ORIENTED COBORDISM: CALCULATION AND APPLICATION

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Abstract. In these notes we give an elementery calculation of the first couple of oriented cobordism groups and then explain Thom's rational calculation. After that we prove the Hirzebruch signature theorem and sketch several other applications.

The classification of oriented compact smooth manifolds up to oriented cobordism is one of the triumphs of 20th century topology. The techniques used are now a part of the foundations of differential topology and stable homotopy theory. These notes gives a quick tour of oriented cobordism, starting with low-dimensional examples and ending with some deep applications.

Convention 0.1. In these notes a manifold always means a smooth compact manifold, possibly with boundary.

Good references are Weston's notes [Wes], Miller's notes [Mil01] or Freed's notes [Fre12]. Alternatively, one can look at Stong's book [Sto68] or the relevant chapters of Hirsch's book [Hir76] or Wall's book [Wal16].

1. The definition of oriented cobordism

Classifying manifolds is a hard problem. In fact, it is impossible to list all of them, or give an algorithm deciding whether two manifolds are diffeomorphic. This follows from the fact that the group isomorphism problem—telling whether two finitely presented groups are isomorphic or not—is undecidable and that any finitely presented group appears as the fundamental group of some manifold of dimension ≥ 4. To prove this, one builds manifolds with specified fundamental groups using handles, which we will do later.

To make the problem tractable, one has two choices:

(i) one either restricts to particular situations, e.g. 3-dimensional manifolds or $(n-1)$-connected $2n$-dimensional manifolds for $n \geq 3$, or
(ii) one can try to classify manifolds up to a coarser equivalence relation than diffeomorphism.

We will pursue the latter. Our challenge is then to find an equivalence relation that is computable, while still being interesting.

In the 50's, Thom discovered a class of equivalence relations that are both interesting and computable [Tho54]. We will look at the representative example of oriented cobordism, which has the advantage of both being relatively easily visualized and relatively easily computed.

To give the definition, we need to pick a convention for the induced orientation on the boundary ∂W of a manifold W with boundary. We say that an ordered basis

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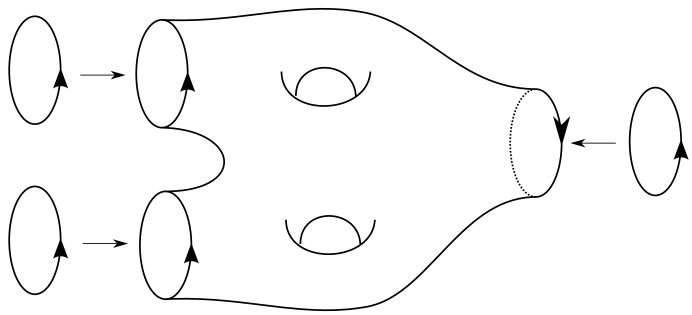

FIGURE 1. A two-dimensional cobordism between two one-dimensional manifolds.

$(v_1, \ldots, v_d)$ of $T_x\partial W$ is *oriented positively* if $(v_1, \ldots, v_d, \nu)$ is a positively oriented basis of T_xW, where ν is the inward pointing normal vector.

Definition 1.1. Let M_1 and M_2 be two d-dimensional oriented manifolds with empty boundary. We say that M_1 and M_2 are *cobordant* if there exists a $(d+1)$-dimensional oriented manifold W with boundary such that ∂W is diffeomorphic as an oriented d-dimensional manifold to $M_1 \sqcup -M_2$, where $-M_2$ is the manifold M_2 with opposite orientation.

A manifold W with boundary like the one that appeared in the previous definition we call a *cobordism* between M_1 and M_2. See Figure 1 for an example.

Remark 1.2. Here are two equivalent ways of defining the relation of cobordism. Firstly, by considering a cobordism from M_1 to M_2 as a cobordism from $M_1 \sqcup -M_2$ to $\varnothing$, we see that M_1 and M_2 are cobordant if and only if $M_1 \sqcup -M_2$ is cobordant to the empty manifold.

Secondly, we may consider W has having its boundary divided into "incoming" and "outgoing" boundary, and incoming boundary is oriented with inward pointing vector and outgoing boundary is oriented with outward pointing vector. Then M_1 and M_2 are cobordant if there is a cobordism W with $\partial_{\mathrm{in}} W \cong M_1$ and $\partial_{\mathrm{out}} W \cong M_2$ as oriented manifolds.

Lemma 1.3. *Bordism is an equivalence relation, i.e. it has the following properties:*

(i) identity: every d-dimensional oriented manifold M is cobordant to itself.
(ii) symmetry: if M_1 is cobordant to M_2, then M_2 is cobordant to M_1.
(iii) transitivity: if M_1 is cobordant to M_2 and M_2 is cobordant to M_3, then M_1 is cobordant to M_3.

Proof. See Figure 2. For (i) we note that $M \times I$ is a cobordism from M to M. For (ii)we remark that if W is a cobordism between M_1 and M_2, then $-W$ is a cobordism between M_2 and M_1. Finally, for (iii) we note that if W_1 is a cobordism between M_1 and M_2 and W_2 is a cobordism between M_2 and M_3, then $W_1 \cup_{M_2} W_2$ is a cobordism between M_1 and M_3. □

Definition 1.4. We define Ω_d^{SO} to be the set of d-dimensional oriented manifolds up to cobordism. These are called the *oriented cobordism groups*.

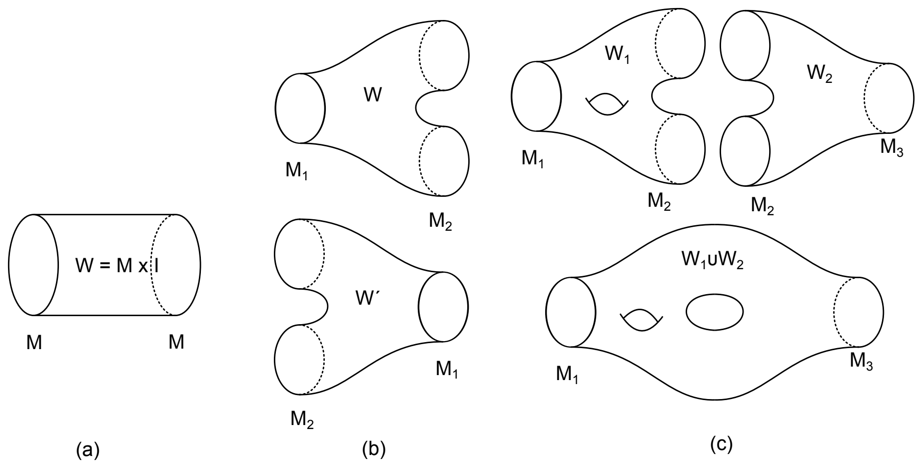

FIGURE 2. The figures (a), (b) and (c) demonstrate the identity, symmetry and transitivity properties of the cobordism relation.

Operations on manifolds give operations on the cobordism groups. In particular, disjoint union makes Ω_d^{SO} into an abelian group and later we will see that cartesian product makes the graded abelian group Ω_*^{SO} into a graded ring. Let us verify the first claim.

Lemma 1.5. *Disjoint union gives Ω_d^{SO} the structure of an abelian group.*

Proof. To see that it is well-defined, it suffices to check that if M_1 is cobordant to M_1' and M_2 is cobordant to M_2', then $M_1 \sqcup M_2$ is cobordant to $M_1' \sqcup M_2'$. To see this just take the disjoint union of the two cobordisms. The identity is $\varnothing$ and the inverse of M is $-M$. □

2. Calculating oriented cobordism groups for dimension ≤ 4

Let us compute the first few groups Ω_d^{SO} defined in the previous section. We will do the cases $d = 0, 1, 2$ with relative ease and the case $d = 3$ with slightly more effort. These early results seem to indicate that the oriented cobordism groups are always trivial, but we show this is not the case by giving a surjective homomorphism $\sigma\colon \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$.

