MSRI Lectures

Anna Marie Bohmann, Alexander Kupers

Contents

CHAPTER 1

The Pontryagin–Thom construction

The goal of this talk is to describe a well-known and important theorem in geometric topology where spectra, the objects that stable homotopy is about, play a crucial role: the classification of manifolds up to cobordism in terms of Thom spectra. My favourite references for this are [Mil01] and [Swi02].

1. Stable framings

In the next section we will give the classification of stably framed manifolds up to cobordism; doing so will naturally lead us to spectra, in particular the sphere spectrum. Before we can do that, we need to explain what a stably framed manifold is.

Stably framed manifolds are manifolds with additional structure, which will simplify their classification up to cobordism. To define this structure, so we use a number of the tools in differential topology. A good reference for these is Wall's recent book [Wal16].

Every m-dimensional closed smooth manifold M has a tangent bundle TM, which is an m-dimensional vector bundle over the topological space M. This is one example of a vector bundle naturally associated to M. Another one, or rather something slightly weaker known as a stable vector bundle, will use the weak Whitney embedding theorem. This says that every compact smooth manifold can be smoothly embedded into some Euclidean space $\mathbb{R}^{m+d}$. Such an embedding $e\colon M \hookrightarrow \mathbb{R}^{m+d}$ of M into Euclidean space has a normal bundle, which we will define to be

$$\nu_e \coloneqq de(TM)^\perp,$$

the orthogonal complement of the subbundle $de(TM) \subset T\mathbb{R}^{m+d}|_M$ with respect to the restriction of the standard Riemannian metric. Let ϵ^d denote the d-dimensional trivial bundle over M.

Definition 1.1. A *normal framing* of $e\colon M \hookrightarrow \mathbb{R}^{m+d}$ is an isomorphism $\phi\colon \nu_e \xrightarrow{\simeq} \epsilon^d$ of d-dimensional vector bundle covering id_M.

The normal bundle by definition depends on the embedding $e\colon M \hookrightarrow \mathbb{R}^{m+d}$, and hence so does the notion of normal framing. However, we can make it independent of this choice once we allow homotopies and stabilisations.

Definition 1.2. A *stable normal framing* of M is an equivalence class of isomorphism $\varphi\colon \epsilon^n \oplus \nu_e \xrightarrow{\simeq} \epsilon^{d+n}$ of vector bundles covering id_M for some e and n, up to equivalence relation generated by homotopy and $\varphi \sim \mathrm{id}_\epsilon \oplus \varphi$

We refer to the operation replacing ν_e and φ with $\epsilon \oplus \nu_e$ and $\mathrm{id}_\epsilon \oplus \varphi$ as *stabilisation.* To see the notion of a stable normal framing is well-defined—that is, does not depend on e—we need to provide compatible homotopy classes of isomorphisms $[\phi_{ee'}]$ between the normal bundles of any two embeddings e and e', after stabilising enough times. Compatibility here means that after stabilising even further we have $[\phi_{ee'}] \circ [\phi_{e'e''}] = [\phi_{ee''}]$.

Proposition 1.3. *After stabilising enough times, there are preferred homotopy classes of isomorphisms between normal bundles of different embeddings, and these are all compatible.*

PROOF. Firstly, suppose that two embedding $e_0, e_1 \colon M \hookrightarrow \mathbb{R}^{m+d}$ are isotopic, in the sense that there is a smooth map

$$e_t \colon M \times [0,1] \longrightarrow \mathbb{R}^{m+d}$$

so that the map $(e_t, \pi_2) \colon M \times [0,1] \to \mathbb{R}^{m+d} \times [0,1]$ is a smooth embedding, Then this yields a preferred homotopy class of isomorphisms $[\psi]$ between the normal bundles of e_0 and e_1. This construction has the property that it takes concatenation of isotopies to composition of homotopy classes of isomorphisms.

Note that normal bundle of (e_t, π_2) is a vector bundle over $M \times [0,1]$, and recall that every vector bundle ξ over $M \times [0,1]$ is isomorphic to $\xi|_{M \times \{0\}} \times [0,1]$. If we impose this is the identity over $M \times \{0\}$ it is unique up to homotopy. This in turn induces an isomorphism between $\xi|_{M \times \{0\}}$ and $\xi|_{M \times \{1\}}$, unique up to homotopy.

Secondly, we observe that if we compose $e \colon M \hookrightarrow \mathbb{R}^d$ with the inclusion $\mathbb{R}^{m+d} \hookrightarrow \mathbb{R}^{m+d+1}$ on the last $m+d$ coordinates to get e', we get an identification $\nu_{e'} = \epsilon \oplus \nu_e$.

Thirdly, we use that if we are allowed to increase the dimension of the Euclidean space, not only are any two embeddings become isotopic but this isotopy is unique up to isotopy. Then any two embedding $e \colon M \to \mathbb{R}^{m+d}$ and $e' \colon M \times \mathbb{R}^{m+d}$ become isotopic after stabilising $m+d$ times. Indeed, the formal

$$\begin{aligned} M \times [0,1] &\longrightarrow \mathbb{R}^{m+d+m+d} \\ (m,t) &\longmapsto (1-t) \cdot e(m) + t \cdot e'(m) \end{aligned}$$

gives an isotopy from e composed with the inclusion of the first $m+d$ terms to e' composed with the inclusion of the last $m+d$ terms, and the latter differs by a rotation from e' composed with the inclusion of the first $m+d$ terms. A similar argument proves that any isotopy after stabilising $m+d$ is isotopic to the one just given after stabilising $2m+2d$ more times. □

Remark 1.4. There is a stronger version of the Whitney embedding theorem, which says that every compact m-dimensional manifold M can be embedded in $\mathbb{R}^{2m}$. There are also relative versions, which say that any two embeddings in $\mathbb{R}^{2m+1}$ are isotopic and in $\mathbb{R}^{2m+2}$ this isotopy is unique up to isotopy. One may use these facts to simplify the above argument.

Example 1.5.

- A point admits a stable normal framing, as any vector bundle over it is trivial.
- A sphere admits a stable normal framing, because the standard embedding of S^d into $\mathbb{R}^{d+1}$ has trivial normal bundle.
- A $\mathbb{R}P^2$ does not admit a stable normal framing. If it t did then some normal bundle for it would be trivial and from the isomorphism $\mathbb{R}^{2+d} \cong T\mathbb{R}P^2 \oplus \nu_e \cong T\mathbb{R}P^2 \oplus \epsilon^n$, we deduce that $T\mathbb{R}P^2$ is orientable, yielding a contradiction. This argument shows that no non-orientable manifold admits a stable normal framing.

