Multiple disjunction lemma

N^n smooth compact

P^p, $\{Q_j^{q_j}\}_{j=1}^{a}$ disjoint, compact proper submanifolds of N and $n-p \geq 3$, $n-q_j \geq 3$ $\forall j$

then the cube $\left(R \subseteq \{1, \dots, a\}\right) \longmapsto C(P, N \setminus Q_R)$, $Q_R = \bigcup_{j \in R} Q_j$

is $\left[(n-p-2) + \sum_{j=1}^{a} (n-q_j-2)\right]$ - cartesian.

i.e. $C(P, N \setminus \bigcup_{j=1}^{a} Q_j) \to \operatorname{holim}_{R \subsetneq \{1,\dots,a\}} C(P, N \setminus \bigcup_{j \in R} Q_j)$ is $(\dots)$-connected.

($a=1$, Morlet): $C(P, N \setminus Q) \to C(P, N)$ $(2n-p-q-4)$-connected.

(This is essentially equivalent to: given $\{A_j\}_{j=0}^{a}$ disjoint, compact submanifolds of M, estimate connectivity of cube

$\left(R \subseteq \{0, \dots, a\}\right) \longmapsto C(M \setminus A_R, N)$.)

Goodwillie's generalisation of Blakers–Massey: X a-cube, and T subcube is K_T-cocartesian

then X is K-connected where

K = minimum, over all partitions $\{T_\alpha\}$ of $\{1,\dots,a\}$, of $(1-a)+\sum K_{T_\alpha}$

$$\left[\begin{array}{c} \begin{matrix} X_{12} & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_\emptyset \end{matrix} \end{array} \text{ is } \min\left(-1+K_1+K_2,\ -1+K_{12}\right)\text{-connected}\right]$$

Main task: a-cube $S \longmapsto C(P, N\setminus Q_S)$ is $(n-p-3)+\sum_{j=1}^{a}(n-q_j-1)$ - cocartesian, i.e.

(1) $\bigcup_{j=1}^{a} C(P, N\setminus Q_j) \longrightarrow C(P,N)$ is $\_$ connected

- The T-subcubes $(R\subseteq T)\mapsto C(P, N\setminus Q_R)$ have similar form ($N \rightsquigarrow N\setminus\bigcup_{k\notin T} Q_k$) so this implies they are $(n-p-3)+\sum_{j\in T}(n-q_j-1)$ - cocartesian

$\Rightarrow$ cube is $(1-a)+\left[(n-p-3)+\sum_{j=1}^{a}(n-q_j-1)\right]$ - cartesian, as required.

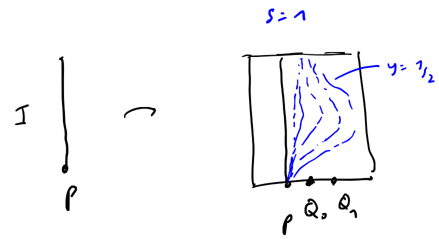
$$\parallel$$

$$n-p-2+\sum_{j=1}^{a}(n-q_j-2)$$

① $(D^s, \partial D^s) \longrightarrow \left(C(P,N), \bigcup_{j=1}^{a} C(P, N \setminus Q_j) \right)$ may be homotoped to land in $\bigcup_{j=1}^{a} C(P, N\setminus Q_j)$

ie every fibred concordance

$$I \times P \times D^s \xrightarrow{F=(h,f,p_3)} I \times N \times D^s$$



such that $\forall y \in \partial D^s$

(187) $\quad \exists j \in \{1,\dots,a\}$ st $F(I\times P \times y) \cap (I \times Q_j \times D^s) = \emptyset$.

WTS: F is isotopic to one for which (187) holds $\forall y \in D^s$.