2.1. **The low-dimensional groups Ω_0^{SO}, Ω_1^{SO} and Ω_2^{SO}.** We will start with Ω_1^{SO} and Ω_2^{SO}, concluding that they are trivial. This proof uses that one-dimensional and two-dimensional manifolds have been classified. We will implicitly assume these classifications, as they are well-known.

Let us think about what it means for Ω_d^{SO} to be trivial: all d-dimensional oriented manifolds are cobordant to each other, or equivalently to the empty manifold $\varnothing$. But that just means they bound a $(d+1)$-dimensional manifold.

We will check this in the case $d = 1$. Every oriented one-dimensional manifold is a finite disjoint union of circles. Since disjoint union gives the abelian group structure, it suffices prove that a single circle is cobordant to $\varnothing$. But the circle S^1 is naturally the boundary of the oriented disk D^2, which can be considered as cobordism from S^1 to $\emptyset$. See part (a) of Figure 3. We conclude the following:

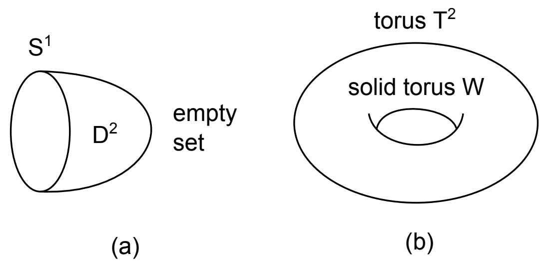

FIGURE 3. Part (a) and (b) of this figure show how a circle S^1 and a torus T^2 respectively bound a higher-dimensional manifold. Equivalently, they are cobordant to the empty set.

Proposition 2.1. $\Omega_1^{\mathrm{SO}} = 0$.

The case $d = 2$ is slightly more difficult because there are more two-dimensional oriented manifolds. The connected ones are classified by their genus $g \geq 0$; for genus 0 we have the sphere S^2, for genus 1 the torus T^2, and for genus $g \geq 2$ the hyperbolic surfaces Σ_g. Each of these is cobordant to $\varnothing$, because they are the boundary of a solid handlebody. For the sphere this is the disk D^3, for the torus the solid torus $S^1 \times D^2$, and for Σ_g the solid handlebody $H_g = \natural_g(S^1 \times D^2)$ (where $\natural$ denotes boundary connected sum). See part (b) of figure 3. We again conclude:

Proposition 2.2. $\Omega_2^{\mathrm{SO}} = 0$.

Let us now do the case $d = 0$. Here we have to admit being slightly sloppy before, as we have to discuss what an oriented 0-dimensional manifold is. An orientation is an everywhere non-zero section of the top exterior power of the tangent bundle, and $\Lambda^0\{0\} = \mathbb{R}$. Thus there are in fact two zero-dimensional oriented manifolds, the positively oriented point $*_+$ and the negatively oriented point $*_-$. By taking disjoint unions of these, we see that Ω_0^{SO} is a quotient of $\mathbb{Z}^2$. It is not hard to see that the oriented interval can be considered as a cobordism from $*_+ \sqcup *_-$ to the empty set. Hence $*_+$ is identified with $-*_-$. All cobordisms from an oriented 0-dimensional manifold to $\varnothing$ are disjoint unions of such intervals, so we conclude that there are no more relations coming from cobordism and thus:

Proposition 2.3. $\Omega_0^{\mathrm{SO}} = \mathbb{Z}$.

2.2. Ω_3^{SO}**.** All of the previous calculations were straightforward, so what about higher dimensions? It turns out that one can still do the case $d = 3$ in a similar geometric fashion. We will prove this vanishes as well. This proof was first given by Rourke [Rou85]. To start this proof we will need surgery decompositions of oriented three-dimensional manifolds.

2.2.1. *Handle decompositions.* Every smooth manifold admits a triangulation, a fact one can prove using Whitney's embedding theorem. This means we obtain any 3-manifold M by glueing together solid tetrahedra along their faces. We will use this to write our manifold as follows (for some $g \geq 0$):

$$M = D^3 \cup \bigcup_{i=1}^{g} (D^1 \times D^2)_i \cup \bigcup_{i=1}^{g} (D^2 \times D^1)_i \cup D^3$$

Let's make this decomposition more precise. It means that M is built as follows:

(i) We start with a disk D^3.
(ii) We glue on g copies of $D^1 \times D^2$ along embeddings of $S^0 \times D^2$ into the boundary of D^3, which is a sphere S^2. The result is a solid handlebody H_g of genus g.
(iii) We glue on g copies of $D^2 \times D^1$ along embeddings of $S^1 \times D^1$ into the boundary of H_g, which is a genus g surface Σ_g.
(iv) Finally one can check that the remaining boundary is a sphere S^2, and we glue in a disk D^3.

This is a special case of a handle decomposition. If M is a n-dimensional manifold and we are given an embedding $\phi\colon S^{i-1} \times D^{n-i} \hookrightarrow \partial M$, then we form the new manifold

$$M' \coloneqq M \cup_\phi D^i \times D^{n-i}$$

This is a manifold with corners, which one may smoothen in an essentially canonical way (an issue we will ignore), which is said to be the *result of a handle attachment*. The number i is called the *index*. A *handle decomposition* of a manifold is a description of it as iterated handle attachments starting with $\varnothing$. Thus the description of the 3-manifold M given above is a special handle decomposition where we start with a single 0-handle, add g 1-handles, then g 2-handles and ending with a single 3-handle.

So how do we get such a handle decomposition of our manifold from the triangulation? See figure 4 for a picture to keep in mind.

(i) Consider the graph obtained by glueing the 1-skeletons of the tetrahedra together and pick a maximal tree T in it. A thickened neighborhood of it is homeomorphic to a disk D^3: D^3_T.
(ii) If we add the thickened remaining edges $\{e\}$, we see that each such edge e contributes a glued on copy of $D^1 \times D^2$: $(D^1 \times D^2)_e$. The union of D^3 with $\bigcup_e (D^1 \times D^2)_e$ is our H_g.
(iii) Now consider the dual graph obtained by taking a vertex for each tetrahedron and an edge for each face of a tetrahedron. Pick a maximal tree T' and then the remaining edges $\{f\}$ correspond to a maximal set $\{f\}$ of faces such that the complement of their union with H_g is a disk. Thickening these gives the glued on copies of $D^2 \times D^1$: $(D^2 \times D^1)_f$.
(iv) Finally the interiors of the tetrahedra with the remaining faces form a disk D^3 by construction. This follows because they correspond to the maximal tree T': $D^3_{T'}$.

So why are the number of edges and faces used above the same? If we do glueing in a slightly different order: indepently first do (i) and (ii), and then (iii) and (iv), resulting in two 3-manifolds with boundary. Glueing these together, we obtain M. So we see that at the final step we glued a handlebody of genus $\#\{e\}$ to a handlebody of genus $\#\{f\}$ along their boundaries. This is only possible if these boundaries have the game genus g, so that $\#\{e\} = \#\{f\}$.

In fact, it is useful to recast our construction in terms of the "middle" boundary surface Σ_g. There are two collections of g curves on this surface: the α_i are the circles $(\{\frac{1}{2}\} \times S^1)_e$ in the thickened edges, the β_i are the circles $(S^1 \times \{\frac{1}{2}\})_f$ in the thickened disks. Note that all of the α_i are disjointly embedded, as are the β_i, and both the α's and the β's cut Σ_g into a disk.

