**THE SUNNY COLLAPSING PROCEDURE**

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Abstract. We explain Goodwillie's sunny collapsing procedure, and a result that constructs its input.

In this talk we explain §I.C of [Goo90].

## 1. Context

Let's recall what we are trying to prove, and what the strategy is. We let $\mathrm{CE}(P,N)$ denote the space (or simplicial set) of concordance embeddings; embeddings $e\colon I\times P\to I\times N$ that are equal to $\mathrm{id}_I\times e_0$ on $\{0\}\times P\cup\partial I\times$ and satisfy $e^{-1}(\{1\}\times N)=\{1\}\times P$.

**Theorem 1.1** (Multiple disjunction)**.** *Suppose that $Q_1,\ldots,Q_k\subset N$ is a collection of $k$ disjoint submanifolds of dimension $q_i$. Writing $\underline{k}=\{1,\ldots,k\}$, then we can produce a cubical diagram of spaces (based at $\mathrm{id}_I\times\mathrm{inc}$)*

$$\underline{k}\supset I\longmapsto \mathrm{CE}(P,N\setminus\cup_{i\notin I}M_i)\in\mathsf{Top}_*$$

*with maps in the diagram given by the inclusions. This cube is $r$-cartesian for $r=n-p-3+\sum_{i=1}^k(n-q_i-1)$, as long as $n-p\geq 3$ and $n-q_i\geq 3$ for all $1\leq i\leq k$.*

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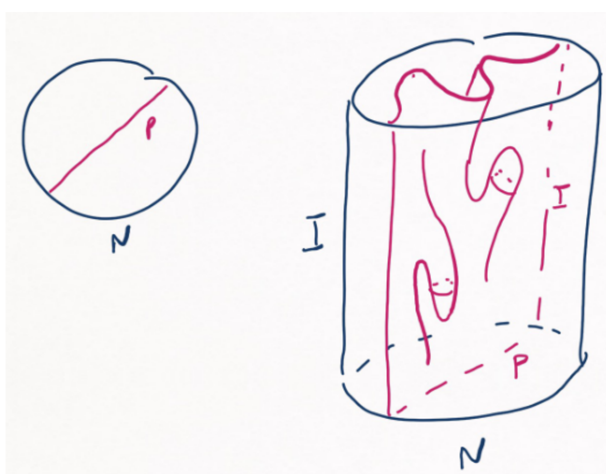


Figure 1. A concordance embedding.

For $k = 1$, this says that $\mathrm{CE}(P, N \setminus M) \to \mathrm{CE}(P, N)$ is $(2n-p-q-4)$-connected. Manuel outlined the argument for it, and Mauricio explained some generalisations of Blakers–Massey which will serve a role in proving the general case.

To motivate the sunny collapsing procedure, we take another look at Manuel's outline, which specialises to $k = 1$. A homotopy class of pairs $(D^s, \partial D^s) \to (\mathrm{CE}(P, N), \mathrm{CE}(P, N \setminus M))$ can be represented by a map

$$F \colon I \times P \times D^s \to I \times P \times N$$

which (a) is smooth and over $D^s$, (b) yields a concordance embedding $F_y$ for fixed $y \in D^s$, (c) $F_y$ avoids $M$ for $y \in \partial D^s$. A map $F$ satisfying just (a) and (b) is called a *fibered concordance*. The definition of a *fibered isotopy of concordances* is the obvious one: replace $D^s$ by $D^s \times [0, 1]$.

**Notation 1.2.** Sometimes, to recall that $D^s$ is the parameter in a fibered concordance, I will denote a point in it by $F_y$ instead of $y$.

The plan is to give a stratification $\{\Sigma\}$ of $P \times D^s$ (morally, pulled back from some universal stratification on $P \times \mathrm{CE}(P, N)$ but no need to do so in practice), with the following properties:

(1) there are good and bad strata,
(2) bad strata have codimension $\geq n - 2$,
(3) good strata admit an inductive construction: say $F$ is "nice" on a stratum $\Sigma$ if $(z, F_y) \in \Sigma \subset P \times D^s$ then $F_y(I \times \{z\}) \cap Q = \varnothing$. The inductive construction says that if $F$ is nice on $\Sigma'$ for all $\Sigma'$ of greater codimension, then we can find a fibered isotopy of concordances so that $F$ is nice on $\Sigma$ as well.

