Purdue START minicourse

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Abstract

These are the collected lecture notes for START minicourse on *Homological stability in high-dimensional manifold theory*, summer 2023.

Contents

Chapter 1

Introduction

In these lectures we discuss the ways in which homological stability arises when studying automorphisms of high-dimensional manifolds.

Our starting point in the first lecture is the mapping class group $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ of certain manifolds $W_{g,1}$, the high-dimensional analogue of a surface of genus g with a single boundary component. This group was determined by Kreck's, and admit a surprisingly simple description in terms of a pair of extensions of an arithmetic group. We use this to deduce homological stability for $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$.

In the second lecture we move on to the diffeomorphism group $\operatorname{Diff}_\partial(W_{g,1})$. We invoke general machinery to reduce homological stability to the connectivity of certain semi-simplicial spaces of destabilisations, and explain how to establish these connectivity results given an analogue connectivity result in an algebraic setting.

In the third lecture we explain how the embedding calculus applied to spaces of self-embeddings $\operatorname{Emb}_{1/2\partial}(W_{g,1})$ can be used to relate the results of the first and second lecture. We end with a general statement that appears in joint work with Krannich and yields novel homological stability results in high dimensions.

Chapter 2

Mapping class groups

In this first lecture we will look at Kreck's determination of the mapping class groups of highly-connected high-dimensional manifolds, and explain how it is an example of the general connection between mapping class groups and arithmetic groups discovered by Sullivan. This is another instance of a general slogan: in high dimensions geometry reduce to algebra.

2.1 Mapping class groups

For a smooth compact d-dimensional manifold M with boundary ∂M, we let $\mathrm{Diff}_\partial(M)$ denote the topological group of diffeomorphisms of M fixing a neighbourhood of the boundary pointwise, in the C^∞-topology.

Definition 2.1.1. The *mapping class group* of M is the group of path-components

$$\pi_0 \,\mathrm{Diff}_\partial(M).$$

Example 2.1.2 (Discs and spheres). It follows from work of Smale (the h-cobordism theorem [Sma62]) and Cerf (the pseudisotopy theorem [Cer70]) that in dimension $d \geq 5$ there is an isomorphism

$$\begin{aligned} \pi_0 \,\mathrm{Diff}_\partial(D^d) &\xrightarrow{\cong} \Theta_{d+1} \\ [f] &\longmapsto [D^{d+1} \cup_{\overline{f}} D^{d+1}] \end{aligned}$$

where $\overline{f}$ is the extension of f from D^d to S^d by the identity, and Θ_{d+1} is the finite abelian group of oriented homotopy $(d+1)$-spheres up to orientation-preserving diffeomorphism. The latter was determined by Kervaire–Milnor [KM63, Lev85] in terms of the stable homotopy groups of spheres and Bernoulli numbers. From this, one easily deduces that $\pi_0 \,\mathrm{Diff}(S^d) \cong \{\pm 1\} \ltimes \Theta_{d+1}$ where -1 is represented by reflection in a hyperplane and acts by negation on Θ_{d+1}. For $d \leq 3$ we have $\pi_0 \,\mathrm{Diff}_\partial(D^d) = \{\mathrm{id}\}$ and $\pi_0 \,\mathrm{Diff}(S^d) \cong \{\pm 1\}$; the case $d = 4$ remains open.

Example 2.1.3 (Tori). Hatcher determined the mapping class group of T^d for $d \geq 5$ as a semi-direct product [Hat78]

$$\mathrm{GL}_d(\mathbb{Z}) \ltimes \left((\mathbb{Z}/2)^{\oplus\infty} \oplus \mathrm{Hom}(\Lambda^2 \mathbb{Z}^d, \mathbb{Z}/2) \oplus \bigoplus_{0 \leq i \leq d} \mathrm{Hom}(\Lambda^i \mathbb{Z}^d, \Theta_{d+1-i}) \right),$$

where the term $(\mathbb{Z}/2)^{\oplus\infty}$ has its origins in algebraic K-theory, the term $\mathrm{Hom}(\Lambda^2\mathbb{Z}^d, \mathbb{Z}/2)$ in triangulation theory, and the remaining terms in smoothing theory. We recently extended this to homotopy tori of the form $T^d \# \Sigma$ for an homotopy d-sphere Σ [BKKT23]. It is open for a general homotopy tori.

2.2 Mapping class groups of $W_{g,1}$'s

One of the most well-studied high-dimensional manifolds is the high-dimensional analogue of a genus g surface with one boundary component:

$$W_{g,1} \coloneqq D^{2n} \# (S^n \times S^n)^{\# g}.$$

Up to smoothing corners, it can also obtained by gluing along disjoint D^{2n-1} in the boundary D^{2n} (also known as "boundary connected sum") a collection of g copies of the *thickened core*

$$W \coloneqq D^{2n-1} \times [0,1] \cup_{D^{2n-1} \times \{1\}} (S^n \times S^n) \backslash \mathrm{int}(D^{2n}).$$

2.2.1 The topology of $W_{g,1}$

Before studying the mapping class group of $W_{g,1}$, we need to compute some of its homotopy-theoretic invariants.

Homology and homotopy groups

The manifold $W_{g,1}$ is homotopy equivalent to a wedge of $2g$ copies of S^n. As such, it is $(n-1)$-connected and the Hurewicz theorem gives an isomorphism

$$\pi_n(W_{g,1}) \xrightarrow{\cong} H_n(W_{g,1}; \mathbb{Z}) \eqqcolon H_n,$$

and the latter can be explicitly identified as $\mathbb{Z}^{2g} = \mathbb{Z}\{e_1, f_1, \ldots, e_g, f_g\}$ with basis vectors represented by the $2g$ spheres S^n, two from each connected summand (or thickened core).

Intersection form

If M is an oriented $2n$-dimensional compact manifold whose boundary is a sphere S^{2n-1} then its middle-dimensional homology acquires a $(-1)^n$-symmetric bilinear intersection form

$$\lambda \colon H_n(M; \mathbb{Z}) \otimes H_n(M; \mathbb{Z}) \longrightarrow \mathbb{Z},$$

counting transverse intersections with sign. It can be written in terms of the cup product, and then Poincaré duality implies it is unimodular: the induced map $H_n(M; \mathbb{Z}) \to$

$\mathrm{Hom}(H_n(M;\mathbb{Z});\mathbb{Z})$ is an isomorphism. By inspection, for $W_{g,1}$ the intersection form with respect to basis $\{e_1, f_1, \ldots, e_g, f_g\}$ is represented by g-fold block sum of the matrix

$$\begin{bmatrix} 0 & 1 \\ (-1)^n & 0 \end{bmatrix}. \tag{2.1}$$

That is, it is either the standard symplectic form on $\mathbb{Z}^{2g}$, or a standard hyperbolic orthogonal form. Both are clearly $(-1)^n$-symmetric and non-degenerate.

Quadratic refinement

Every element of H_n can be represented by an embedding of n n-sphere; for example, $2e_1 + f_1$ is represented by tubing together two parallel copies of the first sphere and a copy of the second sphere. The existence of such an embedded representative is a general fact, a special case of [Hae61, Théorème d'existence]:

Lemma 2.2.1. *If M is an orientable $(n-1)$-connected $2n$-dimensional manifold, then every element of $\pi_n(M) \cong H_n(M;\mathbb{Z})$ is represented by an embedded sphere if $n \geq 3$ and this is unique up to isotopy if $n \geq 4$.*

Proof sketch. For existence, take a generic representative $S^n \to M$; it will be a smooth immersion with finitely many self-intersection points which are transverse double points. By adding Whitney pinches we can guarantee that the number of these, counted with sign, is 0 and then we can cancel them with Whitney tricks. For uniqueness, one uses that concordance implies homotopy in codimension ≥ 3 and applies a similar argument to a generic representative $S^n \times [0,1] \to M \times [0,1]$. □

