# $\mathcal{Z}$ AND THE STRATIFICATION

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ABSTRACT. We complete the last step before the proof of multiple disjunction for concordance embeddings: the construction of $\mathcal{Z}$ and the associated stratification of $P \times D^s$ for a fibered concordance $I \times P \times D^s \to I \times N \times D^s$.

In this lecture we discuss Chapters II.C and II.D of [Goo90].

## 1. Context

As mentioned in the previous talk, recall we are trying to modify a fibered concordance

$$F\colon I \times P \times D^s \longrightarrow I \times N \times D^s$$

through a fibered isotopy of concordances, so as to avoid some submanifold $I \times M$ of $I \times N$ (or at least one in a collection of these). Our plan is to give, after putting $F$ in general position, a stratification of $P \times D^s$ with good and bad strata such that the bad strata have large codimension and on good strata we can inductively remove the intersections with $I \times M$.

In this talk we define this stratification. It will arise from a collection $\mathcal{Z}$ of invariant algebraic sets of complex multijets; these are certain sets of jets of locally holomorphic maps $\mathbb{C}^{p+1} \to \mathbb{C}^n$ near $r$ points that describe these strata in (complexified) local coordinates; they are produced by the four operations A, B, C, and D from a basic one $Z^0$, which records when the images of two points are on the same vertical line in the target.

## 2. Defining $Z$

### 2.1. Recap and the definition of $Z$.

Last lecture we defined *invariant algebraic sets of complex multijets* (IASCMs), which are subsets $Z \subset {}_r\mathcal{J}^\infty_\mathbb{C}(\mathbb{C}^{p+1}, \mathbb{C}^n) =: {}_r\mathcal{J}^\infty$ such that

- $Z = (p^\infty_m)^{-1}(p^\infty_m(Z))$ with $p^\infty_m\colon \mathcal{J}^\infty \to \mathcal{J}^m$,
- $p^\infty_m(Z)$ is closed algebraic defined over $\mathbb{R}$,
- $Z$ is domain-invariant,
- $Z$ is range-invariant.

The number $r$ is called the *rank* and $m$ is called the *level*. The latter two conditions are such that it makes sense to write

$$z \in S(Z, P, N) \iff \text{there exists } \tilde{z} \in Z \text{ such that } r^\infty_j(\psi)(tz) \circ z = \tilde{z} \circ {}_rj^\infty(\mathrm{id} \times \phi)(sz),$$

for charts $\phi$ of $\mathbb{R} \times P$ and $\psi$ of $N$.

The basic IASCM is $Z^0 = \{(z_1, z_2) \in {}_2\mathcal{J}^\infty \mid t(z_1) = t(z_2)\}$ recording vertical coincidence in the target. Further ones are produced from these by operations:

A. "Relegate non-manifold to deeper strata."

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B. "Relegate intersections of strata to deeper strata."
C. "Relegate to deeper strata those points where an outwards pointing vertical tangent vector exists."
D. "Relegate to deeper strata those points obtained by collisions of points in previous strata."

The reader should consult the previous lecture for an outline of their definition.

**Definition 2.1.** The collection $\mathcal{Z}$ of IASCM's is given by iteratively applying operations A, B, C, and D to $Z^0$.

2.2. **Codimension.** Our goal in this section is to define the "codimension" $c(Z)$ of a stratum, and prove that *under the assumption* $n - p \geq 3$ there are only finitely many strata of a given codimension.

2.2.1. *A partial order of strata.* This result will follow by relating codimension to a more combinatorially defined partial order:

**Definition 2.2.** Define a category $\mathsf{C}_{\mathcal{Z}}$ by

- Objects are $Z \in \mathcal{Z}$,
- Morphisms $Z \to Z'$ are injective maps $\phi\colon I(Z) \to I(Z')$ such that $\phi^*(Z') \subset Z$.

All endomorphisms are automorphisms, so the following defines a partial order:

**Definition 2.3.** Let $[Z]$ denote the isomorphism class of $Z$ in $\mathsf{C}_{\mathcal{Z}}$, and write $[Z] \leq [Z']$ if and only if there is a morphism $Z \to Z'$.

**Lemma 2.4.** *For $Z, Z' \in \mathcal{Z}$ we have*

$$[Z] \leq [A(Z)] \qquad [Z], [Z'] \leq [B_{\phi,\phi',i,i'}(Z, Z')] \qquad [Z] \leq [C_i(Z)].$$

*Proof.* By construction of operations A, B, and C. □

*Remark* 2.5. Of course, no analogous statement is true for operation D because it involves collision of points and thus strictly decreases the rank.