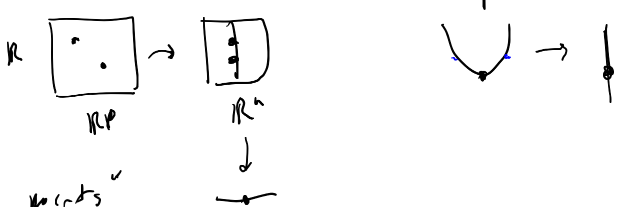
ie that for isotoped F, $\exists$ open cover $\{O_j\}_{j=1}^{a}$ of D^s such that

$$f(I \times P \times \overline{O_j}) \cap Q_j = \emptyset \qquad \forall j.$$

Recall: we "stratify" $I\times$ int $P \times D^s$ according to behaviour of F.

$Z_0, Z_1, \dots$ $\qquad Z_\alpha \subseteq_{I(Z_\alpha)} J^\infty(\mathbb{R}\times\mathbb{R}^p, \mathbb{R}^n)$ multi-jet space

"critical points" $\qquad$ these are invariant under reparametrisations ie define



$$S(Z_\alpha, P, N) \subseteq_r J^\infty(\mathbb{R}\times P, N)$$

"double points"

• $F : I\times P\times D^s \to I\times N\times D^s$ fibered concordance can be put in <u>general position</u> wrt Z's

i.e. up to fibered isotopy, for each Z_α of rank r and level m

$$(I\times \mathrm{int}(P))^{(r)}\times D^s \xrightarrow{{}_r j^m(f)} {}_rJ^m(I\times P, N) \subseteq {}_rJ^m(\mathbb{R}\times P, N)$$

singular set

is $\pitchfork$ to submanifold $p^\infty_m\left(S^*(Z_\alpha, P, N)\right) = p^\infty_m\left(S(Z_\alpha, P, N) \setminus S(A(Z_\alpha), P, N)\right)$

<u>Defn</u> $S^*_\alpha(F) = \pitchfork$ preimage $\subseteq (I\times \mathrm{int}(P))^{(r)}\times D^s$

Similarly, $S_\alpha(F)$ = preimage of $S(Z_\alpha, P, N) \subseteq (I\times \mathrm{int}(P))^{(r)}\times D^s$.

• the Z_α's are constructed from Z_0 by operations: A, B, C, D

These operations typically increase "<u>codimension</u>" (only B can preserve it)

$$c(Z_\alpha) := \mathrm{codim}\left(p^\infty_m(Z_\alpha), {}_rJ^\infty(\mathbb{R}\times\mathbb{R}^p, \mathbb{R}^n)\right) - r(p+1).$$

<u>Lemma</u>: (Assuming $n-p\geq 3$) $\forall s\geq 0$ $\exists$ finitely many Z such that $c(Z)\leq s$.

$\rightsquigarrow$ can order $Z_0, Z_1, Z_2, \dots$ so that $\alpha\leq\beta \Rightarrow c(Z_\alpha)\leq c(Z_\beta)$.

$\leftarrow \mid Z_\alpha \longrightarrow$

<u>Def^n</u> $$Y_\alpha = \bigcup_{\beta \geq \alpha} \bigcup_{j \in I(Z_\beta)} \pi_j S_\beta(\mathcal{F}) \subseteq I \times \mathrm{int}(P) \times D^5.$$

where, for $j \in \{1, \dots, r\}$, $\pi_j : S_\alpha(\mathcal{F}) \longrightarrow I \times \mathrm{int}(P) \times D^5$ is projection of j^{th} point.

The overall strategy is to do a downward induction on the statement

$(189)_\alpha$ $\boxed{\exists \text{ a cover } \{Q_j\}_{j=1}^{a} \text{ of } D^5 \text{ such that } \forall j \in \{1, \dots, a\} \text{ and up to isotopy of } \mathcal{F}: \quad f(I \times (P \times \overline{\theta}_j \cap p_{2,3} Y_\alpha)) \cap Q_j = \phi.}$