From now on we will assume that $n \geq 4$, though there are some tricks that allow one to also deal with the case $n = 3$ (see [Kre79]). By Lemma 2.2.1, the normal bundle of the unique embedded sphere representing $x \in H_n(M;\mathbb{Z})$ is an invariant of x. Since tangent bundles of spheres are stably trivial, if the tangent bundle of M is stable trivial, then so is this normal bundle. In that case, we get a well-defined function (the value of Λ_n is given in [Lev85, Theorem 1.4])

$$\mu\colon H_n(M;\mathbb{Z}) \longrightarrow \ker[\pi_n BO(n) \to \pi_n BO] \cong \mathbb{Z}/\Lambda_n \quad \text{with } \Lambda_n := \begin{cases} 0 & \text{if } n \text{ is even,} \\ \mathbb{Z} & \text{if } n = 3, 7, \\ 2\mathbb{Z} & \text{otherwise.} \end{cases}$$

Wall proved that this is a quadratic refinement of the intersection form λ [Wal63a], which means that it satisfies

$$\mu(ax) = a^2\mu(x) \quad \text{and} \quad \mu(x+y) \equiv \mu(x) + \mu(y) + \lambda(x,y) \mod \Lambda_n.$$

This algebraic structure on the triple $(H_n(M;\mathbb{Z}), \lambda, \mu)$ is known as a $((-1)^n, \Lambda_n)$-*quadratic module.*

Remark 2.2.2. If n is even and λ only takes even values (e.g. for the hyperbolic form), the previous formulas allow one to write μ in terms of λ as $\mu(x) = \lambda(x,x)/2$, but for odd $n \neq 3, 7$ the quadratic refinement μ truly is additional information. In fact, there are two isomorphism classes of quadratic refinements of the standard symplectic form, distinguished by the Arf invariant.

Let us specialise to $W_{g,1}$ again. Since the basis elements e_i, f_i of $H_n = H_n(W_{g,1};\mathbb{Z})$ are represented by embedded spheres with trivial normal bundles, in this case the quadratic refinement satisfies $\mu(e_i) = \mu(f_i) = 0$. That is, in the case of odd $n \neq 3, 7$, it has Arf invariant 0.

2.2.2 Kreck's theorem

Kreck determined the mapping class groups of $W_{g,1}$'s [Kre79], and I will explain some aspects of his proof. The starting point is the observation—clear from the definitions—that any diffeomorphism has to preserve the intersection form λ and its quadratic refinement μ, yielding a homomorphism

$$\pi_0 \operatorname{Diff}_\partial(W_{g,1}) \longrightarrow \operatorname{Aut}(H_n, \lambda, \mu) \cong \begin{cases} \mathrm{O}_{g,g}(\mathbb{Z}) & \text{if } n \text{ is even,} \\ \mathrm{Sp}_{2g}(\mathbb{Z}) & \text{if } n = 3, 7, \\ \mathrm{Sp}^q_{2g}(\mathbb{Z}). & \text{otherwise.} \end{cases}$$

We shall prove that this is surjective and determine its kernel. To do so, we will use self-embeddings as an intermediate: we divide $S^{2n-1} = \partial W_{g,1}$ into two hemispheres D^{2n-1}_+ and D^{2n-1}_-, and let $\operatorname{Emb}_{1/2\partial}(W_{g,1})$ denote the self-embeddings $W_{g,1} \to W_{g,1}$ that are the identity on D^{2n-1}_+, in the C^∞-topology.

Lemma 2.2.3. *There is an exact sequence*

$$1 \longrightarrow \pi_0 \operatorname{Diff}_\partial(D^{2n}) \longrightarrow \pi_0 \operatorname{Diff}_\partial(W_{g,1}) \longrightarrow \pi_0 \operatorname{Emb}_{1/2\partial}(W_{g,1}) \longrightarrow 1.$$

Proof. By isotopy extension there is a fibre sequence

$$\operatorname{Diff}_\partial(D^{2n}) \longrightarrow \operatorname{Diff}_\partial(W_{g,1}) \longrightarrow \operatorname{Emb}_{1/2\partial}(W_{g,1}),$$

so the result follows from the long exact sequence of homotopy groups, once we establish that (i) the right map is surjective and (ii) the left map is injective.

For (i) we consider a self-embedding $e\colon W_{g,1} \to W_{g,1}$ fixing D^{2n-1}_+ and assume that it sends the rest of the boundary into the interior so its complement $C(e)$ is a manifold with corners (at least when we add $D^{2n-1}_- \times [0,1]$ to it); this can always be arranged by an isotopy. Note that e also induces an endomorphism of H_n preserving the intersection form λ and its quadratic refinement μ. It is an exercise in algebra that any endomorphism of H_n that preserves λ must be an automorphism, so e induces an isomorphism on homology. Using Mayer–Vietoris and Seifert–van Kampen, this implies that $C(e)$ is contractible. As its boundary has a preferred identification with S^{2n-1}, the h-cobordism theorem implies that it is diffeomorphic relative to its boundary to $D^{2n}\#\Sigma$ for some homotopy $2n$-sphere Σ. Combining this diffeomorphism with e, we get an orientation-preserving diffeomorphism $W_{g,1} \cong W_{g,1}\#\Sigma$ relative to its boundary. It means, by definition of the so-called first inertia group $I(W_{g,1}) \subseteq \Theta_{2n}$, that Σ lies in $I(W_{g,1})$. But this group is trivial by work of Kosinski [Kos67],[1] so it must have been the case that $C(e)$ was diffeomorphic to D^{2n} relative to its boundary and we can use such a diffeomorphism to produce an isotopy of e to a diffeomorphism.

For (ii) it suffices to prove that $\pi_0 \operatorname{Diff}_\partial(D^{2n})$ injects into the subgroup of those diffeomorphisms of $W_{g,1}$ that act as the identity on homology, that is

$$\pi_0 \operatorname{Tor}_\partial(W_{g,1}) := \ker[\pi_0 \operatorname{Diff}_\partial(W_{g,1}) \to \operatorname{Aut}(H_n, \lambda, \mu)].$$

This uses a trick inspired by work of Birman–Craggs [BC78]: we can use an element of $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ to glue together two solid $(2n+1)$-dimensional handlebodies $(D^{n+1} \times S^n)^{\natural g}$, and if it lies in $\pi_0 \operatorname{Tor}_\partial(W_{g,1})$ then the result is a homotopy $(2n+1)$-sphere. We get a function $\pi_0 \operatorname{Tor}_\partial(W_{g,1}) \to \Theta_{2n+1}$ (not a homomorphism, in general). Restricted to $\pi_0 \operatorname{Diff}_\partial(D^{2n})$ this agrees with the isomorphism of Example 2.1.2, establishing injectivity. □

In the proof of Lemma 2.2.3, we argued that there is a factorisation

$$\pi_0 \operatorname{Diff}_\partial(W_{g,1}) \longrightarrow \pi_0 \operatorname{Emb}_{1/2\partial}(W_{g,1}) \longrightarrow \operatorname{Aut}(H_n, \lambda, \mu).$$

Lemma 2.2.4. *The map $\pi_0 \operatorname{Emb}_{1/2\partial}(W_{g,1}) \to \operatorname{Aut}(H_n, \lambda, \mu)$ is surjective.*

Proof. Given an automorphism $A \in \operatorname{Aut}(H_n, \lambda, \mu)$, we get embedded spheres unique to isotopy representing the elements $A(e_i), A(f_i) \in H_n(W_{g,1}; \mathbb{Z})$ for $i =, 1 \ldots, g$, with trivial normal bundle and satisfying $\lambda(A(e_i), A(f_i)) = 1$. Using Whitney tricks, we can make each pair intersect exactly once and thicken them to get embeddings $(S^n \times S^n)\backslash \operatorname{int}(D^{2n}) \hookrightarrow W_{g,1}$ for $i = 1, \ldots, g$. Since $\lambda(A(e_i)), A(e_j)) = \lambda(A(f_i)), A(f_j)) = \lambda(A(f_i)), A(e_j)) = \lambda(A(e_i)), A(f_j)) = 0$ for $i \neq j$, we can make these disjoint by first making their core spheres disjoint using Whitney tricks and then shrinking the D^n-direction if necessary. If we now fix a small disc D^{2n} near the boundary of $W_{g,1}$ and add disjoint strip from to the boundary of each embedding $(S^n \times S^n)\backslash \operatorname{int}(D^{2n})$ to the boundary of this disc, we get embeddings $W \to W_{g,1}$ of a thickened core to combine into an embedding $W_{g,1} \to W_{g,1}$ relative to D^{2n-1}. □

Remark 2.2.5. [Wal63b, Lemma 10] identifies $\operatorname{Aut}(H_n, \lambda, \mu)$ as the homotopy classes of diffeomorphisms which preserve but do not necessarily fix the boundary.