The construction of the stratification is quite involved, but the idea is that it keeps of whether for $(z_1, z_2, F_y) \in (I \times P)^{(2)} \times D^s$ (here $(-)^{(r)}$ is Goodwillie's notation for ordered configuration spaces, it is the case that $F_y(z_1)$ *is below* $F_y(z_2)$. This means that $F_y(z_i) = (t_i, x_i) \in I \times N$ with $x_1 = x_2$ and $t_1 < t_2$. In fact, the first candidates for strata are $p\pi_i S_0$ with

$$S_0 = \{(z_1, z_2, F_y) \in (I \times P)^{(2)} \times D^s \mid F(z_1) \text{ below } F(z_2)\}$$

and $\pi_i \colon S_0 \to I \times P \times D^s$ and $p \colon I \times P \times D^s \to P \times D^s$ the projections.

*Example* 1.3. $(x, F_y) \in p\pi_1 S_0$ if there is a point $x' \in P$ and $t, t' \in I$ such that $F_y(t, x)$ is below $F_y(t', x')$. For $(x, F_y) \in p\pi_2 S_0$ replace "below" by "above" (defined analogously).

The strata of greater codimension are built as projections $p\pi_i W_\alpha$ of subsets $W_\alpha$ of sets $S_\alpha$ built from $S_0$. These are generated by:

- Operation A: relegate non-manifold points to new strata.
- Operation B: keep track of more complicated "coincidence patterns": more complication in $N$ for example means there are points $x \in P$ such that $F_y(t, x)$ is below $F_y(t', x')$ which in turn is below $F_y(t'', x'')$, and more complication in $P$ for example means there are two distinct points $F_y(t_1, x)$ and $F_y(t_2, x)$ which are below $F_y(t_1', x')$ and $F_y(t_1'', x'')$ respectively.
- Operation C: to be explained momentarily.
- Operation D: add limits points of strata as new strata.

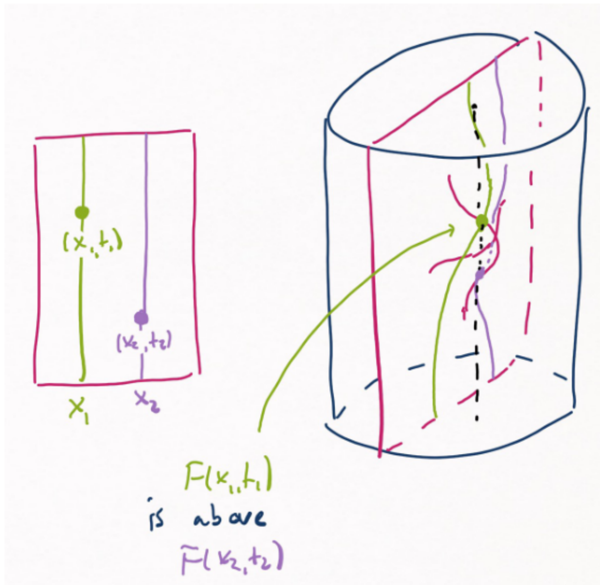


FIGURE 2. An example of $(z_1, z_2, F_y) \in S_0$, $z_i = (x_i, t_i) \in P \times I$.

We obtain a collection $\{S_\alpha\}$ of subsets of $(I \times P)^{(r_\alpha)} \times D^s$. We want to use the projections of $S_\alpha$ as strata, but they may intersect. To amend this, we take

$$W_\alpha = \{z \in S_\alpha \mid \forall i \in \underline{r_\alpha},\, p\pi_i z \text{ is not in any set } p\pi_j S_\beta \subsetneq p\pi_i S_\alpha\},$$

making them evidently disjoint in $P \times D^s$. Intuitively, $W_\alpha \subset S_\alpha$ is the subset of points that satisfy no finer pattern and are not degenerate in the sense addressed by operations A, C, and D.

