MORLET'S LEMMA OF DISJUNCTION AND CONCORDANCE-IMPLIES-ISOTOPY

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ABSTRACT. We outline the proof of Morlet's lemma of disjunction, deduced from concordance-implies-isotopy.

1. CONTEXT

The goal of this seminar will be to understand Goodwillie's proof of multiple disjunction for concordance embeddings [Goo90]. In this talk we discuss an earlier disjunction lemma for cocordance embeddings, due to Morlet and proven in [BLR75]. Some of the techniques will appear in Goodwillie's proof as well, but other do not, as this result unlike Goodwillie's also holds for PL and topological manifolds.

For now, let's work with smooth manifolds. The objects of interest will be manifold triads $M = (M; \partial_0 M, \partial_1 M)$: a manifold with corners M with boundary $\partial M = \partial_0 M \cup \partial_1 M$ the union of two codimension submanifolds, meeting at the corners $\partial_{01} M$. We will consider embeddings of triads $e\colon (M; \partial_0 M, \partial_1 M) \to (N; \partial_0 N_0, \partial_1 N_1)$: embeddings of manifolds with corners such that $e^{-1}(\partial_i N) = \partial_i M$.

Example 1.1. Let M be a manifold with boundary, then we consider $M \times I$ as a manifold triad with $\partial_0(M \times I) = M \times \{0,1\}$ and $\partial_1(M \times I) = \partial M \times I$.

Definition 1.2. Fix an embedding e_0 of manifolds with boundary $M \to N$. Then a *concordance embedding* is an embedding of triads $e\colon M \times I \to N \times I$ such that on $M \times \{0\} \cup \partial M \times I$ the embedding e coincides with $e_0 \times \mathrm{id}$.

We let $\mathrm{CE}(M,N)$ denote the space (or simplicial set) of concordance embeddings. This is not the best notation, since it depends on e_0. This is (partially) justified by identifying M with a submanifold of N through e_0.

Suppose that $M_1, \ldots, M_k \subset N$ is a collection of k disjoint submanifolds. Writing $\underline{k} = \{1, \ldots, k\}$, then we can produce a cubical diagram

$$\underline{k} \supset I \longmapsto \mathrm{CE}(\cup_{i \in I} M_i, N)$$

with maps in the diagram given by restriction. Goodwillie's result tells you this cube is r-cartesian for $r = -2 + \sum_{i=1}^{k}(n - m_i - 1)$, as long as $n - m_i \geq 3$ for all $1 \leq i \leq k$.

1.1. Concordance-implies-isotopy.

For $k = 1$, we obtain that the map

$$\mathrm{CE}(M,N) \longrightarrow *$$

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is $(n-m-3)$-connected. In particular, it is path-connected as long as $n-m \geq 4$. A stronger result was obtained much earlier, by Hudson for smooth and PL manifolds [Hud70] and by Pedersen for topological manifolds [Ped77], and will be used as input:

Theorem 1.3 (Concordance-implies-isotopy)**.** *If $f\colon M \times I \hookrightarrow N \times I$ is an embedding of triads such that $f^{-1}(N \times \{i\}) = M \times \{i\}$ for $i = 0,1$ and $n-m \geq 3$, then f is isotopic to $f|_{M \times \{0\}} \times \mathrm{id}$ rel $M \times \{0\}$.*

2. Morlet's lemma

Morlet's lemma of disjunction for concordance embeddings concerns the case $k=2$:

Theorem 2.1 (Morlet)**.** *The square*

$$\begin{tikzcd} \mathrm{CE}(M_1 \cup M_2, N) \arrow[r] \arrow[d] & \mathrm{CE}(M_1, N) \arrow[d] \\ \mathrm{CE}(M_2, N) \arrow[r] & \ast. \end{tikzcd}$$

is $(2n-p-q-5)$-cartesian if M_1 has handle dimension p and M_2 has handle dimension q, with $n-p \geq 3$ and $n-q \geq 3$.

Remark 2.2. Taking Goodwillie's result for $k=2$ gives Morlet's lemma with slightly improved range: it says this square is $(2n-p-q-4)$-cartesian.

