ON SPACE-FILLING CURVES AND THE HAHN-MAZURKIEWICZ THEOREM

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ABSTRACT. These are notes on space-filling curves, looking at a few examples and proving the Hahn-Mazurkiewicz theorem. This theorem characterizes those subsets of Euclidean space that are the image of the unit interval under a continuous space.

After a historical introduction, we spend two sections constructing space-filling curves. In particular, we treat Peano's example because of its historical significance and Lebesgue's example because of its relevance to the proof of the Hahn-Mazurkiewicz theorem. This theorem is proven in the fourth section, with the exception of some point-set topology lemma's that have been delegated to the appendix. The first three sections only require basic knowledge of real analysis, while the fourth section and the appendix also require some knowledge of point-set topology. The best reference for space-filling curves is Sagan's book [Sag94].[1]

1. WHAT ARE CURVES?

The end of the 19th century was an exciting time in mathematics. Analysis had recently been put on rigorous footing by the combined effort of many great mathematicians, most importantly Cauchy and Weierstrass. In particular, they defined continuity using the well-known ϵ, δ-definition: a function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous if for every $x \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\delta > 0$, such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. We will be more interested in curves: continuous functions $f\colon [0,1] \to \mathbb{R}^n$.

This definition was supposed to capture the intuitive idea of a unbroken curve, one that can be drawn without lifting the pencil from the paper. Continuous curves should not be the same as smooth curves, as they for example can have corners. Though continuity might seem like the correct definition for this, it in fact admits "monstrous" curves: Weierstrass in 1872 wrote down a nowhere differentiable continuous function[2]. By looking at its graph in $\mathbb{R}^2$, one finds a curve having "corners" everywhere!

Shortly after this, in 1878, Cantor developed his theory of sets. One of the (then) shocking consequences was that $\mathbb{R}$ and $\mathbb{R}^2$ had the same cardinality, i.e. the "same number" of points. This means that there exists a bijection between both sets, which Cantor explicitely wrote down. This result questioned intuitive ideas about the (rather elusive) concept of dimension and an obvious question after the previous discovery of badly-behaved curves was: Can we find a bijection between $\mathbb{R}$ and $\mathbb{R}^2$

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[1]Thanks to Anupam Datta for some corrections to an earlier version.

[2]It is given by $\sum_{n=1}^{\infty} a^n \cos(b^n x)$ with $a \in (0,1)$ and b an odd integer satisfying $ab > 1 + \frac{3}{2}\pi$.

which is continuous or maybe even smooth? The answer to this question turns out to be "no" – a 1879 result by Netto – something we will prove later this lecture.

However, it remained open for more than a decade whether it was also impossible to find a surjective continuous map $\mathbb{R} \to \mathbb{R}^2$ or, closely related, $[0,1] \to [0,1]^2$. Peano shocked the mathematical community in 1890 by constructing a surjective continuous function $[0,1] \to [0,1]^2$, a "space-filling curve". It is pictured in figure 1.

This result is historically important for several reasons. Firstly, it made people realize how forgiving continuity is as a concept and that it doesn't always behave as one intuitively would expect. For example, the space-filling curves show it behaves badly with respect to dimension or measure. Secondly, it made clear that result like the Jordan curve theorem and invariance of domain need careful proofs, one of the main motivations for the then new subject of algebraic topology (back then called analysis situs). I think this makes space-filling curves worthwhile to know about, even though they are no longer an area of active research and they hardly have any applications in modern mathematics. If nothing else, thinking about them will teach you some useful things about analysis and point-set topology.

In this lecture we will construct several examples, look at their properties and end with the Hahn-Mazurkiewicz theorem which tells you exactly which subsets of $\mathbb{R}^n$ can be image of a continuous map with domain $[0,1]$.