Any element $e \in \ker[\pi_0 \operatorname{Emb}_{1/2\partial}(W_{g,1}) \to \operatorname{Aut}(H_n, \lambda, \mu)]$ can be assumed to fix the embedded spheres representing $e_i, f_i \in H_n(W_{g,1}; \mathbb{Z})$ pointwise. The derivative of the self-embedding induces a bundle automorphism of their trivial normal bundles, and Wall

[1] The proof in [Kos67] uses the proofs in [KM63]. The idea is that it suffices to prove that the $\operatorname{coker}(J)$-component of Σ is trivial because the bP-term is trivial in even dimensions, and this can be done by associating to frameable oriented closed $2n$-dimensional manifold M like $W_g = W_{g,1} \cup D^{2n}$ a set of elements $p(M) \subseteq \operatorname{coker}(J)_{2n}$ that has three properties: $|p(S^{2n})| = 1$, $p(M\#\Sigma) = p(M) + p(\Sigma)$, $|p(M)| \leq |p(M')|$ when we do primitive framed surgeries. For W_g the latter can be used to reduce W_g to S^{2n} so it must have been the case that $|p(W_g)| = 1$.

proved its homotopy class is independent of choices if we stabilise once [Wal65, Lemma (23)]. We thus get $2g$ elements of the abelian group

$$S\pi_n \mathrm{SO}(n) := \mathrm{im}[\pi_n \mathrm{SO}(n) \to \pi_n \mathrm{SO}(n+1)],$$

which is isomorphic to $\mathbb{Z}$, $\mathbb{Z}/2$, or 0 depending on $n \pmod 8$ (see [Kre79, p. 644] for a table). We can record these elements as a single element of the group of homomorphisms

$$\mathrm{Hom}(H_n, S\pi_n \mathrm{SO}(n)).$$

Remark 2.2.6. Alternatively, we could have defined this invariant for all $x \in H_n$, by looking at how an embedding acts on the normal bundle of an embedded representative, and verified it is a homomorphism.

Lemma 2.2.7. *The homomorphism*

$$\ker[\pi_0 \mathrm{Emb}_{1/2\partial}(W_{g,1}) \to \mathrm{Aut}(H_n, \lambda, \mu)] \longrightarrow \mathrm{Hom}(H_n, S\pi_n \mathrm{SO}(n))$$

is an isomorphism.

Proof. For surjectivity, we modify the identity embedding by twisting the D^n-direction of $S^n \times D^n$ by maps $B\colon S^n \to \mathrm{SO}(n)$ through the formula $(x, y) \mapsto (x, B(x)y)$. For injectivity, we use that if a self-embedding e is in the kernel then up to isotopy we can assume it fixes pointwise only not the core S^n's but also neighbourhood of them. This means that is the identity outside of a disc near the boundary, and as before this means that it is isotopic to the identity. ☐

Removing the self-embeddings from the picture, we get Kreck's formulation:

Theorem 2.2.8 (Kreck)**.** *For $2n \geq 6$ there is a pair of extensions*

$$\begin{tikzcd}
 & 1 \arrow[d] & & & \\
 & \Theta_{2n+1} \arrow[d] & & & \\
1 \arrow[r] & \pi_0 \mathrm{Tor}_\partial(W_{g,1}) \arrow[r] \arrow[d] & \pi_0 \mathrm{Diff}_\partial(W_{g,1}) \arrow[r] & \mathrm{Aut}(H_n, \lambda, \mu) \arrow[r] & 1 \\
 & \mathrm{Hom}(H_n, S\pi_n(\mathrm{SO}(n))) \arrow[d] & & & \\
 & 1 & & &
\end{tikzcd}$$

Analysing these extensions is an interesting problem. The subgroup Θ_{2n+1} is central: it is represented by diffeomorphisms supported in a disc near the boundary and any other diffeomorphism can be isotoped to be supported away from this disc. If we take the quotient by this subgroup, we get a semi-direct product where the action of $\mathrm{Aut}(H_n, \lambda, \mu)$ on $\mathrm{Hom}(H_n, S\pi_n(\mathrm{SO}(n)))$ is through the domain [Kra20, Lemma 1.3]. A complete analysis of these extensions in the case of odd n was done by Krannich [Kra20], but the case of even n remains open.

2.2.3 Homological stability for mapping class groups

One consequence of Theorem 2.2.8 is that the sequence of mapping class groups $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ satisfies homological stability. More precisely, there are stabilisation maps given by extension-by-the-identity

$$\begin{aligned} s\colon \pi_0 \operatorname{Diff}_\partial(W_{g,1}) &\longrightarrow \pi_0 \operatorname{Diff}_\partial(W_{g+1,1}) \\ f &\longmapsto f \natural \mathrm{id}_{W_{1,1}}, \end{aligned}$$

and the maps induced on homology are isomorphisms in a range tending to ∞ with g.

This will be a consequence of the following result for the "arithmetic part" of the mapping class groups. Recall that $((-1)^n, \Lambda_n)$-*quadratic module* is an abelian group with a $(-1)^n$-symmetric bilinear form λ and a $\mathbb{Z}/\Lambda_n$-valued quadratic refinement. Let (H, λ, μ) denote the $((-1)^n, \Lambda_n)$-quadratic module with $H = \mathbb{Z}\{e, f\}$, $\lambda(e, e) = \lambda(f, f) = 0$, $\lambda(e, f) = 1$, and $\mu(e) = \mu(f) = 0$. The homology group $H_n(W_{g,1}; \mathbb{Z})$ is then isomorphic to $H^{\oplus g}$ as a $((-1)^n, \Lambda_n)$-quadratic module. There are stabilisation maps

$$\begin{aligned} s\colon \operatorname{Aut}(H^{\oplus g}) &\longrightarrow \operatorname{Aut}(H^{\oplus g+1}) \\ A &\longmapsto f \oplus \mathrm{id}_H. \end{aligned}$$

Theorem 2.2.9 (Galatius–Randal-Williams)**.** *The relative homology groups*

$$H_*(B\operatorname{Aut}(H^{\oplus g+1}), B\operatorname{Aut}(H^{\oplus g}); \mathbb{Z})$$

of the stabilisation map vanish for $* \leq \frac{g-2}{2}$*. The same is true with polynomial coefficients of degree* r *with ranges* $* \leq \frac{g-2}{2} - r$*.*

This follows from the standard homological stability argument, due to Quillen and formalised by Randal-Williams–Wahl [RWW17]. Given a braided monoidal category $(\mathcal{G}, \oplus, 0)$ satisfying some mild conditions and a stabilising element X, their machinery provides a semi-simplicial set $W_\bullet(X^{\oplus g})$ of "destabilisations by X" of $X^{\oplus g}$, on which $\operatorname{Aut}(X^{\oplus g})$ acts. It is constructed such that the skeletal filtration on $W(X^{\oplus g})$ gives rise upon taking the homotopy quotient by by $\operatorname{Aut}(X^{\oplus g})$ to a spectral sequence

$$E^1_{pq} = H_q(B\operatorname{Aut}(X^{\oplus g-p-1}); \mathbb{Z}) \Longrightarrow H_{p+q}(||W_\bullet(X^{\oplus g})|| /\!\!/ \operatorname{Aut}(X^{\oplus g}); \mathbb{Z}),$$

and if the geometric realisation of the semi-simplicial set is highly-connected, we can use this to deduce homological stability. A more nuanced version of this spectral sequence gives a similar result with coefficients in certain systems $M(g)$ of $\operatorname{Aut}(X^{\oplus g})$-representations, the *polynomial* ones.

To prove Theorem 2.2.9, we take $\mathcal{G}$ the symmetric monoidal groupoid of $((-1)^n, \Lambda_n)$-quadratic modules under orthogonal direct sum and $X = H$, and get:

Definition 2.2.10. The semi-simplicial set $W^{\mathrm{alg}}_\bullet(H^{\oplus g})$ has p-simplices given by maps $H^{\oplus p+1} \to H^{\oplus g}$ of $((-1)^n, \Lambda_n)$-quadratic modules, the face maps act by precomposition and the group $\operatorname{Aut}(H^{\oplus g})$ acts by postcomposition.