Recall that $S_\alpha$ in particular keeps track of various "coincidence patterns": whether $F_y(t,x)$ is below some other $F_y(t',x')$ and whether some such pairs $F_y(t_1,x_1)$ and $F_y(t_2,x_2)$ of overlying points satisfy $x_1 = x_2$. In $W_\alpha$, these patterns can't change when moving through $W_\alpha$; otherwise some points would be relegated to a deeper $W_\beta$. However, there are multiple components because for pairs $F_y(t_1,x_1)$ and $F_y(t_2,x_2)$ which satisfy $x_1 = x_2$, we did not keep track of whether $t_1 < t_2$ or $t_2 > t_1$. This allow us to decompose $W_\alpha$ further into along certain binary relations:

$$W_\alpha^{(D,R)} = \{z \in W_\alpha \mid \forall i,j \in \underline{r_\alpha}\ \pi_i z \text{ is below } \pi_j z \text{ if } iDj \text{ and } \pi_i F_y(z) \text{ is below } \pi_j F_y(z) \text{ if } iRj\}.$$

The strata are then the sets $p\pi_i W_\alpha^{(D,R)}$.

One of the conditions defining good strata will be that $\Sigma = p\pi_i W_\alpha^{(D,R)} \subset P \times D^s$ has an "uppermost lifting" $\tilde{\Sigma} \subset I \times P \times D^s$. A condition that guarantees its existence can be given in terms of $D$ and $R$; it not being satisfied is one of the two ways a stratum can be bad ($W_\alpha^{(D,R)}$ "contains a loop"). We define

$$\tilde{\Sigma}^{\uparrow} := \{(t,x,F_y) \in I \times P \times D^s \mid \exists t' < t \text{ with } (t',x,F_y) \in \tilde{\Sigma}.\}$$

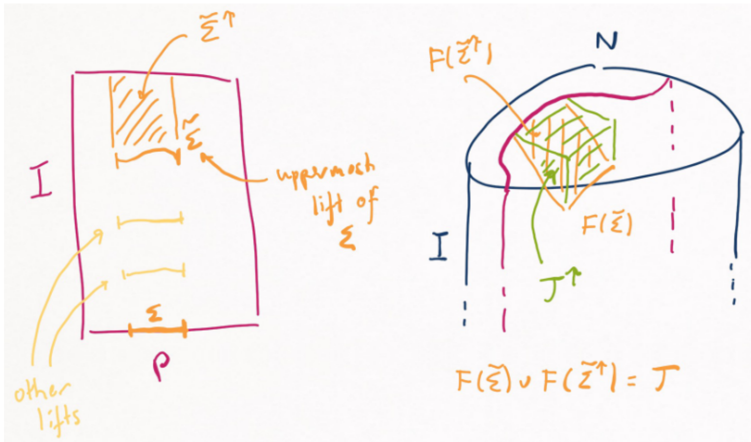


FIGURE 3. $\tilde{\Sigma}$, $\tilde{\Sigma}^\uparrow$, $T$ and $T^\uparrow$.

The fact that $\tilde{\Sigma}$ is an uppermost lifting means that $\tilde{\Sigma} \cup \tilde{\Sigma}^\uparrow$ is disjoint $\pi_1 S_0$, or equivalently that when we define

$$T = \{(F_y(t,x), F_y) \in I \times N \times D^s \mid (t,x,F) \in \tilde{\Sigma} \cup \tilde{\Sigma}^\uparrow\}$$

we have that

$$T^\uparrow = \{(t,x,F_y) \in I \times N \times D^s \mid \exists t' < t \text{ with } (t',x,F_y) \in T\}$$

is disjoint from the image of $F$.