2. Peano's curve

We start by looking at Peano's original example. The main tool used in its construction is the theorem that a uniform limit of continuous function is continuous:

Theorem 2.1. *Let f_n be a sequence of continuous functions $[0,1] \to [0,1]^n$ such that for $\eta > 0$ there is a $N \in \mathbb{N}$ having the property that for all $n, m > N$ we have $\sup_{x\in[0,1]} ||f_n(x) - f_m(x)|| < \eta$. Then $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$ and is a continuous function.*

Proof. Suppose we are given a $x \in [0,1]$ and $\epsilon > 0$, then we want to find a $\delta > 0$ such that $|y - x| < \delta$ implies $||f(y) - f(x)|| < \epsilon$. Note that for all $n \in \mathbb{N}$,

$$||f(y) - f(x)|| \leq ||f(y) - f_n(y)|| + ||f_n(y) - f_n(x)|| + ||f_n(x) - f(x)||$$

By taking n to be a sufficiently large number N, we can make two of the terms in the right hand side be bounded by $\frac{1}{3}\epsilon$. By continuity of f_N, we can find a $\delta > 0$ such that $|y - x| < \delta$ implies $||f_N(y) - f_N(x)|| < \frac{1}{3}\epsilon$. Then if $|y - x| < \delta$ we have

$$\begin{aligned} ||f(y) - f(x)|| &\leq ||f(y) - f_N(y)|| + ||f_N(y) - f_N(x)|| + ||f_N(x) - f(x)|| \\ &\leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon \end{aligned}$$

□

To get Peano's curve, we will define a sequence of continuous curves $f_n\colon [0,1] \to [0,1]^2$ that are uniformly convergent to a surjective function. The previous theorem guarantees that this function will in fact be continuous as well. We will get the f_n through an iterative procedure. The functions f_n are defined by dividing $[0,1]^2$ into 9^n squares of equal size and similarly dividing $[0,1]$ into 9^n intervals of equal length. It is clear what the i'th interval is, but the i'th square is determined by the previous function f_{n-1} by the order in which it reaches the squares. The i'th interval is then mapped into the i'th square in a piecewise linear manner. The basic

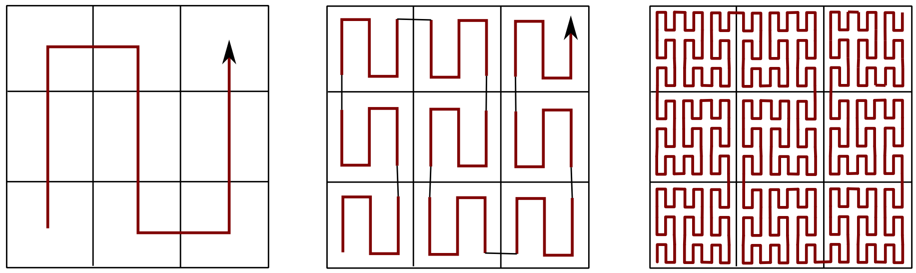

FIGURE 1. The first, second and third iterations f_0, f_1, f_2 of the sequence defining the Peano curve.

shape of these piecewise linear segments is that of f_0, given in figure 1. The next iteration is obtained by replacing the piecewise lienar segements in each square with a smaller copy of f_0, matching everything up nicely.

Clearly each f_n is continuous, but also they become increasingly close.

Lemma 2.2. *For $n, m > N$ we have that $\sup_{x\in[0,1]} ||f_n(x) - f_m(x)|| < \sqrt{2} \cdot 9^{-\min(n,m)}$. Hence the sequence f_n is uniformly convergent.*

Proof. Without loss of generality $n > m$. To get from f_m to f_n we only modify f_m within each of the 9^m cubes. Within such a cube, the points that are furthest apart are in opposite corners and their distance is $\sqrt{2}9^{-m}$. □

The theorem about uniform convergence of continuous functions now tells us that the following definition makes sense.

Definition 2.3. The *Peano curve* is the continuous function $f : [0,1] \to [0,1]^2$ given by

$$f(x) \coloneqq \lim_{n\to\infty} f_n(x)$$

Lemma 2.4. *The Peano curve is surjective.*

Proof. Note that n'th iterate f_n of the Peano curve goes through the centers of all of 9^n squares. Hence any point $(x, y) \in [0,1]^2$ is no more than $\frac{\sqrt{2}}{2}9^{-n}$ away from a point on f_n. Using the uniform convergence, we see that there is a point on the image of f that is less than $\frac{3\sqrt{2}}{2}9^{-n}$ away from (x, y). Hence (x, y) lies in the closure of the image of f. We will see in Proposition 4.3 that the image of the interval under any continuous function is compact and hence closed, as (x, y) in fact lies in the image of f. □

Though each of the f_n is injective, the limit f is not. This is will proven later, as a consequence of one of the intermediate results in the proof of the Hahn-Mazurkiewicz theorem.T

3. Lebesgue's curve

We will now give a second construction of a space-filling curve, due to Lebesgue in 1904. The ideas in this construction will play an important role in the proof of the Hahn-Mazurkiewicz theorem, and along the way we will meet an interesting space known as a Cantor space.