We can then alternatively think of our construction as starting with Σ_g (maybe even slightly thickened to $\Sigma_g \times I$), glueing a disk D^2 to the α_i (on the $\Sigma_g \times \{0\}$ in the thickened version) and β_i (on the $\Sigma_g \times \{1\}$ in the thickened version), and

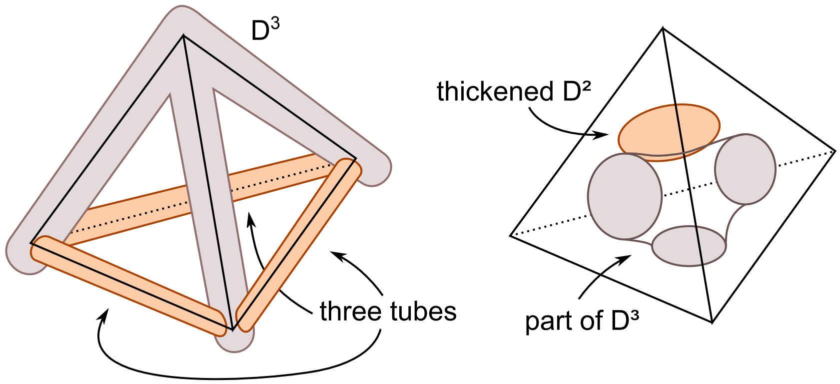

FIGURE 4. This figure demonstrates locally the idea that a triangulation of a connected 3-manifold gives a decomposition into a pair of D^3's, some tubes and some thickened D^2's.

filling the two remaining S^2-boundaries with a disk D^3. To see there are indeed two S^2-boundaries, one computes that the boundary has a single component of genus 0.

Definition 2.4. A *surgery decomposition* of M is a pair α, β of collections of g disjointly embedded curves on Σ_g which cut Σ_g into a disk, such that glueing a disk D^2 to each of the α_i and β_i and then filling the two remaining S^2-boundaries with a disk D^3 gives us M. In this case we say (g, α, β) is a surgery decomposition of genus g. We write $M = M(g, \alpha, \beta)$ if we want to think of M as built from the surgery decomposition.

Remark 2.5. A surgery decomposition is also closely related to a so-called *Heegaard decomposition*. In fact these two types of decompositions are equivalent and differ only in the way they are presented. The data for a Heegaard decomposition is just a single diffeomorphism $\Sigma_g \to \Sigma_g$ and we can create a 3-manifold out of this by taking two copies of a handlebody H_g and glueing their boundaries together using the diffeomorphism. Since all diffeomorphisms of D^2 relative to its boundary are isotopic by Smale's theorem [Sma59], it turns out that the diffeomorphism is uniquely determined up to isotopy by where it sends a collection of g disjointly embedded curves that cut Σ_g into a disk. Hence from (g, α, β) we can construct a unique diffeomorphism $\Sigma_g \to \Sigma_g$ up to isotopy by pretending that it mapped the α_i to the β_i.

2.2.2. *Simplifying surgery decompositions.* A surgery decomposition is not canonical, as it depends on many choices. We take advantage of this by simplifying surgery decompositions. The following cancellation lemma is a special case of a general cancellation lemma in surgery theory, see e.g. Section 1.1 of [LÖ2].

Lemma 2.6. *Let (g, α, β) be a surgery decomposition of M of genus g. If α_1 and β_1 meet transversely in a single point, then we can find surgery decomposition of M of genus $g-1$: $(g-1, \alpha\backslash\{\alpha_1\}, \beta\backslash\{\beta_1\})$.*

Proof. Glue in the disks D^2 to the α_i's and the D^3 corresponding boundary S^2. The situation is then as in figure 5. We see that if we glue in a thickened disk $D^2 \times D^1$ to a neighborhood of β_1, it cancels the handle that α_1 is on: isotoping

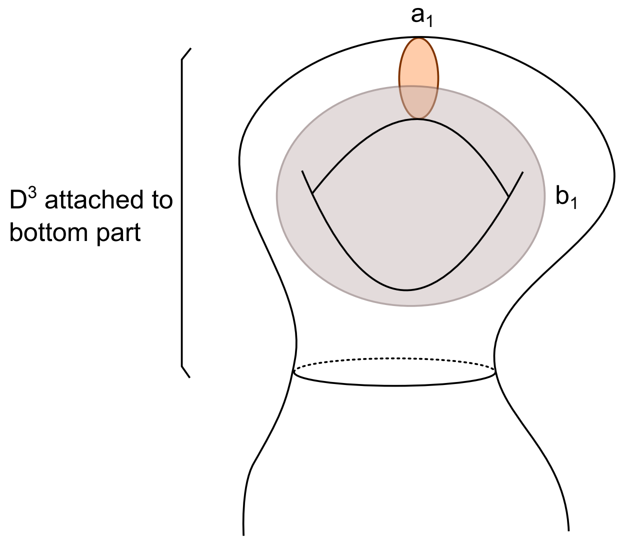

FIGURE 5. If we attach disks to two curves that intersect once transversally, they cancel out, i.e. just form a disk that can be isotoped away.

away the disk D^3 that's indicated in the figure we see that the handlebody is now of genus $g-1$. From this handlebody with the α_i and β_i for $i \geq 2$ in its boundary, we get a surgery decomposition of genus $g-1$. Indeed, isotoping away the disk does not influence the other α_i or β_i as they do not intersect α_1 and β_1 respectively. □

So if we are lucky, we can cancel all of the α's and β's against each other and see that our M is obtained from glueing a D^3 to a D^3 along S^2, i.e. a three-dimensional sphere S^3. This bounds a D^3 and hence is cobordant to the empty set. In that case we are done.

However there is no reason for us to be this lucky. The solution is that ff we are not lucky, we will just make ourselves lucky, by showing by induction that our M is cobordant to oriented 3-manifold that allows for cancellation.

2.2.3. *Surgery modifications.* There is an easy way to construct manifolds $\tilde{M}$ cobordant to M. We start with an identity cobordism $M \times I$ and glue on a $D^2 \times D^2$ to the boundary $M \times \{1\}$ along a $S^1 \times D^2$. This keeps the incoming boundary $M \times \{0\}$ the same, but turns the outgoing boundary into a 3-manifold $\tilde{M} = (M \backslash S^1 \times D^2) \cup_{S^1 \times S^1} (D^2 \times S^1)$. It is just an example of a handle attachment in dimension 4.

Example 2.7. As a first example, let us consider two-dimensional manifolds. Taking a torus T^2, taking the identity cobordism $T^2 \times I$ and glueing on a tube $(D^1 \times D^2)$ along $S^0 \times D^2$ in $T^2 \times \{1\}$, makes the outgoing boundary into $(T^2 \backslash S^0 \times D^2) \cup_{S^0 \times S^1} (D^1 \times S^1)$. The result is a cobordism from T^2 to a surface of genus 2.

Example 2.8. A second example is the so-called *connected sum* operation. Here we start with two n-dimensional manifolds M_1, M_2 and two orientation-preserving embeddings $\phi_1 \colon D^n \hookrightarrow M_1$ and $\phi_2 \colon D^n \hookrightarrow M_2$. Then we can glue $D^1 \times D^n$ to $(M_1 \sqcup M_2) \times I$ along $\phi_1 \sqcup \phi_2 \colon S^0 \times D^n \to (M_1 \sqcup M_2) \times \{1\}$. This is a cobordism group

$M_1 \sqcup M_2$ to a manifold denoted $M_1 \# M_2$ and called the *connected sum* of M_1 and M_2 (if M_1 and M_2 were path-connected, it is indeed a path-connected representative of the sum of the cobordism classes). It is given explicitly by removing $\phi_i(\mathrm{int}(D^n))$ from M_i and identifying the two boundary spheres. A priori it may seem to depend on the choices of ϕ_1 and ϕ_2. However, isotopy extension says that as long as M_1 and M_2 are path-connected, all choices give the same manifold up to orientation preserving diffeomorphism.