We now reach the pictures that Manuel drew before: we want to remove the set $K$ of points on the highest sheet over a good stratum $\Sigma$ of greatest codimension on which $F$ is not yet nice. We intent to do this by pushing up the image of $F$ over $T^\uparrow$, removing $K$ in the process. This is possible since there are no points of the image of $F$ above it, and it is this procedure that I want to describe.

We will see it is given by simultaneously pushing down a surface $G(\phi^u)$ in $I \times P \times D^s$ and $G(\psi^u)$ in $I \times N \times D^s$, so that $F(G(\psi^u))$ is the transverse intersection of $G(\psi^u)$ with the image of $F$.

There is two subtleties. Firstly, this pushing procedure needs a more space: it is performed *in a neighborhood* of $T \cup T^\uparrow$. This will not play a role in this talk. Secondly, we will need more than just that $\tilde{\Sigma} \cup \tilde{\Sigma}^\uparrow$ avoids $\pi_1 S_0$ (so no points lie above it): we also want to avoid an infinitesimal version; the infinitesimal version of lying in $\pi_1 S_0$ is that the image of $F$ is vertical (i.e. $\frac{\partial}{\partial t}$ is tangent to it). Not all such points are a problem, but only those points where the locus in $I \times P \times D^s$ of points whose image under $F$ is vertical is itself vertical. Operation C records derivative data to allow us to introduce second way in which a stratum can be bad: if it is the image under $p$ of $W_\alpha$ with $D(pF)_{x_i}(\frac{\partial}{\partial t}) = 0$ for some $x_i$, with $pF \colon I \times P \times D^s \to I \times N \times D^s \to N \times D^s$.

## 2. The sunny collapsing procedure

I will now give the technical details of the sunny collapsing procedure sketched above.

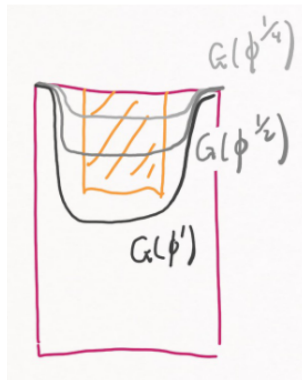


FIGURE 4. The surfaces $G(\phi^u)$ sliding down in $I \times P$.


FIGURE 5. Removing $K$ by restricting to the part below $G(\phi^u)$, and then pushing upwards.

**Definition 2.1.** Suppose that $F = (h, f, p_3)\colon I \times P \times D^s \to I \times N \times D^s$ is a fibered concordance. Then *fibered sunny collapse data* consists of smooth homotopies

$$\phi^u\colon P \times D^s \to (0,1] \qquad \psi^u\colon N \times D^s \to (0,1]$$

satisfying:

(i) $\phi^u(x,y) = 1$ if $u = 0$ or $x \in \partial P$, $\psi^u(z,y) = 1$ if $u = 0$ or $z \in \partial N$,
(ii) if $t = \phi^u(x,y)$, then $h(t,x,y) = \psi^u(f(t,x,y),y)$,
(iii) if $t < \phi^u(x,y)$, then $h(t,x,y) < \psi^u(f(t,x,y),y)$,
(iv) if $t = \phi^u(x,y)$, then $\frac{\partial}{\partial t}(h(t,x,y) - \psi^u(f(t,x,y),y)) > 0$.

From this data, the *fibered sunny collapse* is the fibered isotopy of concordances given by

$$F^u(t,x,y) = \left(\frac{h(t\phi^u(x,y),x,y)}{\psi^u(f(t\phi^u(x,y)),x,y)}, f(t\phi^u(x,y),x,y)\right).$$

That is, we restrict $F$ to $\{(t,x,y) \mid t \leq \phi^u(x,y)\} \subset I \times P \times D^s$, and scale the intervals in the $I$-direction of the domain to have length 1 by dividing by $\phi^u(x,y)$, hence identifying the domain with $I \times P \times D^s$ once more. The result is not a fibered concordance because "it doesn't hit the top," and we scale intervals in the $I$-direction of the target by dividing by $\psi^u(x,y)$ to fix this. By (i) we start at the identity aand nothing happens near the boundary

