POINCARÉ STRUCTURES ON CATEGORIES OF MODULES

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Abstract. We give an overview of Poincaré structures on categories of modules.

1. Introduction

To motivate studying hermitian K-theory, we discussed quadratic forms on finitely-generated projective modules and several variants of these. In the previous two lectures we say the definitions of hermitian and Poincaré categories, as well as hermitian and Poincaré objects in them. Today our goal is to focus on the case of R-modules: we will explain how hermitian structures on $\mathcal{D}^p(R)$ are given by R-modules with genuine involution. We will see that

$$Ϙ^q(X) = \mathrm{Map}(X \otimes X, R)_{hC_2} \qquad \text{and} \qquad Ϙ^s(X) = \mathrm{Map}(X \otimes X, R)^{hC_2}$$

are the extreme cases of a collection $Ϙ^{\geqslant n}(X)$, namely $n = \infty$ and $n = -\infty$. The cases $n = 0, 1, 2$ will be "genuine" variants mentioned later, and we will motivate them from the perspective of non-abelian derived functors.

2. Classification of hermitian structures

2.1. **Recollection.** Recall that a hermitian structure on a stable ∞-category $\mathcal{C}$ is a reduced 2-excisive functor $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$; *reduced* means $Ϙ(0) = 0$ and 2-excisive then means that the *cross effect* functor

$$B_Ϙ\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{p}$$

determined by the formula

$$Ϙ(X \oplus Y) = Ϙ(X) \oplus Ϙ(Y) \oplus B_Ϙ(X, Y),$$

is bilinear, i.e. exact in both entries.

Given such a $Ϙ$, there are C_2-equivariant maps

$$B_Ϙ(X,X) \longrightarrow Ϙ(X \oplus X) \xrightarrow{\Delta^*} Ϙ(X) \xrightarrow{\nabla^*} Ϙ(X \oplus X) \longrightarrow B_Ϙ(X,X)$$

with C_2-action trivial on the middle term and given by flipping on the outer terms. This induces maps

$$B_Ϙ(X,X)_{hC_2} \longrightarrow Ϙ(X) \longrightarrow B_Ϙ(X,X)^{hC_2}$$

where $(-)_{hC_2}$ are the C_2-coinvariants (the left adjoint to the inclusion $\mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}^{BC_2}$ as spectra with trivial action) and $(-)^{hC_2}$ are the C_2-fixed points (the right adjoint). The composition is the *norm map*: in ordinary algebra, for a G-module M with G a finite group there is a map $M_G \to M^G$ sending the class of m to $\sum_{g \in G} gm$ and there is a lift of this construction to G-spectra.

Remark 2.1. Let's demystify the norm a bit, following a construction by Lurie. Let us start with the diagram of spaces or equivalently ∞-groupoids,

$$\begin{tikzcd} BG \arrow[r, "\Delta"] & BG \times BG \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & BG \arrow[d, "p"] \\ & BG \arrow[r, "p"] & * \end{tikzcd}$$

and map it into spectra to get a diagram of stable ∞-categories

$$\begin{tikzcd} \mathcal{S}\mathrm{p}^{BG} & \mathcal{S}\mathrm{p}^{BG\times BG} \arrow[l, "\Delta^*"'] & BG \arrow[l, "\pi_1^*"'] \\ & BG \arrow[u, "\pi_2^*"] & * \arrow[l, "p^*"] \arrow[u, "p^*"'] \end{tikzcd}$$

with $(-)^*$ given by pullback, $(-)_!$ by left Kan extension, and $(-)_*$ by right Kan extension. In particular, $(-)_! = (-)_{hG}$ and $(-)_* = (-)^{hG}$.