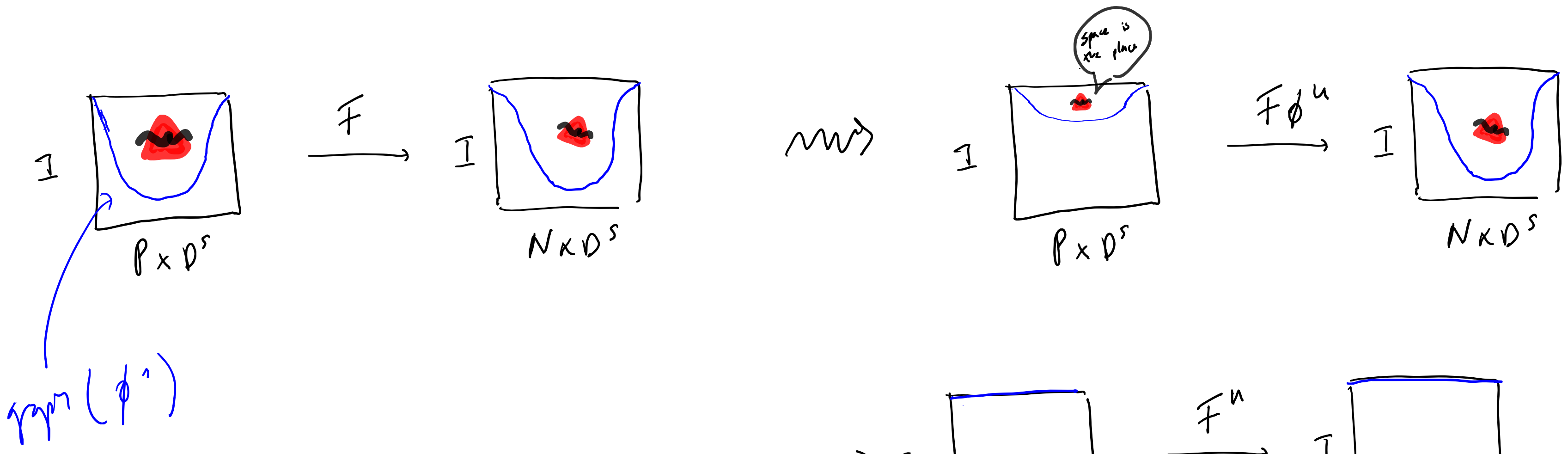
So three steps:

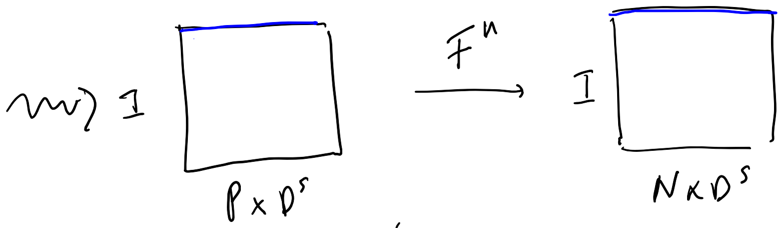
① $(189)_{\alpha_0}$ for some α_0 "largest bad" — general position

② $(189)_{\alpha+1} \Rightarrow (189)_\alpha$ for $\alpha < \alpha_0$ — sunny collapse

③ $(189)_0 \Rightarrow \forall j \ f(I \times P \times \overline{\theta}_j) \cap Q_j = \phi$. — sunny collapse.

③ "sunny collapse" $\phi^u : P\times D^s \longrightarrow [0,1]$, $u\in[0,1]$



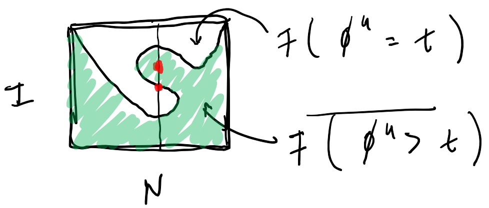


$$\left(F^u(t,x,y) = F\left(t\,\phi^u(x,y), x, y\right) \right)$$

Conditions are:

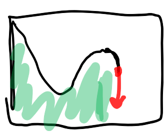
(34) $\phi^u(x,y) = 1$ if $u=0$ or $x\in\partial P$

(40) $0\le t\le \phi^u(x,y) \Rightarrow F\left(\phi^u(x',y), x', y\right)$ is not below $F(t,x,y)$ $\forall\, (u,t,x,x',y)\in I\times I\times P\times P\times D^s$



<u>not</u> allowed. $\simeq$ points above are in the sun.