We will consider a special case of this construction, based on another collection γ of g disjointly embedded curves in Σ_g such that Σ_g cut along the γ is a disk. Let's rethink our reconstruction of M from a surgery decomposition (g, α, β). In the thickened version there is a $\Sigma_g \times I$ in the middle. We think of the γ as lying in $\Sigma_g \times \{\frac{1}{2}\}$ and thicken them to a collection of g disjoint copies of $S^1 \times D^2$ in $\Sigma_g \times (0,1) \subset M$. If we now start with an identity cobordism $M \times I$ and do the previous construction g times for all the thickened γ_i in $M \times \{1\}$, we get a cobordism from M to

$$\tilde{M} = \left(M \backslash \bigcup_{i=1}^{g} (S^1 \times D^2)_{\gamma_i} \right) \cup_{\bigcup_{i=1}^{g} (S^1 \times S^1)_{\gamma_i}} \left(\bigcup_{i=1}^{g} (D^2 \times S^1)_{\gamma_i} \right)$$

We will describe this manifold $\tilde{M}$ in terms of $M(g, \alpha, \gamma)$ and $M(g, \gamma, \beta)$.

Proposition 2.9. *We have that*

$$\tilde{M} = M(g, \alpha, \gamma) \# M(g, \gamma, \beta)$$

Proof. We start by cutting $M \backslash \bigcup_{i=1}^{g} (S^1 \times D^2)_{\gamma_i}$ along the level surface $\Sigma \times \{\frac{1}{2}\}$: we get two components, let's denote by M_1 the one which contains all the disks glued to the α_i and by M_2 the one which contains all the disks glued to the β_i. Similarly it divides each boundary component $(S^1 \times S^1)_{\gamma_i}$ into a copy of $(S^1 \times D^1)_{\gamma_i}$ in both M_1 and M_2, where in both cases the image of S^1 is the curve γ. If to these we glue $(D^2 \times D^1)_{\gamma_i}$ we obtain from M_1 the manifold $M(g, \alpha, \gamma)$ with one of the disks D^3 missing, leaving an S^2 boundary, and from M_2 the manifold $M(g, \gamma, \beta)$, similary with one of the disks D^3 missing. If we now glue both guys together along their common boundary, we exactly obtain $\tilde{M}$. But if one has two three-dimensional manifolds, removes a disk D^3 to both and glues the resulting S^2 boundaries together, then this is by definition the connected sum. $\square$

We can now finish our proof that every path-connected three-dimensional manifold is cobordant to S^3, which in turn implies it is cobordant to the empty set.

Theorem 2.10. *Every path-connected oriented 3-manifold M is cobordant to S^3.*

Proof. Pick a surgery decomposition $M(g, \alpha, \beta)$ and set $r = \min_{i,j} |\alpha_i \cap \beta_j|$ (this is finite as without loss of generality all curves intersect transversally). We will do an induction over g and r, i.e. assume that the statement is true for (i) $g' < g$, and (ii) $g' = g$ and $r' < r$. The case $g = 0$ must have $r = 0$ and then we already know we have a sphere.

- If $r = 1$, then by Lemma 2.6 we have that M can be described by a surgery decomposition of genus $g - 1$ and hence by induction we are done.
- If $r = 0$, we'll find a collection of curves γ such that $|\alpha_1 \cap \gamma_1| = 1$ and $|\beta_1 \cap \gamma|1|| = 1$. Then M is cobordant to $M(g, \alpha, \gamma) \# M(g, \gamma, \beta)$, both of which can be given by a surgery decomposition of genus $g - 1$ and by induction

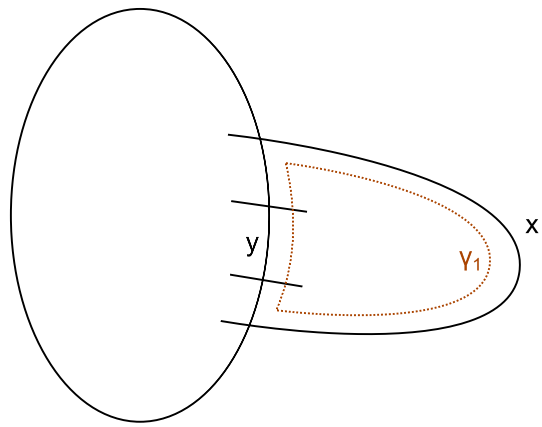

FIGURE 6. A typical case of the construction of a γ_1 intersecting α_1 and β_1 in fewer points.

are cobordant to spheres. By taking connected sum of cobordisms, it is not hard to see that then $M(g, \alpha, \gamma)\#M(g, \gamma, \beta)$ is cobordant to $S^3\#S^3 = S^3$ and we are done.

The collection γ is obtained by finding γ_1 and then randomly picking additional disjointly embedded γ_i to make the γ_i cut Σ_g into a disk. We cut Σ_g along α_1 and glue in disks to get a surface we denote Σ. If β_1 cuts Σ into two path-components, then the disks must lie on opposite sides of β_1 (or β_1 would have cut Σ_g into two pieces) and we can connect them by an arc that intersects β_1 once. This arc corresponds to an embedded curve γ_1 in Σ_g that intersects β_1 and α_1 once. If β_1 does not cut Σ into two path-components, then cut Σ along β_1 and glue in disks to obtain a surface Σ' with four disks on it. Connect pair each disk from α_1 with one from β_1 and connect both of these pairs with an arc. These two arcs corresponds to on Σ_g to an embedded curve γ_1 that intersects β_1 and γ_1 once.

- Finally, if $r > 1$, then we can assume without loss of generality that $|\alpha_1 \cap \beta_1| = r$. We will find a collection of curves γ such that $|\alpha_1 \cap \gamma_1| < r$, $|\beta_1 \cap \gamma_1| < r$. Then M is cobordant to $M(g, \alpha, \gamma)\#M(g, \gamma, \beta)$, both of which have smaller r and by induction are cobordant to spheres.

  To find the collection γ, we again pick γ_1 and complete the collection with random γ_i's. To pick γ_1 we note that we can find two adjacent intersection points along β_1 with α_1. Take the arc x on β_1 between these. Then one of the two arcs y, y' on α_1 between the intersection points has the property that $x \cup y$ or $x \cup y'$ does not cut the surface into two pieces, without loss of generality it is y. The curve γ will be obtained by pushing x and y a bit to the side and connecting them. See figure 6.

□

This concludes our calculation of Ω_3^{SO}. The reader may want to think about the unoriented case.

2.3. **Signature and the non-triviality of Ω_4^{SO}.** Until now oriented cobordism groups have turned out to be quite boring, so we will give an example of non-trivial geometric data that it can detect. To do this, we will define a surjective homomorphism $\sigma\colon \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$ called the signature. It is actually an isomorphism, something that we will see rationally in the next section.

To define the signature, we need to know something about cup products in cohomology. There are several ways to think about these. Most geometrically one can think of cohomology as de Rham cohomology, i.e. closed forms modulo exact forms, and then we define the cup product $a \cup b$ of $a = [\alpha]$ and $b = [\beta]$ as the cohomology class $[\alpha \wedge \beta]$, where $\wedge$ is the exterior product of forms. More algebraically one can think of cohomology as coming from maps from the singular simplices to $\mathbb{Z}$. Then $a \cup b$ for $a = [f]$ and $b = [g]$ of degree k_1 and k_2 respectively is given by $(f \smile g)(\gamma\colon \Delta^n \to M) = f(\gamma|_{\Delta^{k_1}})g(\gamma|_{\Delta^{k_2}})$ where the first restriction is to the first face of dimension k_1 and the second restriction is to the last face of dimension k_2.

Theorem 2.11 (Poincaré duality in middle dimension)**.** *Let M be a path-connected oriented manifold of even dimension $2d$ with empty boundary. Then the cup product induces a non-degenerate bilinear form*

$$H^d(M;\mathbb{R}) \otimes H^d(M;\mathbb{R}) \to H^{2d}(M;\mathbb{R}) \cong \mathbb{R}$$

which is symmetric if d is even and skew-symmetric if d is odd.