To obtain the range in Theorem 2.2.9, we need to prove that the geometric realisation $||W_\bullet^{\mathrm{alg}}(H^{\oplus g})||$ is $\frac{g-5}{2}$-connected, which was done in [GRW18]. We deduce from Theorem 2.2.9 the following homological stability result for mapping class groups:

Corollary 2.2.11. *The relative homology groups*

$$H_*(B\pi_0 \operatorname{Diff}_\partial(W_{g+1,1}), B\pi_0 \operatorname{Diff}_\partial(W_{g,1}); \mathbb{Z})$$

of the stabilisation map vanish for $* \leq \frac{g-2}{2}$.

Proof. In summary, we run two Serre spectral sequences for the two extensions in Theorem 2.2.8. For simplicity, let us take $n \equiv 3 \pmod 8$ and $n > 3$, so that $\operatorname{Aut}(H^{\oplus g}) = \operatorname{Sp}_{2g}^q(\mathbb{Z})$ and $S\pi_n \operatorname{SO}(n) = \mathbb{Z}$; the other cases are analogous.

It is also useful to make the identification $\operatorname{Hom}(H_n, \mathbb{Z}) \cong H^{\oplus g}$ through the intersection form, which is compatible with the actions of $\operatorname{Aut}(H^{\oplus g})$ and the stabilisation maps. Kreck's theorem then gives a pair of extension

$$1 \longrightarrow H^{\oplus g} \longrightarrow \pi_0 \operatorname{Diff}_\partial(W_{g,1})/\Theta_{2n+1} \longrightarrow \operatorname{Sp}_{2g}^q(\mathbb{Z}) \longrightarrow 1$$
$$1 \longrightarrow \Theta_{2n+1} \longrightarrow \pi_0 \operatorname{Diff}_\partial(W_{g,1}) \longrightarrow \pi_0 \operatorname{Diff}_\partial(W_{g,1})/\Theta_{2n+1} \longrightarrow 1$$

with a pair of associated Serre spectral sequences

$$^{I}E^2_{pq} = H_p(B\operatorname{Sp}_{2g}^q(\mathbb{Z}), H_p(BH^{\oplus g})) \Longrightarrow H_{p+q}(B\pi_0 \operatorname{Diff}_\partial(W_{g,1})/\Theta_{2n+1})$$
$$^{II}E^2_{pq} = H_p(B\pi_0 \operatorname{Diff}_\partial(W_{g,1})/\Theta_{2n+1}; H_q(B\Theta_{2n+1})) \Longrightarrow H_{p+q}(B\pi_0 \operatorname{Diff}_\partial(W_{g,1})).$$

Varying g, these extensions are compatible with the stabilisation maps, which hence induce maps between the spectral sequences.

We now analyse these spectral sequences, starting with $^{I}E^2_{pq}$. Note that the coefficients are given by $H_*(BH^{\oplus g}; \mathbb{Z}) \cong \Lambda^*(H^{\oplus g})$, and in degree q this is a polynomial coefficient system of degree $\leq q$ (cf. [Dwy80, Lemma 4.3], it is one of the prototypical examples). This implies that the maps induced by stabilisation are isomorphisms on $^{I}E^2_{pq}$ for $p \leq \frac{g-4}{2} - q$ and surjections for $p \leq \frac{g-2}{2} - q$. By the diagram chase, the same is true for the abutment does for $p + q \leq \frac{g-4}{2}$ and $p + q \leq \frac{g-2}{2}$ respectively.

Once we know this is stability for the group $\pi_0 \operatorname{Diff}_\partial(W_{g,1})/\Theta_{2n+1}$ with $\mathbb{Z}$-coefficients, it is true with coefficients in any abelian group by the universal coefficients theorem. A similar but easier argument for the spectral sequence $^{II}E^2_{pq}$ then completes the proof. □

2.3 Mapping class groups are almost arithmetic groups

One crucial feature of Theorem 2.2.8 is that it shows the mapping class group is almost an arithmetic group.

The prototypical example of an arithmetic group is as follows: if $G \subset \operatorname{GL}_n(\mathbb{Q})$ defined by finitely many polynomial equations with rational coefficients in the entries x_{ij} and $\det^{-1}$, then

$$G_\mathbb{Z} \coloneqq G \cap \operatorname{GL}_n(\mathbb{Z})$$

is an arithmetic group. For example, $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{O}_{g,g}(\mathbb{Z})$ are of this form. We say Γ is *arithmetic* if it is equivalent to such a group under the equivalence relation on groups generated by isomorphism and passage to finite index subgroups. For example, $\mathrm{Sp}^q_{2g}(\mathbb{Z})$ is of this form. More generally, Γ is *almost arithmetic*[2] if it is equivalent to such a group under the equivalence relation generated by isomorphism, passage to finite index subgroups, and quotients by finite normal subgroups.

Example 2.3.1. From Theorem 2.2.8, it follows that $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ is almost arithmetic: the equivalence relation allows us to ignore the finite abelian group Θ_{2n+1} as well as the term $\operatorname{Hom}(H, S\pi_n \operatorname{SO}(n))$ when $S\pi_n \operatorname{SO}(n)$ is finite. If $S\pi_n \operatorname{SO}(n) = \mathbb{Z}$, which can only happen for odd n, then there is an injective homomorphism $\pi_0 \operatorname{Diff}_\partial(W_{g,1})/\Theta_{2n+1} \to \mathrm{Sp}_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ with finite index by [Kra20, Section 2] and it is an easy exercise that the latter is arithmetic.

Remark 2.3.2. Allowing finite quotients is necessary, as in [KRW20] it is proven using work of Deligne that some $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ are not arithmetic. Their examples are in dimension $2n$ with n odd, and I am not aware of examples in other dimensions though they surely exist.

This almost arithmeticity is a general phenomenon, proven using rational homotopy theory, surgery theory, and pseudo-isotopy theory [Sul77, Tri95, BKKT23]:

Theorem 2.3.3 (Sullivan, Triantafillou, Bustamante–Krannich–K.)**.** *If M is a closed smooth manifold of dimension $d \geq 6$ with $\pi_1 M$ finite then the mapping class group $\pi_0 \operatorname{Diff}(M)$ is almost arithmetic.*

Such qualitative results are important for more quantitative ones, as arithmetic groups have excellent cohomological properties. For example, if torsion-free they have classifying spaces which are finite CW-complexes [BS73] and if the ambient algebraic group is semisimple Borel proved vanishing results for coefficients in a non-trivial rational representation [Bor74, Bor81] (with useful improvements to the ranges in [Tsh19, LS19]). Both of these play a crucial role in recent work on Torelli groups.

[2]There is no satisfying terminology in the literature: I have seen "differs by finite groups from an arithmetic group", "is S-commensurable to an arithmetic group", "is commensurable up to finite kernel to an arithmetic group". None are aesthetically pleasing so I hope the reader will forgive me.

Chapter 3

Diffeomorphism groups

Last lecture we discussed the mapping class groups $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ of the manifolds

$$W_{g,1} \coloneqq D^{2n} \#(S^n \times S^n)^{\# g},$$

proved that they are extensions of the arithmetic groups of automorphisms of $((-1)^n, \Lambda_n)$-quadratic modules, and used this to prove they satisfy homological stability. Today we discuss analogous results for the diffeomorphism groups $\operatorname{Diff}_\partial(W_{g,1})$ themselves. The proof will suggest a direct link between homological stability for mapping class groups and for diffeomorphisms groups, and we will make this precise in the next lecture.