(41)



not allowed.

typically, $\phi^u = 1 - u(1-\phi)$ for $\phi : P\times D^s \to [0,1]$ satisfying analogous conditions

) Let $X = p_{2,3} Y_0 \cup \partial P \times D^s \subseteq P \times D^s$. $\quad f(I \times P \times \bar\Theta_j \cap I \times X) \cap Q_j = \emptyset$ $\quad$ | $(189)_0 \Rightarrow \checkmark$

$\mathcal{F}$ satisfies $f^{-1}(Q_j) \cap (I \times P \times \bar\Theta_j \cap I \times X) = \emptyset \quad \forall j \,]$ ⊛

WTS: $Y := \bigcup_{j=1}^{\ell} \left(f^{-1}(Q_j) \cap I \times P \times \bar\Theta_j\right) = \emptyset$, up to isot. of $\mathcal{F}$.

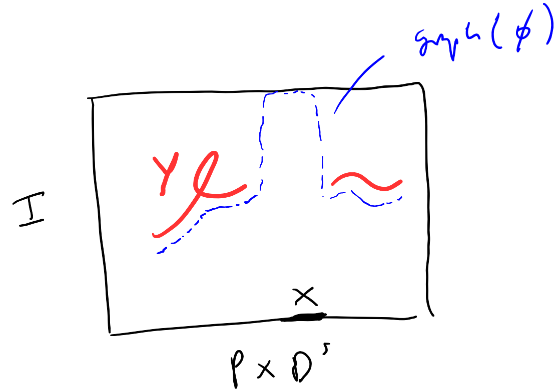
$$\subset \; I \times P \times D^s$$

- $(t,x,y) \in Y \Rightarrow (x,y) \notin X$ and $(t,x,y) \in Y \Rightarrow t > 0$ $\square$

and X, Y are compact $\Rightarrow \exists$ smooth function

$$P \times D^s \xrightarrow{\phi} [0,1]$$

with
- $\phi(x,y) < t$ for $(t,x,y) \in Y$
- $\phi(x,y) = 1$ for $(x,y) \in X$



and $\phi^u = 1 - u(1-\phi)$ sunny collapse

Let $\mathcal{F}^u$ be the associated isotopy.

for $y \in \bar\Theta_j$, $f^1(t,x,y) \in Q_j \Rightarrow f(t\phi(x,y), x, y) \in Q_j$

$\Rightarrow (t\phi(x,y), x, y) \in Y$

$\Rightarrow \phi(x,y) < t\phi(x,y) < \phi(x,y)$ ↯

eg. condition (no) follows from:

[If $\mathcal{F}(t,x,y)$ is below $\mathcal{F}(t',x',y)$ then either $\begin{cases} \phi(x,y) > t & \text{or} \\ (1-t)(1-\phi(x',y)) > (1-t') \end{cases}$

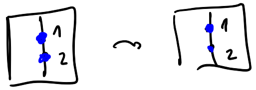
holds since $(t,x,t',x',y) \in S_0(\mathcal{F})$ or

other way $(x,y) \in X \Rightarrow \phi(x,y$

But $t < 1 \Rightarrow \phi(x,y) > t$.

saw some of these are called "bad" and some "good"

eg $\left\{(z_1,z_2)\in J^\infty_2 \;\middle|\; \begin{array}{l} p_2 s z_1 = p_2 s z_2 \\ t z_1 = t z_2 \end{array}\right\}$



$\Rightarrow 1 \sim 2 \simeq 1$

$1\sim 2$ $\qquad$ $1 \simeq 2$

admissible (D,R) pair of relations on $\{1,\dots,r\}$, $r = I(Z)$

$i \sim j \iff (\, i D j$ or $i = j$ or $j D i\,)$

$i \simeq j \iff (\, i R j$ or $i = j$ or $j R i\,)$

Z good $\Rightarrow \exists\, i_0$ which is maximal wrt D within its $\sim$ class & wrt R within its $\simeq$ class

$c(\text{bad } Z) \geq n-p-2$

α_0 is largest bad one — smaller codim.

define
$$V_j = \left\{ y \in D^s \;\middle|\; \begin{array}{l} \forall \alpha \geq \alpha_0,\ x \in S_\alpha(F),\ t \in I,\ i \in I(Z_\alpha), \\ p_3(x) = y \implies F(t, p_2\pi_i x, y) \notin I \times Q_j \times D^s \end{array} \right\}$$

parameters for which $(189)_{\alpha_0}$ holds.