In our discussion of this result, it will helpful to observe that the following are equivalent

- The square is r-cartesian.
- The map $\mathrm{CE}(M_1 \cup M_2, N) \to \mathrm{CE}(M_1, N) \times \mathrm{CE}(M_2, N)$ is r-connected.
- The map $\mathrm{CE}(M_1, N \setminus M_2) \to \mathrm{CE}(M_1, N)$ is r-connected.
- The map $\mathrm{CE}(M_2, N \setminus M_1) \to \mathrm{CE}(M_2, N)$ is r-connected.

The last two of these use isotopy extension.

Suppose that we add a handle, so $M_1' = M_1 \cup_{\partial D^i \times D^{m-i}} D^i \times D^{m-i}$. This is considered as a triad, with $\partial_0 M_1' = \partial M_1 \setminus \partial D^i \times D^{m-i}$ and $\partial 1 M_1' = D^i \times \partial D^{m-i}$. Its product with an interval $M_1' \times I$ is a 4-ad: the part $\partial_1 M_1' \times I$ is considered "free-moving." Then there is a map of fibration sequences

$$\begin{tikzcd} \mathrm{CE}(D^i \times D^{m-i}, N \setminus (M_2 \cup M_1)) \arrow[r] \arrow[d] & \mathrm{CE}(D^i \times D^{m-i}, N \setminus M_1) \arrow[d] \\ \mathrm{CE}(M_1', N \setminus M_2) \arrow[r] \arrow[d] & \mathrm{CE}(M_1', N) \arrow[d] \\ \mathrm{CE}(M_1, N \setminus M_2) \arrow[r] & \mathrm{CE}(M_1, N) \end{tikzcd}$$

so the the middle map is r-connected when the top and bottom map are. Since the maps

$$\mathrm{CE}(D^i \times D^{m-i}, N) \longrightarrow \mathrm{CE}(D^i \times \{0\}, N)$$

is a weak equivalence and similarly for $N \setminus D^q$ in place of N, we see that it suffices to consider the case that M_1 is a disc. Using the equivalent formulation in terms of a square, we may then also assume that M_2 is a disc. Once we resolve this case, the result for general M_1 and M_2 in terms of their handle dimensions follows:

Proposition 2.3. *If $n-p \geq 3$ and $n-q \geq 3$, then the square*

$$\begin{tikzcd} \mathrm{CE}(D^p \cup D^q, N) \arrow[r] \arrow[d] & \mathrm{CE}(D^p, N) \arrow[d] \\ \mathrm{CE}(D^q, N) \arrow[r] & *. \end{tikzcd}$$

is $(2n-p-q-5)$-cartesian.

Proof. The proof will be an inductive argument: the statement (i,p,q) says that the square is i-cartesian. We intend to prove this for $i \leq 2n-p-q-5$, $n-p \geq 3$, $n-q \geq 2$. Note that our discussion of equivalent statement tells us that $(i,p,q) \Leftrightarrow (i,q,p)$.

Initial case. The initial cases $(0,p,q)$ follow by looking at the equivalent statement concerning

$$\mathrm{CE}(D^p, N \setminus D^q) \longrightarrow \mathrm{CE}(D^p, N)$$

where by symmetry we may assume $p \leq q$. Observe that both terms are path-connected by concordance-implies-isotopy. As no condition is imposed on q, we may also take $n-q=2$.

Reduction under a hypothesis. We will now prove that the case $(i-1,p+1,q)$ implies the case (i,p,q) under the condition that $g\colon D^p \to N \setminus D^q$ extends to a map $D^{p+1}_+ \to N \setminus D^q$ where $D^{p+1}_+ = D^{p+1} \cap \{x \in \mathbb{R}^{p+1} \mid x_1 \geq 0\}$ is the upper hemisphere and $D^{p+1}_+ \cap S^p$ goes to the boundary $\partial N \setminus \partial D^q$. In this case, we can compare the fibration sequences