The set of stable normal framings is a torsor for the group of isomorphisms of the trivial bundle up to homotopy and stabilising. To use the latter for computations, we need that linear isomorphisms for n large are given by the colimit $\mathrm{GL}(\mathbb{R}) := \mathrm{colim}_{n \to \infty} \mathrm{GL}_n(\mathbb{R})$ over the inclusion $\mathrm{GL}_n(\mathbb{R}) \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{R})$ in the bottom-right corner.

Example 1.6.

- The point $*$ has two stable normal framings. To see this, observe that homotopy classes of linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ for n large are given by $\pi_0 \operatorname{GL}(\mathbb{R}) \cong \{\pm 1\}$ using the normalised determinant.
- The circle S^1 has four stable normal framings. To see this, observe that homotopy classes of S^1-indexed families of linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ for n large are given by $\pi_0 \operatorname{Map}(S^1, \operatorname{GL}(\mathbb{R})) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.
- In general, if M admits a stable normal framings then the set of stable normal framings is in bijection with $\pi_0 \operatorname{Map}(M, \operatorname{GL}(\mathbb{R}))$.

2. Stably framed manifolds and the sphere spectrum

We now proceed by giving an invariant of stably framed manifolds, and then understand which equivalence relation to impose on these so that this invariant yields a classification. The sphere spectrum will appear naturally.

2.1. The Pontryagin–Thom construction. Recall that the normal bundle ν_e of an embedding $e\colon M \hookrightarrow \mathbb{R}^n$ was defined to be the subbundle $de(TM)^\perp \subseteq T\mathbb{R}^{m+d}|_M$. Now consider the map

$$\begin{aligned} \nu_e &\longrightarrow \mathbb{R}^{m+d} \\ v &\longmapsto p(v) + v, \end{aligned}$$

with $p\colon \nu_e \to M$ the projection, addition in $\mathbb{R}^{m+d}$, and the tangent spaces of $\mathbb{R}^{m+d}$ identified with $\mathbb{R}^{m+d}$ in the standard manner. Its derivative at the 0-section can be identified with the isomorphism $TM \oplus \nu_e \cong T\mathbb{R}^{m+d}|_M$, so by the inverse function theorem it is a diffeomorphism near the 0-section. Using compactness of M, one then proves that there exists a real number $r > 0$ such that the restriction

$$g\colon \{v \in \nu_e \mid ||v|| < r\} \longrightarrow \mathbb{R}^{m+d}$$

is a smooth embedding. Identifying the domain with ν_e via the homeomorphism $w \mapsto \frac{2rw}{1+||w||}$ we obtain a *tubular neighbourhood*: an embedding $g\colon \nu_e \hookrightarrow \mathbb{R}^n$ extending e whose derivative at the 0-section is the inclusion. Tubular neighbourhoods are unique up to isotopy; this is the reason for including the condition on the derivative.

Given a tubular neighbourhood g for $e\colon M \hookrightarrow \mathbb{R}^{m+d}$, we can collapse the complement of its image to a point to get a Thom collapse map whose target will be the Thom space $\operatorname{Th}(\nu_e)$:

Definition 2.1. Let ξ be a vector bundle over a compact space X, then the *Thom space* $\operatorname{Th}(\xi)$ is the one-point compactification of the total space of ξ.

Exercise 1. Prove that $\operatorname{Th}(\xi)$ is homeomorphic to both of the following two alternative constructions:

(i) Take the fibrewise one-point compactification of ξ and collapse the section at ∞ to a point.
(ii) Pick a Riemannian metric of ξ and take the quotient of the unit disc bundle by the sphere bundle.

Remark 2.2. Definition 2.1 is not the correct definition when X is not compact. Instead, one may use either of the two constructions in the previous exercise.

Identifying S^{m+d} with one-point compactification of $\mathbb{R}^{m+d}$, we have that $\mathrm{Th}(\nu_e) \cong S^{m+d}/(S^{m+d} - g(\nu_e))$ and we call the quotient map the collapse map. Explicitly, it is given by

$$\begin{aligned} S^{m+d} &\longrightarrow \mathrm{Th}(\nu_e) \\ x &\longmapsto \begin{cases} g^{-1}(x) & \text{if } x \in g(\nu_e), \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

If now M is stably framed, then by increasing d if necessary we may assume that there is an isomorphism $\varphi\colon \nu_e \xrightarrow{\simeq} \epsilon^d$ and use this isomorphism to map out of $\mathrm{Th}(\nu_e)$ via the map

$$\mathrm{Th}(\nu_e) \xrightarrow{\mathrm{Th}(\varphi^{-1})} \mathrm{Th}(\epsilon^d) = S^d \wedge M_+ \xrightarrow{\pi} S^d. \tag{2}$$

Composing (1) with (2) yields the *Pontryagin–Thom collapse map*. Explicitly, it is given by

$$\begin{aligned} PT(M,\varphi,e,g)\colon S^{m+d} &\longrightarrow S^d \\ x &\longmapsto \begin{cases} \pi\varphi^{-1}g^{-1}(x) & \text{if } x \in g(\nu_e), \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2.3. *The basepoint preserving homotopy class of $PT(M,\varphi,e,g)$ only depends on the homotopy class of φ and isotopy class of e.*

PROOF. The isomorphism φ only enters through $\mathrm{Th}(\varphi^{-1})$ in (2) and its basepoint preserving homotopy class—and hence $PT(M,\varphi,e,g)$—only depends on the homotopy class of φ. An isotopy of tubular neighbourhoods induces a basepoint preserving homotopy of (1), so $PT(M,\varphi,e,g)$ is independent of g. The isotopy extension theorem implies that we can move a tubular neighbourhood along with an isotopy, inducing a homotopy of (1), so $PT(M,\varphi,e,g)$ only depends on the isotopy class of e. □

Thus we have a well-defined element $[PT(M,\varphi,e)] \in \pi_{m+d}(S^d)$. What do we need to do make it independent of e? Given that embeddings are stably unique up to isotopy, it is a good idea to stabilise. If in the construction of the Pontryagin–Thom collapse map we replace $e\colon M \hookrightarrow \mathbb{R}^{m+d}$ with $e'\colon M \hookrightarrow \mathbb{R}^{m+d+1}$ obtained by composing e with the inclusion on the last factors, then we get a canonical isomorphism $\nu_{e'} \cong \epsilon \oplus \nu_e$. Using $g' = \mathrm{id}_\epsilon \oplus g$ and $\varphi' = \mathrm{id}_\epsilon \oplus \varphi$, we get that $g'(\nu_{e'}) = \mathbb{R} \times \nu_e$ and that on the (s,x) in this subset of $\mathbb{R}^{n+d+1} \cong \mathbb{R} \times \mathbb{R}^{n+d}$ the Pontryagin–Thom collapse map takes the value $(s, \pi_2\varphi^{-1}g^{-1}(x))$. We conclude that $[PT(M,\varphi,e')] \in \pi_{m+d+1}(S^{d+1})$ is obtained by taking the suspension:

$$S^{m+d+1} \cong \Sigma S^{m+d} \xrightarrow{\Sigma PT(M,\varphi,e)} \Sigma S^d \cong S^{d+1}.$$

