n\rangle} & \longrightarrow & BO(2n)\langle n\rangle \\ \downarrow & & \downarrow \\ * & \longrightarrow & BO\langle n\rangle \end{array}$$

is cartesian with bottom and right legs n-, resp. $2n$-connected. Hence by dual Blakers–Massey [MV15, Theorem 6.2.2] is it about $3n$-cocartesian: this is useful because Thom spectra are a homotopy colimit, and implies that when taking Thom spectra for minus the canonical bundle we get an about n-cocartesian diagram in spectra

$$\begin{array}{ccc} \Sigma^{\infty-2n}\frac{O}{\mathrm{O}(2n)}_+ & \longrightarrow & \mathrm{MT}\Theta(2n) \\ \downarrow & & \downarrow \\ \Sigma^{\infty-2n}*_+ & \longrightarrow & \Sigma^{-2n}\mathrm{MO}\langle n\rangle \end{array}$$

and hence there is a fibre sequence up to about degree n

$$\Sigma^{\infty-2n}\frac{\mathrm{O}}{\mathrm{O}(2n)} \longrightarrow \mathrm{MT}\Theta(2n) \longrightarrow \Sigma^{-2n}\mathrm{MO}\langle n\rangle.$$

Both the left and right sides have been studied. For the left, by the Freudenthal suspension theorem, up to about degree $2n$, its homotopy groups are those of the infinite Stiefel manifold $\mathrm{O}/\mathrm{O}(2n)$; we care about degree $2n+1$ and this can be looked up in [HM65] (it is 2-periodic in n, roughly as a consequence of James periodicity). For the right, highly-connected bordism groups were studied by Stolz [Sto85]: he constructed the fibre sequences

$$\mathbb{S} \longrightarrow \mathrm{MO}\langle n\rangle \longrightarrow \overline{\mathrm{MO}}\langle n\rangle \qquad \text{and} \qquad A[n+1] \longrightarrow \overline{\mathrm{MO}}\langle n\rangle \longrightarrow \mathrm{ko}\langle n\rangle$$

and computed the homotopy groups of $A[n+1]$ in degree just above $2n$ (it is 8-periodic in n). In the final answer there are a few cases to consider, some extensions to resolve, and some stable homotopy groups of spheres to insert as a black box. For example one can get

$$H_1(B\mathrm{Diff}_\partial(W_{g,1}^{12});\mathbb{Z}) \xrightarrow{\cong} \pi_1\,\mathrm{MT}\Theta(6) \cong (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 \qquad \text{for } g \geqslant 5.$$

Question 3.3.6. Can we generalise the computations of [GRW16] to compute the homology group $H_1(B\mathrm{Diff}_\partial(M\#W_{g,1});\mathbb{Z})$, i.e. the stable abelianisation of the mapping class groups? What about other tangential structures, boundaries, etc.?

Question 3.3.7. What is the $\mathbb{Z}$- or $\mathbb{F}_p$-(co)homology of $\Omega^\infty_0\mathrm{MT}\Theta(2n)$? Can we find interesting families of elements, or find useful qualitative properties?

Question 3.3.8. There is an equivariant version of the computation of the classifying space of the cobordism category in [GS21]. Is there also an equivariant generalisation for the homological stability result?

3.4 Stabilisation as a philosophy

Let us take a step back and look at the big picture. The two most powerful methods to access the classifying space $B\mathrm{Diff}_\partial(M)$ use the philosophy of *stabilisation*: there is an operation that induces an isomorphism on homotopy or homology groups in a range, and applying it infinitely many times brings the situation within the purview of homotopy theory or algebra.

The modern method was the focus of this lecture: it keeps the dimension fixed but makes more serious modifications to the manifolds. For simplicity we will focus on even dimension $d = 2n$, in which case the stabilisation operation is given by replacing M with the connected sum $M\#(S^n \times S^n)$. This induces an isomorphism on homology groups of $B\mathrm{Diff}_\partial(M)$ in a range depending on the number of $S^n \times S^n$ connected summands. Once we stabilise infinitely many times, the homology of stable diffeomorphism groups can be described in terms of certain Thom spectra.

There is also a classical method: it first builds an approximation to the diffeomorphism group in terms of *block diffeomorphisms*. It is designed to be understood in terms of homotopy theory and hermitian K-theory, by *surgery theory*. The difference between diffeomorphisms and block diffeomorphisms can be described in terms of *concordance diffeomorphisms* $\mathrm{C}(M)$. On these, there is a stabilisation operation replacing M by $M \times I$, and it induces an isomorphism on homotopy groups of $B\mathrm{C}(M)$ in a range depending on the dimension. Once we stabilise infinitely many times, the homotopy of stable concordance diffeomorphisms can be described in terms of algebraic K-theory of spaces. See [WW01] for an overview and references.

Chapter 4

Applications to diffeomorphisms of discs

In this last lecture we do an extended example, by combining the embedding calculus of the second lecture with the cobordism categories of the third lecture using the fibre sequences of the first lecture, to prove some results about diffeomorphisms of even-dimensional discs.