In the case that d is even, i.e. the dimension of the manifold is divisible by 4, the symmetry tells us that the eigenvalues of the matrix representing the bilinear form are real. Non-degeneracy tells us they are non-zero. The *signature* $\sigma(M)$ of M is the signature of this bilinear form: the number of positive eigenvalues minus the number of negative eigenvalues. We can extend this to manifolds which are not path-connected by

$$\sigma(M_1 \sqcup M_2) = \sigma(M_1) + \sigma(M_2).$$

Let us now specialize to the case of 4-dimensional manifolds. The signature extends easily to oriented 4-dimensional manifolds that are not connected by taking the sum of the signatures of each of the components. This makes σ a homomorphism $\Omega_4^{\mathrm{SO}} \to \mathbb{Z}$, if it is well-defined, which we prove in the following lemma:

Lemma 2.12. *If M_1 and M_2 are cobordant 4-dimensional oriented manifolds, then $\sigma(M_1) = \sigma(M_2)$.*

Proof. It is enough to prove that the signature of $M_1 \sqcup -M_2$ is zero, where $-M_2$ is M_2 with the opposite orientation. Take a cobordism W from M_1 and M_2 and consider it as a manifold with boundary $M = M_1 \sqcup -M_2$. For simplicity we suppose that M_1, M_2 and W are path-connected and we will prove that $\sigma(M) = 0$.

The fundamental diagram in our argument is the following one, which by relative Poincaré duality is commutative and has exact rows

$$\begin{tikzcd}
\ldots \arrow[r] & H^2(W;\mathbb{R}) \arrow[r, "f^*"] \arrow[d, "\cong"'] & H^2(M;\mathbb{R}) \arrow[r, "\delta^*"] \arrow[d, "\cong"'] & H^3(W,M;\mathbb{R}) \arrow[r] \arrow[d, "\cong"'] & \ldots \\
\ldots \arrow[r] & H_3(W,M;\mathbb{R}) \arrow[r, "\delta_*"'] & H_2(M;\mathbb{R}) \arrow[r, "f_*"'] & H_2(W;\mathbb{R}) \arrow[r] & \ldots
\end{tikzcd}$$

Let f be the inclusion of M into W. We start by noting that the bilinear form, which denote by $\langle -,- \rangle$, vanishes on $f^*(H^2(W))$ in $H^2(M)$. To see this we write

$[M]$ for the fundamental class of M and $[W,M]$ for the relative fundamental class of W. Then we have that

$$\begin{aligned}\langle f^*a \cup f^*b, [M]\rangle &= \langle f^*(a\cup b), [M]\rangle = \langle f^*(a\cup b), \delta_*[W,M]\rangle \\ &= \langle \delta^* f^*(a\cup b), [W,M]\rangle = \langle 0, [W,M]\rangle\end{aligned}$$

where in order we used naturality of the cup product, (relative) fundamental classes and the cap product, followed by exactness of the top row of the diagram.

We will now prove that $\dim H^2(M;\mathbb{R}) = 2\dim f^*(H^2(W;\mathbb{R}))$. To do this we note that

$$\dim H^2(M) = \dim f^*(H^2(W;\mathbb{R})) + \dim \ker(\delta^*)^\perp$$

By the diagram $\ker(\delta^*)^\perp$ is isomorphic to $\ker(f_*)^\perp$ and by considering the following commutative duality diagram coming from the universal coefficient theorem

$$\begin{tikzcd} H^2(W;\mathbb{R}) \arrow[r, "f^*"] \arrow[d, "\cong"'] & H^2(M;\mathbb{R}) \arrow[d, "\cong"] \\ H_2(W;\mathbb{R}) & H_2(M;\mathbb{R}) \arrow[l, "f_*"] \end{tikzcd}$$

we see that $\ker(f_*)^\perp$ is isomorphic to $\mathrm{im}(f^*)$.

Now the proof is straightforward. Diagonalize the matrix for the bilinear form to get a decomposition $H^2(M;\mathbb{R}) = P \oplus N$, where P is the subspace on which the bilinear form is positive definite and N is the subspace on which it is negative definite. We claim these have the same dimension. For suppose that say P has dimension $\geq \frac{1}{2}\dim H^2(M;\mathbb{R}) + 1 = \dim f^*(H^2(W;\mathbb{R})) + 1$, then $f^*(H^2(W;\mathbb{R}))$ and P must intersect in a subspace of dimension at least 1 and hence the bilinear form can not vanish on $f^*(H^2(W;\mathbb{R}))$. This gives a contradiction and thus P and N have the same dimension, which implies that the signature of M is zero. $\square$

So to check that σ is surjective, we just need to find a 4-dimensional oriented manifold with signature 1.

Lemma 2.13. *With its standard orientation, $\sigma(\mathbb{C}P^2) = 1$.*

Proof. To see it is non-zero, one simply needs to compute that $H^2(\mathbb{C}P^2) = \mathbb{Z}$, for example using the standard cell decomposition with a single cell in degrees $0, 2, 4$. Let x be the generator in degree 2, then by Poincaré duality we can't have $x \cup x = 0$ and hence the bilinear form used in the signature is non-zero. To see that the signature is 1 instead of -1, one need to compute that the cohomology ring is in fact isomorphic to $\mathbb{Z}[x]/(x^3)$. $\square$

Corollary 2.14. *Ω_4^{SO} surjects onto $\mathbb{Z}$.*

3. Thom's calculation of the rational oriented cobordism groups

So what is Ω_*^{SO} in general? Since the signature is an interesting invariant of 4-dimensional manifolds, this is an interesting question to answer. In this section we will describe the rational answer using an additional ring structure on Ω_*^{SO} and then sketch its proof using the Pontryagin-Thom construction.

3.1. **The oriented cobordism ring.** A basic operation on oriented manifolds which we mentioned before, is taking Cartesian products of these, and such a product has a canonical orientation. We want to import this structure to the oriented cobordism groups. If everything works out, a k-dimensional oriented manifold M and an l-dimensional oriented manifold N give us a $(k+l)$-dimensional oriented manifold $M \times N$, so we expect to get a graded commutative ring structure on Ω_*^{SO} (the sign comes in when one defines the orientation of a product of oriented manifolds). The only thing we need to check is that this is independent of the choice of representatives of the cobordism classes.

Lemma 3.1. *If M is cobordant to M' and N is cobordant to N', then $M \times N$ is cobordant to $M' \times N'$.*

Proof. If N is cobordant to N' by W and M is cobordant to M' by V, then $W \times N$ is a cobordism from $M \times N$ to $M' \times N$ and $M' \times V$ is a cobordism from $M' \times N$ to $M' \times N'$. Thus $M \times N$ and $M' \times N'$ are cobordant. $\square$

Hence the product structure is well defined. Thom calculated the oriented cobordism ring. As is customary, we want to split the data into its free and primary parts by taking the tensor product with $\mathbb{Q}$ or $\mathbb{F}_p$ respectively. In this note, we will just look at the rational part and that case the result is as follows.

Theorem 3.2 (Thom)**.** *We have an isomorphism of rings*

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[x_{4i} \mid i \geq 1]$$

with $|x_{4i}| = 4i$. Furthermore x_{4i} is a non-zero multiple of $\mathbb{C}P^{2n}$.

Remark 3.3. The complex projective spaces $\mathbb{C}P^{2n}$ do not generate the oriented cobordism ring integrally. However, Wall has constructed more complicated generators for the integral case.

Note that the oriented cobordism ring is concentrated in degrees divisible by four. This implies that, for example, for every five-dimensional oriented manifold M there exists an integer $N \geq 1$ such that $\bigsqcup_N M$ bounds a six-dimensional oriented manifold. We must have $N > 1$ when the five-dimensional oriented manifold represents a non-zero torsion class in the cobordism group.