2.2.2. *Equivalence relations for vertical coincidence.* The result about the number of strata of a given codimension will to proceed by induction over the partial order $\leq$, by an internal induction over a stratification of source points of the target $Z'$. To that end, we introduce a pair of equivalence relations and a subset. (These will also play a role when defining bad strata.) Recall that if $Z$ has rank $r$, it consists of jets at $r$ ordered points, that is, its source is indexed by the set $\{1, \ldots, r(Z)\}$. Because injections between these sets will play an important role, we will introduce the notation:

$$I(Z) \coloneqq \{1, \ldots, r(Z)\}.$$

Recall we write $z \in Z$ as $(z_1, \ldots, z_{r(Z)})$ and $s$, resp. $t$, denote the source or target of a jet.

**Definition 2.6.** Given $Z \in \mathcal{Z}$, we define two equivalence relation $\sim$ and $\approx$ on $I(Z)$, and a subset $\Delta \subset I(Z)$ by

$$i \sim j \iff \text{for all } z \in Z,\, p_2 s z_i = p_2 s z_j,$$
$$i \approx j \iff \text{for all } z \in Z,\, t z_i = t z_j,$$
$$i \in \Delta \iff \text{for all } z \in Z,\, \ker(D z_i) \neq 0.$$

**Lemma 2.7.**

(i) *The equivalence relation generated by* $\sim$ *and* $\approx$ *is the unique equivalence relation with one equivalence class.*
(ii) *For each* $i \in I(Z)$ *either (I)* $i \in \Delta$ *or (II) there exists* $j \in I(Z)$ *such that* $j \neq i$ *and* $j \approx i$*.*

*Sketch of proof.* This is true for $Z^0$ and preserved by operations $A$, $B$, $C$, and $D$. □

2.2.3. *Codimension.* If $Z$ is an IASCM then by definition we have $Z = (p^\infty_m)^{-1}(p^\infty_m(Z))$ with $p^\infty_m(Z)$ a closed algebraic subset of some ${}_r\mathcal{J}^m$. As such $p^\infty_m(Z)$ has a codimension, and it is easy to verify that this is independent of $m$.

**Definition 2.8.** For a IASCM $Z$ of rank $r$, we set

$$c(Z) \coloneqq \operatorname{codim}(p^\infty_m(Z)) - r(p+1).$$

*Remark* 2.9. The motivation is that later, when we use $Z$ to define the singular set $S^*_\alpha$ for fibered concordance $F\colon I \times P \times D^s \to I \times N \times D^s$, this will generically have dimension $s - c(Z)$.

**Lemma 2.10.** *Suppose* $n - p \geq 3$*. If* $[Z] \leq [Z']$ *then* $c(Z) \leq c(Z')$*. Moreover either* $c(Z) < c(Z')$ *or* $r(Z) = r(Z')$ *and* $Z'$ *is isomorphic to a subset of* $Z$*.*

*Proof.* If $[Z] \leq [Z']$ there is an injection $\phi\colon I(Z) \hookrightarrow I(Z')$ such that $\phi^*(Z') \subset Z$. As a consequence of Lemma 2.7, there is a sequence

$$\phi(I(Z)) = I_0 \subset I_1 \subset \cdots \subset I_h = I(Z)$$

for $h \geq 0$, where in each step on the following holds:

(1) $I^k$ is obtained from $I^{k-1}$ by adding $i$ with $i \sim j$ for $j \in I^{k-1}$ and $i \in \Delta$.
(2) $I^k$ is obtained from $I^{k-1}$ by adding $i, i'$ with $i' \approx i \sim j$ for $j \in I^{k-1}$.
(3) $I^k$ is obtained from $I^{k-1}$ by adding $i$ with $i \approx j$ for $j \in I^{k-1}$.