What we have shown is that to each stably framed manifold (M,φ) we can assign a well-defined element $[PT(M,\varphi)] \in \mathrm{colim}_{d\to\infty}\pi_{m+d}(S^d)$. Here the colimit is taken over suspensions, and this colimit can be computed by taking the quotient of $\bigoplus_d \pi_{m+d}(S^d)$ by the equivalence relation generated by $\varphi \sim \Sigma\varphi$. This only depends on the isomorphism class of (M,φ) in the following sense: any diffeomorphism $\rho\colon M \to M'$ induces an isomorphism $\nu_\rho\colon \nu_M \to \nu_{M'}$, and we define an *isomorphism of stably framed manifolds* to be a diffeomorphism such that $\varphi' \circ \nu_\rho$ is homotopic to φ.

In other words, we have defined an invariant

$$\mathrm{PT}\colon \frac{\{\text{stably framed } m\text{-dimensional manifolds}\}}{\text{isomorphism}} \longrightarrow \operatorname*{colim}_{d\to\infty} \pi_{m+d}(S^d)$$

known as the *Pontryagin–Thom construction*.

2.2. A glimpse of spectra. Let us put the *stable homotopy group* $\operatorname{colim}_{d\to\infty}\pi_{m+d}(S^d)$ is a more general context.

Definition 2.4. A *pre-spectrum* E is a sequence $\{E_n\}_{n\geqslant 0}$ of based spaces with basepoint preserving maps $\Sigma E_n \to E_{n+1}$.

Remark 2.5. Note that equivalently, we can exchange the suspension on the domain for a based loops on the target. Thus an equivalent definition of a pre-spectrum is as a sequence $\{E_n\}_{n\geqslant 0}$ of based spaces with basepoint preserving maps $E_n \to \Omega E_{n+1}$. This perspective will appear naturally in the next lecture, but for now using suspensions is more convenient.

The following is the example relevant to this discussion.

Definition 2.6. The *sphere spectrum* $\mathbb{S}$ is the pre-spectrum $\{S^n\}_{n\geqslant 0}$ with maps $S^n \to \Omega S^{d+1}$ as above.

Given a pre-spectrum E we can define its stable homotopy groups as

$$\pi_m(E) \coloneqq \operatorname*{colim}_{n\to\infty} \pi_{m+n}(E_n) \qquad \text{for } m \in \mathbb{Z}.$$

Then we can rephrase the Pontryagin–Thom construction as an invariant

$$\mathrm{PT}\colon \frac{\{\text{stably framed } m\text{-dimensional manifolds}\}}{\text{isomorphism}} \longrightarrow \pi_m(\mathbb{S}).$$

The target is very difficult to compute, and we only know a bit more than the first 100 groups. Here are some examples:

$$\pi_0(\mathbb{S}) \cong \mathbb{Z}, \quad \pi_1(\mathbb{S}) \cong \mathbb{Z}/2, \quad \pi_2(\mathbb{S}) \cong ....$$

There are no non-zero negative homotopy groups in this case because $\pi_*(S^n) = 0$ for $* < n$.

2.3. The Pontryagin–Thom isomorphism. The Pontryagin–Thom construction as stated is *not* a bijection. One way to see this, is to observe that there are infinitely many isomorphism classes of stably framed 0-manifolds—one can take finite disjoint unions of stably framed points—but only two elements in $\pi_0(\mathbb{S})$. A second way to see this, is to note that disjoint union makes the domain into a commutative monoid with identity give by the empty manifold, so that there are few inverses. However, PT preserves addition and $\pi_*(\mathbb{S})$ is an abelian group. This raises the following question: is it possible to take a further quotient of the domain of PT so that it becomes an isomorphism of abelian groups?

To see the answer, we will attempt to construct an inverse. This will show PT is surjective and the failure of injectivity comes from a choice we need to make; understanding the effect of this choice will tell us which quotient we must take. Observe that $M = PT(M,\varphi,e,g)^{-1}(0) \subset \mathbb{R}^{m+d} \subset S^{m+d}$. Moreover, the map $PT(M,\varphi,e,g)$ is smooth on the preimage of a neighbourhood of 0 and this restriction is transverse to 0. The latter implies that $PT(M,\varphi,e,g)$ induces an isomorphism between the normal bundle of e and the pullback along $PT(M,\varphi,e,g)$ of the normal bundle of a point in $\mathbb{R}^d$. Since this normal bundle is trivial this is a normal framing of M, which represents the stable normal framing f.

To prove surjectivity of PT, we try to repeat this for a general element of $\pi_m(\mathbb{S})$. First, we represent it by a map $h\colon S^{m+d} \to S^d$ for some large $d \in \mathbb{N}$. Second, we make it smooth on the preimage of a neighbourhood of 0. Third, we make it transverse to 0. Having done so, we obtain a m-dimensional compact manifold $h^{-1}(0)$ together with a normal framing. We made two choices here: (1) an integer d and (2) a particularly nice representative h. The first choice matters little; if we replace h by Σh the preimage of 0 remains the same and the normal framing gets replaced by its sum with id_ϵ. The second choice is much more important. If we have two homotopic nice representatives $h_0, h_1 \colon S^{m+d} \to S^d$, we can modify the entire homotopy $h_t \colon S^{m+d} \times [0,1] \to S^d$ to be smooth on the preimage of a neighbourhood of 0 and transverse to it. Then the preimage $h_t^{-1}(0) \subset S^{m+d} \times [0,1]$ will be a $(m+1)$-dimensional compact submanifold with boundary and normal framing. In other words, it is a stably framed cobordism between (M_0, φ_0) and (M_1, φ_1). This turns out to be appropriate equivalence relation on the domain of PT. The following is originally due to Thom [Tho54].