$$\begin{tikzcd} \mathrm{CE}(D^{p+1}, N \setminus (D^p \cup D^q)) \arrow[r] \arrow[d] & \mathrm{CE}(D^{p+1}, N \setminus D^p) \arrow[d] \\ \mathrm{CE}(D^{p+1}_+, N \setminus D^q) \simeq * \arrow[r] \arrow[d] & \mathrm{CE}(D^{p+1}_+, N) \simeq * \arrow[d] \\ \mathrm{CE}(D^p, N \setminus D^q) \arrow[r] & \mathrm{CE}(D^p, N) \end{tikzcd}$$

to get that

$$\pi_i(\mathrm{CE}(D^p, N), \mathrm{CE}(D^p, N \setminus D^q)) = \pi_{i-1}(\mathrm{CE}(D^{p+1}, N \setminus D^p), \mathrm{CE}(D^{p+1}, N \setminus (D^p \cup D^q)))$$

for $i \geq 1$. The inductive hypothesis $(i-1,p+1,q)$ gives the desired case (i,p,q). The remainder of this proof will be removing concerned with removing the hypothesis.

A weaker hypothesis. We say $\overline{D}^p \subset N$ is a *parallel copy of* $D^p \subset N$ if there is an embedding $D^p \times [0,1] \hookrightarrow N \setminus D^q$ restricting to D^p at 0 and $\overline{D}^p$ at $t=1$. We will also refer to $\overline{D}^p \times I \subset N \times I$ as a *parallel copy of* $D^p \times I$.

Recall that a $\alpha \in \pi_i(\mathrm{CE}(D^p, N), \mathrm{CE}(D^p, N \setminus D^q)$ is represented by a map of pairs $(\Delta^i, \partial\Delta^i) \to (\mathrm{CE}(D^p, N), \mathrm{CE}(D^p, N \setminus D^q))$. Suppose we have a representative such that there exists a parallel copy $\overline{D}^p$ of D^p so that α avoids $\overline{D}^q \times I$. Then $\alpha \in \pi_i(\mathrm{CE}(D^p, N), \mathrm{CE}(D^p, N \setminus D^q))$ is in the image of

$$\pi_i(\mathrm{CE}(D^p, N \setminus \overline{D}^p), \mathrm{CE}(D^p, N \setminus (\overline{D}^p \cup D^q))$$

and we are done, as the desired extension of $g\colon D^p \setminus N \setminus (\overline{D}^p \cup D^q)$ to D^{p+1}_+ obviously exists, and hence α is the image of 0 by the previous step.

Given a representative of α as a map of pairs $(\Delta^i, \partial\Delta^i) \to (\mathrm{CE}(D^p, N), \mathrm{CE}(D^p, N \setminus D^q))$ we may be not be able to find a parallel copy for all $s \in \Delta^i$. However, they do exist locally up to an isotopy of α, using concordance-implies-isotopy and isotopy extension. The strategy will now be to use these *local parallel copies* and the case (j, p, q) for $j < i$ to write α as a sum of elements which have a parallel copy. More precisely, we will give a subdivision of Δ^i, and construct an isotopy α^t of α by induction over the skeleton which avoids D^q for $s \in \partial\Delta^i$ and that on each m-simplex σ of the subdivision we have

(1) $\alpha^1|_{\partial\sigma}$ lands in $N \setminus D^q$,
(2) $\alpha^1|_\sigma$ admits a parallel copy.

We can then invoke the case (i, p, q) in presence of a parallel copy.

Invoking isotopy extension We start with some preparation: we claim that $\alpha\colon \Delta^i \times D^p \times I \to \Delta \times N \times I$ is the restriction to $\Delta^i \times D^p \times I$ of a concordance diffeomorphism $A\colon \Delta^i \times N \times I \to \Delta \times N \times I$. That is, A is the identity on $\Delta^i \times (N \times \{0\} \cup \partial N \times I)$.

For $i = 0$, we note that concordance-implies-isotopy implies α is isotopic to $\mathrm{id}_{D^p \times I}$ and hence the restriction of a diffeomorphism; by applying isotopy extension, so is α. For the general case, we now that α restricted to a vertex of Δ^i is the restriction of a diffeomorphism and by isotopy extension the same result follows for α.