Also note that our calculation of the signature shows that the homomorphism $\sigma\colon \Omega_4^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$ is an isomorphism. However, it is more natural to use that the Pontryagin numbers as maps $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$ which distinguish all cobordism classes. These Pontryagin classes are geometrically constructed characteristic classes $p_i(M) \in H^{4i}(M;\mathbb{Q})$, which can be multiplied to $p_I(M) = \prod_{i \in I} p_i(M)$. These distinguish all the cobordism classes up to torsion, in the sense that if we consider all ordered non-decreasing tuples I of positive integers with $\sum_{i \in I} i = n$ given by

$$\Omega_{4n}^{\mathrm{SO}} \ni M \mapsto \langle p_I(M), [M] \rangle \in \mathbb{Q}$$

we get a basis for the dual space to $\Omega_{4n}^{\mathrm{SO}} \otimes \mathbb{Q}$. Part of this statement is that the Pontryagin numbers are invariant under oriented cobordism.

The reader can now continue with a sketch of Thom's calculation or take the calculation of the oriented cobordism ring on faith and skip to the next section about the Hirzebruch signature theorem,. This theorem tells one how to compute the signature of a $4n$-dimensional manifold in terms of its Pontryagin numbers.

3.2. **The Pontryagin-Thom construction.** To compute the oriented cobordism ring, we do one of the few things an algebraic topologists can do: convert the problem into computing homotopy groups of some object. This object will be a spectrum. One should think of this a space up to suspension or roughly equivalently an infinite loop space:

Definition 3.4. A (naive) *spectrum* E is a sequence $\{E_n\}_{n\geq 0}$ of pointed spaces together with structure maps $\Sigma E_n \to E_{n+1}$ or equivalently $E_n \to \Omega E_{n+1}$.

A map of spectra $E \to F$ is a sequence of pointed maps $E_n \to F_n$ compatible with the structure maps. A weak equivalence of spectra is defined as a map inducing an isomorphism on stable homotopy groups: these are the homotopy groups of spectra discussed earlier and they are defined as

$$\pi_i^s E \coloneqq \mathrm{colim}_{k\to\infty} \pi_k E_{k+i}.$$

To explain our previous remark about infinite loop spaces, we note that every spectrum is weakly equivalent to an Ω-spectum, i.e. a spectrum such that all the maps $E_n \to \Omega E_{n+1}$ are weak equivalences.

Example 3.5. For any pointed space X we may define the *suspension spectrum* $\Sigma^\infty X$: $(\Sigma^\infty X)_n = \Sigma^n X$ and the structure maps $\Sigma\Sigma^n X \to \Sigma^{n+1} X$ are the identity. A good example of a suspension spectrum is the *sphere spectrum* $\mathbb{S}$, which is the suspension spectrum of S^0. Its homotopy groups are the stable homotopy groups of spheres.

Thom's theorem expresses Ω_*^{SO} as homotopy groups of the *Thom spectrum* MSO. To define this we introduce the Thom space of a finite-dimensional vector bundle ξ over a compact space B. This is simply the one-point compactification of the total space of ξ:

$$\mathrm{Thom}(\xi) \coloneqq \xi \cup \{\infty\}$$

If B is not compact, one takes the one-point compactification of each fiber of ξ and identifies all the points at infinity to a single point. The classifying space $BSO(n)$ for n-dimensional oriented real vector bundles is the universal example of a space with a finite-dimensional vector bundle over it: the universal bundle ξ_n, which is n-dimensional, real, and oriented. Every other n-dimensional real oriented vector bundle over some space B is obtained by pullback along some map $f\colon B \to BSO(n)$. It is not surprising that if Thom spaces are interesting at all, $\mathrm{Thom}(\xi_n)$ is the most interesting. Note that there is a natural map $\mathrm{Thom}(f^*\xi_n) \to \mathrm{Thom}(\xi_{n+1})$.

Definition 3.6. The *Thom spectrum* MSO is given by $MSO_n \coloneqq \mathrm{Thom}(\xi_n)$. The structure maps are induced by the inclusion $BSO(n) \to BSO(n+1)$, since the pullback of ξ_{n+1} is $\xi_n \oplus \mathbb{R}$ and $\mathrm{Thom}(\xi_n \oplus \mathbb{R}) \cong \Sigma\mathrm{Thom}(\xi_n)$.

We can now state Thom's theorem.

Theorem 3.7 (Thom)**.** *We have an isomorphism of rings*

$$\Omega_*^{\mathrm{SO}} \cong \pi_*(MSO)$$

Sketch of proof. We will just describe two maps

$$\Omega_n^{\mathrm{SO}} \to \pi_n(MSO) \qquad \text{and} \qquad \pi_n(MSO) \to \Omega_n^{\mathrm{SO}}$$

and leave it to the reader or the references to check that these are mutually inverse and ring homomorphisms.

Let's start with the map $\Omega_n^{\mathrm{SO}} \to \pi_n(MSO)$, which is called the *Pontryagin-Thom construction*. The star here is the normal bundle. If we take an oriented n-dimensional manifold M, we can embed it into some $\mathbb{R}^{n+k}$ by the Whitney embedding theorem and it has a k-dimensional normal bundle ν there. The tubular neighborhood theorem gives an embedding $\nu \hookrightarrow \mathbb{R}^{n+k}$ such that the restriction to the zero-section is the embedding of M into $\mathbb{R}^{n+k}$. Note that if we collapse the complement of the image of ν to a point, we get something that is homeomorphic to the Thom space of ν and we can extend this collapse map to the point at ∞ in $\mathbb{R}^{n+k}$ by sending it to $\{\infty\}$ in $\mathrm{Thom}(\nu)$. Now consider the following sequence of maps

$$S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} \backslash \mathrm{im}(\nu)) = \mathrm{Thom}(\nu) \to \mathrm{Thom}(\xi_k)$$

where only the last map remains to be explained: it is the map induced by a classifying map $M \to BSO(k)$ for ν. Passing to homotopy classes gives us an element in $\pi_{n+k}(MSO_k)$ and hence in $\pi_n(MSO)$.

We must check that is independent of the choice of embedding, tubular neighborhood and classifying map. The latter two are easy: every two tubular neighborhoods are isotopic and every two classifying maps are homotopic, hence the two induced maps $S^{n+k} \to \mathrm{Thom}(\xi_k)$ are homotopic. To check that our construction does not depend on the choice of embedding, one notes that if we use the standard embedding of $\mathbb{R}^{n+k}$ into $\mathbb{R}^{n+k+1}$, the same construction gives an element of $\pi_{n+k+1}(MSO_{k+1})$ that equal to the image of our original element of $\pi_{n+k}(MSO_k)$ under suspension. Hence we can assume that the two embeddings are into the same $\mathbb{R}^{n+k}$ and that $k \geq n+1$. In that case any two embeddings are isotopic and the induced maps homotopic. Applying the same construction to a cobordism between two manifolds gives a homotopy between their corresponding elements of $\pi_n(MSO)$, so this construction factors over Ω_n^{SO}.

For the map $\pi_n(MSO) \to \Omega_n^{\mathrm{SO}}$ we will use transversality. Any element of $\pi_n(MSO)$ is represented by some map $S^{n+k} \to MSO_k$ for $k \geq n+1$. By compactness, the latter factors through $\mathrm{Thom}(\xi_{k,l})$ for some sufficiently large l, where $\xi_{k,l}$ is the canonical k-dimensional vector bundle over the Grassmannian of oriented k-planes in $\mathbb{R}^l$. Because $k \geq n+1$, we can assume the map from S^{n+k} is smooth and by transversality we can assume it transverse to the zero-section. Thus we obtain a smooth map from an open subset of $\mathbb{R}^{n+k}$ (the complement of the inverse image of the point at ∞ in the Thom space) to the total space the total space of the canonical vector bundle, both manifolds. This map is transverse to the zero section, a codimension k submanifold. The inverse image of the zero-section is then a n-dimensional manifold $M \subset \mathbb{R}^{n+k}$, which can be shown to be canonically oriented.