THEOREM 2.7 (Pontryagin–Thom). *There is a well-defined induced map*

$$\overline{\mathrm{PT}} \colon \Omega_m^{\mathrm{fr}} := \frac{\{\textit{stably framed } m\textit{-dimensional manifolds}\}}{\textit{stably framed cobordism}} \longrightarrow \pi_m(\mathbb{S})$$

which is an isomorphism of abelian groups.

Example 2.8. Using the classification of stably framed 0- and 1-dimensional manifolds we can actually compute $\pi_0(\mathbb{S})$ and $\pi_1(\mathbb{S})$. We can considering the cylinder $M \times [0,1]$ as a cobordism from $M \sqcup M$ to $\varnothing$ instead of a cobordism from M to M. Keeping track of stably framings, this saying that $-(M, f) = (M, -\varphi)$ where $-\varphi$ is φ composed with the non-orientation preserving map $\epsilon \to \epsilon$ of stable vector bundles. (As a hint: imagine bending $\mathbb{R}^{m+d} \times [0,1]$ into a U-shape to embed it into $\mathbb{R}^{m+d} \times [0,\infty)$ and compare the two resulting embeddings $\mathbb{R}^{m+d} \hookrightarrow \mathbb{R}^{m+d}$.)

Looking at 0-dimensional manifolds, we have any 0-dimensional stably framed manifolds is a disjoint union of m copies of $(*, \mathrm{std})$ and m' copies of $(*, -\mathrm{std})$. As $(*, -\mathrm{std}) = -(*, \mathrm{std})$, we see sending this $m - m'$ gives an isomorphism between this groups of stably framed cobordism classes and the integers $\mathbb{Z}$. Using the Pontryagin–Thom isomorphism, we compute that $\pi_0(\mathbb{S}) \cong \mathbb{Z}$.

We can similar use the classification of 1-dimensional stably framed manifolds to recoevr $\pi_1(\mathbb{S}) = \mathbb{Z}/2$. The reason we get $\mathbb{Z}$ rather than $\mathbb{Z}/2$ is that there is a pair of pants cobordism; writing (S^1, std) from the stable normal framing induced from the standard inclusion $S^1 \hookrightarrow \mathbb{R}^2$ and the (S^1, Lie) for the other, this gives $(S^1, \mathrm{std}) = 2(S^1, \mathrm{std})$ and $(S^1, \mathrm{Lie}) = (S^1, \mathrm{Lie}) + (S^1, \mathrm{std})$.

EXERCISE 2. Understand the details of the computation Ω_1^{fr} outlined at the end of the previous example.

EXERCISE 3. Use the strong Whitney embedding theorem to deduce the Freudenthal suspension theorem, which says that $\pi_{m+d}(S^d) \to \pi_{m+d+1}(S^{d+1})$ is an isomorphism for $d \geqslant m + 2$.

3. Stable tangential structures and Thom spectra

Next we generalise this to other types of manifolds, in particular ordinary manifolds and almost-complex manifolds. This leads to Thom spectra.

3.1. The Pontryagin–Thom construction for ordinary manifolds. If M is an ordinary manifold, it does not come with a stable normal framing and we can still define (1) but no longer (2). Thinking of the sphere S^d as the Thom space of the trivial d-dimensional bundle over $*$ suggests that the appropriate replacement is a different Thom space.

Let $\mathrm{Gr}_d(\mathbb{R}^{m+d})$ be the Grassmannian of d-planes in $\mathbb{R}^{m+d}$. This is homeomorphic to the quotient of the Stiefel manifold $V_d(\mathbb{R}^{m+d}) := \mathrm{GL}_{m+d}(\mathbb{R})/\mathrm{GL}_m(\mathbb{R})$ of parametrised d-planes in $\mathbb{R}^{m+d}$ by the action of $\mathrm{GL}_d(\mathbb{R})$. The Grassmannian carries a canonical $\mathbb{R}^d$-bundle $\gamma_{d,m}$ whose fibre over d-plane V is V itself, and in terms of the Stiefel manifolds this is

$V_d(\mathbb{R}^{m+d}) \times_{\mathrm{GL}_d(\mathbb{R})} \mathbb{R}^d$. The inclusion of $\mathbb{R}^{m+d}$ into $\mathbb{R}^{m+d+1}$ onto the first $m+d$ components induces an inclusion $\mathrm{Gr}_d(\mathbb{R}^{m+d}) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^{m+d+1})$ and taking the colimit over these we obtain the infinite Grassmannian $\mathrm{Gr}_d(\mathbb{R}^{\infty})$ with d-dimensional vector bundle $\gamma_{d,\infty}$.

There is a map of vector bundles from ν_e to $\gamma_{d,m}$ given on base spaces by sending m to ν_e and on fibres by the identity. This induces a map of Thom spaces

$$\mathrm{Th}(\nu_e) \longrightarrow \mathrm{Th}(\gamma_{d,m}) \longrightarrow \mathrm{Th}(\gamma_{d,\infty}). \tag{3}$$

Composing (1) with (3) then gives a map

$$PT(M,e,g)\colon S^{d+m} \longrightarrow \mathrm{Th}(\gamma_{d,\infty}).$$

Under stabilisation of the embedding we get that $PT(M,e')$ is the map

$$S^{d+m+1} = \Sigma S^{d+m} \xrightarrow{\Sigma PT(M,e)} \Sigma\mathrm{Th}(\gamma_{d,\infty}) \longrightarrow \mathrm{Th}(\gamma_{d+1,\infty}).$$

Here right map is induced by noting that under the inclusion $\epsilon \oplus -\colon \mathrm{Gr}_d(\mathbb{R}^{m+d}) \to \mathrm{Gr}_{d+1}(\mathbb{R}^{m+d+1})$ the vector bundle $\gamma_{m,d+1}$ restricts to $\epsilon \oplus \gamma_{m,d}$, that $\mathrm{Th}(\epsilon \oplus \xi) \cong \Sigma\mathrm{Th}(\xi)$, and letting $m \to \infty$. We recognise a pre-spectrum.

Definition 3.1. The *Thom spectrum* MO is the pre-spectrum $\{\mathrm{Th}(\gamma_{d,\infty})\}_{n\geqslant 0}$ with maps $\Sigma\mathrm{Th}(\gamma_{d,\infty}) \to \mathrm{Th}(\gamma_{d+1,\infty})$ as above.