Arranging the local parallel copies. Take A is as in the previous part. We pick one embedding $D^p \times [0,1] \hookrightarrow N \setminus D^q$ and we subdivide Δ^i finely enough so that:

- $A(\{s\} \times D^p \times [0, 1/2] \times I)$ and $A(\{s'\} \times D^p \times \{0\} \times I)$ are disjoint for s, s' in the same i-simplex.
- $A(\{s\} \times D^p \times \{1/2 + j/3^{i-1}\} \times I)$ is disjoint from $A(\{s'\} \times D^p \times \{1/2 + j'/3^{i-1}\} \times I)$ when $1 \leq j \neq j' \leq 3^{i-1}$ and s, s' are in the star of the same vertex.
- $A(\{s\} \times D^p \times [0, 1] \times I) \subset (N \setminus D^q) \times I$ when s is in the star of a vertex in the induced subdivision of $\partial\Delta^i$.

The number 3^{i-1} is chosen so that there exists an assignment of numbers $1 \leq j \leq 3^{i-1}$ to the vertices of the triangulation so that no two vertices in the star of a given vertex share the same number. Pick such an assignment and denote $1/2 + j(x)/3^{i-1}$ by $r(x)$.

As suggested before, we want for each vertex x of the subdivision an isotopy ϕ^x_t of $A(\{x\} \times D^p \times \{r(x)\} \times I)$ to $\mathrm{id}_{D^p \times \{r(x)\} \times I}$. Thus we refer to the former as *candidate parallel copies*. These isotopies will be constructed to have the following properties:

- $\phi^x_t = \phi^1_t$ for $t \geq \frac{1}{i+1}$.
- If a candidate parallel copy $A(\{x\} \times D^p \times \{r(x)\} \times I)$ is disjoint from another one $A(\{y\} \times D^p \times \{r(y)\} \times I)$, then this remains true during simultaneous isotopies.
- If x is in the star of a vertex in the induced subdivision of $\partial\Delta^i$, then the image of ϕ^t_x remains within $N \setminus D^q$.

For now we assume these exist, and finish the proof.

The desired isotopy of α will be constructed inductively over the skeleta of the subdivision. That is, we will construct inductively for $0 \leq m \leq i$ an isotopy

$$\alpha_t^m : \mathrm{sk}_m(\text{subdivision of } \Delta^i) \longrightarrow \mathrm{CE}(D^p, N)$$

starting at $\alpha|_{\mathrm{sk}_m}$ which satisfies

(a) $\alpha_t^m = \alpha_1^m$ for $t \geq \frac{m+1}{i+1}$.
(b) remains within $(N \setminus D^q) \times I$ when x is the star of a vertex of $\partial\Delta^i$,
(c) lands in $(N \setminus D^q) \times I$ at $t = 1$,
(d) on a simplex $\sigma \in \mathrm{sk}_m$ it remains disjoint from the image of ϕ_t^x for all $x \in \mathrm{Star}(\sigma)$.

Supposing α_t^{m-1} exists, we construct α_t^m over σ. To do so, we apply isotopy extension to $\alpha_t^{m-1}|_{\partial\sigma} \cup \{\phi_t^x \mid x \in \mathrm{Star}(\sigma)\}$. This yields in particular an isotopy of α_t^m for $t \leq \frac{m}{i+1}$. At $t = \frac{m}{i+1}$ we are in the situation that on σ we have an element of $\pi_m(\mathrm{CE}(D^p, N), \mathrm{CE}(D^p, N \setminus D^q))$ with has many disjoint parallel copies, one for any vertex in the star of σ. Thus we can consider it is an element of $\pi_m(\mathrm{CE}(D^p, N \setminus \bigcup_x D^p \times \{r(x)\}), \mathrm{CE}(D^p, N \setminus D^q \bigcup_x D^p \times \{r(x)\}))$. We may use one of these to exhibit this element as 0; this gives the desired extension of the isotopy, which may be performed for $\frac{m}{i+1} \leq t \leq \frac{m+1}{i+1}$.