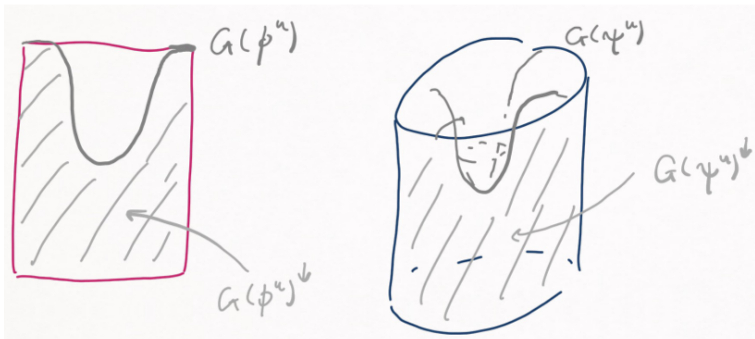


FIGURE 6. $G(\phi^u)$, $G(\phi^u)^\downarrow$, $G(\psi^u)$, and $G(\psi^u)^\downarrow$.

of $P$. Using (ii) this hits the top, and using (iii) nothing gets pushed out the top. It remains to interpret (iv).

Geometrically, we consider the graphs

$$G(\phi^u) = \{(t, x, y) \in I \times P \times D^s \mid t = \phi^u(x, y)\}$$
$$G(\psi^u) = \{(t, x, y) \in I \times N \times D^s \mid t = \psi^u(x, y)\}$$

and the regions

$$G(\phi^u)^\downarrow = \{(t, x, y) \in I \times P \times D^s \mid t < \phi^u(x, y)\}$$
$$G(\psi^u)^\downarrow = \{(t, x, y) \in I \times N \times D^s \mid t < \psi^u(x, y)\}.$$

Then (ii) is equivalent to $F(G(\phi^u)) \subset G(\psi^u)$ and (iii) to $F(G(\phi^u)^\downarrow) \subset G(\psi^u)^\downarrow$. Moreover, (iv) says that $F$ is transverse to $G(\psi^u)$.

We want to *construct* $\psi^u$ given by $\phi^u$. On the way to this, we observe:

**Lemma 2.2.** *Conditions (i)–(iv) on $\phi^u$ and $\psi^u$ imply*

*(i') For $0 \leq t \leq \phi^u(x, y)$, no $F(\phi^u(x', y), x', y)$ is below $F(t, x, y)$.*

*(ii') If $v$ is a tangent vector to $(t, x, y) \in I \times P \times D^s$ such that $t = \phi^u(x, y)$, and $D(\phi^u p_{2,3})(v) \geq D(p_1)(v)$, then $DF(v)$ is not upward vertical.*

Geometrically, (i') says no point in $F(G(\phi^u))$ is below a point of $\mathrm{cl}(F(G(\phi^u)^\downarrow)) = F(G(\phi^u) \cup G(\phi^u)^\downarrow)$; if $F(t, x, y)$ is below $F(t', x', y)$ then the moving surface $G(\phi^u)$ may not sweep through $(t, x, y)$ unless it has already swept through $(t', x', y)$. Similarly, (ii') says that no inward pointing vector of $\mathrm{cl}(F(G(\phi^u))^\downarrow))$ at $F(G(\phi^u))$ may be upward vertical. These are of course necessary consequences of the fact that $G(\psi^u)$ is a graph transverse to $F$.

*Remark* 2.3. Let us give the sunny interpretation: think of $\mathrm{cl}(F(M(\phi^u)))$ as subsurface of $I \times N \times D^s$ melting away, then (i') says only parts in the sun may melt, and (ii') says you are not in the sunshine if the ray of light hitting a point is outward tangent to this subsurface.

*Remark* 2.4. In terms of good/bad strata: to eventually verify (i') we don't want points over the uppermost sheet, and to eventually verify (ii') we want to rule out certain type of vertical points. (It does not seem to me that the condition defining the "bad strata" in the latter case is the same as (ii'), but presumably the link between them will become clear in the proof.)

