THE FIRST TWO k-INVARIANTS OF Top/O

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Abstract. We show that the first two k-invariants of Top/O vanish and give some applications.

In this note we investigate the k-invariants of the infinite loop space Top/O, which plays an important role in smoothing theory of topological manifolds. Its homotopy groups $\pi_k(\mathrm{Top}/\mathrm{O})$ are given by $\mathbb{Z}/2$ if $k = 3$ and the group Θ_k of oriented homotopy k-spheres otherwise. We will prove that its first two k-invariants vanish, which amounts to:

Theorem A. *There is a* 9*-connected map*

$$\mathrm{Top}/\mathrm{O} \longrightarrow K(\mathbb{Z}/2, 3) \times K(\mathbb{Z}/28, 7) \times K(\mathbb{Z}/2, 8).$$

Remark.

- It is a direct consequence that the map

  $$[M, \mathrm{Top}/\mathrm{O}] \longrightarrow [M, K(\mathbb{Z}/2, 3) \times K(\mathbb{Z}/28, 7) \times K(\mathbb{Z}/2, 8)]$$

  is a bijection if $\dim(M) \leqslant 8$, and a surjection if $\dim(M) = 9$. Since the map

  $$[M, \mathrm{Top}/\mathrm{O}] \longrightarrow [M, \mathrm{Top}/\mathrm{PL}] \tag{1}$$

  agrees under this bijection with projection to $[M, K(\mathbb{Z}/2, 3)]$, (1) is surjective if dimension $\dim(M) \leqslant 9$). Through smoothing theory, this is a rephrasing of [Mor75, Corollary 6.3], *"smoothability of 8 and 9 PL manifolds is topologically invariant"*, which means that if two PL-manifolds M_α and M_β are homeomorphic then M_α admits a compatible smooth structure if and only if M_β does.
- On the other hand [Mor75, Theorem 6.1] says that there are smooth manifolds of dimension $\geqslant 22$ with a PL-structure on their underlying topological manifolds that does not admit a compatible smooth structure. In fact, Morita's proof uses that the 10-skeleton of this 22-dimensional manifold can not be smoothed. Applying the above reasoning, this implies that the next k-invariant of Top/O must be non-zero.
- This is related to [BT21, Remark 1.12], which asks whether the map $[M, \mathrm{Top}/\mathrm{O}] \to [M, \mathrm{Top}/\mathrm{PL}]$ admits a splitting when $\dim(M) = 7$; it does because (1) is not just surjective but even split surjective when $\dim(M) \leqslant 8$. This may help in removing the restriction in [BT21, Theorem 1.2] that the order of G must be odd.

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1. Preliminaries and strategy

The space Top/O is 1-connected and has finite homotopy groups, so as a consequence of the arithmetic fracture square it is the product of its p-localisations. Moreover, in the range of interest for Theorem A the k-invariants are 2-local so it suffices to work 2-locally.

Convention 1.1. In the remainder of this note all spaces and maps will be 2-localised.

There is a fibre sequence of infinite loop spaces

$$\mathrm{PL/O} \longrightarrow \mathrm{Top/O} \longrightarrow \mathrm{Top/PL}.$$

As $\tau_{\leqslant 8}\mathrm{PL/O}$ is a product $K(\mathbb{Z}/4,7) \times K(\mathbb{Z}/2,8)$ [Jah73, BKMS23], the first k-invariant of Top/O lies in in the cohomology group $H^8(K(\mathbb{Z}/2,3);\mathbb{Z}/4)$.

Proposition 1.2. *The first k-invariant $k_1 \in H^8(K(\mathbb{Z}/2,3);\mathbb{Z}/4)$ is trivial.*

Given this, the second k-invariant lies in $H^9(K(\mathbb{Z}/2,3) \times K(\mathbb{Z}/4,8);\mathbb{Z}/2)$, which is equal to $H^9(K(\mathbb{Z}/2,3);\mathbb{Z}/2) \oplus H^9(K(\mathbb{Z}/4,8);\mathbb{Z}/2)$ by the Künneth theorem.