For simplicity, we assume that $I^k = \{1, \ldots, r_k\}$ and $\phi$ is the standard inclusion. (The general case follows by modifying notation.) We then define $Z_k$, starting with $Z_0 = Z$ and obtaining $Z_k$ from $Z_{k-1}$ by (depending on the above case)

(1) $Z_k$ consists of $(z_1, \ldots, z_{r_{k-1}+1})$ with $(z_1, \ldots, z_{r_{k-1}}) \in Z_{r-1}$ and $\ker Dz_{r_{k-1}+1} \neq 0$, $p_2 s z_{r_{k-1}+1} = p_2 s z_j$.
(2) $Z_k$ consists of $(z_1, \ldots, z_{r_{k-1}+1}, z_{r_{k-1}+2})$ with $(z_1, \ldots, z_{r_{k-1}}) \in Z_{k-1}$ and $tz_{r_{k-1}+1} = tz_{r_{k-1}+2}$, $p_2 s z_{r_{k-1}+1} = p_2 s z_j$.
(3) $Z_k$ consists of $(z_1, \ldots, z_{r_{k-1}+1})$ with $(z_1, \ldots, z_{r_{k-1}}) \in Z_{r-1}$ and $tz_{r_{k-1}+1} = tz_j$.

Then we can estimate that

(1) $c(Z_k) = \operatorname{codim}(Z_k) - (r_{k-1}+1)(p+1) = \operatorname{codim}(Z_{k-1}) + (n-p) + p - r_{k-1}(p+1) - (p+1) = c(Z_{k-1}) + (n-p-1)$.
(2) $c(Z_k) = \operatorname{codim}(Z_k) - (r_{k-1}+2)(p+1) = \operatorname{codim}(Z_{k-1}) + n + p - r_{k-1}(p+1) - 2(p+1) = c(Z_{k-1}) + (n-p-2)$.
(3) $c(Z_k) = \operatorname{codim}(Z_k) - (r_{k-1}+1)(p+1) = \operatorname{codim}(Z_{k-1}) + n - r_{k-1}(p+1) - (p+1) = c(Z_{k-1}) + (n-p-1)$.

Since $Z' \subset Z_h$, we have that $c(Z') \geq c(Z^h)$.

The result follows once we observe that equality can only occur if $h = 0$, and then $\phi$ is a bijection and $\phi^*(Z') \subset Z$ with $\phi^*$ an isomorphism. $\square$

How do the operations A, B, C, and D, affect codimension?

**Lemma 2.11.** *Operations A, C and D strictly increase codimension. For operation B we have*

$$c(B_{\phi,\phi',i,i'}(Z,Z')) \geq c(Z) \quad \text{with equality only if the } r(B_{\phi,\phi',i,i'}(Z,Z')) = r(Z)$$

*and similarly for $Z'$.*

*Sketch of proof.* For A this is trivial, for B this follows from the previous lemma. For C and D this is more involved (this uses various facts about algebraic varieties).

Let us outline the proof for D. To do so, we need to recall its precise description: you give a surjection $\phi\colon \{1,\ldots,r(Z)\} \to \{1,\ldots,r'\}$ which is not a bijection, and let $K^n$ denote the space of all complex polynomials maps $\mathbb{C}^{p+1} \to \mathbb{C}^n$ of degree $< r\binom{p+1+m}{m}$. Then you let $\overline{X}$ be the Zariski closure in $(\mathbb{C}^{p+1})^r \times K^n$ of $X = \{(x,f) \in (\mathbb{C}^{p+1})^{(r)} \times K^n \mid {}_rj^m(f)(x) \in p^\infty_m(Z)\}$, and set $Y = \{(y,f) \in (\mathbb{C}^{p+1})^{r'} \times K^n \mid (y \circ \phi, f) \in \overline{X}\}$. Finally $D_\phi(Z) \subset {}_r\mathcal{J}^\infty$ is given by those $(z_1,\ldots,z_{r'})$ such that $p^\infty_{(m+1)\#\phi^{-1}(i')-1}(z_{i'}) = j^{(m+1)\#\phi^{-1}(i')-1}(f)(y_{i'})$.

Let $j'\colon (\mathbb{C}^{p+1})^{r'} \times K^n \to \prod_{i'=1}^{r'} \mathcal{J}^{(m+1)\#\phi^{-1}(i')-1}$ be the projection onto the corresponding jets. If last time we had done the proof that $D_\phi(Z)$ is an IASCM, we would have learned that this is a submersion/ Then we have that

$$\begin{aligned}
c(D_\phi(Z)) &= \operatorname{codim}(j'(Y), \prod_{i'=1}^{r'} \mathcal{J}^{(m+1)\#\phi^{-1}(i')-1}) - r'(p+1) \\
&= \operatorname{codim}(Y, (\mathbb{C}^{p+1})^{r'} \times K^n) - r'(p+1) \\
&= \dim(K^n) - \dim(Y) \\
&\geq \dim(K^n) - \dim(\overline{X} - X) \\
&> \dim(K^n) - \dim(X) \\
&= \operatorname{codim}(Z, {}_r\mathcal{J}^m) - r(p+1) \\
&= c(Z),
\end{aligned}$$

where we used that $Y$ is isomorphic to a subset of $\overline{X} - X$, and that $X$ is isomorphic to $p^\infty_m(Z)$. $\square$