Lebesgue's curve will be clearly surjective, because it will be given by connecting the point in the image of a surjective map from a subset of $[0,1]$ by linear segments. This subset is the Cantor set:

Definition 3.1. The *Cantor set* $\mathcal{C}$ is the subset of $[0,1]$ obtained by deleting all elements that have a 1 in their ternary decimal expansion.

Alternatively, $\mathcal{C}$ is given by the repeatedly removing middle thirds of intervals. One starts with $\mathcal{C}_0 = [0,1]$, $\mathcal{C} = [0,1/3] \cup [2/3,1]$, etc., and sets $\mathcal{C} = \bigcap \mathcal{C}_n$. Lebesgue then defined the following function from $\mathcal{C}$ to $[0,1]^2$:

$$l(0._3(2a_1)(2a_2)(2a_3)\ldots) = \begin{pmatrix} 0._2 a_1 a_3 a_5 \ldots \\ 0._2 a_2 a_4 a_6 \ldots \end{pmatrix}$$

This is easily seen to be surjective. We will show it is continuous.

Lemma 3.2. *The function l is continuous.*

Proof. We will prove that if $|y-x| < \frac{1}{3^{2n+1}}$ (i.e. our δ), then $||l(y)-l(x)|| < \frac{\sqrt{2}}{2^n}$ (i.e. our ϵ). The idea is to note that if $|y-x| < \frac{1}{3^{2n+1}}$, then the first $2n$ ternary decimals are the same. Hence the first n binary decimals of the x and y-coordinate of $l(x)$ are the same. This means that they are no further that $\frac{\sqrt{2}}{2^n}$ apart. □

Now Lebesgue extends this function linearly over the gaps in $[0,1]$ left by the Cantor set. We denote such a gap by (a_n, b_n). An example is $(1/3, 2/3)$. We obtain $\tilde{l} : [0,1] \to [0,1]^2$ as follows

$$\tilde{l}(x) = \begin{cases} l(x) & \text{if } x \in \mathcal{C} \\ (l(b_n) - l(a_n))\frac{x-a_n}{b_n-a_n} + l(a_n) & \text{if } x \in (a_n, b_n) \subset ([0,1]\backslash \mathcal{C}) \end{cases}$$

We claim that this function is continuous. Clearly it is continuous on the gaps (a_n, b_n) in the complement of the Cantor set because it is linear there. It is however not clear it continuous at the points of the Cantor set. However, by continuity of l and the fact that linear interpolations don't get far from their endpoints, we will also be able to prove continuity there.

Lemma 3.3. *The function $\tilde{l}$ is continuous.*

Proof. By the remarks above it suffices to prove continuity at $x \in \mathcal{C}$. Since $\mathcal{C}$ has empty interior or equivalently every point is a boundary point, there are three situations: (i) x is both on the left and the right a limit of points in $\mathcal{C}$, (ii) x borders a gap (a_n, b_n) on the left, (iii) x borders a gap (a_n, b_n) on the right. We will do the last case and skip the first two, because they are very similar to the last case.

We do not need to worry about continuity from the right, as $\tilde{l}$ is linear there. We can take a δ satisfying $\frac{1}{3^{2n+1}} > \delta > 0$ and such that if $y < x$ and $|y-x| < \delta$, then y lies in $\mathcal{C}$ or in a gap (a_k, b_k) such that $|a_k - x| < \frac{1}{3^{2n+1}}$. The idea is just to take δ a bit smaller than $\frac{1}{3^{2n+1}}$ such that $[x - \frac{1}{3^{2n+1}}, x-\delta]$ contains an element of $\mathcal{C}$.