4.1 An easier problem: finiteness

Our goal will be to prove the following result [Kup19]:

Theorem 4.1.1. *If $2n \geqslant 6$, then the homotopy groups of $B\mathrm{Diff}_\partial(D^{2n})$ are degree-wise finitely-generated.*

Example 4.1.2. We learned in Example 1.2.8 that $\pi_0\,\mathrm{Diff}_\partial(D^d) \cong \pi_0\,\mathrm{Diff}^+(S^d)$ where $(-)^+$ denotes the restriction to orientation-preserving diffeomorphism. Such a diffeomorphism $f\colon S^d \to S^d$ can be used to produce an oriented homotopy $(d+1)$-sphere

$$\Sigma_f := D^{d+1} \cup_f D^{d+1}.$$

Its diffeomorphism and hence h-cobordism class only depends on the isotopy class of f so we get a map $\pi_0\,\mathrm{Diff}_\partial(D^d) \to \Theta_{d+1}$, the group of oriented homotopy $(d+1)$-spheres up to h-cobordism. If $d \geqslant 5$, this is a bijection: by the h-cobordism theorem it is a surjection and by Cerf's theorem it is a bijection. These groups Θ_{d+1} were computed by Kervaire–Milnor in terms of the stable homotopy groups of spheres [KM63]; they in particular showed that the Θ_{d+1} are finite.

Example 4.1.3. Historically, this was known only in a range using the classical methods described at the end of the previous lecture. A clean way to state the results is that there is an approximation $\widetilde{\mathrm{Diff}}_\partial(D^d)$ to the diffeomorphism having by design the property

$$\pi_i \widetilde{\mathrm{Diff}}_\partial(D^d) \cong \Theta_{d+i+1}$$

and then Weiss–Williams [WW01] gave an approximately $d/3$-connected map

$$\frac{\widetilde{\mathrm{Diff}}_\partial(D^d)}{\mathrm{Diff}_\partial(D^d)} \longrightarrow \Omega^\infty(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)_{hC_2})$$

where $A(X) \simeq \mathrm{Wh}^{\mathrm{Diff}}(X) \vee \Sigma^\infty X_+$ is Waldhausen's algebraic K-theory of spaces, and $(-)_{hC_2}$ are its homotopy orbits with respect to an involution that depends on the dimension d. In a range, the result then follows from the previous example and the fact that $A(*) = K(\mathbb{S})$ has finitely-generated homotopy groups by a result of Dwyer [Dwy80].

4.1.1 Some facts about Serre classes

To prove Theorem 4.1.1, we need to transfer various finiteness properties between spaces. This may be done by the tools of Serre class, in this case the Serre class of finitely-generated abelian groups. We will suffice for us to consider the following class of spaces:

Using the long exact sequence of homotopy groups and Serre spectral sequence, one may prove:

Lemma 4.1.4. *Suppose that $F \to E \to B$ is a fibre sequence of path-connected spaces with abelian fundamental groups. If any two have finitely-generated homotopy groups in degrees $i \leqslant k$ then the remaining one does in degrees $i \leqslant k-1$.*

Lemma 4.1.5. *Suppose that $F \to E \to B$ is a fibre sequence of path-connected spaces so that $\pi_1(B)$ acts trivially on $H_*(F;\mathbb{Z})$. If any two have finitely-generated homology groups in degrees $i \leqslant k$, then the remaining one does in degrees $i \leqslant k-1$.*

Exercise 7. What can go wrong if the assumptions on the fundamental groups in these lemma's are not satisfied?

The following is a great tool for proving that a simple space has degreewise finitely-generated homotopy groups:

Definition 4.1.6. A space is *simple* if each path-component has abelian fundamental group acting trivially on the higher homotopy groups.

Example 4.1.7. Every 1-connected space is simple, as as is any 0-connected H-space.

Lemma 4.1.8. *If X is a simple space then the following are equivalent:*

1. *The homotopy groups $\pi_i X$ for $i \leqslant k$ are finitely generated.*
2. *The homology groups $H_i(X;\mathbb{Z})$ for $i \leqslant k$ are finitely generated.*

Example 4.1.9. All spheres are simple and have degreewise finitely-generated homology groups, so also have degreewise finitely-generated homotopy groups. By induction over the fibre sequences $S^{d-1} \to B\mathrm{O}(d-1) \to B\mathrm{O}(d)$ it follows that $B\mathrm{O}(d)$ has degreewise finitely-generated homotopy groups.

4.1.2 Setup

It is helpful to add framings to the problem, as they make various computations smaller. However, they do add a few minor issues involving restricting to certain path components—or the classifying space equivalent, passing to certain covers—which I will ignore for brevity.[1]

Let us start by making an interesting observation. Thinking of D^{2n} as sitting inside $\mathbb{R}^{2n}$ as usual, the standard framing of $\mathbb{R}^{2n}$ restricts to one near $\partial D^{2n} \subset \mathbb{R}^{2n}$, which will serve as our boundary condition. We can do the same for the annulus $S^{n-1} \times [0,1]$. Then the inclusion gives a map

$$\begin{gathered} B\mathrm{Diff}^{\mathrm{fr}}_\partial(S^{2n-1} \times [0,1]) \simeq \mathrm{Map}_\partial(S^{2n-1} \times [0,1], \mathrm{O}(2n)) /\!\!/ \mathrm{Diff}_\partial(D^{2n}) \\ \downarrow \\ B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^{2n}) \simeq \mathrm{Map}_\partial(D^{2n}, \mathrm{O}(2n)) /\!\!/ \mathrm{Diff}_\partial(D^{2n}). \end{gathered}$$

Its homotopy fibre can be computed using the isotopy extension as $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^{2n}, D^{2n})$, the space of embeddings together with a homotopy from their derivative to the identity. But since embeddings of Euclidean spaces are determined by their derivative, this space is contractible! We conclude that:

Lemma 4.1.10. $B\mathrm{Diff}^{\mathrm{fr}}_\partial(S^{2n-1} \times [0,1]) \simeq B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^{2n})$.