3.1 Homological stability for diffeomorphisms of $W_{g,1}$

As on path components, extension-by-the-identity induces a stabilisation map

$$\begin{aligned} s\colon B\operatorname{Diff}_\partial(W_{g,1}) &\longrightarrow B\operatorname{Diff}_\partial(W_{g+1,1}) \\ f &\longmapsto f \natural \mathrm{id}_W. \end{aligned}$$

The machinery of Randal-Williams and Wahl [RWW17] for homological stability of automorphism groups in a braided monoidal groupoid, was extended to the topological setting by Krannich [Kra19]. This takes as input a graded E_1-module $\mathcal{M}$ over an E_2-algebra with stabilising object X, satisfying certain conditions, and constructs a semi-simplicial space $W_\bullet(\mathcal{M}_n)$ so that if it is highly-connected then the $\mathcal{M}_n$ satisfy homological stability.

We can apply this to the E_{2n}-algebra (by forgetting structure also an E_2-algebra)

$$\mathcal{M} = \bigsqcup_{g \geq 0} B\operatorname{Diff}_\partial(W_{g,1})$$

as a module over itself. The resulting semi-simplicial space is weakly equivalent to the following one, denoted $W_\bullet^{\mathrm{top}}(W_{g,1})$. To define it, we fix an open collar neighbourhood $\mathbb{R}^{2n-1} \times [0,\infty) \hookrightarrow W_{g,1}$. Recall that a *thickened core* is the manifold given by

$$W \coloneqq D^{2n-1} \times [0,1] \cup_{D^{2n-1} \times \{1\}} (S^n \times S^n) \backslash \operatorname{int}(D^{2n})$$

and a *normalised* embedding $W \hookrightarrow W_{g,1}$ is one that agrees with $(t,x) \mapsto (\lambda \cdot x + \alpha, t)$ for $\lambda > 0$ and $\alpha \in \mathbb{R} \times \{0\}$ near $\mathbb{R}^{2n-1} \times \{0\}$.

Definition 3.1.1. The semi-simplicial space $W_\bullet^{\mathrm{top}}(W_{g,1})$ has p-simplices given by the subspace of $\mathrm{Emb}(\sqcup_{p+1} W, W_{g,1})$ of an ordered $(p+1)$-tuple of disjoint normalised embeddings $(e_0, \ldots, e_p)$ so that $\alpha_0 < \cdots < \alpha_p$, and the ith face map forgets the ith embedding.

The goal of the first part of this lecture is to explain the proof of the following connectivity result due to Galatius–Randal-Williams:

Proposition 3.1.2. *$||W_\bullet^{\mathrm{top}}(W_{g,1})||$ is $\frac{g-5}{2}$-connected.*

Krannich's homological stability machinery deduces from this a result originally due to Galatius–Randal-Williams [GRW18]:

Theorem 3.1.3 (Galatius–Randal-Williams)**.** *The relative homology groups*

$$H_*(B\mathrm{Diff}_\partial(W_{g+1,1}), B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Z})$$

of the stabilisation map vanish for $* \leq \frac{g-2}{2}$. The same is true with polynomial coefficients of degree r with ranges $* \leq \frac{g-2}{2} - r$.

Returning to the proof of Proposition 3.1.2, the starting point is that embeddings induce maps of $((-1)^n, \Lambda_n)$-quadratic modules so we get a map of semi-simplicial spaces

$$W_\bullet^{\mathrm{top}}(W_{g,1}) \longrightarrow W_\bullet^{\mathrm{alg}}(H^{\oplus g}).$$

The strategy will be to leverage the connectivity of the right term, through the following diagram

$$\begin{tikzcd}
W_\bullet^{\mathrm{top}}(W_{g,1}) \arrow[r, "③"] & rW_\bullet^{\mathrm{top}}(W_{g,1}) & rW_{\bullet,\bullet}(W_{g,1}) \arrow[l, "②a"'] \arrow[d, "②b"] & \\
 & & rW_\bullet^{\mathrm{top},\delta}(W_{g,1}) \arrow[ul, "②"] \arrow[r, squiggly] \arrow[d] & rW_\circ^{\mathrm{top},\delta}(W_{g,1}) \arrow[d, "①"] \\
 & & W_\bullet^{\mathrm{alg}}(W_{g,1}) \arrow[r, squiggly] & W_\circ^{\mathrm{alg}}(W_{g,1})
\end{tikzcd}$$

where the notation is as follows:

- a prefix r indicates we use a more relaxed variant of where the images of the standard embeddings $e_0, \ldots, e_p$ are not necessarily disjoint but they are disjoint near the cores $\{0\} \times [0,1] \cup (S^n \vee S^n)$,
- a superscript δ indicates we use the discrete topology of spaces of embeddings rather than using the C^∞-topology,
- a subscript $\circ$ means we consider the analogous simplicial complex instead of semisimplicial set, obtained by forgetting orderings,
- finally, $rW_{\bullet,\bullet}(W_{g,1})$ will be a bi-semisimplicial space with (p,q)-simplices given as a set by $rW_{p+q+1}^{\mathrm{top}}(W_{g,1})$ but topologised so that the first $p+1$ thickened cores have the C^∞-topology and the last $q+1$ ones have the discrete topology.

The first step is to use the map ① to prove that the geometric realisation $||rW_\circ^{\mathrm{top},\delta}(W_{g,1})||$ is highly-connected. This will use that in addition to $||W_\circ^{\mathrm{alg}}(H^{\oplus g})||$ being $\frac{g-5}{2}$-connected, also the link of each p-simplex is $(\frac{g-5}{2}-p)$-connected (in fact even $\frac{g-5-p}{2}$-connected) because it is isomorphic to $W_\circ^{\mathrm{alg}}(H^{\oplus g-p-1})$: one says that this simplicial complex is *weakly Cohen-Macaulay of dimension* $\geq \frac{g-3}{2}$.

Lemma 3.1.4. $||rW_\circ^{\mathrm{top},\delta}(W_{g,1})||$ *is* $\frac{g-5}{2}$*-connected.*

Proof. This is a so-called "lifting argument". Our goal is to prove that any map f with $i \leq \frac{g-5}{2}$ as in

$$\begin{tikzcd} S^i \arrow[r, "f"] \arrow[d] & ||rW_\circ^{\mathrm{top},\delta}(W_{g,1})|| \arrow[d] \\ D^{i+1} \arrow[r, "F"'] \arrow[ur, dotted, "G"] & ||W_\circ^{\mathrm{alg}}(W_{g,1})||, \end{tikzcd}$$

is null-homotopic. To do this, we use the connectivity of $||W_\circ^{\mathrm{alg}}(W_{g,1})||$ to obtain the extension F and then try to lift it to G.

To find G, we start with some preparation. By simplicial approximation, we can homotope the entire diagram so that both horizontal maps are simplicial with respect to some simplicial triangulation of D^{i+1}. Next, by an application of [GRW18] using the fact that $W_\circ^{\mathrm{alg}}(W_{g,1})$ is weakly Cohen–Macaulay (a "badness argument"), we can further homotope the bottom map so that it is *simplex-wise injective* in the interior: if there are adjacent vertices x, x' in D^{i+1} satisfying $F(x) = F(x')$ then $x, x' \in S^i$.

After this preparation, we create the lift G one vertex v at a time. Those in the boundary already have a lift, so we may assume v lies in the interior of D^{i+1}. The element $F(v)$ is represented by a map $F(v)\colon H = \mathbb{Z}\{e, f\} \to H^{\oplus g}$ of $((-1)^n, \Lambda_n)$-quadratic modules. By simplex-wise injectivity, for any v' in the link of v in D^{i+1} we have another such map $F(v')\colon H \to H^{\oplus g}$ whose image is orthogonal to that of $F(v)$. Some of these are already been lifted to standard embeddings $G(v')\colon W \to W_{g,1}$.

To also lift $F(v)$, we use the embeddings theorems from the previous lecture to represent $F(v)(e)$ and $F(v)(f)$ by embedded spheres with trivial normal bundle. Since $\lambda(F(v)(e), F(v)(f)) = 1$, we can use Whitney tricks make these two spheres intersect transversally in a single point, and by plumbing these together and adding a strip to the boundary we obtain a candidate standard embedding $G(v)\colon W \to W_{g,1}$ of a thickened core. The core of $G(v)$ might not be disjoint from the cores of $G(v')$ for $v' \in \mathrm{Link}(v)$, but since their algebraic intersection numbers are 0 we can isotopy $G(v)$ using Whitney tricks so that they are.