Finding the ϕ_t^x. It remains to prove we can find the isotopies ϕ_x^t straightening the candidate parallel copies $A(\{x\} \times D^p \times \{r(x)\} \times I)$, which we will shorten to W_i for $i \in I$ the set of vertices. This would be easy if all were disjoint; use concordance-implies-isotopy and isotopy extension to construct them inductively.

In general, we need to handle the intersections. Then if it were possible to show that after simultaneous isotopies preserving disjointness, they are all transverse and in the image of the single concordance diffeomorphims $H : N \times I \to N \times I$ we would be done: to straighten all $W_i \times I$ in this case, we observe that their union is stratified by concordance embeddings of lower-dimensional manifolds, which can be straightened inductively using concordance-implies-isotopy.

Making the intersections products. Thus it remains to show that given transverse $W_i \times I \subset N \times I$ we can isotope these simultaneously preserving disjointness so that at they are all transverse and in the image of a single concordance. As the above argument suggests, we want to inductively isotope the $W_i \times I$ so that the intersections $\bigcup_{i \in I} W_i \times I$ look like $(\bigcup_{i \in I} W_i) \times I$ for $|I| \leq s$. This is done by embedded Whitney and half-Whitney moves, and as it is complicated let me just explain the case that $|I| = 2$.

That is, we have concordance embeddings $W_1 \times I, W_2 \times I \hookrightarrow N \times I$ which we may assume transverse. Suppose that $W_1 \cap W_2 \neq \varnothing$, necessarily of dimension $w_1 + w_2 - n$ (this is less familiar than the case $W_1 \cap W_2 = \varnothing$). Then we want to modify the intersection $(W_1 \times I) \cap (W_2 \times I)$ until they are the product. It is a $(w_1 + w_2 - n + 1)$-dimensional cobordism, and we are trying to remove all its handles rel $(W_1 \cap W_2) \times \{0\}$. This is by done by exchanging i-handles for $(i+1)$-handles below the middle dimension and $(i+1)$-handles for i-handles in the middle dimension.

For example, for j is the smallest such that $\pi_k((W_1 \times I) \cap (W_2 \times I), (W_1 \cap W_2) \times \{0\})$ is non-zero then we can represent a generator of it by an embedded (D^k, S^{k-1}). These may then be extended to embeddings of $D^{k+1} \to W_i \times I$ with S_+^k hitting the $D^k \subset (W_1 \times I) \cap (W_2 \times I)$ and

S^k_- hitting $W_i \times \{0\}$. These glue to a map $(D^{k+1}, S^k) \to ((W_1 \times I) \cup (W_2 \times I), (W_1 \cup W_2) \times \{0\})$ which in turn extends to a map $D^{k+2} \to N \times I$ intersecting $(W_1 \times I) \cup (W_2 \times I)$ only in S^{k+1}_+ hitting the D^{k+1} and S^{k+1}_- in $N \times \{0\}$. This is the guide for a half-Whitney move sliding $W_1 \times I$ over $W_2 \times I$ to kill the generator. As you can imagine, this requires a careful bookkeeping of various connectivities and a number of embedding/disjunction results. $\square$

Remark 2.4. There are improvements to concordance-implies-isotopy when N is 1-connected, due to Rourke and Perron. These allow you to prove the square is $(2n - p - q - 4)$-cartesian.

Question 2.5. One approach for getting multiple disjunction for topological or PL embeddings involves puncturing handles so as to get a smoothable manifold. Can the techniques in the proof of Morlet's lemma be used for this?

2.1. **A recasting in more modern terms.** We shall give a strategy for a different proof, starting at the point of the weaker hypothesis. There we make one further observation: let us call a concordance embedding $\eta\colon \overline{D}^p \times I \to (N \setminus D^q) \times I$ which avoids D^p starting at a parallel copy of D^p a *bent parallel copy.* It is enough to find these: by concordance-implies-isotopy and isotopy extension there is a concordance diffemorphism $N \times I \to N \times I$ fixing $D^q \times I$ sending the bent parallel copy to a parallel copy.