[1] The source of problems is that in general one should expect the set of framings up to homotopy and diffeomorphism, if non-empty, to have more than one element. However, due to a coincidence this does not occur for $W_{g,1}$ in many dimensions $2n$ when g is sufficiently large, by [KRW21b].

This means that we may as well study $B\mathrm{Diff}^{\mathrm{fr}}_\partial(S^{2n-1}\times[0,1])$ if we care about $B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^{2n})$. The latter will suffice to prove Theorem 4.1.1, since there is a fibre sequence

$$\mathrm{Map}_\partial(D^{2n},\mathrm{O}(2n)) \simeq \Omega^{2n}\mathrm{O}(2n) \longrightarrow B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^{2n}) \longrightarrow B\mathrm{Diff}_\partial(D^{2n}),$$

and we can invoke the long exact sequence of homotopy groups as in Lemma 4.1.4 using that $\Omega^{2n}\mathrm{O}(2n)$ has degreewise finitely-generated homotopy groups by Example 4.1.9.

Recall that for a manifold M with boundary ∂M, an elaboration of the isotopy extension gave us a Weiss fibre sequence $B\mathrm{Diff}_\partial(M) \to B\mathrm{Emb}^{\cong}(M,M) \to B^2\mathrm{Diff}_\partial(\partial M\times[0,1])$ and we can add in framings to get a variation

$$B\mathrm{Diff}^{\mathrm{fr}}_\partial(M) \longrightarrow B\mathrm{Emb}^{\mathrm{fr},\cong}(M,M) \longrightarrow B^2\mathrm{Diff}_\partial(\partial M\times[0,1]).$$

Taking $M = W_{g,1}$ with $\partial W_{g,1} = S^{2n-1}$, this yields

$$B\mathrm{Diff}^{\mathrm{fr}}_\partial(W_{g,1}) \longrightarrow B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1},W_{g,1}) \longrightarrow B^2\mathrm{Diff}_\partial(S^{2n-1}\times[0,1]). \tag{4.1}$$

Our strategy will to prove that the homology of the fibre and total space are both degreewise finitely-generated, and then use a Serre spectral sequence argument to draw conclusions about the base.

4.1.3 The cobordism category step

By the framed version of Theorem 3.3.1 there is a map

$$B\mathrm{Diff}^{\mathrm{fr}}_\partial(W_{g,1}) \longrightarrow \Omega^\infty_0\mathbb{S}^{-2n} \tag{4.2}$$

which induces an isomorphism on homology in degrees $\leqslant \frac{g-3}{2}$. From this we will conclude:

Proposition 4.1.11. *The homology group $H_i(B\mathrm{Diff}^{\mathrm{fr}}_\partial(W_{g,1};\mathbb{Z}))$ is finitely-generated for $i \leqslant \frac{g-3}{2}$.*

Proof. Using the fact that (4.2) induces an isomorphism on homology in this range, it suffices to prove that $H_*(\Omega^\infty_0\mathbb{S}^{-2n};\mathbb{Z})$ is degreewise finitely-generated. It is a simple space because it is a path-connected loop space (in fact an infinite loop space) and its homotopy groups are $\pi_i\,\Omega^\infty_0\mathbb{S}^{-2n} = \pi_{i+2n}\mathbb{S}$ for $i>0$; these stable homotopy groups of spheres are finitely-generated by a theorem of Serre. Now apply Lemma 4.1.8. □

4.1.4 The embedding calculus step

For the homology groups of $B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1},W_{g,1})$, we will need to separately look at its fundamental group and higher homotopy groups, and then draw the following conclusion about its homology:

Proposition 4.1.12. *The homology groups $H_*(B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1},W_{g,1});\mathbb{Z})$ are degreewise finitely-generated.*

The higher homotopy groups

We first study its homotopy groups and prove:

Proposition 4.1.13. *The homotopy groups $\pi_i\,\mathrm{Emb}^{\mathrm{fr}}(W_{g,1},W_{g,1})$ are finitely-generated for $i>0$, with any basepoint.*

Proof. There will be three steps.

Step (1): reduction to a finite embedding calculus stage. To understand these homotopy groups we use the embedding calculus. The framings change the setup little: as a $\mathcal{J}_1$-sheaf, they do not affect convergence nor any layers except the first: bundle maps are replaced by maps. Thus the tower converges when $2n \geqslant 3$, since $\mathrm{hdim}(W_{g,1}) = n$ and $\dim(W_{g,1}) = 2n$. To see the former, first build $W_{1,1}$ by attaching two n-handles to a disc. Next observe that $W_{g,1}$ is a g-fold iterated boundary connected sum $\natural_g W_{g,1}$ and that $\mathrm{hdim}(M \natural N) = \max(\mathrm{hdim}(M), \mathrm{hdim}(N))$. Moreover, the framed embedding calculus tower convergences quantitatively, in the sense that the layers of the embedding calculus tower, given by the framed analogue of (2.4), become more highly-connected. This means that when computing

$$\pi_i \,\mathrm{Emb}^{\mathrm{fr}}(W_{g,1}, W_{g,1}) \xrightarrow{\cong} \pi_i \, T_k \mathrm{Emb}^{\mathrm{fr}}(W_{g,1}, W_{g,1})$$

for k sufficiently large. We will prove the latter is finitely-generated by induction over k.