**Corollary 2.12.** *For any $N$ there are only finitely many $Z \in \mathcal{Z}$ such that $c(Z) \leq N$.*

*Proof.* This is a proof by contradiction. Since $c(Z) \geq c(Z^0)$, there is a smallest $N$ where this fails. By construction, we may write the $Z \in \mathcal{Z}$ with $c(Z) \leq N$ as a sequence $Z_0, Z_1, \ldots$ such that $Z_0 = Z^0$ and for all $\nu > 0$ either $Z_\nu$ is obtained from $Z_\mu$ for $\mu < \nu$ by operations A, C, and D, or $Z_\nu$ is obtained from $Z_{\mu_1}, Z_{\mu_2}$ for $\mu_1, \mu_2 < \nu$ by operation B.

Since operations A, C, and D strictly increase codimension, we may upon deleting elements from the sequence suppose that either $Z_{\nu+1}$ is obtained from $Z_\nu$ by operation A, C, or D, or $Z_{\nu+1}$ is obtained from $Z_\mu$ for $\mu < \nu$ and $Z_\nu$ by operation B.

Among the $Z$ with $c(Z) < N$ only finitely many levels (the degree of the Taylor polynomials) $m$ appear. Operations A, C, and D strictly increase codimension, only operations C and D increase level, and operation B takes the maximum of the input levels; this means that there is a uniform bound on the levels in the sequence. By further deleting finitely many elements from the sequence we may assume that D does not occur and $c(Z_\nu) = N$ for all $\nu$. Since operation A and C do not increase rank (the number of particles) and operation B strictly increases codimension when it increases rank, by deleting finitely many elements from the sequence we may further assume that the rank is constant with value $r$. When the rank and codimension are constant, so is the level with value $m$.

By construction, there is an injection $\phi\colon I(Z_\nu) \to I(Z_{\nu+1})$ such that $\phi^*(Z_{\nu+1}) \subset Z_\nu$. Since $I(Z_\nu)$ and $I(Z_{\nu+1})$ have the same cardinality, $\phi$ is a bijection and we may assume that the $Z_\nu$ are an infinite descending sequence of Zariski closed subsets of ${}_m\mathcal{J}^r$. By Noetherianness we get that it is eventually constant; this is a contradiction. □

2.3. **Goodness and badness.** There are two ways in which an IASCM can be "bad": it has a cycle of vertical coincidences in domain and target (there is nowhere to start resolving this by sunny collapsing) or its derivative annihilates $\frac{\partial}{\partial x_1}$ (the sunny collapsing procedure can not be performed).

**Definition 2.13.** Let $\widetilde{Z}_k$ for $k \geq 1$ be given by those $(z_1, \ldots, z_{2k}) \in {}_{2k}\mathcal{J}^\infty$ such that $p_2 s z_{2i+1} = p_2 s z_{2i+2}$ and $t z_{2i} = t z_{2i+1}$ (where the indices wrap around).

**Definition 2.14.** Let $\widetilde{Z}$ be given by those $z \in {}_1\mathcal{J}^\infty$ such that $Dz \circ \frac{\partial}{\partial x_1}$.

Clearly, these lie in $\mathcal{Z}$ ($\widetilde{Z}_k$ is obtained from $Z^0$ by just operation B, and $\widetilde{Z} = D_\phi(Z^0)$, called $Z^1$ in the previous lecture) and satisfy $c(\widetilde{Z}_k) = k(n-p-2)$ and $c(\widetilde{Z}) = n-p-1$.

**Definition 2.15.** We say that $Z \in \mathcal{Z}$ is *bad* if $[Z] \geq [\widetilde{Z}_k]$ or $[Z] \geq [\widetilde{Z}]$. Otherwise $Z$ is *good*.

**Notation 2.16.** We will order the $Z$ in $\mathcal{Z}$ as $Z_0, Z_1, \ldots$ with $Z_0 = Z^0$, $Z_1 = \widetilde{Z}$, and otherwise such that $[Z_\alpha] \gneqq [Z_\beta]$ implies $\alpha < \beta$ and $\alpha \leq \beta$ implies $c(Z_\alpha) \leq c(Z_\beta)$.