Making the last step precise, one realises that there is a subtlety: to make sure that removing a pair of intersection points by the Whitney trick does not introduce new intersections you need all $G(v')$ to be transverse. We can guarantee this for the new lifts we produce by choosing them generically, but for the lift on the boundary S^i it requires another homotopy of the diagram, and to prove such a homotopy exists we need to use the induction hypothesis. □

Sketch of proof of Proposition 3.1.2. Since the standard embeddings $\{e_0, \dots, e_p\}$ of thickened cores in a p-simplex of $rW_\circ^{\mathrm{top},\delta}(W_{g,1})$ have a unique preferred ordering—the order in which their strips attach to the boundary—there is a homeomorphism of geometric realisations

$$||rW_\bullet^{\mathrm{top},\delta}(W_{g,1})|| \cong ||rW_\circ^{\mathrm{top},\delta}(W_{g,1})||.$$

Let us now outline how to deduce from this discrete variant the desired connectivity of $||W_\bullet^{\mathrm{top}}(W_{g,1})||$.

Firstly, the diagonal map in

$$\begin{tikzcd} rW_\bullet^{\mathrm{top}}(W_{g,1}) & rW_{\bullet,\bullet}(W_{g,1}) \arrow[l, "2a"'] \arrow[d, "2b"] \\ & rW_\bullet^{\mathrm{top},\delta}(W_{g,1}) \arrow[ul, "2"] \end{tikzcd}$$

is the identity, the horizontal map forgets the cores with discrete topology and the vertical map forgets the cores with C^∞-topology. One proves by induction on g that the map 2a is $\frac{g-4}{2}$-connected because the map

$$||rW_{p,\bullet}(W_{g,1})|| \longrightarrow rW_p^{\mathrm{top}}(W_{g,1})$$

on the space of p-simplices is $\frac{g-4-p}{2}$-connected: (1) it is a Serre microfibration, (2) the fibres are essentially[1] of the form $||rW_\bullet^{\mathrm{top},\delta}(W_z)||$ for the complement W_z of $p+1$ cores in $W_{g,1}$, which turns out to be diffeomorphic to $W_{g-p-1,1}$ when the connectivity bounds are non-vacuous, so the fibres are are $\frac{g-6}{2}$-connected, (3) Serre microfibrations with highly-connected fibres are highly-connected maps.

Next, upon geometric realisation the composition 2 $\circ$ 2b is homotopic to 2a; given a (p,q)-simplex in $||rW_\bullet^{\mathrm{top}}(W_{g,1})$ we can consider the discrete thickened cores as topologised ones to get a $(p+q+1)$-simplex in $||W_\bullet^{\mathrm{top}}(W_{g,1})||$. Transferring simplicial weights to the first $p+1$ or last $q+1$ gives the homotopy. Thus the $\frac{g-4}{2}$-connected map 2a factors up to homotopy through a $\frac{g-5}{2}$-connected space, and hence its target is also $\frac{g-5}{2}$-connected.

Finally, the inclusion

$$W_\bullet^{\mathrm{top}}(W_{g,1}) \xrightarrow{3} rW_\bullet^{\mathrm{top}}(W_{g,1})$$

is a levelwise equivalence by shrinking embeddings ones to a small neighbourhood of their core. □

Remark 3.1.5. This strategy for proving homological stability for high-dimensional diffeomorphism groups has been used for other stabilisations maps by Perlmutter [Per18, Per16b, Per16a].

Remark 3.1.6. Recently, Sierra has used E_k-cell techniques to improve the stability ranges for $W_{g,1}$ of dimension $2n$ with n odd [Sie22].

[1] Actually, only the cores of the W need to lie in W_z and one ought to repeat the lifting giving above in this setting. This is a general lesson: during a connectivity argument you often learn you need to go back to prove that some more semi-simplicial sets are highly-connected.

3.2 The stable homology of diffeomorphisms of $W_{g,1}$

A homological stability result becomes much more useful when you also know the stable homology. For diffeomorphism groups of surfaces, this is the Madsen–Weiss theorem. For diffeomorphism groups of high-dimensional manifolds, the corresponding result was proven by Galatius–Randal-Williams [GRW14] using surgery on cobordism categories, building on earlier work on the homotopy type of cobordism categories due to Galatius–Madsen–Tillmann–Weiss [GTMW09]. Specialised to the case of $W_{g,1}$, the answer is as follows:

Theorem 3.2.1 (Galatius–Randal-Williams)**.** *There is an acyclic scanning map*

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}_\partial(W_{g,1}) \longrightarrow \Omega_0^\infty MT\Theta_{2n},$$

where $MT\Theta_{2n}$ is the Thom spectrum for the inverse of the canonical bundle over $B\mathrm{O}(2n)\langle n\rangle$.

The cohomology of the spectrum $MT\Theta_{2n}$ can be computed by the Thom isomorphism, so that rationally we get

$$H^*(\Omega_0^\infty MT\Theta_{2n};\mathbb{Q}) \cong \mathbb{Q}[\kappa_c \mid |c|-2n>0]$$

where c runs over the monomial in $e, p_{n-1}, \cdots, p_{\lceil\frac{n+1}{4}\rceil}]$ and κ_c is the generalised MMM-class of degree $|c|-2n$. This assigns to a $W_{g,1}$-bundle $\pi\colon E\to B$ the cohomology class

$$\kappa_c(\pi) := \pi_*(c(T_\pi E)),$$

the pushforward of the characteristic class c applied to the vertical tangent bundle.

Example 3.2.2. It is instructive to consider the effect of the map $B\mathrm{Diff}_\partial(W_{g,1}) \to B\mathrm{Aut}(H^{\oplus g},\lambda,\mu)$ on stable cohomology. Borel proved that

$$\lim_{g\to\infty} H^*(B\mathrm{Aut}(H^{\oplus g},\lambda,\mu)) \cong \mathbb{Q}[x_i \mid 4i-2n>0]$$

where $|x_i| = 4i-2n$ and Atiyah proved the family signature theorem saying that x_i pulls back to κ_{L_i}, where L_i is the Hirzebruch L-polynomial:

$$\begin{aligned}\lim_{g\to\infty} H^*(B\mathrm{Aut}(H^{\oplus g},\lambda,\mu)) &\longrightarrow \lim_{g\to\infty} H^*(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q})\\ x_i &\longmapsto \kappa_{L_i}.\end{aligned}$$

Example 3.2.3. The *tautological ring* is the subalgebra of $H^*(B\mathrm{Diff}_\partial(W_{g,1};\mathbb{Q})$ generated by κ_c (or equivalently, the image of the stable cohomology under pullback). Unlike for mapping class groups, this is in general not finite-dimensional [GGRW17]. However, the subalgebra of the tautological ring generated by classes κ_{L_i} *is* finite-dimensional: these classes are pulled back from $B\mathrm{Aut}(H^{\oplus g})$ and this has finite-dimensional cohomology. In the previous lecture we said that one reason to prove a group is arithmetic, is that such groups have finite index subgroups that admit classifying spaces which are a finite CW-complex, and the rational cohomology of a group injects into that of finite index subgroup by a transfer argument.

Chapter 4

Self-embeddings

In the previous two lectures we discussed homological stability for the mapping class groups $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ and the diffeomorphism groups $\operatorname{Diff}_\partial(W_{g,1})$. Today I want to illuminate the connection between these, and I will do so through embedding calculus.

4.1 A user's guide to embedding calculus

Embedding calculus attempts to capture spaces of embedding between manifolds by how they act on the spaces of embeddings of open discs into these manifolds, preserving the natural maps between these. It is due to Weiss [Wei99], and alternative models were given in [GKW03, BdBW13, Tur13]; this discussion follows [KK22].