Then, if $y \in \mathcal{C}$, then we know that $||\tilde{l}(y) - \tilde{l}(x)|| = ||l(y) - l(x)|| < \frac{\sqrt{2}}{2^n}$ by the previous lemma. If $y \notin \mathcal{C}$, then y lies in another gap (a_k, b_k) with $x \leq a_k$ and $|a_k - x| < \frac{1}{3^{2n+1}}$. That $||\tilde{l}(y) - \tilde{l}(x)|| < \frac{\sqrt{2}}{2^n}$ now follows by the convexity of balls in Euclidean space: if the endpoints of a line segment lie in a ball, then the entire line

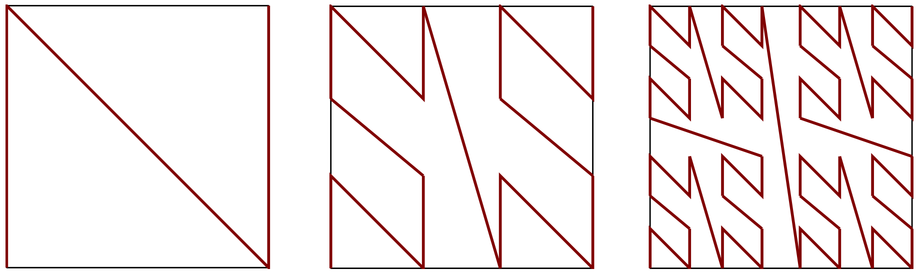

FIGURE 2. The first, second and third iterations of a sequence converging to hte Lebesgue curve.

segment lies in the ball. Alternatively, we can simply estimate

$$\begin{aligned}||\tilde{l}(y) - \tilde{l}(x)|| &= ||(l(b_k) - l(a_k))\frac{y - a_k}{b_k - a_k} + l(a_k) - l(y)|| \\ &= \frac{1}{b_k - a_k}(||l(b_k) - l(y)||(y - a_k) + ||l(a_k) - l(y)||(b_k - y)) \\ &< \frac{1}{b_k - a_k}(\frac{\sqrt{2}}{2^n}(y - a_k + b_k - y)) = \frac{\sqrt{2}}{2^n}\end{aligned}$$

where we have used that $(a_k, b_k) \subset [x - \frac{1}{3^{2n+1}}, x]$ implies that $||l(b_k) - l(y)|| < \frac{\sqrt{2}}{2^n}$ and similarly for $||l(a_k) - l(y)||$. □

There is an iterative construction of Lebesgue's curve similar to Peano's curve. It is given in figure 2.

4. THE HAHN-MAZURKIEWICZ THEOREM

In this section we prove the Hahn-Mazurkiewicz theorem for subsets of Euclidean space.

Definition 4.1. A subset A of $\mathbb{R}^n$ is said to be *compact* if every sequence has a convergent subsequence with limit in A. Equivalently, by Bolzano-Weierstrass, it is bounded and closed.

The set A is said to be *connected* if we can't write it $A = (U \sqcup V) \cap A$, where U and V are non-empty disjoint open subsets of $\mathbb{R}^n$.

Finally, it is said to be *weakly locally-connected* if for all $a \in A$ and $\eta > 0$ we can find a smaller $\epsilon > 0$ such that $x \in B_\epsilon(a)$, then x and a lie in the same connected component of $A \cap B_\eta(a)$.

The last definition, that of weakly locally-connectedness, is in fact equivalent to the more widely used notion of locally-connectedness. However, weakly locally-connectedness at a point is not equivalent to locally-connectedness at that point! This is proven in the appendix.

Theorem 4.2. *A subset $A \subset \mathbb{R}^n$ is the image of some continuous map $f\colon [0,1] \to \mathbb{R}^n$ if and only if it is compact, connected and weakly locally-connected.*

We will split the proof into two parts. The easier implication is that the image of a space-filling curve is compact, connected and locally-connected, i.e. that our conditions are necessary. For the converse we will have to work a bit harder. The trick is to mimic Lebesgue's construction in general, using Hausdorff's theorem on the images of the Cantor set.

4.1. **The conditions are necessary.** We will start by proving that the conditions of compactness, connectedness and weakly locally-connectedness are necessary.

Proposition 4.3. *The image of a continuous map $f : [0,1] \to \mathbb{R}^n$ is compact, connected and weakly locally-connected.*

Proof. Let's call the image A. One of the equivalent definitions of being compact is that every sequence in A has a convergent subsequence with limit in A. Let's prove this. Let a_n be a sequence in A, then by picking a point in each preimage of an a_n we get a sequence x_n in $[0,1]$. The interval $[0,1]$ is closed and bounded, hence compact by Bolzano-Weierstrass. Thus there is a convergent subsequence x_{n_k} in $[0,1]$, with limit x. We claim that the a_{n_k} converge for $f(x)$. But this is a direct consequence of the continuity of f: it sends convergent sequences to convergent sequences.