We now define a semisimplicial resolution $\mathrm{CE}(D^p, N \setminus D^q)_\bullet$ of $\mathrm{CE}(D^p, N \setminus D^q)$ and $\mathrm{CE}(D^p, N)_\bullet$ of $\mathrm{CE}(D^p, N)$. In both cases, the k-simplices will be a disjoint union indexed by $(k+1)$-tuples of $a_0, \ldots, a_k \in [1/2, 1]$ and bent parallel copies $\eta_0, \ldots, \eta_k\colon \overline{D}^p_{a_i} \times I \to (N \setminus D^q) \times I$ which are all disjoint. Then we set

$$\mathrm{CE}(D^p, N \setminus D^q)_k = \bigsqcup_{\vec{a}, \vec{\eta}} \{x \in \mathrm{CE}(D^p, N \setminus D^q) \text{ disjoint from } \eta_0, \ldots, \eta_k\}$$

$$\mathrm{CE}(D^p, N)_k = \bigsqcup_{\vec{a}, \vec{\eta}} \{x \in \mathrm{CE}(D^p, N) \text{ disjoint from } \eta_0, \ldots, \eta_k\}$$

There are evident face maps and augmentations, as well as a semisimplicial map

$$\mathrm{CE}(D^p, N \setminus D^q)_\bullet \longrightarrow \mathrm{CE}(D^p, N)_\bullet$$

which is levelwise i-connected due to the existence of at least one bent parallel copy. Thus it remains to show that in

$$\begin{tikzcd} ||\mathrm{CE}(D^p, N \setminus D^q)_\bullet|| \arrow[r] \arrow[d] & ||\mathrm{CE}(D^p, N)_\bullet|| \arrow[d] \\ \mathrm{CE}(D^p, N \setminus D^q) \arrow[r] & \mathrm{CE}(D^p, N) \end{tikzcd}$$

the vertical maps are weak equivalences. They are microfibrations, so it suffices to prove their fibers are weakly contractible. This is trivial for the left term (just use a small push-off from the given $x \in \mathrm{CE}(D^p, N \setminus D^q)$). For the right term, we need to prove that given an $x \in \mathrm{CE}(D^p, N)$ and a collection

$$\eta_0, \ldots, \eta_N$$

of bent parallel copies disjoint from x and $D^q \times I$, we can find a further one disjoint from all of them. This is difficult only when x and $D^q \times I$ intersect. Using the usual tricks, we may assume $\eta_0, \ldots, \eta_N$ are transverse to each other. Here the argument gets stuck though.

3. Concordance-implies-isotopy

Let us recall the statement of concordance-implies-isotopy:

Theorem 3.1 (Concordance-implies-isotopy)**.** *If* $f\colon M \times I \hookrightarrow N \times I$ *is an embedding of triads such that* $f^{-1}(N \times \{i\}) = M \times \{i\}$ *for* $i = 0, 1$ *and* $n - m \geq 3$*, then* f *is isotopic to* $f|_{M\times\{0\}} \times \mathrm{id}$ *rel* $M \times \{0\}$*.*

This is of course equivalent to $\pi_0(\mathrm{CE}(M, N)) = 0$ (with boundary condition $f|_{M\times\{0\}}$).

Remark 3.2. All the early arguments for this work in the PL category, using various approximation tricks to get the smooth case. For a direct proof in the smooth category, I would suggest following Rourke's outline using embedded handle theory [**?**]. Below I will give the PL proof since I believe some tools may be useful in Goodwillie's argument. Note that it is enough to prove the PL case for the disjunction lemma, since the square

$$\begin{array}{ccc} \mathrm{CE}^{\mathrm{Diff}}(M_1 \cup M_2, N) & \longrightarrow & \mathrm{CE}^{\mathrm{Diff}}(M_1, N) \times \mathrm{CE}^{\mathrm{Diff}}(M_2, N) \\ \downarrow & & \downarrow \\ \mathrm{CE}^{\mathrm{PL}}(M_1 \cup M_2, N) & \longrightarrow & \mathrm{CE}^{\mathrm{PL}}(M_1, N) \times \mathrm{CE}^{\mathrm{PL}}(M_2, N) \end{array}$$

is ∞-cartesian by smoothing theory.