Proposition 1.3. *The second k-invariant $k_2 \in H^9(K(\mathbb{Z}/2,3) \times K(\mathbb{Z}/4,7);\mathbb{Z}/2)$ is trivial.*

Once we establish these propositions Theorem A follows. The strategy is to start with the pullback square of infinite loop spaces

$$\begin{array}{ccc} \mathrm{Top/O} & \longrightarrow & \mathrm{G/O} \\ \downarrow & & \downarrow \\ \mathrm{Top/PL} & \longrightarrow & \mathrm{G/PL}. \end{array}$$

It will follow from the work of Madsen–Milgram that the horizontal maps are surjective on π_3 and passing to 2-connective covers we get another pullback square of infinite loop spaces

$$\begin{array}{ccc} \mathrm{Top/O} & \longrightarrow & \mathrm{G/O}\langle 2\rangle \\ \downarrow & & \downarrow \\ \mathrm{Top/PL} & \longrightarrow & \mathrm{G/PL}\langle 2\rangle. \end{array}$$

This implies that the k-invariants in the cohomology of Top/PL are pulled back from the cohomology of $\mathrm{G/PL}\langle 2\rangle$. In the Section 2 we will understand the bottom map, in Section 3 we will compute some relevant cohomology groups, and in Section 4 we will combine these to prove Propositions 1.2 and 1.3.

2. The work of Madsen–Milgram

There is a fibre sequence

$$\mathrm{Top/PL} \longrightarrow \mathrm{G/PL} \longrightarrow \mathrm{G/Top}$$

whose right map (2-locally, per convention) admits a simple description:

Proposition 2.1. *The map $p\colon \mathrm{G/PL} \to \mathrm{G/Top}$ is equivalent to the product of a map*

$$f\colon Y = K(\mathbb{Z}/2,2) \times_{\delta \mathrm{Sq}^2} K(\mathbb{Z},4) \longrightarrow K(\mathbb{Z}/2,2) \times K(\mathbb{Z},4)$$

with the identity map of $\prod_{k \geqslant 2} K(\mathbb{Z}/2,4k+2) \times \prod_{\ell \geqslant 2} K(\mathbb{Z},4\ell)$.

Proof. By [MM79, Remark 4.36], there are cohomology classes

$$K_{4k+2} \in H^{4k+2}(\mathrm{G/Top}; \mathbb{Z}/2) \quad \text{and} \quad \widetilde{K}_{4\ell} \in H^{4\ell}(\mathrm{G/Top}; \mathbb{Z}/2)$$

and these yield a (2-local) equivalence

$$\mathrm{G/Top} \xrightarrow{\simeq} \prod_{k \geqslant 1} K(\mathbb{Z}/2, 4k+2) \times \prod_{\ell \geqslant 1} K(\mathbb{Z}, 4\ell).$$

The case of G/PL is more complicated [MM79, Theorem 4.8] (the $K_{4\ell}$ are not unique there, but [MM79, Theorem 4.32] makes the specific choice $\widetilde{K}_{4\ell}$): taking only K_{4k+2} for $k \geqslant 2$ and $\widetilde{K}_{4\ell}$ for $\ell \geqslant 2$ yields a map

$$\mathrm{G/PL} \longrightarrow \prod_{k \geqslant 2} K(\mathbb{Z}/2, 4k+2) \times \prod_{\ell \geqslant 2} K(\mathbb{Z}, 4\ell)$$

whose fibre is $Y \simeq K(\mathbb{Z}/2, 2) \times_{\delta \mathrm{Sq}^2} K(\mathbb{Z}, 4)$. The obstructions to existence of a section of this map lie in cohomology groups of the base of degrees $\leqslant 5$, which vanish since it is 5-connected. Thus there exists a section

$$s \colon \prod_{k \geqslant 2} K(\mathbb{Z}/2, 4k+2) \times \prod_{\ell \geqslant 2} K(\mathbb{Z}, 4\ell) \longrightarrow \mathrm{G/PL}.$$

Let i denote the inclusion Y into G/PL, j the inclusion of $K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)$ into G/Top, and f be the map induced by p on fibres, so that $jf = pi$. Taking the addition with respect to using the infinite loop space structures of G/PL or G/Top, we get horizontal equivalences fitting into a commutative square

$$\begin{array}{ccc} Y \times \prod_{k \geqslant 2} K(\mathbb{Z}/2, 4k+2) \times \prod_{\ell \geqslant 2} K(\mathbb{Z}, 4\ell) & \xrightarrow[\simeq]{(i+s)} & \mathrm{G/PL} \\ \downarrow{\scriptstyle f \times \mathrm{id}} & & \downarrow{\scriptstyle p} \\ K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4) \times \prod_{k \geqslant 2} K(\mathbb{Z}/2, 4k+2) \times \prod_{\ell \geqslant 2} K(\mathbb{Z}, 4\ell) & \xrightarrow[\simeq]{(j+ps)} & \mathrm{G/Top}. \end{array}$$