The image of a path-connected set is clearly path-connected. But path-connected implies connected, so A is in fact connected.

For weakly locally-connectedness we have to a bit more careful. Suppose that for $a \in A$ the condition of weakly locally-connectedness fails. Then we can find an $\epsilon > 0$ and $\eta_n < \epsilon$ with $\eta_n \to 0$ and $a_n \in B_{\eta_n}(a) \cap A$ such that a_n and a don't lie in the same connected component of $B_\epsilon(a)$. Note that by construction the sequence $\{a_n\}$ converges to a. Pick an element x_n in the preimage of a_n. By compactness of $[0,1]$ this has a convergent subsequence x_{n_k} with limit x, which necessarily is mapped to a. There is a $\delta > 0$ such that $|y - x| < \delta$ implies $||f(y) - f(x)|| < \epsilon$. Take K sufficiently large such that $|x - x_{n_K}| < \delta$. Then the entire line segment $[x_{n_K}, x]$ (or $[x, x_{n_K}]$, depending on what makes sense), is mapped into $B_\epsilon(a) \cap A$ and hence a_{n_K} and a lie in the same connected component of $B_\epsilon(a) \cap A$, contradicting our assumption on a_{n_K}. □

This theorem may be generalized by proving that the image of any compact, resp. connected, space is compact, resp. connected.

4.2. **Intermezzo: Netto's theorem.** In the first part of the proof of the previous proposition, we actually proved the image of each closed subset $B \subset [0,1]$ under a continuous map $f\colon [0,1] \to \mathbb{R}^n$ is closed. We will use this to prove that no space-filling curve can be injective. This is known as Netto's theorem. It is a special version of the theorem that a continuous bijection from a compact space to a Hausdorff space is homeomorphism.

Proposition 4.4. *A continuous surjective map $f\colon [0,1] \to [0,1]^2$ can not be injective.*

Proof. We claim that the inverse $g \coloneqq f^{-1}\colon [0,1]^2 \to [0,1]$ is continuous and hence f is a homeomorphism (a continuous bijective function with continuous inverse). Let $C \subset [0,1]$ be a closed subset. It is also bounded, so compact. We have that $g^{-1}(C) = f(C)$ and every continuous function sends compact sets to compact sets, as implicitly proven in proposition 4.3. Every compact set in $[0,1]^2$ is in particular closed. So g^{-1} of closed sets are closed, hence by looking at complements under g^{-1} of open sets we can conclude that these are open. Hence $g = f^{-1}$ is continuous.

Connectedness is preserved by homeomorphisms, because if X and Y are homeomorphic, a decomposition of one of them into a disjoint union of non-empty open sets can be We can find a $y \in \text{int}([0,1]^2)$ whose preimage x is not equal to 0 or 1. Then the restriction of f to $[0,x) \cup (x,1]$, gives a homeomorphism of the set $[0,x) \cup (x,1]$, having two connected components, with the connected set $[0,1]^2 \backslash \{y\}$. This gives a contradiction. □

4.3. **Hausdorff's theorem and a theorem on pathconnectedness.** In this subsection we will prove Hausdorff's theorem about the possible images in Euclidean space under a continuous map of the Cantor set. This was proven in 1927 by Hausdorff and independently in 1928 by Alexandroff. It says, in some sense, that the Cantor set is the universal compact subset of Euclidean space of any dimension.

Theorem 4.5. *Every compact subset of* $\mathbb{R}^n$ *is a continuous image of the Cantor set.*

Proof. Let A be our compact set. Consider the cover of A given by $B_1(a)$ for $a \in A$. By compactness there is a finite subcover, which by duplicating elements of the cover if necessary is of the form $B_1(a_i)$ for $1 \leq i \leq 2^{n_1}$. Now cover $A_i = \bar{B}_1(a_i) \cap A$ by $B_{1/2}(a)$ for $a \in B_1(a_i) \cap A$ and by duplicating elements of the finitely many covers if necessary we can assume it has a finite subcover $B_{1/2}(a_{i,j})$ for $1 \leq i \leq 2^{n_1}$ and $1 \leq j \leq 2^{n_2}$ with n_2 independent of i. We set $A_{i,j} = \bar{B}_{1/2}(a_{i,j}) \cap \bar{B}_1(a_i) \cap A$. Continue this procedure, making the radius of the balls half as large each step.