Step (2): the initial case and the first stage. The initial case is the first stage, given by the space of maps $\mathrm{Map}(W_{g,1}, W_{g,1})$. We will prove that its homotopy groups are degreewise finitely-generated.

Recalling that $W_{g,1}$ is homotopy equivalent to $\vee_{2g} S^n$, it is 1-connected and hence simple. Moreover, its homology groups are be degreewise finitely-generated, so by Lemma 4.1.8 so are its homotopy groups. Now apply Lemma 4.1.4 to the fibre sequence

$$\mathrm{Map}_*(W_{g,1}, W_{g,1}) \longrightarrow \mathrm{Map}(W_{g,1}, W_{g,1}) \longrightarrow W_{g,1},$$

using that the left term is equivalent to $\prod_{2g} \Omega^n W_{g,1}$.

Step (3): the induction step and the layers. The induction step will follow from the long exact sequence of the fibration sequence

$$\{\text{section space (2.5) for cardinality } k\} \longrightarrow T_k \mathrm{Emb}^{\mathrm{fr}}(W_{g,1}, W_{g,1}) \longrightarrow T_{k-1} \mathrm{Emb}^{\mathrm{fr}}(W_{g,1}, W_{g,1}),$$

once we establish that the section space has degreewise finitely-generated homotopy groups.

To do so, recall from Remark 2.3.2 that the pair $(C_k(W_{g,1}), C_k(W_{g,1}))$ is equivalent to the pair $(\overline{C}_k(W_{g,1}), \partial \overline{C}_k(W_{g,1}))$ of a compact topological manifold and its boundary, which admits the structure of a finite CW-pair. By an induction over the cells, it suffices to prove that fibres of the fibration (2.5) have degreewise finitely-generated homotopy groups.

Recall that these may be written as the total homotopy fibre

$$\mathrm{tohofib}_x(\underline{k} \supset T \mapsto \mathrm{Conf}_{\underline{k} \setminus T}(W_{g,1}))$$

of a k-cube of configuration spaces. As a total homotopy fibres can be computed by taking iterated homotopy fibres in each of the k directions, it suffices to prove that the configuration spaces $\mathrm{Conf}_k(W_{g,1})$ has degreewise finitely-generated homotopy groups. This follows by induction over the fibre sequence

$$W_{g,1} \backslash \{k{-}1 \text{ points}\} \longrightarrow \mathrm{Conf}_k(W_{g,1}) \longrightarrow \mathrm{Conf}_{k-1}(W_{g,1})$$

and the observation that $W_{g,1} \backslash \{k{-}1 \text{ points}\} \simeq (\vee_{2g} S^n) \vee (\vee_{k-1} S^{2n-1})$. Having completed the induction step, this finishes the proof. □

Exercise 8. Fill in the details of Step (3) of the previous proof.

The fundamental group

We will need the following notion:

Definition 4.1.14. A group Γ is of *type* F_∞ if it has a classifying space that is a CW complex with finitely many cells in each dimension.

These groups have the following property, crucial for us: if M is a $\mathbb{Z}[\Gamma]$-module that is finitely-generated as an abelian group—which we can think of a local system on $B\Gamma$—then the homology groups $H_*(B\Gamma; M)$ with local coefficients are degreewise finitely-generated.

Exercise 9. Prove this fact.

Example 4.1.15.

- The group $\mathbb{Z}$ is of type F_∞ since it has the circle as a classifying space.
- Any finite group is of type F_∞ since its bar construction is a finite CW complex.
- If $\mathbb{G}$ is a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ cut out by finitely many polynomial equation in the entries and the determinant, then $\mathbb{G} \cap \mathrm{GL}_n(\mathbb{Z})$ is of type F_∞ [BS73] (see also [Ser79]).

This notion behaves well with respect to passing to finite subgroups and quotients.

Lemma 4.1.16. *If $\Gamma \to \Gamma'$ has finite index and kernel, then Γ is of type F_∞ if and only if Γ' is of type F_∞.*

Exercise 10. Prove Lemma 4.1.16.

Every self-embedding $W_{g,1} \to W_{g,1}$ acts on middle-dimensional homology preserving the intersection form, yielding a map of groups

$$\alpha_g \colon \pi_1 \, B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1}) \longrightarrow \begin{cases} \mathrm{Sp}_{2g}(\mathbb{Z}) & \text{if } n \text{ is odd,} \\ \mathrm{O}_{g,g}(\mathbb{Z}) & \text{if } n \text{ is even.} \end{cases}$$

Here $\mathrm{Sp}_{2g}(\mathbb{Z})$ is the *symplectic group* of matrices with integer entries preserving the g-fold direct sum of the antisymmetric bilinear form encoded by the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathrm{O}_{g,g}(\mathbb{Z})$ is the *(hyperbolic) orthogonal group* of matrices with integer entries preserving the g-fold direct sum of the symmetric bilinear form encoded by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Note both are instances of groups as in the third part of Example 4.1.15, so of type F_∞.