This is independent of the choices made: the representative map $S^{n+k} \to MSO_k$ and the perturbation of this. Increasing k just increases the dimension of the Euclidean space M is embedded in. Picking two different homotopic representatives gives by a similar transversality argument a cobordism between the two manifolds, as does a different perturbation of the map. □

3.3. **The rational homotopy groups of MSO.** The computation of the rational homotopy groups of MSO is surprisingly easy, given several standard results in algebraic topology.

Proposition 3.8. *We have that*

$$\pi_*(MSO) \otimes \mathbb{Q} \cong H_*(MSO; \mathbb{Q}) \cong H_*(BSO; \mathbb{Q}) \cong \mathbb{Q}[p_i \mid i \geq 1]$$

where the p_i of degree $4i$ are the Pontryagin classes mentioned before.

One of the terms in this sequence of equalities has not been defined yet: the stable rational homology groups $H_i(MSO; \mathbb{Q}) \coloneqq \mathrm{colim}_{k \to \infty} H_{i+k}(MSO(k); \mathbb{Q})$.

Proof. We will start by proving that the Hurewicz map induces an isomorphism

$$\mathrm{colim}_{k \to \infty} \pi_{i+k}(MSO(k)) \otimes \mathbb{Q} \cong \mathrm{colim}_{k \to \infty} H_{i+k}(MSO(k); \mathbb{Q}).$$

To prove this, note that the natural map $S^n_{\mathbb{Q}} \to K(\mathbb{Q}, n)$ is either a weak equivalence or $(2n-2)$-connected depending on whether n is odd or even; this is a result of Serre. We conclude that that if X is k-connected, then $\pi_i(X) \otimes \mathbb{Q} \to \tilde{H}_i(X; \mathbb{Q})$ is an isomorphism in degrees $0 \leq i \leq 2k-2$. We remarked before $MSO(k)$ is k-connected, so $\pi_{i+k}(MSO(k)) \otimes \mathbb{Q} = H_{i+k}(MSO(k); \mathbb{Q})$ for $0 \leq i \leq k-2$. As $k \to \infty$, we get the first isomorphism in the statement of the theorem.

Next we use that $H_{i+k}(MSO(k); \mathbb{Q}) = H_i(BSO(k); \mathbb{Q})$ by the Thom isomorphism. Since the inclusion $BSO(k) \to BSO$ is k-connected, we have that $H_i(BSO(k); \mathbb{Q}) = H_i(BSO; \mathbb{Q})$ for k sufficiently large, proving the second isomorphism. The final isomorphism is a standard computation, e.g. by noting that direct sums makes BSO into an H-space and its rational homotopy groups are $\mathbb{Q}$ in positive degrees divisible by 4 by Bott periodicity, so that the rational cohomology of BSO by the Milnor–Moore theorem is that free graded-commutative algebra on generators in positive degrees divisible by 4. □

4. The Hirzebruch signature theorem

We will now explain a classical application of the computation of the oriented cobordism groups: a formula of the signature of a $4n$-dimensional manifold in terms of its Pontryagin numbers. This is a result by Hirzebruch, explained nicely in [Hir66]. To set it up, note that any homomorphism

$$\Omega^{\mathrm{SO}}_{4n} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$

can be written as a linear combination of Pontryagin numbers, as they span the dual space to $\Omega^{\mathrm{SO}}_{4n} \otimes \mathbb{Q}$. We can find the coefficients by evaluating our map on products of $\mathbb{C}P^{2n}$'s. If the map actually comes from a ring homomorphism

$$\Omega^{\mathrm{SO}}_{*} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$

its value is actually determined by its value on the $\mathbb{C}P^{2n}$'s, as these represent multiplicative generators.

The application we have in mind is the signature. We previously only defined it for 4-dimensional manifolds, but the same construction can be used to give a homomorphism $\sigma \colon \Omega^{\mathrm{SO}}_{4n} \to \mathbb{Z}$ for all $n \geq 1$. Using the Künneth formula one may prove that

$$\sigma(M \times N) = \sigma(M)\sigma(N),$$

so the signature is a ring homomorphism. We will find the coefficients for its expression in terms of Pontryagin numbers in this section, but only after defining genera in general.

4.1. **Genera and the L-genus as an example.** We start with the definition of a genus, directly copying the properties of the signature.

Definition 4.1. A *genus* ϕ with values in a ring R is a homomorphism of R-modules

$$\phi\colon \Omega_*^{\mathrm{SO}} \otimes R \longrightarrow R$$

In the case where 2 is invertible in R, we have that $\Omega_*^{\mathrm{SO}} \otimes R = R[p_i \mid i \geq 1]$ (this follows from the fact that the only torsion is $H_*(BSO)$ is 2-torsion). Under this condition we will give a general construction of a genus from a power series $Q(t) =\in R[[t]]$ with leading coefficient 1.

Let's first think about how we would go about obtaining a genus from a formal power series. Our goal will be to find some polynomial expression $P_n^Q(z_1, \ldots, z_n)$ obtained from the power series such that if we substitute Pontryagin numbers of M for a monomial z_I—that is, take $z_I = \langle p_I(M), [M]\rangle$—we get the value of our genus on M. Since $\langle p_I(M), [M]\rangle$ is additive under disjoint union, this is an additive homomorphism.

Thus the main restriction on the P_n^Q's is that they have to be compatible with cartesian product. For this we need to use the product formula, which says that (if 2 is invertible) $p(E \oplus F) = p(E) \cup p(F)$ for the total Pontraygin class

$$p(E) \coloneqq 1 + \sum_{i=1}^{\infty} p_i(E).$$

Specializing to tangent bundles of manifolds we obtain $p(M \times N) = p(M) \cup p(N)$. It turns out to be useful to package the P_n^Q into a single expression $1 + \sum_{n=1}^{\infty} P_n^Q$. We then conclude that our P^Q must satisfy that if

$$1 + \sum_{i=1}^{\infty} z_i t^i = (1 + \sum_{i=1}^{\infty} x_i t^i)(1 + \sum_{i=1}^{\infty} w_i t^i)$$

then we must have

$$P^Q(z) = P^Q(x) P^Q(w).$$

Let us now give the definition of P_n^Q in a way that forces $P^Q(z) = P^Q(x)P^Q(w)$. We write the coefficients of Q by q_i. We define for a sequence I of numbers $i_1 \leq \ldots \leq i_k$ with $\sum i_j = n$ the element $Q(I)$ to be $\prod_{j=1}^k q_{i_j}$. Furthermore we define a polynomial $s_I(z_1, \ldots, z_n)$ to be the unique polynomial such that $s_I(\sigma_1(t), \ldots, \sigma_n(t)) = \sum t^I$, where the σ_i are the elementary symmetric polynomials and the sum is over distinct permutations of the indices of the t_i. These polynomials satisfy $\sum_I s_I(z) = \sum_{I_1 \cup I_2 = I} s_{I_1}(x) s_{I_2}(w)$. We then define

$$P_n^Q \coloneqq \sum_I Q(I) s_I(z_1, \ldots, z_n)$$

It is now easy to check that $P^Q(z) = P^Q(x)P^Q(w)$ and we have thus defined a genus which we call ϕ_Q by defining for a $4n$-dimensional manifold M

$$\phi_Q(M) \coloneqq P_n^Q(\langle p_I(M), [M]\rangle) = \langle P_n^Q(p_1(M), \ldots, p_n(M)), [M]\rangle$$

Remark 4.2. There is an inverse to this construction, giving a power series Q_ϕ for each genus ϕ such that $\phi_{Q_\phi} = \phi$ and $Q_{\phi_Q} = Q$. It involves the log-series of a formal power series with leading coefficient 1.

Remark 4.3. It is not hard to see that the coefficient of z_1^n in P_n^Q is exactly q_n.