Continuing the argument as in the stably framed case, one obtains an isomorphism of abelian groups

$$\overline{\mathrm{PT}}\colon \Omega_m^{\mathrm{O}} := \frac{\{m\text{-dimensional manifolds}\}}{\text{cobordism}} \longrightarrow \pi_m(\mathrm{MO}).$$

The target was computed by Thom, and is given by

$$\pi_*(MO) = \mathbb{F}_2[x_i \mid i \neq 2^k - 1].$$

3.2. The Pontryagin–Thom construction in general. One can repeat the story for stably framed and ordinary manifolds for other structures on manifold. Obvious choices are orientations or spin structures, yielding Thom spectra MSO and MSpin figuring in isomorphisms

$$\Omega_m^{\mathrm{SO}} := \frac{\{\text{oriented } m\text{-dimensional manifolds}\}}{\text{cobordism}} \xrightarrow{\cong} \pi_m(\mathrm{MSO})$$
$$\Omega_m^{\mathrm{Spin}} := \frac{\{\text{spin } m\text{-dimensional manifolds}\}}{\text{cobordism}} \xrightarrow{\cong} \pi_m(\mathrm{MSpin}).$$

However, for algebraic topology almost-complex structures are of crucial importance, especially after work of Quillen relating it formal group laws [**Qui69**]; this is the starting point of chromatic homotopy theory [**Rav86**]. The reason is that the homotopy groups of its Thom spectrum MU are very easy to describe. That is, the target of the Pontryagin–Thom isomorphism

$$\Omega_m^{\mathrm{U}} := \frac{\{\text{almost-complex } m\text{-dimensional manifolds}\}}{\text{cobordism}} \xrightarrow{\cong} \pi_m(\mathrm{MU})$$

is given by $\pi_*(MU) = \mathbb{Z}[y_{2i} \mid i > 0]$.

CHAPTER 2

Atiyah duality and Poincaré duality

1. Duality in symmetric monoidal categories

A slogan is that spectra are a more algebraic replacement of spaces. In particular, we should be able to apply our experience with the category of vector spaces to the stable homotopy category. In this lecture we do so to discover a notion of duality for spectra, and deduce from this two classical duality theorems: Alexander duality and Poincaré duality. Our exposition is inspired by the elegant short paper of Dold and Puppe [**DP83**].

1.1. Duals in linear algebra. A useful concept in linear algebra over a field $\mathbb{F}$ is the *linear dual* vector space

$$DV := \mathrm{Lin}(V, \mathbb{F}),$$

whose elements are the linear functionals on V. The linear dual has the following universal property, using the Hom-$\otimes$ adjunction: there is a natural bijection

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Vect}_\mathbb{F}}(W, DV) = \mathrm{Hom}_{\mathsf{Vect}_\mathbb{F}}(W, \mathrm{Lin}(V,\mathbb{F})) &\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Vect}_\mathbb{F}}(V \otimes W, \mathbb{F}) \\ \alpha &\longmapsto \Big(v \otimes w \longmapsto \alpha(w)(v)\Big). \end{aligned} \tag{4}$$

Then taking $W = DV$, the identity on the left goes to the evaluation map

$$\begin{aligned} \mathrm{ev}\colon V \otimes DV &\longrightarrow \mathbb{F} \\ (v, \phi) &\longmapsto \phi(v). \end{aligned}$$

In terms of this evaluation map, the bijection (4) is given by $\alpha \mapsto \mathrm{ev} \circ (\mathrm{id}_V \otimes \alpha)$.

If V is finite-dimensional there is also a coevaluation map

$$\begin{aligned} \mathrm{coev}\colon \mathbb{F} &\longrightarrow DV \otimes V \\ 1 &\longmapsto \sum_i De_i \otimes e_i \end{aligned}$$

with $\{e_i\}_{1 \leqslant i \leqslant \dim V}$ a basis of V and $\{De_i\}_{1 \leqslant i \leqslant \dim V}$ the dual basis characterised by $De_j(e_i) = \delta_{ij}$ with δ_{ij} the Kronecker function.

Exercise 4. Prove that this is independent of the choice of basis.

The evaluation and coevaluation maps satisfy the triangle equations, which say that both of the following compositions are the identity linear maps

$$\begin{aligned} V \cong V \otimes \mathbb{F} \xrightarrow{\mathrm{id}_V \otimes \mathrm{coev}} V \otimes DV \otimes V \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_V} \mathbb{F} \otimes V \cong V, \\ DV \cong \mathbb{F} \otimes DV \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_{DV}} DV \otimes V \otimes DV \xrightarrow{\mathrm{id}_{DV} \otimes \mathrm{ev}} DV \otimes \mathbb{F} \cong DV. \end{aligned} \tag{5}$$

In the language of category theory, the triangle equalities (5) say that the functor $V \otimes -\colon \mathsf{Vect}_\mathbb{F} \to \mathsf{Vect}_\mathbb{F}$ is the left adjoint of the functor $DV \otimes -\colon \mathsf{Vect}_\mathbb{F} \to \mathsf{Vect}_\mathbb{F}$. See e.g. [**Mac71**, Chapter IV] for details about adjunctions. Using an alternative criterion for two functors to be adjoint, this is equivalent to the map

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Vect}_{\mathbb{F}}}(W, DV \otimes U) &\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Vect}_{\mathbb{F}}}(V \otimes W, U) \\ \beta &\longmapsto (\mathrm{ev} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes \beta) \end{aligned} \tag{6}$$

being a natural bijection, visibly extending (4). As should be clear from our formulation, the map in (6) always exists: the coevaluation is used in its inverse

$$\gamma \longmapsto (\mathrm{id}_{DV} \otimes \gamma) \circ (\mathrm{coev} \otimes \mathrm{id}_W),$$

and hence requires finite-dimensionality. At this point we have proven that (5) $\iff$ (6). Moreover, since adjoints are unique up to natural isomorphism, either determines DV up to isomorphism.

The relationship between coevaluation and finite-dimensionality becomes clearer once we recall that V is finite-dimensional if and only if $V \to D(DV)$ is an isomorphism. This map is produced by using the symmetry of the tensor product to get from the evaluation map $\mathrm{ev}\colon V \otimes DV \to \mathbb{F}$ a linear map $DV \otimes V \to \mathbb{F}$ and in turn using (4) to construct from this a linear map $V \to D(DV)$. Moreover, we can construct a map $DV \otimes DW \to D(W \otimes V)$ from the pair of evaluations $W \otimes V \otimes DV \otimes DW \to W \otimes DW \to \mathbb{F}$.

Exercise 5. Prove that (5) $\iff$ (6) $\iff$ the maps $V \to D(DV)$ and $V \otimes DV \to D(DV) \otimes DV \to D(DV \otimes V)$ are isomorphisms.