Combining the Weiss fibre sequence (4.1) with a result of Kreck [Kre79], one may prove that the map α_g has finite index and finite kernel. Using Lemma 4.1.16, we conclude that:

Proposition 4.1.17. *The group $\pi_1 \, B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1})$ is of type F_∞.*

Remark 4.1.18. Kreck's result can be replaced by one of Sullivan [Sul77] saying that for general closed 1-connected manifolds M the group $\pi_0 \, \mathrm{Diff}(M)$ is of type F_∞.

Question 4.1.19. Can Kreck's or Sullivan's result be proven by embedding calculus?

The homology groups

At this point we know two facts about the space of framed self-embeddings of $W_{g,1}$:

- The higher homotopy groups of $B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1})$ are degreewise finitely-generated.
- The fundamental group Γ of $B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1})$ is of type F_∞.

Proof of Proposition 4.1.12. The first point implies, using Lemma 4.1.8, that the universal cover $\overline{B\mathrm{Emb}}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1})$ of $B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1})$ has degreewise finitely-generated homology groups. Thus when we run a Serre spectral sequence for the fibre sequence

$$\overline{B\mathrm{Emb}}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1}) \longrightarrow B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1}) \longrightarrow B\Gamma$$

given by the universal cover, which has the form

$$E^2_{pq} = H_p(B\Gamma; H_q(\overline{B\mathrm{Emb}}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1}); \mathbb{Z})) \Longrightarrow H_{p+q}(B\mathrm{Emb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1}); \mathbb{Z}),$$

all entries on the left are finitely-generated, and hence so is the abutment on the right. □

4.1.5 Conclusions

What have we done so far? We have constructed a fibre sequence (4.1)

$$B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(W_{g,1}) \longrightarrow B\mathrm{Emb}^{\cong,\mathrm{fr}}(W_{g,1}, W_{g,1}) \longrightarrow B^2\mathrm{Diff}^{\mathrm{fr}}_{\partial}(S^{2n-1} \times [0,1])$$

whose fibre has degreewise finitely-generated homology groups by Proposition 4.1.11 and whose total space has finitely-generated homology in degrees $\leqslant \frac{g-3}{2}$ by Proposition 4.1.12.

As the base is visibly 1-connected, we can use Lemma 4.1.5 to deduce that the homology groups of the base $B^2\mathrm{Diff}^{\mathrm{fr}}_{\partial}(S^{2n-1} \times [0,1])$ are finitely-generated in degrees $\leqslant \frac{g-5}{2}$. By Lemma 4.1.8 its homotopy groups are finitely-generated in the same range; looping once we get that

$$\pi_i\, B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(S^{2n-1} \times [0,1]) \text{ is finitely generated for } i \leqslant \tfrac{g-7}{2}.$$

Now comes a neat trick: this space does not depend on g, so we can let it go to ∞ and conclude that *all* the homotopy groups of $B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(S^{2n-1} \times [0,1])$ are finitely-generated. As explained in Section 4.1.2, this implies Theorem 4.1.1.

4.2 A harder problem: rational homotopy

I now want to outline what happens if you try to make the previous arguments quantitative, at least over the rationals. This is contained in [KRW20c, KRW20a, KRW20b, KRW21c]; an extension to the odd-dimensional case—or rather, the concordances of the even-dimensional case—can be found in [KRW21a].

4.2.1 Examples

Before stating the answer, I will describe some known rational homotopy groups of $B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(D^{2n})$.

Example 4.2.1 (Clasper bundles and configuration space integrals). Kontsevich constructed invariants of homology 3-spheres Σ as follows [Kon94] (see [Les20] for a comprehensive discussion). These are always orientable because w_1 lies in a trivial group, and hence parallellisable. The analogous compactification $\overline{\mathrm{Conf}}_2(\Sigma)$ of $\mathrm{Conf}_2(\Sigma)$ to the one for unordered configuration spaces mentioned in Remark 2.3.2, has the property that $\partial\overline{\mathrm{Conf}}_2(\Sigma)$ is homeomorphic to the unit sphere bundle $S(T\Sigma)$, whence there is a map $p\colon S(T\Sigma) \cong \Sigma \times S^2 \to S^2$. A *propagator* for Σ is a closed 2-form on $\overline{\mathrm{Conf}}_2(\Sigma)$ extending $p^*\mathrm{vol}_{S^2}$.

Then, given at least-trivalent finite graph Γ without self-loops, set of vertices V with given bijection to $\underline{k}$, and set of directed edges E, we can write a closed differential form

$$\omega_\Gamma \coloneqq \bigwedge_{e \in E} p^*_{s(e)t(e)}\omega \in \Omega^{2|E|}(\overline{\mathrm{Conf}}_k(\Sigma); \mathbb{R}).$$

If $2|E| = 3|V|$, we can then integrate this. This may depend on choices, but it does not for certain Γ that are linear combinations of graphs: the condition is that $d\Gamma = 0$ where d sends a graph to the sum over all graphs obtained by collapsing an edge, with certain signs (having to do with the order of the vertices and the direction of the edges). This can be phrased in terms of a complex $\mathrm{GC}_3^\vee$ of graphs: there is a map

$$H^0(\mathrm{GC}_3^\vee) \longrightarrow \{\mathbb{R}\text{-valued invariants of integral homology 3-spheres}\}.$$

That these invariants are non-trivial was proven by evaluating them on the results of clasper surgeries.