Notice that for a sequence $k_i \in \{1, \ldots, 2^{n_i}\}$ we get a sequence of nested closed sets with radius going to 0. This determines a unique element of A as follows: a is the unique element of $\bigcap_i A_{k_1,k_2,\ldots,k_i}$. Since we are dealing with increasingly fine covers and every element of A lies in some intersection of elements of the cover, we obtain each element of A this way. For convenience, we define a way of getting from a sequence $\{a_n\}_{n\in\mathbb{N}}$ of $0, 1$'s to a sequence of numbers $\{K_n\}_{n\in\mathbb{N}}$ in $\{1, \ldots, 2^{n_i}\}$. They are given by $K_1 = \sum_{j=1}^{n_1} a_j 2^{j-1}$, $K_2 = \sum_{j=1}^{n_2} a_{j+n_1} 2^{j-1}$ and in general by $K_i = \sum_{j=1}^{n_i} a_{j+n_1+\ldots+n_{i-1}} 2^{j-1}$.

We will define a function $f : \mathcal{C} \to A$. It is given by

$$f(0._3(2a_1)(2a_2)(2a_3)\ldots) \coloneqq \text{unique element of} \bigcap_i A_{K_1,K_2,\ldots,K_i}$$

This is surjective by the previous remark about every element lying in an intersection of some elements of the cover. We also claim it is continuous. This is actually rather simple: if $|y - x| < 3^{n+1}$ then the first n digits in the ternary expansion are equal. We can find a j such that $n_1 + n_2 + \ldots + n_j \leq n \leq n_1 + n_2 + \ldots + n_{j+1}$. Thus $f(y)$ and $f(x)$ lie in the same $A_{k_1,k_2,\ldots,k_j}$, which lies in a ball of radius $\frac{1}{2^{k-1}}$. Hence $||f(y) - f(x)|| < \frac{1}{2^{k-1}}$ and we conclude that f is continuous. □

We want to construct our space-filling curve using this map, by connecting using paths in A over the gaps in the Cantor set. However, to be able to do this we need find such paths and show they can be chosen sufficiently short for points in A that are close together. Let's make definitions for these notions.

Definition 4.6. We say that a subset A of $\mathbb{R}^n$ is *path-connected* if any two points can be joined a continuous path with image in A.

The subset A is said to be *uniformly locally path-connected* if for all $\epsilon > 0$ there exists a $\eta \in (0, \epsilon)$ such that if $a, a' \in A$ satisfy $||a' - a|| < \eta$, then there is a continuous path with image in $B_\epsilon(a') \cap B_\epsilon(a) \cap A$ connecting a' and a.

Theorem 4.7. *If A is compact, connected and weakly locally-connected, it is path-connected and uniformly locally path-connected.*

This is proven in the appendix.

4.4. **The conditions are sufficient.** Now we'll combine Hausdorff's theorem and our theorem on pathconnectedness to construct a surjective map $[0, 1] \to A$ under the conditions listed before. As said before, the idea is to copy the idea of Lebesgue's construction of a space-filling curve: use Hausdorff's theorem to get a surjective map from the Cantor set into A and use our theorem on pathconnectedness to prove that we can connect the image by paths our the gaps in the Cantor set.

Theorem 4.8. *If $A \subset \mathbb{R}^n$ is compact, connected and weakly locally-connected, then there exists a surjective continuous function $f \colon [0, 1] \to A$.*

Proof. Since A is compact, by Hausdorff's theorem there exists a continuous surjective function $g \colon \mathcal{C} \to A$.