(up to replacing V_j by slightly smaller θ_j, $\overline{\theta}_j \subseteq$

$\boxed{\text{Claim: For a generic } F,\ \{V_j\} \text{ form an open cover of } D^s.}$

Suppose $\forall j\ \exists\, \alpha_j \geq \alpha_0$, $x_j \in S_{\alpha_j}(F)$, $t_j \in I$, $i_j \in I(Z_{\alpha_j})$ such that $p_3(x) = y$ but

$F(t, p_2 \pi_{i_j} x_j, y) \in I \times Q_j \times D^s$.

Choose among these pairs

$$(\alpha^1, x^1), \dots, (\alpha^b, x^b)$$

so as to minimise # distinct (α_j, x_j). (The $\psi : \{1, \dots, a\} \to \{1, \dots, b\}$ surjection st $(\alpha_j, x_j) = (\alpha^{\psi(j)}, x^{\psi(j)})$ for each

define

$$x := (\rho_{12} x^1, \dots, \rho_{12} x^b, y) \in (I \times P)^{(r_1)} \times \dots \times (I \times P)^{(r_b)} \times D^s \underset{\text{minimality.}}{\subseteq} (I \times P)^{(r)} \times D^s$$

In fact, $\boxed{x \in {}_r j(f)^{-1}(S(Z, P, N)) \cap (I \times \mathrm{int}(P))^{(r)} \times D^s}$

for the following IASEM Z (not in our collection)

$$Z = \{ (z_1, \dots, z_r) \in {}_r J^\infty \mid (z_1, \dots, z_{r_1}) \in Z_{\alpha^1}, (z_{r_1+1}, \dots, z_{r_1+r_2}) \in Z_{\alpha^2}, \dots \}$$

then $c(Z) = \sum_{k=1}^{b} c(Z_{\alpha^k})$ and $\boxed{F(t_j, \rho_2 \pi_{i_j'} x, y) \in I \times Q_j \times D^s}$ $\left(i_j' := i_j + \sum_{k=1}^{\psi(j)-1} r_k, \quad 1 \le j \le a \right)$

$\Rightarrow$ for F in general position, this cannot happen :

$Z, A(Z), A^2(Z), \dots$ $\quad \Sigma :=$ transverse preimage of $S^*(A^h(Z), P, N) \subseteq (I \times \mathrm{int}(P))^{(r)} \times D^s$ $(h \ge 0)$

$$\dim \Sigma = s - c(A^h(Z)) \le s - c(Z) = s - \sum_{k=1}^{b} c(Z_{\alpha^k}) \le s - (n-p-2) < \sum_{j=1}^{a} (n - q_j - 1)$$

$\alpha^k \ge \alpha_0 \Rightarrow c(Z^k) \ge n-p-2$ $\qquad$ assumption.

$\therefore \quad I^a \times \Sigma \longrightarrow N^a$

$(s_1, \dots, s_a, x) \longmapsto \left(f(s_1, p_2 T_{i_1} x, p_2 x), \dots \right)$

generically has disjoint image from $Q_1 \times \dots \times Q_a$ since

$$\dim (I^a \times \Sigma) < \sum_{j=1}^{a} n - q_j = \operatorname{codim} (Q_1 \times \dots \times Q_a, N^a)$$

if $\Rightarrow x \in \Sigma$ for some large enough h (since eventually $A^h(Z) = \emptyset$)

but $\Rightarrow T(t_1, \dots, t_a, x) \in Q_1 \times \dots \times Q_a$ $\lightning$