Kontsevich also observed that this generalises to a map

$$H^*(\mathrm{GC}_d^\vee) \longrightarrow H^*(B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(\Delta); \mathbb{R})$$

for Δ an integral homology d-ball; in families the framings and a choice of chart near a point become important. Watanabe then generalised the clasper surgery construction to give bundles over spheres on which the invariants coming from the trivalent part of graph homology are non-trivial, with a few exceptions in low degrees. The informal way of stating this is that there is a nearly injective map

$$H_*(\mathrm{GC}_d)_{\mathrm{trivalent}} \longrightarrow \pi_*(B\mathrm{Diff}^{\mathrm{fr}}_\partial(\Delta);\mathbb{R}).$$

Question 4.2.2. Do these configuration space integral invariants factor over the limit of the embedding calculus Taylor tower?

Example 4.2.3 (Morlet's theorem and topological Pontryagin classes)*.* The space of smooth structures on a topological manifold is a $\mathcal{J}_1$-sheaf when $d \neq 4$: this is *smoothing theory* [KS77]. More precisely, it concerns the presheaf $\mathrm{Sm}\colon \mathcal{M}\mathrm{an}_d^{\mathrm{op}} \to \mathcal{S}$ that sends U to the simplicial set whose k-simplices are commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad\cong\quad} & U \times \Delta^k \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & \Delta^k & \end{array}$$

where p is a smooth submersion and the horizontal map is a homeomorphism. Picking a representative of each smooth structure on U up to diffeomorphism, using the Kirby–Siebenmann bundle theorem this is non-canonically equivalent to $\bigsqcup_\sigma \mathrm{Homeo}(U)/\mathrm{Diff}(U_\sigma)$. For $U = \mathbb{R}^d$ we thus get $\mathrm{Top}(d)/\mathrm{O}(d)$ (using the non-obvious fact that $\mathbb{R}^d$ admits a unique smooth structure when $d \neq 4$). Using this for $U = D^d$ relatively to the boundary, we get

$$\bigsqcup_\sigma \frac{\mathrm{Homeo}_\partial(D^d)}{\mathrm{Diff}_\partial(D^d_\sigma)} \simeq \mathrm{Sm}_\partial(D^d) \simeq \Omega^d \frac{\mathrm{Top}(d)}{\mathrm{O}(d)}.$$

Observing that $\mathrm{Homeo}_\partial(D^d) \simeq *$ by the Alexander trick, and restricting to the component of the standard smooth structure, we recover Morlet's theorem that

$$B\mathrm{Diff}_\partial(D^d) \simeq \Omega^d_0 \frac{\mathrm{Top}(d)}{\mathrm{O}(d)} \qquad \text{for } d \neq 4.$$

In particular, any difference in rational homotopy groups between $B\mathrm{O}(d)$ and $B\mathrm{Top}(d)$ in degrees $> d$, gives non-trivial rational homotopy groups of $B\mathrm{Diff}_\partial(D^d)$.

Such differences were found by Weiss. You can use smoothing theory to prove $BO \to B\mathrm{Top}$ is a rational equivalence, so $H^*(B\mathrm{Top};\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \cdots]$ and we can define rational Pontryagin classes in $H^*(B\mathrm{Top}(d);\mathbb{Q})$ by pullback. In [Wei21], Weiss proved that many of these are non-trivial and detected on the rational Hurewicz image. In fact, all these cohomology classes are algebraically independent [GRW22] and all are detected on the rational Hurewicz image (forthcoming joint work with Krannich). Taking into account framings, an informal way of stating this is that there is a surjection

$$\pi_* \, B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^d) \longrightarrow \pi_{*+d}\,\mathrm{Top} \qquad \text{for } d \geqslant 6.$$

Question 4.2.4. What happens in dimension $d = 4, 5$?

4.2.2 Statement

The main theorem of [KRW20b] is that to up degree about $4n$, these are the *only* rational homotopy groups, and above degree about $4n$ that is the case outside of bands of degrees just below multiples of $2n$. We strongly encourage the reader to look at the visual summary in Fig. 4.1, but the precise statement is:

Theorem 4.2.5 (K.–Randal-Williams)**.** *Let* $2n \geqslant 6$*.*

- *In degree $k < 2n-1$ we have*

$$\pi_k\, B\mathrm{Diff}_\partial(D^{2n}) \otimes \mathbb{Q} = 0$$

- *In degrees $k \geqslant 2n-1$ we have*

$$\pi_k\, B\mathrm{Diff}_\partial(D^{2n}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \textit{if } k \equiv 2n{-}1 \textit{ mod } 4,\ k \notin \bigcup_{r\geqslant 2} [2r(n{-}2){-}1, 2r(n{-}1){+}1], \\ 0 & \textit{if } k \not\equiv 2n{-}1 \textit{ mod } 4,\ k \notin \bigcup_{r\geqslant 2} [2r(n{-}2){-}1, 2r(n{-}1){+}1], \\ ? & \textit{otherwise.} \end{cases}$$

Remark 4.2.6. I have only included the results whose proofs have been currently written up. The expectation, to be proven in joint work with Oscar Randal-Williams and Thomas Willwacher, is that the map

$$B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^{2n}) \simeq \Omega^{2n}\mathrm{Top}(2n) \longrightarrow \Omega^{2n}\mathrm{Top} \times \Omega^{2n}\mathrm{Aut}(E^{\mathbb{Q}}_{2n})$$

is a rational equivalence. By [FTW17, FW20], the rational homotopy groups of $\mathrm{Aut}(E^{\mathbb{Q}}_{2n})$ are given by the homology of the graph complex GC_{2n}.