## 3. Obtaining the stratification of $P \times D^s$

We can now apply the transversality results described by Nils to $\mathcal{Z}$. Recall that if $Z \in \mathcal{Z}$ had level $m$ and rank $r$, $S(Z, P, N) \subset {}_r\mathcal{J}^\infty(P \times \mathbb{R}, N)$ meant those jets that lie in $Z$ with respect to some chart. Similarly, last time we saw $S^*(Z, P, N) \subset {}_r\mathcal{J}^\infty(P \times \mathbb{R}, N)$, defined similarly but replacing $Z$ by $Z \setminus \Sigma(Z)$ (where $\Sigma(Z) \subset Z$ is the singular part). This is definition so as the open stratum on which $p^\infty_m(S(Z, P, N))$ is a submanifold.

**Proposition 3.1.** *Each fibered concordance*

$$F = (h, f, p_3)\colon I \times P \times D^s \longrightarrow I \times N \times D^s$$

*can be isotoped so that for each $Z \in \mathcal{Z}$ of level $m$ and rank $r$, ${}_r j^m(f) \pitchfork p^\infty_m S^*(Z, P, N)$.*

The singular sets are the locations in configuration spaces of the domain where the jets lie in $S(Z, P, N)$ or $S^*(Z, P, N)$:

**Definition 3.2.** For $Z_\alpha \in \mathcal{Z}$ and $F$ as above, we set

$$S_\alpha = ({}_r\mathcal{J}^\infty(f))^{-1}(S(Z_\alpha, P, N)) \subset (I \times P)^{(r)} \times D^s,$$
$$S^*_\alpha = ({}_r\mathcal{J}^\infty(f))^{-1}(S(Z^*_\alpha, P, N)) \subset (I \times P)^{(r)} \times D^s.$$

The latter is a submanifold (up to some issues near the boundary that are easily resolved by conditioning $F$ beforehand). We again proceed to replace $S_\alpha$ by $W_\alpha$ by removing points in earlier strata. Let $\pi_i \colon S_\alpha \to I \times P$ denote the projection to its $i$th.

**Definition 3.3.** $W_\alpha \subset S_\alpha$ consists of those $x$ such that for all $\beta > \alpha$ and $i \in I(Z_\alpha)$, $j \in I(Z_\beta)$ we have $\pi_i(x) \notin \pi_j(S_\beta)$.

This has the following properties:

- Our inclusion of operation A implies $W_\alpha \subset S^*_\alpha$.
- Our inclusion of operation D implies that $W_\alpha$ is open.
- Our inclusion of operation B implies that $p_{2,3}\pi_i W_\alpha \cap \pi_{2,3}\pi_j W_\beta = \varnothing$ with $p_{2,3} \colon I \times P \times D^s \to I \times P$ the projection.
- Our inclusion of operation C implies that $p_{2,3}\pi_i \colon W_\alpha \to P \times D^s$ is an immersion.

The vertical coincidences of $W_\alpha$ in the domain and target are encoded by $\sim$ and $\approx$. They however do not encode which point lies *above* which other point. To do so, we add a pair $(D, R)$ of binary relations on $I(Z)$ satisfying $i \sim j$ if and only if $iDj$, $i = j$, or $jDi$, and $i \approx j$ if and only if $iRj$, $i = j$, or $jRi$; such pairs of binary relations are called *admissible*.

**Definition 3.4.** For $(D, R)$ admissible on $I(Z_\alpha)$, we let $W^{(D,R)}_\alpha \subset W_\alpha$ consist of those $x$ where $iDj$ if and only if $\pi_i x$ is below $\pi_j x$ in $I \times P \times D^s$ and $iRj$ if and only if $F(\pi_i x)$ if below $F(\pi_j x)$.

This is open and *as long as $Z_\alpha$ is good* has the property that

$$p_{2,3}\pi_i \colon W^{(D,R)}_\alpha \to P \times D^s$$

is injective and upon varying $i$ and $(D, R)$ either yields a disjoint subsets or there is an automorphism of $I(Z_\alpha)$ relating them. We can now finally give the stratification of $P \times D^s$: it will by the images of $W^{(D,R)}_\alpha$.

## References

[Goo90] T. G. Goodwillie, *A multiple disjunction lemma for smooth concordance embeddings*, Mem. Amer. Math. Soc. **86** (1990), no. 431, viii+317. 1