Fixing a dimension d for the remainder of this section, let Man be the ∞-category whose objects are d-dimensional manifolds and whose morphism spaces are the spaces of embeddings

$$\operatorname{Map}_{\mathsf{Man}}(P, Q) = \operatorname{Emb}(P, Q).$$

This is constructed as the coherent nerve of a topologically enriched category, and if you are unfamiliar with ∞-categories you can also in that setting if you are willing to deal with model-categorical complications. For $k \in \mathbb{N} \cup \{\infty\}$, we let $\mathsf{Disc}_{\leq k} \subset \mathsf{Man}$ be the full subcategory whose objects are the manifolds $\sqcup_\ell \mathbb{R}^d$ for finite $\ell \leq k$. Assigning to a manifold M the presheaf

$$\begin{aligned} E_M \colon \mathsf{Disc}_{\leq k}^{\mathrm{op}} &\longrightarrow \mathsf{Top} \\ \sqcup_\ell \mathbb{R}^d &\longmapsto \operatorname{Emb}(\sqcup_\ell \mathbb{R}^d, M) \end{aligned}$$

is natural in M, yielding a restricted Yoneda functor

$$E \colon \mathsf{Man} \longrightarrow \mathsf{PSh}(\mathsf{Disc}_{\leq k}).$$

Part of the data of such a functor in particular are maps

$$\operatorname{Emb}(P, Q) \longrightarrow T_k\operatorname{Emb}(P, Q) := \operatorname{Map}_{\mathsf{PSh}(\mathsf{Disc}_{\leq k})}(E_P, E_Q),$$

which are the "embedding calculus approximations" to spaces of embeddings; think of them as the best approximation you can construct to embeddings of P into Q if you are only given information about embeddings of at most k discs in P into Q. As we vary k, these assemble to a tower

$$\mathrm{Emb}(P,Q) \to \begin{matrix} T_\infty \mathrm{Emb}(P,Q) \\ \simeq \\ \operatorname*{holim}_{k\to\infty} T_k\mathrm{Emb}(P,Q) \end{matrix} \to \cdots \to T_2\mathrm{Emb}(P,Q) \to T_1\mathrm{Emb}(P,Q).$$

There are four properties that make these formal constructions useful:

1. The first is a *Yoneda* property: the map

$$\mathrm{Emb}(\sqcup_\ell \mathbb{R}^d, Q) \longrightarrow T_k\mathrm{Emb}(\sqcup_\ell \mathbb{R}^d, Q)$$

   is an equivalence for $\ell \leq k$.

2. The second is a *homotopy sheaf* property. A *Weiss k-cover* $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of P such that every subset of cardinality $\leq k$ is contained in some U_i. Then if $N\mathcal{U}$ denote the nerve of such a Weiss k-cover with objects the finite intersections $U_I = U_{i_1} \cap \cdots \cap U_{i_r}$ the map $\mathrm{hocolim}_{U_I \in N\mathcal{U}} E_{U_I} \to E_P$ is an equivalence and hence so is

$$T_k\mathrm{Emb}(P,Q) \longrightarrow \operatorname*{holim}_{U_I\in N\mathcal{U}} T_k\mathrm{Emb}(U_I, Q).$$

   In particular, $T_1\mathrm{Emb}(P,Q)$ is a homotopy sheaf for ordinary open covers and combining this with the Yoneda property we get an equivalence $T_1\mathrm{Emb}(P,Q) \simeq \mathrm{Bun}(TP, TQ)$, whose target is the space of bundle maps.

3. The third is a *convergence* property. One way of formulating Goodwillie–Klein's multiple disjunction theorem for embedding spaces [GK15], is that $P \mapsto \mathrm{Emb}(P,Q)$ has the homotopy sheaf property in a range for certain Weiss k-covers. Here "certain" means those k-covers by removing k parallel cocores of a handle of index $\leq d-3$, and "in a range" means that the map to the homotopy limit has connectivity increasing with k. By induction over handles one proves that if P has a handle decomposition with only $(\leq d-2-h)$-handles then the map

$$\mathrm{Emb}(P,Q) \longrightarrow T_k\mathrm{Emb}(P,Q)$$

   is about kh-connected, with initial case $P = \sqcup_\ell \mathbb{R}^d$ given by the Yoneda property. This is the main result of [GW99].

4. The fourth is a *recollement* property. We can restrict presheaves on $\mathsf{Disc}_{\leq k}$ to the subcategory $\mathsf{Disc}_{=k}$ with only object $\sqcup_k \mathbb{R}^d$ and only morphisms those embeddings that are π_0-isomorphisms; this functor shall be denoted I. Moreover, we can compose this with the restriction map $\mathsf{PSh}(\mathsf{Disc}_{\leq k}) \to \mathsf{PSh}(\mathsf{Disc}_{\leq k-1})$ followed by its left or right adjoint; these functors shall be denoted L and R. The unit and counit natural transformations for these adjunctions yield a commutative diagram

$$\begin{tikzcd} \mathsf{PSh}(\mathsf{Disc}_{\leq k}) \arrow[r, "L\to I\to R"] \arrow[d] & \mathsf{Fun}([2], \mathsf{PSh}(\mathsf{Disc}_{=k})) \arrow[d] \\ \mathsf{PSh}(\mathsf{Disc}_{\leq k-1}) \arrow[r, "L\to R"] & \mathsf{Fun}([1], \mathsf{PSh}(\mathsf{Disc}_{=k})). \end{tikzcd}$$

This is a pullback, and on morphism spaces this yields Weiss' description of the layers of the embedding calculus tower: we have that

$$\mathrm{hofib}_e[T_k\mathrm{Emb}(P,Q) \to T_{k-1}\mathrm{Emb}(P,Q)] \simeq \mathrm{Sect}_\partial^{\Sigma_k}(E_k \to \mathrm{Conf}_k(P)/\Sigma_k)$$

where the right term is the space of sections of a bundle over the unordered configuration spaces $\mathrm{Conf}_k(P)/\Sigma_k$ with fibres given by total homotopy fibre $\mathrm{tohofib}_{I\subset\{1,\ldots,k\}}\mathrm{Conf}_I(Q)$ relative to the boundary, where it is a section determined by e.

4.2 Serre class arguments

For the space $\mathrm{Emb}(M,M)$ of self-embeddings of a manifold M, the upshot is that if M has handle dimension $\leq d-3$ then the maps

$$\mathrm{Emb}(M,M) \longrightarrow T_k\mathrm{Emb}(M,M)$$

increasing in connectivity with k, and the target can be computed iteratively in terms of bundle maps of TM and section spaces involving configuration spaces of M. Getting definitive answers in these computations are difficult, but getting qualitative consequences is straightforward. A convenient framework is the following:

Definition 4.2.1. Recall that *Serre class* of abelian groups is a collection $\mathcal{C}$ so that:

- if $0 \to A \to B \to C \to 0$ is short exact then $A, C \in \mathcal{C}$ if and only if $B \in \mathcal{C}$,
- if $A, B \in \mathcal{C}$ then $A \otimes B, \mathrm{Tor}(A,B), \mathrm{Hom}(A,B), \mathrm{Ext}(A,B) \in \mathcal{C}$,
- If $K \in \mathcal{C}$ then $H_j(K(A,1)) \in \mathcal{C}$ for all $j \geq 1$.

Example 4.2.2. An example of a Serre class is that of finitely-generated abelian groups.

One of the reasons for the second and third points is the following lemma:

Lemma 4.2.3. *For* 1*-connected space* X*, we have* $H_*(X;\mathbb{Z}) \in \mathcal{C}$ *if and only if* $H^*(X;\mathbb{Z}) \in \mathcal{C}$ *if and only if* $\pi_* X \in \mathcal{C}$.

The same definition of a Serre class can be made for $\mathbb{Z}[G]$-modules instead of $\mathbb{Z}$-modules. We can thus apply this for the group $G = \pi_0 \mathrm{Diff}(M)$ with $\mathbb{Z}[G]$-module given by the higher homotopy groups $\pi_i \mathrm{Emb}(M,M)$.

Proposition 4.2.4. *Let* $\mathcal{C}_M$ *be the Serre class of* $\mathbb{Z}[\pi_0 \mathrm{Diff}(M)]$ *generated by the homology groups* $H_*(M;\mathbb{Z})$ *(with the induced action). If* M *is* 1*-connected and has handle dimension* $\leq d-3$*, then* $\pi_i \mathrm{Emb}(M,M) \in \mathcal{C}_M$.