Question 4.2.7. Can we use the study of diffeomorphism groups to prove results about the graph homology groups $H_*(\mathrm{GC}_{2n})$?

4.2.3 Strategy

The main difficulty with using (4.1) to comput the rational homotopy groups of $B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^{2n})$, or rather those of $B\mathrm{Diff}^{\mathrm{fr}}_\partial(S^{2n-1} \times [0,1])$, is that the fundamental group of $B\mathrm{Diff}^{\mathrm{fr}}_\partial(W_{g,1})$ is quite large. Indeed, by the result of Kreck mentioned in Section 4.1.4, it differs from $\mathrm{Sp}_{2g}(\mathbb{Z})$ or $\mathrm{O}_{g,g}(\mathbb{Z})$ by finite groups. The map from the fundamental group to these arithmetic groups was obtained by letting diffeomorphisms act $H_n(W_{g,1};\mathbb{Z})$, so we can get rid of this arithmetic part by passing to *Torelli groups*: the subgroups of those path components acting as the identity on this homology group. The variant of (4.1) obtained this way looks like

$$B\mathrm{Tor}^{\mathrm{fr}}_\partial(W_{g,1}) \longrightarrow B\mathrm{TorEmb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1}) \longrightarrow B^2\mathrm{Diff}^{\mathrm{fr}}_\partial(S^{2n-1} \times [0,1]).$$

Let us rotate it once to get instead:

$$B\mathrm{Diff}^{\mathrm{fr}}_\partial(S^{2n-1} \times [0,1]) \longrightarrow B\mathrm{Tor}^{\mathrm{fr}}_\partial(W_{g,1}) \longrightarrow B\mathrm{TorEmb}^{\mathrm{fr},\cong}(W_{g,1}, W_{g,1}).$$

We will now split $B\mathrm{Tor}^{\mathrm{fr}}_\partial(W_{g,1})$ into two pieces. It turns out that by both restricting to diffeomorphisms acting as the identity on homology and adding framings, we did "too much." Borel computed the stable rational cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{O}_{g,g}(\mathbb{Z})$ as $g \to \infty$. They are given polynomial algebras on generators x_{4i-2n} and the family signature theorem tells one that the pullback of x_{4i-2n} to $B\mathrm{Diff}_\partial(W_{g,1})$ is equal to the generalised Miller–Morita–Mumford class $\kappa_{\mathcal{L}_i}$ with $\mathcal{L}_i \in H^{4i}(B\mathrm{O}(2n)\langle n\rangle;\mathbb{Q})$ the ith Hirzebruch L-polynomial. Passing to Torelli groups kills x_{4i-2n} and adding framings kills $\kappa_{\mathcal{L}_i}$, so passing to framed Torelli groups kills the same classes twice yielding secondary classes. Let us define a space X_0 as a product of Eilenberg–MacLane spaces $\prod_i K(\mathbb{Q}, 4i-2n+1)$ and the map

$$B\mathrm{Tor}^{\mathrm{fr}}_\partial(W_{g,1}) \longrightarrow X_0$$

is given by these secondary classes. We then define a space $X_1(g)$ as its homotopy fibre; think of it as a "sanitised" version of $B\mathrm{Tor}^{\mathrm{fr}}_\partial(W_{g,1})$. This extends our fibre sequence to a fundamental diagram

$$\begin{array}{ccccc} & & X_1(g) & \dashrightarrow & \\ & & \downarrow & & \\ B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(S^{2n-1}\times[0,1]) & \longrightarrow & B\mathrm{Tor}^{\mathrm{fr}}_{\partial}(W_{g,1}) & \longrightarrow & B\mathrm{TorEmb}^{\mathrm{fr},\cong}(W_{g,1},W_{g,1}) \\ & \dashrightarrow & \downarrow & & \\ & & X_0 = \prod_i K(\mathbb{Q}, 4i-2n+1) & & \end{array}$$

Our strategy is to compute the rational homotopy groups of $X_1(g)$ and $B\mathrm{TorEmb}^{\mathrm{fr},\cong}(W_{g,1},W_{g,1})$ as $g\to\infty$.

Exercise 11. Prove that the homotopy fibres of the two dashed maps are equivalent.

The embedding calculus step

To get at $B\mathrm{TorEmb}^{\mathrm{fr},\cong}(W_{g,1},W_{g,1})$ we use the same strategy as for the proof of Proposition 4.1.13 but proceed with more care. Here are the two ideas that go into this:

- It is helpful to consider instead embeddings that fix half the boundary. Even though these have the same homotopy type, the embedding calculus tower is different and has slightly smaller layers.
- It is important to keep track of the action of the fundamental group on the higher homotopy groups, because the connecting homomorphism to $\pi_{*-1}\, B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(S^{2n-1}\times[0,1])$ must be zero on anything but the trivial representations.
- We know a lot about the homology of configuration spaces, making it possible to compute the homology and homotopy groups of $C_k(W_{g,1})$ explicitly.