) $(189)_{\alpha+1} \Rightarrow (189)_\alpha$, $0 \le \alpha \le \alpha_0$

Let $F : I \times P \times D^s \longrightarrow I \times N \times D^s$ satisfy $(189)_{\alpha+1}$ (and conditions on ∂D^s)

From last time,

$$W_\alpha(F) := \{ x \in \underset{(I\times\mathrm{int}(P))^{(\iota)}_\times D^s}{\overset{}{S_\alpha}}(F) \mid \forall \beta > \alpha,\ i \in I(Z_\alpha), j \in I(Z_\beta),\ \pi_i(x) \notin \pi_j S_\beta(F) \}$$

Refine this by introducing, for an admissible pair (D,R) of relations on $I(Z_\alpha)$,

$$W_\alpha^{(D,R)}(F) = \left\{ x \in W_\alpha(F) \;\middle|\; \begin{array}{l} \forall i,j \in I(Z_\alpha), \quad iDj \Leftrightarrow \pi_i x \text{ is below } \pi_j x \text{ in } I\times P\times D^s \\ iRj \Leftrightarrow F\pi_i x \text{ is below } F\pi_j x \text{ in } I\times N\times D^s \end{array} \right.$$

(choice of "ordering of points".)

<u>Lemma</u> $W_\alpha(F) = \coprod_{(D,R)} W_\alpha^{(D,R)}(F)$ and if Z_α is <u>good</u>, then for each $i \in I(Z_\alpha)$

$$p_{2,3}\pi_i : W_\alpha^{(D,R)}(F) \longrightarrow P \times D^s \text{ is an embedding}$$

and image is a <u>closed</u> subset of $(\mathrm{int}\,P)\times D^s \setminus p_{2,3} Y_{\alpha+1}$.

For each admissible pair of orderings (O,R) for Z_α (rank r), set

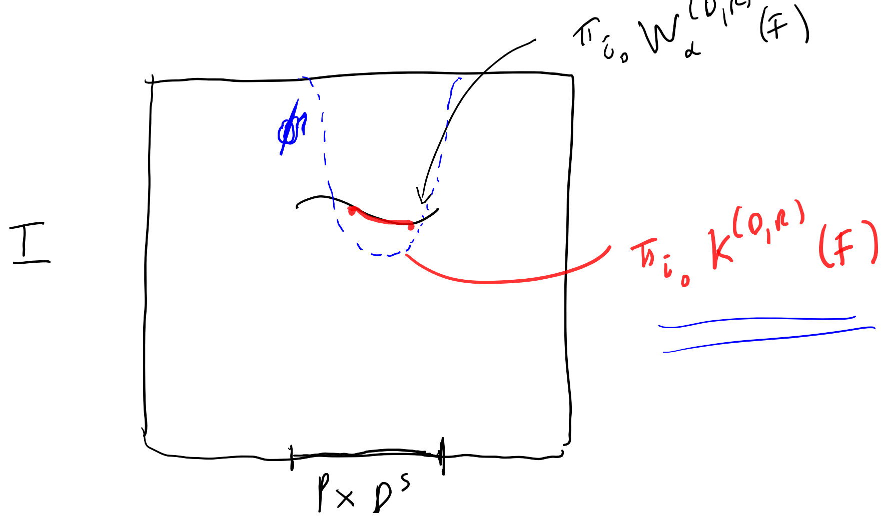
$$K^{(O,R)}(F) := \bigcup_{j=1}^{a} \bigcup_{i=1}^{r} \left\{ x \in W_\alpha^{(O,R)}(F) \;\middle|\; \begin{array}{l} p_3 \pi_i x \in \overline{O_j} \\ \text{for some } t \in I,\ f(t, p_{2,3}\pi_i x) \in Q_j \end{array} \right\}$$

if all $K^{(O,R)}(F) = \emptyset \implies (189)_\alpha$ holds.