By uniformly locally path-connectedness, we can find a decreasing sequence of η_n such that $||a' - a|| < \eta_n$ implies that there is a continuous path with image in $B_{1/2^n}(a) \cap B_{1/2^n}(a') \cap A$. By continuity of g and compactness of $\mathcal{C}$, we can find a decreasing sequence of $\delta_n > 0$ such that $|y - x| < \delta_n$ implies $||g(y) - g(x)|| < \eta_n$. If (a_k, b_k) is a gap in $\mathcal{C}$ such that $\delta_{n+1} \leq |b_k - a_k| < \delta_n$, then we can find a continuous path $\gamma_k : [a_k, b_k] \to A$ such that γ_k lies in $B_{1/2^n}(g(a_k)) \cap B_{1/2^n}(g(b_k)) \cap A$ connecting $g(a_k)$ to $g(b_k)$. We define f as follows:

$$f(x) := \begin{cases} g(x) & \text{if } x \in \mathcal{C} \\ \gamma_k(x) & \text{if } x \in (a_k, b_k) \subset ([0,1] \backslash \mathcal{C}) \end{cases}$$

This function is clearly continuous at all points of $([0, 1]\backslash\mathcal{C})$ and as before the points $x \in \mathcal{C}$ comes in three types: (i) x is both on the left and the right a limit of points in $\mathcal{C}$, (ii) x borders a gap (a_n, b_n) on the left, (iii) x borders a gap (a_n, b_n) on the right. Again we will do the last case and skip the first two, because they are very similar to the last case.

Continuity from the right is obvious, as f is just a continuous path there. For continuity from the left pick $\delta > 0$ satisfying $\delta_n > \delta > 0$ and such that if $y < x$ and $|y - x| < \delta$, then $y \in \mathcal{C}$ or it lies in a gap (a_k, b_k) such that $|a_k - x| < \delta_n$.

If $y \in \mathcal{C}$, then $|y - x| < \eta_n$ and hence $||f(y) - f(x)|| = ||g(y) - g(x)|| \leq \eta_n$. If $y \notin \mathcal{C}$, then y lies on a path that doesn't get further than $\frac{1}{2^n}$ from $g(x)$ and hence $||f(y) - f(x)|| \leq \frac{1}{2^n}$. So given an $\epsilon > 0$ just take a $n \in \mathbb{N}$ so that $\min(\{\eta_n, \frac{1}{2^n}\}) < \epsilon$ and set $\delta = \delta_n$. Then $|y - x| < \delta_n$ implies $||f(y) - f(x)|| < \epsilon$. □

Appendix A. Different notions of local connectedness

In this appendix we prove several relations between different notion of connectedness. Let's recall their definitions. We will use open sets instead of open balls for convenience, except in the definition about uniform path-connectedness, which needs explicit ϵ's.

Definition A.1. (1) A subset $A \subset \mathbb{R}^n$ is said to be *weakly locally-connected* if for all $a \in A$ and open neighborhoods U of a in A we can find open neighborhood $V \subset U$ containing a such that $x \in V$, then x and a lie in the same connected component of U.

(2) The subset A is said to be *uniformly weakly locally-connected* if for all $\epsilon > 0$ there exists a $\eta \in (0, \epsilon)$ such that if $a, a' \in A$ satisfy $||a - a'|| < \eta$ then a, a' lie in the same connected component of $B_\epsilon(a) \cap B_\epsilon(a') \cap A$.

(3) $A \subset \mathbb{R}^n$ is said to be *locally connected* if for all $a \in A$ and all open neighborhoods U of a in A we can find a neighborhood $V \subset A$ of a which is connected.

(4) A is said to be *locally path-connected* if for all $a \in A$ and all open neighborhoods U of a in A we can find a neighborhood $V \subset A$ of a which is path-connected.

(5) The subset A is said to be *uniformly locally path-connected* if for all $\epsilon > 0$ there exists a $\eta \in (0, \epsilon)$ such that if $a, a' \in A$ both satisfy $||a - a'|| < \eta$ then there is a continuous path with image in $B_\epsilon(a) \cap B_\epsilon(a')$ connecting a' and a.

Lemma A.2. *The subspace A is weakly locally-connected if and only if it is locally connected.*

Proof. The implication $\Leftarrow$ is clear. For the implication $\Rightarrow$, we claim that the connected components of all open subsets of A are open. For if this is true, then take V to be the connected component containing a in U.