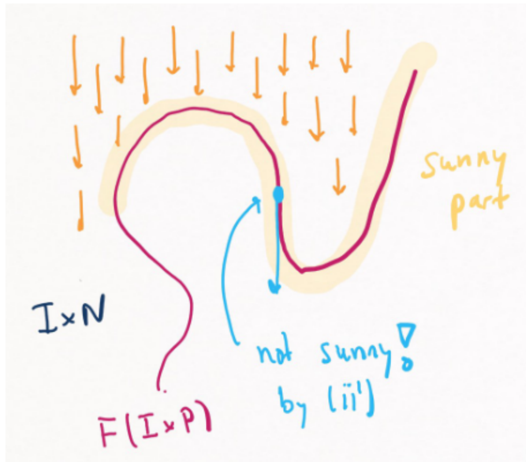


Figure 7. Sunny parts of a concordance embedding.

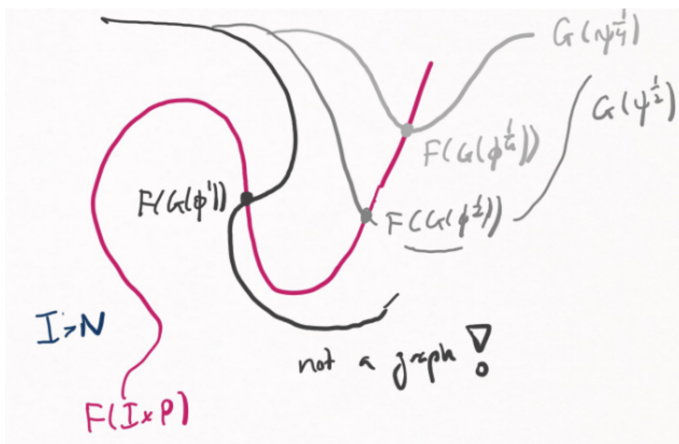


Figure 8. A reason for including condition (ii').

The converse is that given $\phi^u$ satisfying (i), (i'), and (ii') we can construct $\psi^u$:

**Proposition 2.5.** *If* $\phi^u\colon P \times D^s \to (0,1]$ *satisfies (i), (i'), and (ii'), there exists a* $\psi^u\colon N \times D^s \to (0,1]$ *satisfying (i)–(iv).*

*Proof.* We first reduce to a local problem, in $I \times N \times D^s$ (note $I$ is now the time parameter of the homotopy, so we use the notation $u \in I$). We also write $\psi^u(x,y) = \psi(u,x,y)$. The crucial observation is that (i), (ii), and (iii) and preserved by convex combinations. More precisely, let $\{U_\alpha\}$ be an open cover $I \times N \times D^s$ and $\{\rho_\alpha\}$ be a subordinate partition of unity. Then we have $\psi_\alpha\colon U_\alpha \to (0,1]$ satisfies (i), (ii), and (iii), so does $\sum_\alpha \mu_\alpha \psi_\alpha$. This is obvious for (i) and (ii). For (iii) we note that for $t < \phi^u(x,y)$ we have

$$\sum_\alpha \mu_\alpha(f(t,x,y),x,y)\psi_\alpha(u,f(t,x,y),x,y) > \sum_\alpha \mu_\alpha(f(t,x,y),x,y)h(t,x,y) = h(t,x,y)$$

since in the first inequality all terms are non-negative, and at least one is smaller. The argument for (iv) is similar.

Now we solve the local problem. Fix $(u_0, z_0, y_0) \in I \times N \times D^s$, then we will construct $\psi^u$ in a neighborhood of this set (suppressing $\alpha$). Of course (i) is easy to arrange and we'll ignore it. We start with an observation simplifying the verification of (iii) and (iv). Since (iii) and (iv) are equivalent to

$$g(u,t,x,y) = \begin{cases} \frac{h(t,x,y)-\psi(u,f(t,x,y),y)}{t-\phi^u(x,y)} & \text{if } t \neq \phi^u(x,y) \\ \frac{\partial}{\partial t}(h(t,x,y) - \psi(u,f(t,x,y),y) & \text{if } t = \phi^u(x,y) \end{cases}$$

being positive on $(u,z,y)$ of the form $(u, f(t,x,y), y)$ with $t \leq \phi^u(x,y)$, this is true near $(u_0, z_0, y_0)$ if it is true at this point.