<u>Claim 113</u> For each (O,R), there is an F' isotopic to F such that

$$K^{(O,R)}(F') = \emptyset$$

and if $K^{(O',R')}(F) = \emptyset$ for some $(O',R') \implies K^{(O',R')}(F') = \emptyset$.



i_0 maximal (exists by goodness)

Claim 116 $\exists$ smooth fibered sunny collapse ϕ

$$P \times D^s \xrightarrow{\phi^u} (0,1]$$

satisfying: $\quad \forall x \in K^{(D,R)}(F), \quad \phi^1(p_{2,3}\pi_{i_0} x) < p_1 \pi_{i_0} x$.

$\boxed{\text{Claim } 116 \Rightarrow \text{Claim } 113}$

$F \sim F^1$ through ϕ^u

F satisfies $(189)_{\alpha+1} \Rightarrow F^1$ also: $\pitchfork$ well behaved wrt sunny collapse and

$$x = ((t_1, x_1), \dots, (t_{r'}, x_{r'}), y) \in S_\beta(F^1) \subseteq (I \times \mathrm{int}(P))^{(r')} \times D^s$$

$$\Updownarrow$$

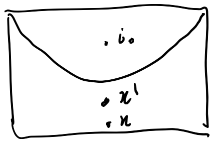
$$x' = (t_1 \phi^1(x_1, y), \dots, t_{r'} \phi^1(x_{r'}, y), y) \in S_\beta(F).$$

if $x \in K^{(D,R)}(F^1) \Rightarrow x' \in K^{(D,R)}(F)$

$$\Rightarrow \phi^1(p_{2,3}\pi_{i_0} x') < p_1 \pi_{i_0} x'$$
$$= p_1 \pi_{i_0} x \cdot \phi^1(p_{2,3}\pi_{i_0} x)$$
$$\leq \phi^1(p_{2,3}\pi_{i_0} x') \quad \lightning$$

since if $f^1(t, p_{2,3}\pi_i x) \in Q_j$ for some t and $p_3 \pi_i x \in \overline{\Theta}_j$.

$$\begin{gathered} f^1(t, p_{2,3}\pi_i x) \\ \| \\ f(t\phi^1(p_{2,3}\pi_i x), p_3 \pi_i x) \\ \shortparallel \\ (t', p_{23}\pi_i x') \end{gathered}$$



To prove Claim 116, do another downward induction on the statement: for $0 \leq \beta \leq \alpha$

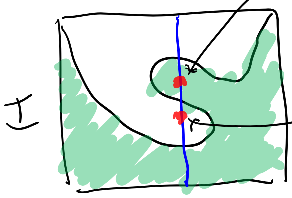
$(206)_\beta$ $\exists$ smooth homotopy $P \times D^s \xrightarrow{\phi_\beta^u} [0,1]$ satisfying condition of Claim 116

<u>and</u> which is a sunny collapse "rel β", i.e. satisfies sunny conditions (40), (41) for $p_{2,3} Y_\beta$

$$\boxed{(206)_0 \Rightarrow \text{Claim } 116}$$

<u>i.e.</u> sunny on $p_{2,3} Y_0 \Rightarrow$ sunny on $P \times D^s$.

Suppose (40) fails,



and so $((t,x),(\phi_u(x',y),x'),y)$

$\cap$

$_2j^\infty(f)^{-1} S(Z_0, P, N)$

$\Downarrow$

$(x',y) \in p_{2,3} Y_0$ $\lightning$

(or $x' \in \partial P \Rightarrow \phi_0^u(x',y) = 1$ $\lightning$)

Then
Ⓐ verify $(206)_\alpha$ (remember, the assumption is that $\mathcal{F}$ satisfies $(189)_{\alpha+1}$) for $0 \le \alpha \le \alpha_0$

Ⓑ show $(206)_{\beta+1} \Rightarrow (206)_\beta$.

Ⓐ use a partition of unity to construct

$$\phi : P_\alpha D^s \to [0,1]$$

satisfying sunny conditions for $p_{2\beta} Y_0$ and "above" $K^{(D,R)}(\mathcal{F})$.