Let C be a connected component of U. We will cover C by open sets $V_c \subset A$, then the expression $C = (\cup_c V_c) \cap A$ shows it's open and we are done. Pick $c \in C$ and an open neighborhood V_c of it such that $U_c \subset U$. Using weakly locally-connectedness, take a smaller open neighborhood V_c such that if $x \in V_c$, then c and x lie in the same connected component of U_c. In particular c and x must then both lie in C. This means that $V_c \subset C$ and we are done. □

Lemma A.3. *Connected and locally path-connected implies path-connected. Uniformly locally path-connected implies locally path-connected.*

Proof. Let's start with the first statement. Let $a \in A$ and P_a be the subset of A of all points that can be connected by a path to A. It is open by locally path-connectedness. To see it is closed as well, use that "being connected be a path" is an equivalence relation, hence $A = \sqcup P_{a_i}$. Hence the complement of P_a is open and thus P_a is closed. An open and closed set is a connected component and since A was assumed connected, we conclude $A = P_a$.

The second statement is trivially true. □

Lemma A.4. *If A is compact and weakly locally-connected it is uniformly weakly locally-connected.*

Proof. Suppose that it is not uniformly weakly locally-connected. Then there exists an $\epsilon > 0$, $\eta_i \to 0$ and $||a_i - a'_i|| < \eta_i$ such that a_i and a'_i lie in different connected components of $B_\epsilon(a) \cap B_\epsilon(a') \cap A$. By compactness, we can assume that a_i and a'_i converge, necessarily to the same element a. Now we apply weakly locally-connectedness at a to $B_\epsilon(a)$: we get a $\eta > 0$ such that $x \in B_\eta(a)$ implies that x, a lie in the same connected component of $B_\epsilon(a)$. Take i large enough such $\eta_i < \frac{1}{2}\eta$. Then a_i and a'_i lie in the same connected component of $B_\epsilon(a)$ at a and hence lie in the same connected component. This gives a contradiction. □

Theorem A.5. *If A is compact, connected and weakly locally-connected, it is path-connected, locally path-connected and in fact uniformly locally path-connected.*

Proof. By the previous lemma's, it suffices to prove uniform locally path-connectedness. Take $\epsilon > 0$ and set $\epsilon_k = \frac{1}{4}\frac{\epsilon}{2^k}$. Take η_k to be number coming out of uniformly weakly locally-connectedness for ϵ_k and set $\eta = \eta_1$. Let $a, a' \in A$ be such that $||a - a'|| < \eta$.

We claim that by connectedness we can find a $n_1 \geq 1$ and a sequence of points $x_i^{(1)}$ for $0 \leq i \leq 2^{n_1}$ such that $x_0^{(1)} = a$, $x_{2^{n_1}}^{(1)} = a'$ and $||x_i^{(1)} - x_{i+1}^{(1)}|| < \eta_1$. This is because the set of points in A that can be connected by a η_1-chain of points is both open and closed, hence equal to A. Each of the $x_i^{(1)}$ lie the same connected component of $B_{\epsilon_1}(x_i^{(1)}) \cap B_{\epsilon_1}(x_{i+1}^{(1)}) \cap A$ and hence we can find n_2 and sequences $x_{i,j}^{(2)}$ in $B_{\epsilon_1}(x_i^{(1)}) \cap B_{\epsilon_1}(x_{i+1}^{(1)}) \cap A$ for $0 \leq i \leq 2^{n_1}$, $0 \leq j \leq 2^{n_2}$ with $x_{i,0}^{(2)} = x_i^{(1)}$ and $x_{i,2^{n_2}}^{(2)} = x_{i+1}^{(1)}$. Continue this process, getting n_k and $x^{(k)}$.

Let $J \subset [0,1]$ be the set of points with binary expansion that can be written as $\frac{i}{2^{n_1+\cdots+n_k}}$ for some i and k. This is dense. Then we define $g : J \to A$ by sending $\frac{i}{2^{n_1+\cdots+n_k}}$ to the point of the $x^{(n_k)}$ that i corresponds to. This can be checked to be uniformly continuous, since $\eta_k < \epsilon_k$ and the ϵ_k go to zero. Hence g extends to a continuous function $\gamma : [0,1] \to A$. The image lies in A since A is compact, hence closed.

The only thing left to check is that γ stays within $B_\epsilon(a) \cap B_{\epsilon'}(a) \cap A$. But it stays within $\sum_{k=1}^{\infty} \epsilon_k < \epsilon$ of a and a' by construction. □

References

[Sag94] Hans Sagan, *Space-filling curves*, Universitext Series, Springer-Verlag, 1994. 1