□

From the long exact sequence of homotopy groups and the identification of the fibre of p with Top/PL, we see that on homotopy groups the composition

$$Y \longrightarrow \mathrm{G/PL} \xrightarrow{p} \mathrm{G/Top} \longrightarrow K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)$$

is given by identity on π_2 and multiplication-by-2 on π_4. It follows that the map $\mathrm{G/PL}\langle 2 \rangle \to \mathrm{G/Top}\langle 2 \rangle$ is equivalent to the product of the multiplication-by-2 map $K(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4)$ with the identity map of $\prod_{k \geqslant 2} K(\mathbb{Z}/2, 4k+2) \times \prod_{\ell \geqslant 2} K(\mathbb{Z}, 4\ell)$. This leads to the following description of the map $\mathrm{Top/PL} \to \mathrm{G/PL}\langle 2 \rangle$.

Corollary 2.2. *The map* $\mathrm{Top/PL} \to \mathrm{G/PL}\langle 2 \rangle$ *is equivalent to the product of*

$$\delta \mathrm{Sq}^1 \iota_3 \colon K(\mathbb{Z}/2, 3) \longrightarrow K(\mathbb{Z}, 4),$$

the integral lift of the first Steenrod square of the fundamental class $\iota_3 \in H^3(K(\mathbb{Z}/2, 3); \mathbb{Z}/3)$, *with the inclusion of the basepoint into* $\prod_{k \geqslant 2} K(\mathbb{Z}/2, 4k+2) \times \prod_{\ell \geqslant 2} K(\mathbb{Z}, 4\ell)$.

3. Some cohomology groups of $K(\mathbb{Z},4)$

Given Corollary 2.2, it will be crucial to understand some cohomology groups of the Eilenberg–Mac Lane space $K(\mathbb{Z},4)$. We will need two reduction maps

$$\overline{(-)}\colon H^*(X;\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z}/4)$$
$$\widetilde{(-)}\colon H^*(X;\mathbb{Z}/4) \longrightarrow H^*(X;\mathbb{Z}/2)$$

We will also use $\widetilde{(-)}$ for the composition $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}/2)$ of these maps.

Lemma 3.1. *We have that*

$$H^8(K(\mathbb{Z},4);\mathbb{Z}/4) \cong \mathbb{Z}/4 \cdot \{\bar\iota_4^2\} \quad \textit{and} \quad H^9(K(\mathbb{Z},4);\mathbb{Z}/2) = 0.$$

Proof. We start with $\mathbb{Z}/2$-cohomology. By [MT68, Chapter 9, Theorem 3], the cohomology ring $H^*(K(\mathbb{Z},4);\mathbb{Z}/2)$ is the polynomial ring on generators $\mathrm{Sq}^I(\iota_4)$ with $I = (i_1, \cdots, i_r)$ an admissible sequence of excess $e(I) < 4$ and $i_r \neq 1$. This shows that

$$H^7(K(\mathbb{Z},4);\mathbb{Z}/2) = \mathbb{Z}/2 \cdot \{\mathrm{Sq}^3\tilde\iota_4\},$$
$$H^8(K(\mathbb{Z},4);\mathbb{Z}/2) = \mathbb{Z}/2 \cdot \{\tilde\iota_4^2\},$$
$$H^9(K(\mathbb{Z},4);\mathbb{Z}/2) = 0$$

with $\iota_4 \in H^4(K(\mathbb{Z},4);\mathbb{Z})$ the generator and $\tilde\iota_4$ its reduction modulo 2. To determine the $\mathbb{Z}/4$-cohomology in degree 8 we use that $H^8(K(\mathbb{Z},4);\mathbb{Q}) = \mathbb{Q}$ generated by ι_4^2. The universal coefficient theorem then tells us that there must be a $\mathbb{Z}/4 \subset H^8(K(\mathbb{Z},4);\mathbb{Z}/4)$, generated by an element y satisfying $c \cdot y = \bar\iota_4^2$ for some $c \geqslant 1$. Working 2-locally, we may assume that c is a power of 2. Since the Bockstein homomorphism associated to the long exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ of coefficients agrees with Sq^1 and hence vanishes on H^7 and H^8 using the Adem relations, the Bockstein long exact sequence gives a short exact sequence

$$0 \longrightarrow H^8(K(\mathbb{Z},4);\mathbb{Z}/2) \longrightarrow H^8(K(\mathbb{Z},4);\mathbb{Z}/4) \longrightarrow H^8(K(\mathbb{Z},4);\mathbb{Z}/2) \longrightarrow 0$$

with right map given by the reduction homomorphism $H^*(X;\mathbb{Z}/4) \to H^*(X;\mathbb{Z}/2)$. This shows that the cohomology group $H^8(K(\mathbb{Z},4);\mathbb{Z}/4)$ has 4 elements, so must be equal to $\mathbb{Z}/4$. Moreover, the reduction modulo 2 maps its generator y to $\tilde\iota_4^2$ and we must have $c = 1$. □