(i_0 maximal, suppose $\pi_{i_0} x$ not critical pt)

Let $X_i^{(D',R')}$ for the image of $p_{2,3}\pi_i : W_\alpha^{(D',R')} \hookrightarrow P_\alpha D^s$.

$$X_i^{(D',R')} \subseteq (\mathrm{int}\, P) \times D^s \setminus p_{2,3} Y_{\alpha+1} \quad \underline{\text{closed}}.$$

$\underline{\underline{\text{and}}}$ $X_{i_0}^{(D,R)} \cap X_i^{(D',R')} = \emptyset$ unless they are equal.

$\underline{\text{Set}}$

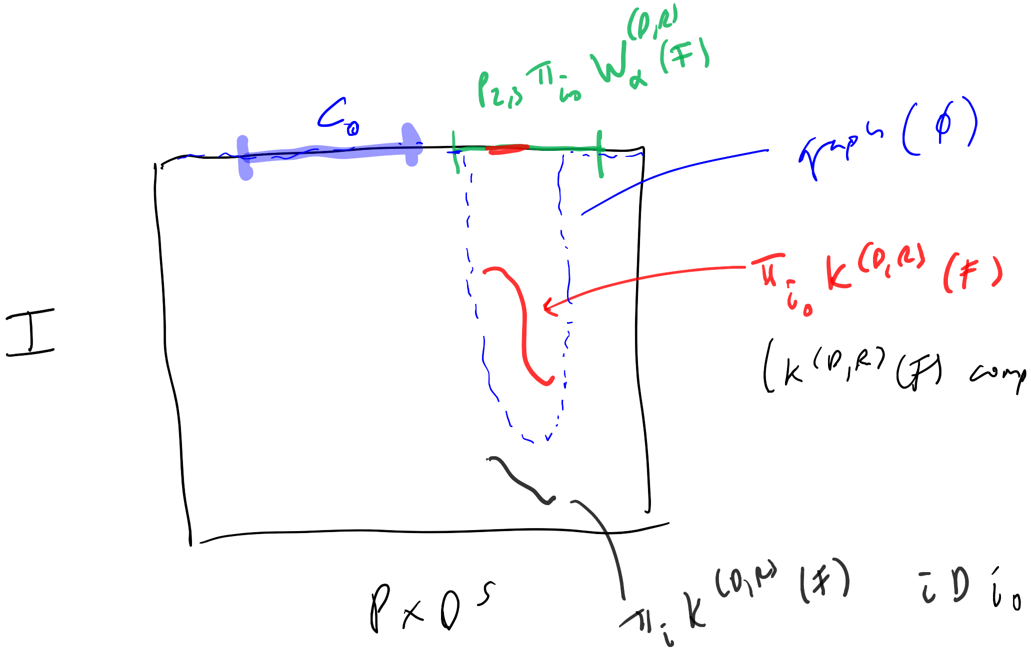
$$C_0 = \partial P \times D^s \cup p_{2,3} Y_{\alpha+1} \cup \bigcup_{X_i^{(D',R')} \neq X_{i_0}^{(D,R)}} X_i^{(D',R')} \subseteq P \times D^s \quad \text{closed}.$$

Pick $\quad \phi : P \times D^s \longrightarrow [0,1]$ satisfying

(i) $\phi \equiv 1$ near C_0

(ii) $\phi(p_{2,3}\pi_{i_0} x) > p_1 \pi_i x \quad \forall x \in W^{(D,R)}(\mathcal{F})$ with $i \triangleright i_0$

(iii) $\phi(p_{2,3}\pi_{i_0} x) < p_1 \pi_{i_0} x \quad \forall x \in K^{(D,R)}(\mathcal{F})$



$(K^{(D,R)}(\mathcal{F})$ compact$)$

$\pi_i K^{(D,R)}(\mathcal{F}) \quad i \triangleright i_0$

below by maximality of i_0.

(If $\pi_i x$ critical point need to be more careful...)

(ii) $p_1\pi_i x$ is in the shade of $\phi(p_{2,3}\pi_{i_0} x)$, $i \triangleright i_0$

(iii) $\pi_{i_0} K^{(D,R)}(\mathcal{F})$ is in the sun.

(B) I'm just skimmed through but the arguments are similar