1.2. Duals in symmetric monoidal categories. The relationship between a vector space V and its linear dual DV only involves concepts that make sense in any symmetric monoidal category, and hence can be made into a definition there. Recall that a symmetric monoidal category $(\mathsf{C}, \otimes, \mathbb{1})$ is a category C with functor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ with natural associativity, commutativity, and unit isomorphisms; for example, the latter are natural isomorphism $\mathbb{1} \otimes A \xrightarrow{\simeq} A$ and $A \otimes \mathbb{1} \xrightarrow{\simeq} A$. For the details, see [**Mac71**, Chapter XI].

Example 1.1.
- $\mathsf{Vect}_{\mathbb{F}}$ is symmetric monoidal with $\otimes$ the tensor product of vector spaces and $\mathbb{1} = \mathbb{F}$.
- Top is symmetric monoidal with $\times$ the product and $\mathbb{1} = *$.
- SHC is symmetric monoidal with $\otimes$ the smash product of spectra and $\mathbb{1} = \mathbb{S}$.

Definition 1.2. Let A be an object of a symmetric monoidal category $(\mathsf{C}, \otimes, \mathbb{1})$, then a *dual of* A is an object B together with a pair of morphisms $\mathrm{ev}\colon A \otimes B \to \mathbb{1}$ and $\mathrm{coev}\colon \mathbb{1} \to B \otimes A$ with the property that both compositions

$$A \cong A \otimes \mathbb{1} \xrightarrow{\mathrm{id}_A \otimes \mathrm{coev}} A \otimes B \otimes A \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_A} \mathbb{1} \otimes A \cong A,$$

$$B \cong \mathbb{1} \otimes B \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_B} B \otimes A \otimes B \xrightarrow{\mathrm{id}_B \otimes \mathrm{ev}} B \otimes \mathbb{1} \cong B$$

are the identity morphisms.

Moreover, the arguments used in the previous section to construct various maps and prove equivalences between various properties of these maps, go through in any symmetric monoidal category. Thus we have:

Lemma 1.3. *The following are equivalent:*
1. *B is the dual of A.*
2. *There is a natural bijection* $\mathrm{Hom}_{\mathsf{C}}(A \otimes C, E) \cong \mathrm{Hom}_{\mathsf{C}}(C, B \otimes E)$.

Example 1.4. From Lemma 1.3 (2) it is clear that the dual of $\mathbb{1}$ is $\mathbb{1}$.

Suppose that C next that closed symmetric monoidal, that is, we have an internal mapping object functors $\mathrm{Map}(A,-)$ which are left adjoint to $A\otimes -$.

Example 1.5.

- $\mathsf{Vect}_{\mathbb{F}}$ is closed symmetric monoidal with internal mapping objects given by vector spaces of linear maps $\mathrm{Hom}(V,-)$.
- SHC is closed symmetric monoidal with internal mapping object given by function spectra $F(X,-)$.

Then taking $E=\mathbb{1}$ in Lemma 1.3 (2) shows that if A has a dual, it must be given by $\mathrm{Map}(A,\mathbb{1})$ which we abbreviate as

$$DA := \mathrm{Map}(A,\mathbb{1}).$$

Then we have a natural isomorphism

$$\mathrm{Map}(A,E)\cong DA\otimes E \tag{7}$$

as well as a generalisation of Exercise 5:

Lemma 1.6. *The two conditions in Lemma 1.3 are equivalent to the maps $A\to D(DA)$ and $A\otimes DA\to D(DA\otimes A)$ being isomorphisms.*

Definition 1.7. If DA is the dual of A we say that A is *dualisable*.

Exercise 6. Prove that if A and B are dualisable then so is $A\otimes B$. What is its dual?

Exercise 7. Prove that if C is closed then $\otimes$ preserves colimits in each entry. Suppose that C has biproducts $A\oplus B$ (coproducts which coincide with products). Prove that if A and B are dualisable then so is $A\oplus B$. What is its dual?

1.3. Duals in stable homotopy theory. As explained in Lecture 3, the stable homotopy category SHC has all structure required to make sense of duality as explained above: it is closed symmetric monoidal, with tensor product given by the smash product, unit given by the sphere spectrum, and internal mapping objects given by function spectra. Thus there is a theory of duality in spectra, which is known as *Spanier–Whitehead duality* [**SW55**]. Two questions now arise: What are examples of spectra that have duals? What use is Spanier–Whitehead duality? We start with some exercises:

Exercise 8. Prove that the dual of $\mathbb{S}$ is $\mathbb{S}$ itself.

Exercise 9. Prove that if X is dualisable with dual Y then so is $\Sigma^n X$ with dual $\Sigma^{-n}Y$.

Exercise 10. Prove that if X and X' are dualisable with duals Y and Y' then so is $X\otimes X'$ with dual $Y\otimes Y'$.

Exercise 11. Prove that if X and X' are dualisable with duals Y and Y' then so is $X\vee X'$ with dual $Y\vee Y'$.

2. The dual of the suspension spectrum of a finite complex

Our first goal is will be to show that for any finite CW-complex X, its suspension spectrum $\Sigma^\infty_+ X$ is dualisable. We will do so by constructing a candidate for its dual as well as evaluation and coevaluation maps, and verifying the triangle identities.

Using an argument similar to that for the weak Whitney embedding theorem, by induction over the number of cells one proves that every finite CW-complex can be embedded into some Euclidean space $\mathbb{R}^N$. We may assume for convenience that its image is contained in open unit ball $B \subset \mathbb{R}^N$. Recalling that the mapping cone of a pair (A, B) of a $B \subseteq A$ is given by

$$\mathrm{Cone}(A, B) = (A \times \{1\}) \cup (B \times [0, 1])/(B \times \{0\}),$$

which canonically pointed by the image of $B \times \{0\}$. If $A \hookrightarrow B$ is a cofibration (e.g. the inclusion of a subcomplex of a CW-complex), then the map $\mathrm{Cone}(A, B) \to B/A$ is a homotopy equivalence; you should think of the mapping cone as a more well-behaved replacement of the quotient. We take the mapping cone of the pair $(\mathbb{R}^N, \mathbb{R}^N \backslash X)$, take the suspension spectrum of the corresponding pointed space, and desuspend it N times to get the spectrum

$$\Sigma^{\infty - N}\mathrm{Cone}(\mathbb{R}^N \backslash X \hookrightarrow \mathbb{R}^N).$$

Since $\mathbb{R}^N$ is contractible, this is homotopy equivalent to $\Sigma^{\infty-N+1}(\mathbb{R}^N \backslash X)_+$ but it is better not to use this.