Now there are two cases:

**Case 1.** If $z_0 \neq f(\phi^{u_0}(x_0,y_0), x_0, y_0)$, then we can just set $\psi(u,z,y) = 1$: (ii) and (iv) are vacuous, and (iii) is true because $t < \phi^u(x,y) \implies t < 1 \Rightarrow h(t,x,y) < 1$.

**Case 2.** If $z_0 = f(\phi^{u_0}(x_0,y_0), x_0, y_0)$ consider the map

$$\begin{aligned} E\colon I \times P \times D^s &\longrightarrow I \times N \times D^s \\ (u,x,y) &\longmapsto (u, f(\phi^u(x,y),x,y),y). \end{aligned}$$

We claim it is a proper injective immersion, so an embedding (its image is the graph swept out by the projection of $F(G(\phi^u))$ to $N$). We will use (i') to prove it is an injection: if $E(u',x',y') = E(u,x,y)$ then it must be true $u' = u$ and $y' = y$ but $x' \neq x$, and also $f(\phi^u(x',y),x',y) = f(\phi^u(x,y),x,y) \in N$. Since $F$ is injective, it must be true without loss of generality that $h(\phi^u(x',y),x',y) < h(\phi^u(x,y),x,y) \in I$. That is, $F(\phi^u(x',y),x',y)$ is below $F(t,x,y)$ with $t = \phi^u(x,y)$. By (i') this is only possible if $\phi^u(x,y) = t > \phi^u(x,y)$, contradiction. Similarly, (ii') is used to prove it is an immersion.

Since $E$ is an embedding and $(u_0, z_0, y_0)$ is in the image of $E$ there is a smooth function $\psi$ defined near $(u_0, z_0, y_0)$ so that $\psi(E(u,x,y)) = h(\phi^u(x,y),x,y)$. This definition guarantees that (ii) holds:

$$\psi^u(f(\phi^u(x,y),x,y),y) = \psi(u, f(\phi^u(x,y),x,y),y) = \psi(E(u,x,y)) = h(\phi^u(x,y),x,y).$$

Next we claim that (iii) and (iv) are true at $(u_0,z_0,y_0)$, i.e. when $(u,z,y) = (u, f(\phi^u(x,y),x,y),y)$ is equal to $(u_0,z_0,y_0)$. For (iii), note that if $t < \phi^u(x,y) = \phi^{u_0}(x,y_0)$ and $f(t,x,y_0) = z_0$ then firstly we must have that

$$(t,x,y_0) \neq (\phi^{u_0}(x_0,y_0),x_0,y_0)$$

because if this were true then $x = x_0$ and we get a contradiction from $t < \phi^{u_0}(x,y_0) = \phi^{u_0}(x_0,y_0)$. Since $F$ is injective and $f(t,x,y_0) = z_0 = f(t_0,x_0,y_0)$ we must have that $h(t,x,y_0) \neq h(\phi^{u_0}(x_0,y_0),x_0,y_0)$. By (i'), we then must have

$$h(t,x,y_0) < h(\phi^{u_0}(x_0,y_0),x_0,y_0) = \psi(f(\phi^{u_0}(x_0,y_0),x_0,y_0).$$

which is (iii). We'll leave (iv) to the motivated reader. $\square$

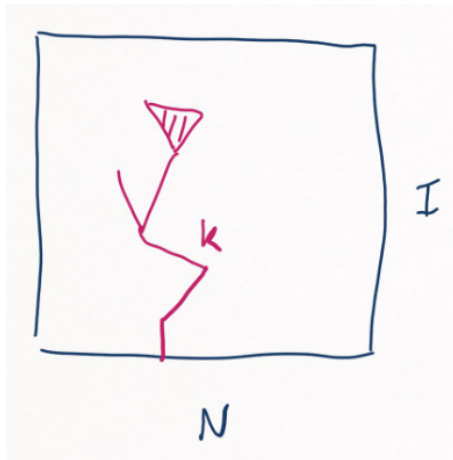


FIGURE 9. An example of $K \subset N \times I$ which sunny collapses.