The cobordism category step

It is a qualitative application of embedding calculus similar to Theorem 4.1.1 that $X_1(g)$ is a nilpotent space. This means that there is an usntable Adams spectral sequence

$$E^2_{pq} = H^r_{\mathrm{Com}}(H^*(X_1(g);\mathbb{Q}))_s \Longrightarrow \mathrm{Hom}(\pi_{s-r-1}\, X_1(g),\mathbb{Q}),$$

where H^*_{Com} denotes commutative algebra cohomology, also known as Harrison or André–Quillen cohomology. This reduces the computation to two steps:

- Another qualitative application of embedding calculus is that the cohomology of $X_1(g)$ consists of algebraic representations of the fundamental group Γ' of $B\mathrm{Tor}^{\mathrm{fr}}_{\partial}(W_{g,1})$. Recalling that this is essentially $\mathrm{Sp}_{2g}(\mathbb{Z})$ or $\mathrm{O}_{g,g}(\mathbb{Z})$, an algebraic representation is one that is a subquotient of a sums of tensor powers of the standard representation V_1. Thus as $g\to\infty$, we can read off $H^*(B\mathrm{Tor}^{\mathrm{fr}}_{\partial}(W_{g,1});\mathbb{Q})$ from
$$(H^*(B\mathrm{Tor}^{\mathrm{fr}}_{\partial}(W_{g,1});\mathbb{Q})\otimes V_1^{\otimes m})^{\Gamma'}.$$
- It follows from work of Borel that the cohomology of Γ' with coefficients in a non-trivial algebraic representations vanishes as $g\to\infty$. By a Serre spectral sequence, we thus have
$$H^*(B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(W_{g,1}); V_1^{\otimes m}) \cong (H^*(B\mathrm{Tor}^{\mathrm{fr}}_{\partial}(W_{g,1});\mathbb{Q})\otimes V_1^{\otimes m})^{\Gamma'}.$$
- The left side in this isomorphism can be accessed as $g\to\infty$ by an application of the variant of Theorem 3.3.1 for a custom tangential structure, given by $B\mathrm{O}(2n)\langle n\rangle \times K(V,n) \longrightarrow B\mathrm{O}(2n)$ considered as a functor of a finite-dimensional vector space V.

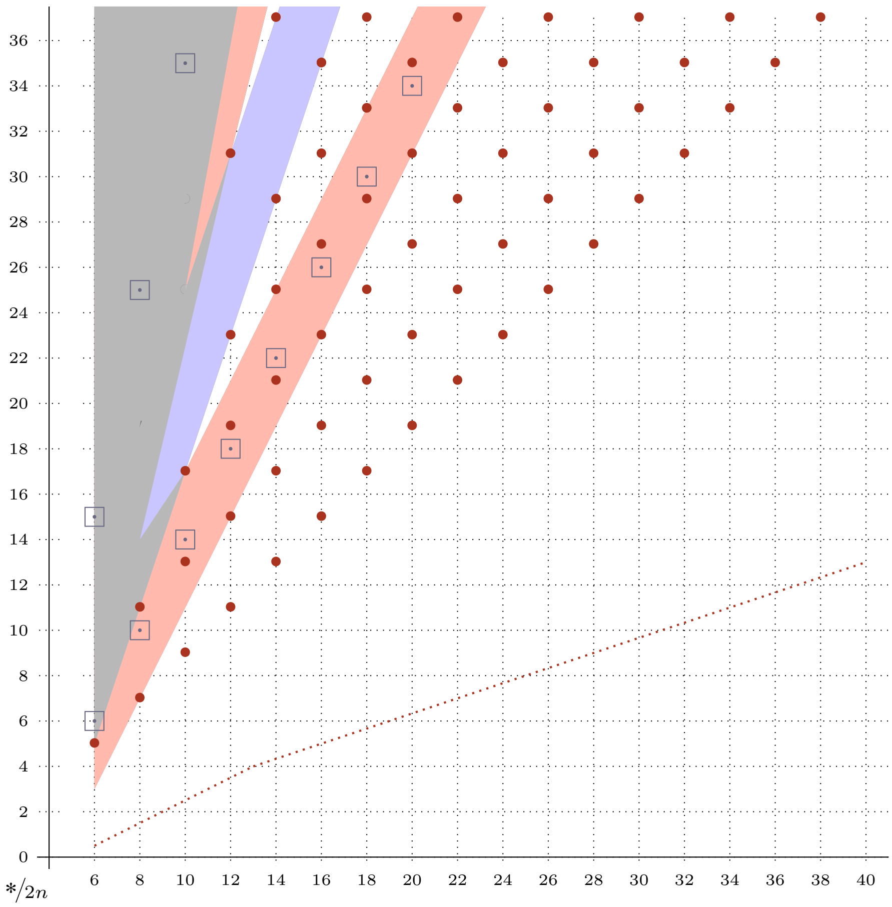

Figure 4.1 The rational homotopy groups $\pi_* B\mathrm{Diff}_\partial(D^{2n}) \otimes \mathbb{Q}$. In this figure, we have: (i) Dots are $\mathbb{Q}$'s, given by Weiss' "surreal" Pontryagin classes (work-in-progress, joint with Krannich: every four degrees). (ii) Boxes are $\mathbb{Q}$'s Watanabe's classes. (iii) Bands of uncertainty (work-in-progress, joint with Randal-Williams and Willwacher: these only contain graph homology. (iv) Colours indicate how the involution on $\mathrm{Diff}_\partial(D^{2n})$ given by reflection acts.

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