In the proof of the next theorem we will use that $\mathrm{Cone}(-)$ sends homotopy equivalences of pairs to based homotopy equivalences, sends products $(A, B) \times (A, B) = (A \times A', B \times A' \cup A \times B')$ of pairs to wedge products of pointed spaces, and satisfies a version of excision. These require minor point-set topological assumptions; we will not spell these out but see [**DP83**].

Theorem 2.1. *The spectrum* $\Sigma^{\infty-N}\mathrm{Cone}(\mathbb{R}^N \backslash X \hookrightarrow \mathbb{R}^N)$ *is the dual of* $\Sigma^\infty X_+$.

Proof. To construct an evaluation map, it suffices to construct a map

$$X_+ \wedge \mathrm{Cone}(\mathbb{R}^N \backslash X \hookrightarrow \mathbb{R}^N) \longrightarrow S^N$$

of pointed spaces; taking suspension spectra and desuspending N times gives the desired map of spectra. Identifying S^N, up to based homotopy equivalence, with the mapping cone of the pair $(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$, we define the map of pairs

$$\begin{aligned} S \colon (X, \varnothing) \times (\mathbb{R}^N, \mathbb{R}^N \backslash X) &\longrightarrow (\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) \\ (x, y) &\longmapsto y - x \end{aligned}$$

and take mapping cones.

For the coevaluation map we pick an open neighbourhood V of X with retraction $r \colon V \to X$, which exists since finite CW-complexes are ANR's. Then it suffices to construct a map

$$S^N \longrightarrow \mathrm{Cone}(\mathbb{R}^N \backslash X \hookrightarrow \mathbb{R}^N) \wedge X_+,$$

up to inverting some based homotopy equivalences (as we already implicitly did above). This is given by the induced map on mapping cones of the zigzag of maps of pairs

$$\begin{array}{ccc}
(\mathbb{R}^N, \mathbb{R}^N \backslash \{0\}) & & (\mathbb{R}^N, \mathbb{R}^N \backslash X) \times (X, \varnothing) \\
{\scriptstyle \simeq}\uparrow & & \| \\
(\mathbb{R}^N, \mathbb{R}^N \backslash B) & & (\mathbb{R}^N, \mathbb{R}^N \backslash X) \times (X, \varnothing) \\
\downarrow & & {\scriptstyle \simeq}\uparrow \\
(\mathbb{R}^N, \mathbb{R}^N \backslash X) & & (\mathbb{R}^N \times X, (\mathbb{R}^N \backslash X) \times X) \\
{\scriptstyle \simeq}\uparrow & & {\scriptstyle \mathrm{id} \times r}\uparrow \\
(V, V \backslash X) & \xrightarrow{\ \Delta\ } & (\mathbb{R}^N \times V, (\mathbb{R}^N \backslash X) \times V)
\end{array}$$

where $\simeq$ denote maps that induce based homotopy equivalences on mapping cones, and $\Delta(v) = (v, v)$ is the diagonal map.

We will prove one of the triangle identities, verifying that $(\mathrm{ev}\wedge\mathrm{id})\circ(\mathrm{id}\wedge\mathrm{coev})\colon \Sigma^\infty X_+ \to \Sigma^\infty X_+$ is the identity. It suffices to prove that the zigzag of maps of pairs

$$\begin{array}{ccc}
(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash\{0\}) & & (X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash\{0\}) \\
\| & & \cong\uparrow \\
(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash\{0\}) & & (\mathbb{R}^N,\mathbb{R}^N\backslash\{0\})\times(X,\varnothing) \\
\simeq\uparrow & & {\scriptstyle S\times\mathrm{id}}\uparrow \\
(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash B) & & (X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash X)\times(X,\varnothing) \\
\downarrow & & \simeq\uparrow \\
(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash X) & & (X,\varnothing)\times(\mathbb{R}^N\times X,(\mathbb{R}^N\backslash X)\times X) \\
\simeq\uparrow & & {\scriptstyle \mathrm{id}\times r}\uparrow \\
(X,\varnothing)\times(V,V\backslash X) & \xrightarrow{\Delta} & (X,\varnothing)\times(\mathbb{R}^N\times V,(\mathbb{R}^N\backslash X)\times V)
\end{array}$$

induces a map on mapping cones that is based homotopic to the identity map $S^N\wedge X_+ \to S^N\wedge X_+$. The composed map $(X,\varnothing)\times(V,V\backslash X)\to(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash\{0\})$ is given by $(x,v)\mapsto(r(v),v-x)$ from the bottom-left to the top-right, and it suffices to prove it is homotopic to (x,v).

To do so, we first show that we may replace it by $(x,v-x)$. This follows from the commutative diagram

$$\begin{array}{ccccc}
(X,\varnothing)\times(V,V\backslash X) & \xrightarrow{\subset} & (V\times V,V\times V\backslash\Delta) & \xleftarrow{\supset} & (W,W\backslash\Delta) \\
& {\scriptstyle (x,v)\mapsto(v-x,v)}\searrow & \downarrow{\scriptstyle (v',v)\mapsto(v-v',v)} & & \\
& & (\mathbb{R}^N,\mathbb{R}^N\times\{0\})\times(V,\varnothing) & &
\end{array}$$

with $W\subset V\times V$ the subspace of (v',v) such that the line segment $\{tv+(1-t)v' \mid t\in[0,1]\}\subset V$. The top-left map induces a based homotopy equivalence on mapping cones, but on its image the middle vertical map is homotopic to $(v',v)\mapsto(v-v',v')$ by linear interpolation in the second coordinate.

Making this replacement, the resulting composed map extends to all of $(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash X)$ and hence by precomposition to the map $(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash B)\to(X,\varnothing)\times(\mathbb{R}^N,\mathbb{R}^N\backslash\{0\})$ given by $(x,y)\mapsto(x,y-x)$. This is homotopic to the swap map by linear interpoliation in the second coordinate, using that $X\subset B$. ☐

EXERCISE 12. Prove the other triangle identity.

Example 2.2. If $X=S^q\subset\mathbb{R}^{q+1}\subset\mathbb{R}^N$, then $\mathbb{R}^N\backslash S^q\simeq S^{N-1}\vee S^{N-q-1}$ and hence $\mathrm{Cone}(\mathbb{R}^N\backslash S^q\hookrightarrow\mathbb{R}^N)\simeq S^N\vee S^{N-q}$ as a pointed space. Taking suspension spectra and desuspending N times we get $\mathbb{S}^0\vee\mathbb{S}^{-q}$. This is indeed the dual of $\Sigma^\infty S^q_+=\mathbb{S}^0\vee\mathbb{S}^q$.