The PL proof of concordance-implies-isotopy has two main ingredients:

- *Unknotting of balls* [Zee63]: any PL-embedding $D^m \hookrightarrow D^n$ that is the standard inclusion on the boundary is isotopic to the standard inclusion when $n - m \geq 3$.
- *Sunny collapsing* [Hud69]: given $f\colon M \times I \hookrightarrow N \times I$, there is a homeomorphism of $N \times I$ fixed on $N \times \{0\} \cup \partial N \times I$ and arbitrary close to the identity such that (i) the projection $hf(M \times N) \to N \times \{0\}$ is non-degenerate, (ii) $hf(M \times I)$ can be triangulated as K_n allowing a sequence of elementary collapse $M = K_n \searrow K_{n-1} \searrow \cdots \searrow K_0 = M \times \{0\} \cup \partial M \times I$ so that $\mathrm{sh}(K_i) \cap K_n \subset K_{i-1}$ with $\mathrm{sh}(K_i) = \{(x,t) \in N \times I \mid (x, t') \in K_i \text{ for some } t' > t\}$. That is, we only collapse things visible from above.

Sketch of proof. As usual, it suffices to prove this when M is a PL-disc D^m by induction over a triangulation; in particular, on $\partial D^m \times I$ the concordance embedding is already a product. There is a subtlety here: for the induction to work you need to rather straighten a neighborhood of D^m.

Then there are essentially two steps, which are similar in flavor. (The reason there are two steps is that in the sunny collapsing it is not guaranteed to you that h is isotopic to the identity.)

Moving into a tube. First, we will prove that there is a neighborhood $U \subset N$ such that $U \cong D^m \times D^{n-m} = D^n$ and there is an isotopy of concordance embeddings moving $f(D^m \times I)$ into $U \times I$. We will do so by not straightening $f(D^m \times I)$ but $hf(D^m \times I)$ for h as in sunny collapsing. Doing so is enough; since h is small if we straighten $hf(D^m \times I)$ then $h^{-1}hf(D^m \times I) = f(D^m \times I)$ ends up in a neighborhood $U \times I$. (Of course, this requires me to keep track of some "smallness', which I shall forego.)

Let $U_i \subset K_i$ be obtained by adding a small closed collar on K_{i-1} with boundary $\mathrm{fr}(U_i)$. and set $X_i \coloneqq U_i \cup (\mathrm{fr}(U_i))^\uparrow$ with $(A)^\uparrow = \{(x,t) \in N \times I \mid (x,t') \in A \text{ and } t \geq t'\}$ for $A \subset N \times I$. The sunny part of sunny collapsing implies that X_i is a PL submanifold of $N \times I$ again. We also set $K_{n+1} = hf(D^m \times I)$ and $K_{-1} = (D^m \times \{0\})^\uparrow$. Then we will construct a sequences of PL isotopies fixing $N \times \{0\} \cup \partial N \times I$ moving X_i onto X_{i-1}.

This is done as follows: take a neighborhood V in K_n of the simplex added to K_{i-1} to obtain K_i. Then X_i and X_{i-1} only differ within the interior of $(V)^\uparrow$. Furthermore, $(V)^\uparrow$ is a ball by the non-degeneracy of the projection of the simplex and the collapsing part of sunny collapsing (to see how this might be used, recall that Whitehead proved that a PL manifold is collapsible to a point if and only it is a PL ball), as are $X_i \cap (V)^\uparrow$ and $X_{i-1} \cap (V)^\uparrow$. Now first use local unknotting in $(V)^\uparrow \cap (N \times \{1\})$ and then local unknotting in $(V)^\uparrow$ to move $X_i \cap (V)^\uparrow$ onto $X_{i-1} \cap (V)^\uparrow$ fixing the boundary.

Straightening within tube. Once this is achieved, we can first apply local unknotting of $f(D^m \times \{1\})$ in $U \times \{1\}$ and then local unknotting of $f(D^m \times I)$ in $U \times I$. $\square$

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