For an infinite loop space X, or more generally an H-space, the product map

$$\mu\colon X \times X \longrightarrow X$$

induces a map $\Delta\colon H^*(X;A) \to H^*(X \times X;A)$ on cohomology with coefficients in A. If A is a commutative ring, then the universal coefficients theorem provides an injective map $H^*(X;A) \otimes_A H^*(X;A) \to H^*(X \times X;A)$. We say that $x \in H^*(X;A)$ is *primitive* if $\Delta(x)$ is the image of $x \otimes 1 + 1 \otimes x$ under this inclusion.

Lemma 3.2. *The primitive elements in $H^8(K(\mathbb{Z},4);\mathbb{Z}/4)$ are 0 and $2 \cdot \bar\iota_4^2$.*

Proof. Since Δ is a homomorphism with respect to the cup product, we have that $\Delta(\bar\iota_4^2) = \bar\iota_4^2 \otimes 1 + 2\bar\iota_4 \otimes \bar\iota_4 + 1 \otimes \bar\iota_4^2$, so $\bar\iota_4^2$ is not primitive. However, a 2-multiple of it visibly is. □

4. Proof of Propositions 1.2 and 1.3

To compute the first Postnikov k-invariant k_1, our starting point is the map of fibre sequences, up to degree 8, of infinite loop spaces

$$\begin{array}{ccccc} \mathrm{Top/O} & \longrightarrow & \mathrm{Top/PL} \simeq K(\mathbb{Z}/2,3) & \xrightarrow{\pi_1} & K(\mathbb{Z}/4,8) \\ \downarrow & & \downarrow \phi & & \| \\ \mathrm{G/O}\langle 2\rangle & \longrightarrow & \mathrm{G/PL}\langle 2\rangle & \xrightarrow{\pi_2} & K(\mathbb{Z}/4,8). \end{array}$$

Then we have $k_1 = \pi_1^*(\iota_8)$, the pull back of the identity element $\iota_8 \in H^8(K(\mathbb{Z}/4,8);\mathbb{Z}/4)$ along the right map in the fibre sequence of infinite loop spaces. By naturality $\pi^*(\iota_8) = \phi^*\pi_2^*(\iota_8)$. Note that $x \coloneqq \pi_2^*(\iota_8)$ is primitive because (i) pulling back along a map of infinite loop spaces, or more generally H-spaces, preserves primitive elements and (ii) for degree reasons, the identity element $\iota_8 \in H^4(K(\mathbb{Z}/4,8);\mathbb{Z}/4)$ is primitive.

By Corollary 2.2 the map ϕ factors as

$$\phi\colon \mathrm{Top/PL} \simeq K(\mathbb{Z}/2,3) \xrightarrow{\delta \mathrm{Sq}^1} K(\mathbb{Z},4) \xrightarrow{i'} \mathrm{G/PL}\langle 2\rangle$$

so $k_1 = (\delta\mathrm{Sq}^1)^*(i')^*(x)$. The right map $i'\colon K(\mathbb{Z},4) \to \mathrm{G/PL}\langle 2\rangle$ is obtained as a connective cover of $i\colon Y \to \mathrm{G/PL}$, which is a map of H-spaces by [MM79, Theorem 4.34] (in fact, it deloops twice by [MM79, Theorem 7.25]), and hence so is i'. Hence $(i')^*(x)$ is primitive, and thus the element k_1 is obtained by pulling back a primitive element in $H^8(K(\mathbb{Z},4);\mathbb{Z}/4)$ along $\delta\mathrm{Sq}^1$. In Lemma 3.2 we saw all primitive elements in $H^8(K(\mathbb{Z},4);\mathbb{Z}/4)$ are 2-multiples, and hence must vanish upon pulling back to $H^8(K(\mathbb{Z}/2,3);\mathbb{Z}/4)$, as the latter is 2-torsion by the universal coefficients theorem and [Cle02, Table C.2].

To compute the second Postnikov k-invariant k_2, we argue similarly that it is the pullback of an element of $H^9(K(\mathbb{Z},4);\mathbb{Z}/2)$. But this group vanishes by Lemma 3.1.

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