Let us explain how a classical result in algebraic topology follows from this: Alexander duality.

Corollary 2.3. *If $X\subset\mathbb{R}^N$ is a finite CW complex, then there is an isomorphism*

$$H^{N-1-*}(X;\mathbb{Z})\cong\tilde{H}_*(\mathbb{R}^N\backslash X;\mathbb{Z}).$$

PROOF. From the long exact sequence of a pair, we get an isomorphism

$$\tilde{H}_{*+N-1}(\mathbb{R}^N\backslash X;\mathbb{Z})\cong\tilde{H}_{*+N}(\mathrm{Cone}(\mathbb{R}^N\backslash X\hookrightarrow\mathbb{R}^N)).$$

We now use duality to argue

$$\begin{aligned}\tilde{H}_{*+N}(\mathrm{Cone}(\mathbb{R}^N\backslash X \hookrightarrow \mathbb{R}^N)) &\cong \pi_*(\Sigma^{\infty-N}\mathrm{Cone}(\mathbb{R}^N\backslash X \hookrightarrow \mathbb{R}^N)) \wedge H\mathbb{Z}) \\ &\cong \pi_*(D(\Sigma^\infty X_+) \wedge H\mathbb{Z}) \\ &\cong \pi_*(F(\Sigma^\infty X_+, H\mathbb{Z})) \\ &\cong H^{-*}(X;\mathbb{Z}),\end{aligned}$$

the first writing reduced homology in terms of spectra, the second using Theorem 2.1, the third an instance of (7), and the fourth writing cohomology in terms of spectra. Then we rearrange the result to get the statement of the theorem. ☐

3. Atiyah duality and Poincaré duality

We will use the computation of the dual of a finite complex X to compute the dual of a closed m-dimensional manifold M in terms of a Thom spectrum, circling back to the first lecture. The following result is due to Atiyah [**Ati61**, Section 3].

THEOREM 3.1 (Atiyah duality). *$\Sigma^{\infty-m-d}\mathrm{Th}(\nu_e)$ is the dual of $\Sigma^\infty M_+$.*

PROOF. It suffices to observe that if $e\colon M \hookrightarrow \mathbb{R}^{m+d}$ is an embedding with tubular neighbourhood g, then the map of pairs $(g(\nu_e), g(\nu_e)\backslash e(M)) \to (\mathbb{R}^{m+d}, \mathbb{R}^{m+d}\backslash e(M))$ induces a based homotopy equivalence on mapping cones. On the other hand, we can also identify the mapping cone of the pair $(g(\nu_e), g(\nu_e)\backslash e(M))$ with the one-point compactification of $g(\nu_e)$, i.e. the Thom space $\mathrm{Th}(\nu_e)$. This identifies the dual of $\Sigma^\infty M_+$ as $\Sigma^{\infty-m-d}\mathrm{Th}(\nu_e)$. ☐

Remark 3.2. Given a manifold M, we could define the Thom spectrum of its stable normal bundle as follows: pick an embedding $e\colon M \hookrightarrow \mathbb{R}^{m+d}$ and take $\Sigma^{\infty-m-d}\mathrm{Th}(\nu_e)$. (The idea is that ν_e has formal dimension $-m$ so the Thom class (which generates the lowest degree non-zero reduced homology of a Thom space) should be in degree $-m$.) This is well-defined up to preferred equivalence by the discussion in the first lecture, when we discussed the well-definedness of stable normal framings.

From this we can deduce Poincaré duality, once we know the Thom isomorphism.

Proposition 3.3 (Thom isomorphism)**.** *If ξ is a k-dimensional orientable vector bundle over X, then $\tilde{H}_{*+k}(\mathrm{Th}(\xi));\mathbb{Z}) \cong H_*(X;\mathbb{Z})$.*

This appears in a related talk, where it is explained that an orientation amounts to an equivalence $\Sigma^\infty\mathrm{Th}(\xi) \wedge H\mathbb{Z} \simeq \Sigma^{\infty+k}X_+ \wedge H\mathbb{Z}$; generalising this from $H\mathbb{Z}$ to other spectra gives generalises the notion of orientation. Let us proceed with our goal:

Corollary 3.4 (Poincaré duality)**.** *If M is a closed orientable m-dimensional manifold, then $H^{m-*}(M;\mathbb{Z}) \cong H_*(M;\mathbb{Z})$.*

PROOF. If M is orientable, then so is ν_e because $\nu_e \oplus TM \cong \epsilon^{m+d}$. We thus have that

$$H_*(M;\mathbb{Z}) \cong \tilde{H}_{*+d}(\mathrm{Th}(\nu_e);\mathbb{Z}).$$

We now interpret this in terms of spectra and invoke Atiyah duality:

$$\begin{aligned}\tilde{H}_{*+d}(\mathrm{Th}(\nu_e);\mathbb{Z}) &\cong \pi_*(\Sigma^{\infty-d}\mathrm{Th}(\nu_e) \wedge H\mathbb{Z}) \\ &\cong \pi_*(\Sigma^m D(\Sigma^\infty M_+) \wedge H\mathbb{Z})) \\ &\cong \pi_*(\Sigma^m F(\Sigma^\infty X_+, H\mathbb{Z})) \\ &\cong H^{m-*}(X;\mathbb{Z}),\end{aligned}$$

the first writing reduced homology in terms of spectra, the second using Theorem 3.1, the third an instance of (7), and the fourth writing cohomology in terms of spectra. □

Bibliography

[Ati61] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc. (3) **11** (1961), 291–310. MR 131880 18

[DP83] A. Dold and D. Puppe, *Duality, trace and transfer*, vol. 154, 1983, Topology (Moscow, 1979), pp. 81–97. MR 733829 13, 16

[Mac71] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York-Berlin, 1971. MR 0354798 14

[Mil01] H. Miller, *Notes on cobordism*, `https://math.mit.edu/~hrm/papers/cobordism.pdf`, 2001. 5

[Qui69] D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298. MR 253350 11

[Rav86] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR 860042 11

[SW55] E. H. Spanier and J. H. C. Whitehead, *Duality in homotopy theory*, Mathematika **2** (1955), 56–80. MR 74823 15

[Swi02] R. M. Switzer, *Algebraic topology—homotopy and homology*, Classics in Mathematics, Springer-Verlag, Berlin, 2002, Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)]. MR 1886843 5

[Tho54] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86. MR 61823 10

[Wal16] C. T. C. Wall, *Differential topology*, Cambridge Studies in Advanced Mathematics, vol. 156, Cambridge University Press, Cambridge, 2016. MR 3558600 5