Sketch of proof. We use that by convergence the map

$$\pi_i \mathrm{Emb}(M,M) \longrightarrow \pi_i T_k\mathrm{Emb}(M,M)$$

is an isomorphism for some large enough k. Abbreviating

$$L_k\mathrm{Emb}(M,M) \coloneqq \mathrm{hofib}_{\mathrm{id}}[T_k\mathrm{Emb}(M,M) \longrightarrow T_{k-1}\mathrm{Emb}(M,M)],$$

the long exact sequences of homotopy groups

$$\cdots \longrightarrow \pi_i\, L_k\mathrm{Emb}(M,M) \longrightarrow \pi_i\, T_k\mathrm{Emb}(M,M) \longrightarrow \pi_i\, T_{k-1}\mathrm{Emb}(M,M)] \longrightarrow \cdots$$

implies that $\pi_i\, T_k\mathrm{Emb}(M,M) \in \mathcal{C}$ if we can prove that $\pi_i\, L_k\mathrm{Emb}(M,M) \in \mathcal{C}$ and $\pi_1\, L_k\mathrm{Bun}(TM,TM)$. These are accessible through homotopy-theoretic methods. Both are essentially mapping spaces, and these can be accessed through variants of the Federer spectral sequence [Fed56]

$$H^p(X;\pi_q(Y)) \Longrightarrow \pi_{q-p}\,\mathrm{Map}(X,Y).$$

In our case X and Y will be of the form M, $\mathrm{Conf}_k(M)$, or $\mathrm{BO}(n)$. The homology groups of the first are in $\mathcal{C}_M$ by construction, those of the second are by the Totaro spectral sequence [Tot96], and the third because $\mathcal{C}_M$ contains all finitely-generated abelian groups because it contains $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$. $\square$

Example 4.2.5. Noting that $\mathcal{C}_M$ is contained in the Serre class of $\mathbb{Z}[\pi_0\,\mathrm{Diff}(M)]$-modules whose underlying abelian group is finitely-generated, we see that $\pi_i\,\mathrm{Emb}(M,M)$ is finitely generated.

4.3 Relationship to homological stability

I want to explain how the techniques of the previous section to deduce the following result relating homological stability for diffeomorphism groups and mapping class groups:

Theorem 4.3.1. *The topological groups $\mathrm{Diff}_\partial(W_{g,1})$ have homological stability with finite degree coefficients if and only if the groups $\pi_0\,\mathrm{Diff}_\partial(W_{g,1})$ do.*

To prove this, we observe that the notion of a Serre class can be extended not just from $\mathbb{Z}$-modules to $\mathbb{Z}[G]$-modules, but even to $\mathbb{Z}[\Omega]$-modules where Ω is a category. Here Ω will be the category with objects $\mathbb{N}$ and $\mathrm{Hom}_\Omega(g,g')$ empty if $g > g'$ and $\pi_0\,\mathrm{Diff}_\partial(W_{g',1})/\pi_0\,\mathrm{Diff}_\partial(W_{g'-g},1)$; it is really the construction $\mathcal{U}$ of [RWW17, Section 1.2] applied to a symmetric monoidal groupoid of mapping class groups. Coefficient system can arise from $\mathbb{Z}[\Omega]$-modules and it follows from work of Church, Ellenberg, Farb, Nagpal, and Djament that the finite degree coefficient systems form a Serre class $\mathcal{C}_{\mathrm{fd}}$.

Lemma 4.3.2. *Theorem 4.3.1 follows if we prove that $g \mapsto H_*(B\mathrm{Diff}^{\mathrm{id}}_\partial(W_{g,1});\mathbb{Z}) \in \mathcal{C}_{\mathrm{fd}}$, where $\mathrm{Diff}^{\mathrm{id}}_\partial(W_{g,1}) \subset \mathrm{Diff}_\partial(W_{g,1})$ is the identity component.*

Proof. For $\Leftarrow$, we pick a finite degree $\mathbb{Z}[\Omega]$-module A and use the spectral sequence of $\mathbb{Z}[\Omega]$-modules sending $g \in \Omega$ to

$$E^2_{pq} = H_p(B\pi_0\,\mathrm{Diff}_\partial(W_{g,1}); H_q(B\mathrm{Diff}^{\mathrm{id}}_\partial(W_{g,1});A)) \Longrightarrow H_{p+q}(B\mathrm{Diff}_\partial(W_{g,1});A)$$

If $g \mapsto H_*(B\mathrm{Diff}^{\mathrm{id}}_\partial(W_{g,1});\mathbb{Z})$ is of finite degree so is $g \mapsto H_*(B\mathrm{Diff}^{\mathrm{id}}_\partial(W_{g,1});A)$ by the universal coefficient theorem. Thus if the groups $\pi_0\,\mathrm{Diff}_\partial(W_{g,1})$ have homological stability with finite degree coefficients, the E^2-page stabilises and hence so does the abutment.

For $\Rightarrow$, we proceed by induction over k that the groups $\pi_0 \operatorname{Diff}_\partial(W_{g,1})$ have homological stability with finite degree coefficients in degrees $* \leq k$. The initial case $* = 0$ follows since $H_0(B\mathrm{Diff}_\partial(W_{g,1}); A) \to H_0(B\pi_0, \mathrm{Diff}_\partial(W_{g,1}); A)$ is an isomorphism. For the induction step we use the same spectral sequence as above. In it, the entries E^2_{pq} stabilise for $p \leq k$ and since the abutment stabilises in all degrees, one concludes that $E^2_{p+1,0} = H_{p+1}(B\pi_0 \operatorname{Diff}_\partial(W_{g,1}); A)$ must also stabilise. □

Proof of Lemma 4.3.2. By the properties of Serre classes, to prove that the $\mathbb{Z}[\Omega]$-module $g \mapsto H_*(B\mathrm{Diff}^{\mathrm{id}}_\partial(W_{g,1}); \mathbb{Z})$ lies $\mathcal{C}_{\mathrm{fd}}$, it suffices to prove that $g \mapsto \pi_* \operatorname{Diff}^{\mathrm{id}}_\partial(W_{g,1}) \in \mathcal{C}_{\mathrm{fd}}$ for $* > 0$. We reduce this from diffeomorphisms to self-embeddings: in the first lecture we used the fibre sequence

$$B\mathrm{Diff}_\partial(D^{2n}) \longrightarrow B\mathrm{Diff}_\partial(W_{g,1}) \longrightarrow B\mathrm{Emb}_{1/2\partial}(W_{g,1}).$$

As this is compatible with stabilising by g, the homotopy groups $\pi_* \operatorname{Emb}_{1/2\partial}(W_{g,1})$ are $\mathbb{Z}[\Omega]$-modules and as action on the homotopy groups $\pi_i \operatorname{Diff}_\partial(D^{2n})$ is trivial, it suffices to prove that these lie in $\mathcal{C}_{\mathrm{fd}}$. Performing embedding calculus as is the proof of Proposition 4.2.4, it suffices to prove that $g \mapsto H_*(W_{g,1}; \mathbb{Z}) \in \mathcal{C}_{\mathrm{fd}}$; it is indeed the prototypical example of a coefficient system of degree 1. □

A more general result will appear in forthcoming work with Manuel Krannich:

Theorem 4.3.3 (Krannich–K.)**.** *Suppose that M and P are 2-connected manifolds of dimension $d \geq 6$ with $\partial M = S^{d-1}$ and $\partial P = \varnothing$. Then the following are equivalent:*

1. $\operatorname{Diff}_\partial(M \# P^{\# g})$ *has homological stability with finite degree coefficients,*
2. $\pi_0 \operatorname{Diff}_\partial(M \# P^{\# g})$ *has homological stability with finite degree coefficients,*
3. $\operatorname{im}[\pi_0 \operatorname{Diff}_\partial(M \# P^{\# g}) \to \operatorname{Aut}(H_*(M \# P^{\# g}))]$ *has homological stability with finite degree coefficients.*

Example 4.3.4. It is a straightforward computation that for $M = D^{4n}$ and $P = \mathbb{H}P^n$, the groups in (3) are given by $\mathfrak{S}_g$ if n is odd and $\mathfrak{S}_g \wr \{\pm 1\}$ if n is even. By [RWW17, Section 5.1] these have homological stability with finite degree coefficients. We conclude that the groups $\operatorname{Diff}_\partial(D^{4n} \# \mathbb{H}P^n)$ have homological stability with finite degree coefficients as well, so in particular with constant coefficients.

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