Example 4.4. A very simple power series over $\mathbb{Q}$ is $1+t$. To figure out what the corresponding genus is, we note that only $I=(1,\ldots,1)$ has a non-zero coefficient and $s_{1,\ldots,1}$ exactly is the elementary symmetric polynomial σ_n. So the corresponding genus is given on a $4n$-dimensional manifold by the n'th Pontryagin number $\langle p_n(M),[M]\rangle$.

Example 4.5. Another nice example of a genus is the Todd genus Td, which plays an important in Atiyah-Singer and Hirzebruch-Riemann-Roch. It is the unique genus with values in $\mathbb{Q}$ such that $\mathrm{Td}(\mathbb{C}P^{2n})=1$ for all n. The corresponding power series is the expansion of $\frac{t}{1-\exp(-t)}$.

4.2. **The L-genus gives signature.** We will be concerned with the genus coming from the power series $Q(t)$ obtained by expanding $\frac{\sqrt{t}}{\tanh(\sqrt{t})}$ at 0. We call it the L-genus.

Theorem 4.6 (Hirzebruch signature theorem)**.** *For all $4n$-dimensional manifolds we have*

$$\sigma(M)=\phi_L(M)$$

Proof. It suffices to prove the equality on the $\mathbb{C}P^{2n}$. Their signature is 1, so all the difficulty will be in proving that ϕ_L also takes the value 1. We will use that $p(\mathbb{C}P^{2n})=(1+x^2)^{2n+1}$, if x denotes the generator of the cohomology ring of $\mathbb{C}P^{2n}$. As P^L is multiplicative, we have that

$$P^L((1+x^2)^{2n+1})=P^L(1+x^2)^{2n+1}.$$

So what is $P^L(1+x^2)$? The only relevant term is x^2, so it is $\sum P^L_n(x^2,0,\ldots)$. We know the coefficient of z_1^n is exactly the n'th Taylor coefficient of $\frac{\sqrt{x}}{\tanh(\sqrt{x})}$, so we get

$$P^L(1+x^2)=\frac{\sqrt{x^2}}{\tanh(\sqrt{x^2})}=\frac{x}{\tanh(x)}.$$

We conclude that

$$\phi_L(\mathbb{C}P^{2n})=\left\langle\left(\frac{x}{\tanh(x)}\right)^{2n+1},[\mathbb{C}P^{2n}]\right\rangle$$

so that our goal is to prove that the coefficient a_{2n} of x^{2n} in $(\frac{x}{\tanh(x)})^{2n+1}$ is 1. We can do this using complex analysis:

$$a_{2n}=\frac{1}{2\pi i}\oint\frac{1}{\tanh(z)^{2n+1}}dz$$

This can be computed using the substitution $u=\tanh(z)$, which has the property that $du=(1-\tanh(z)^2)dz=(1-u^2)dz$, so that we may write

$$a_{2n}=\frac{1}{2\pi i}\oint\frac{du}{(1-u^2)u^{2n+1}}=\frac{1}{2\pi i}\oint\frac{(1+u^2+\ldots+u^{2n}+\ldots)du}{u^{2n+1}}=1. \qquad\square$$

5. More applications

Finally, we sketch some other applications of oriented cobordism. The first two are related to particular genera and their properties, and the last one was the actual motivation for Thom to introduce cobordism: figuring out which homology classes are represented by the image of the fundamental class of a manifold mapping into your space.

5.1. **Positive scalar curvature and the $\hat{A}$-genus.** The scalar curvature of a manifold with metric is the trace of the Riemann curvature tensor. Geometrically it is related to the difference between the volumes of small spheres in the manifold and Euclidean volume of spheres. The result is a real-valued function on M which captures some properties of the geometry of M.

Related to this is another example of a genus; the one coming from the rational power series

$$Q(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}.$$

This is called that $\hat{A}$-genus. The exact expression is not important, but the following result is:

Theorem 5.1 (Lichnerowicz)**.** *Let M be a spin-manifold (i.e. the classifying map $M \to BSO(n)$ of the tangent bundle lifts to $B\mathrm{Spin}(n)$, the 2-connected cover of $BSO(n)$). Then M can only have a positive scalar curvature metric if $\hat{A}(M) = 0$.*

In other words, we have found a topological invariant giving an obstruction to M possessing a metric with the scalar curvature being positive at each point of M. This opens up an interesting direction for studying the question whether a manifold admits a positive scalar curvature metric. Conversely, there is a positive result concerning cobordism and positive scalar curvature by Gromov-Lawson [GL80]: if M_0 admits a positive scalar curvature metric and there exists a cobordism W from M_0 to M_1 which can be constructed by handle attachments of codimension ≥ 3, then M_1 admits a positive scalar curvature metric as well.

Using the explicit determination of the oriented and spin cobordism rings, Gromov-Lawson and Stolz completely answered the question which simply-connected manifolds admit a positive scalar curvature metric. A good survey about these and related results is [RS01].

Theorem 5.2 (Gromov-Lawson, Stolz)**.** *If M is a simply-connected non-spin manifold of dimension ≥ 3, then M admits a metric of positive scalar curvature. If M is a simply-connected spin-manifold of dimension ≥ 3, then M admits positive scalar curvature metric if and only if an invariant $\alpha(M)$ (a refined version of the $\hat{A}$-genus) vanishes.*

5.2. **Elliptic genera.** Both the L-genus and $\hat{A}$-genus are special cases of the elliptic genus. It is a genus depending on particular δ and ϵ

$$\mathrm{Ell}\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Z}[1/2, \delta, \epsilon]$$

defined by $Q(z) = \frac{\sqrt{z}}{f(\sqrt{z})}$ for f given by

$$f(z) = \int_0^z \frac{du}{\sqrt{1 - \delta u^2 + \epsilon u^4}}.$$

Though the motivation for this is clearer if one looks at complex cobordism (closely related to oriented cobordism by complexification), the following examples should show it is interesting: if we set $\delta = \epsilon = 1$, we get $f(z) = \int_0^z \frac{du}{1-u^2} = \tanh(z)$, and if we set $\delta = -\frac{1}{8}$ and $\epsilon = 0$, we get $f(z) = \int_0^z \frac{du}{\sqrt{1+\frac{1}{8}u^2}} = 2\sinh(z/2)$.

Definition 5.3. An *elliptic genus* is a genus obtained from the elliptic genus Ell by specifying δ and ϵ, i.e. as a composite

$$\Omega_*^{\mathrm{SO}} \xrightarrow{\mathrm{Ell}} \mathbb{Z}[1/2, \delta, \epsilon] \xrightarrow{\mathrm{ev}} \mathbb{Z}[1/2] .$$

So in particular the L- and $\hat{A}$-genus are elliptic. There is an interesting geometric characterisation of elliptic genera due to Ochanine [Och91].

Theorem 5.4 (Ochanine)**.** *A genus with values* $\mathbb{Z}[1/2]$ *is elliptic if and only if it vanishes on projectivizations of complex vector bundles.*

Pursuing the connection between elliptic curves and topology further leads to elliptic cohomology theories [Tho99] and eventually TMF [Lur09]. It also led Witten to define the Witten genus and conjecture relations to the scalar curvature of loop spaces [Wit87].

5.3. **Geometric representatives for homology classes.** Finally, we go back all the way to the origins of oriented cobordism and look at the question that Thom wanted it to answer. Given a manifold M and a map f from an oriented manifold N into it, we can obtain an element of $H_{\dim N}(M)$ by taking the image $f_*([N])$ of the fundamental class of N. When the definition of homology was still in flux, people wanted to know which homology classes of M can realized in this way.

Thom solved this question by proving all homology classes are when one works with $\mathbb{F}_2$-coefficients, but not all homology classes are with $\mathbb{F}_p$-coefficients for primes $p \geq 3$ or with $\mathbb{Z}$-coefficients. However, in the latter case it is possible up to an odd multiple. See [Sul04] for a survey.

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