In practice, $\phi^u(x,y) = 1 - u(1-\phi(x,y))$ for some $\phi\colon P \times D^s \times (0,1]$, so we are linearly interpolating between 1 and $\phi(x,y)$. In this case, the assumptions (i), (i'), and (ii') are satisfied if:

(i") $\phi(x,y) = 1$ if $x \in \partial P$,

(ii") if $F(t,x,y)$ is below $F(t',x',y')$, then $\phi(x,y) > t$ or $(1-t)(1-\phi(x',y)) > (1-t')(1-\phi(x,y))$,

(iii") if $\phi(x_0,y_0) \leq t_0 < 1$ and $DF(v) = \frac{\partial}{\partial t}$ for $v$ a tangent vector at $(t_0,v_0,y_0)$, then $(d\phi)(Dp_{2,3}(v)) < \frac{1-\phi(x_0,y_0)}{1-t_0} dp_1(v)$.

## 3. SUNNY COLLAPSING IN PL TOPOLOGY

In PL topology, the role of "sunny collapse data" is the following:

**Definition 3.1.** We say $K \subset N \times I$ *sunny collapses* if $N \times I$ can be triangulated such that $K$ is a subcomplex and there is a sequence of elementary collapse $M = K_n \searrow K_{n-1} \searrow \cdots \searrow K_0 = M \times \{0\} \cup \partial M \times I$ so that $\mathrm{sh}(K_i) \cap K_n \subset K_{i-1}$ with $\mathrm{sh}(K_i) = \{(x,t) \in N \times I \mid (x,t') \in K_i \text{ for some } t' > t\}$.

The closed to a "sunny collapsing procedure" I have seen is when $K$ is the image of a concordance embedding, and we take iterated application of the disc unknotting theorems make the concordance increasingly straight. This occurs in the proof of concordance-implies-isotopy [Hud70]. Generically, $K$ sunny collapses, by combining [Hud69, Lemma's 4.9 and 5.4]:

**Theorem 3.2.** *Given a PL embedding $f\colon X \times I \hookrightarrow N \times I$ of codimension $\geq 2$ so that $f^{-1}(N \times \{0\} \cup \partial N \times I) = X \times \{0\}$. Then there is a homeomorphism $h\colon N \times I \to N \times I$ which is the identity on $N \times \{0\} \cup \partial N \times I$ and arbitrary close to the identity such that $hf(X \times I)$ sunny collapses.*

Let me illustrate its use by a different result in PL-topology, which is input to proving sphere and disc unknotting in codimension $\geq 3$.

**Proposition 3.3.** *If $K \subset N \times I$ sunny collapses, then $N \times I$ collapses to $N \times \{0\} \cup \partial N \times I \cup K$.*

*Proof.* Step 1: collapse downwards onto $K \cup \mathrm{sh}(K)$. Step 2: collapse away the pieces $\mathrm{sh}(K_i) \setminus \mathrm{sh}(K_{i-1})$ "sideways", starting with $i = n$. □

## References


[Goo90] T. G. Goodwillie, *A multiple disjunction lemma for smooth concordance embeddings*, Mem. Amer. Math. Soc. **86** (1990), no. 431, viii+317. 1

[Hud69] J. F. P. Hudson, *Piecewise linear topology*, University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees, W. A. Benjamin, Inc., New York-Amsterdam, 1969. 9

[Hud70] ———, *Concordance, isotopy, and diffeotopy*, Ann. of Math. (2) **91** (1970), 425–448. 9