Since the square of spaces was a pullback, by a version of Lurie's proper base-change theorem the Beck–Chevalley transformation $p^*p_* \to (\pi_1)_*\pi_2^*$ is an equivalence; informally, M^G considered as a trivial G-module can also be obtained by considering M as a $G\times G$-space through π_2 and taking the fixed points with respect to $\ker(\pi_1\colon G\times G\to G)$ to get a G-module. Now we compute that $\Delta_! \to \Delta_*$ is an equivalence if G is finite: on a G-spectrum X it is given by $\bigoplus_{g\in G} X \to \prod_{g\in G} X$ and finite coproducts are finite products in $\mathcal{S}\mathrm{p}$. Now write the natural transformation

$$(\pi_1)^* \to \Delta_*\Delta^*(\pi_1)^* = \Delta_* \simeq \Delta_! \to \Delta_!\Delta^*(\pi_2)^* \to (\pi_2)^*,$$

in turn adjoint to $\mathrm{id} \to (\pi_1)_*(\pi_2)^*$. This is equivalent to $\mathrm{id} \to p^*p_*$, which is in turn equivalent to $p_! \to p_*$.

We define

$$\Lambda_{\mathfrak{Q}}(X) := \mathrm{cofib}(B_{\mathfrak{Q}}(X,X)_{hC_2} \longrightarrow \mathfrak{Q}(X)).$$

Because this map induces an equivalences on cross effects, the functor $\Lambda_{\mathfrak{Q}}$ is exact. It is in fact the initial exact functor under $\mathfrak{Q}$, so should be thought of as the linear part of $\mathfrak{Q}$ in analogy with $B_{\mathfrak{Q}}$ being the quadratic part. It remains to understand how the part is glued to the bilinear part. This is the content of the following theorem:

Proposition 2.2. *There is a pullback square*

$$\begin{tikzcd} \mathfrak{Q}(X) \arrow[r] \arrow[d] & \Lambda_{\mathfrak{Q}}(X) \arrow[d] \\ B_{\mathfrak{Q}}(X,X)^{hC_2} \arrow[r] & B_{\mathfrak{Q}}(X,X)^{tC_2}, \end{tikzcd}$$

where $B_{\mathfrak{Q}}(X,X)^{tC_2} := \mathrm{cofib}(\mathrm{Nm}\colon B_{\mathfrak{Q}}(X,X)_{hC_2} \to B_{\mathfrak{Q}}(X,X)^{hC_2})$ *is the* Tate construction*.*

Proof. Both horizontal fibres are $B_{\mathfrak{Q}}(X,X)_{hC_2}$. □

This is natural in X, yielding an pullback of functors of $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$:

$$\begin{tikzcd} \mathfrak{Q} \arrow[r] \arrow[d] & \Lambda_{\mathfrak{Q}} \arrow[d] \\ (B_{\mathfrak{Q}}\Delta)^{hC_2} \arrow[r] & (B_{\mathfrak{Q}}\Delta)^{tC_2}. \end{tikzcd}$$

In other words, the data of a hermitian structure is the same as a triple (B,Λ,α) where $B\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is symmetric bilinear, $\Lambda\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is exact, and $\alpha\colon \Lambda \to (B\Delta)^{tC_2}$ is a natural transformation.

Remark 2.3. $X \mapsto (X\otimes X)^{tC_2}$ is exact. It is clearly reduced and 2-excisive, the latter because the category of 2-excisive functors is closed under finite limits and colimits. This means it suffices to see that its cross effect vanishes: we compute $(X\oplus Y)\otimes(X\oplus Y) = X\otimes X \oplus Y\otimes Y \oplus B(X,Y)[C_2]$ as a C_2-space, and the Tate construction preserves finite sums and vanishes on induced C_2-objects; informally for an induced module $M[C_2]$ the norm map is the isomorphism $(-)\cdot(e+\sigma)\colon M\{e\sim\sigma\} \to M\{e+\sigma\}$. More generally, this argument yields that $X \mapsto B_{\mathfrak{Q}}(X,X)^{tC_2}$ is exact.

The fact that is exact has the following consequence, using that evaluating at the sphere gives an equivalence between the category of exact functors $\mathcal{S}\mathrm{p}^\omega \to \mathcal{S}\mathrm{p}$ and $\mathcal{S}\mathrm{p}$. Thus, natural transformations from $\mathrm{id}_{\mathcal{S}\mathrm{p}}$ to $X \mapsto (X \otimes X)^{tC_2}$ are determined by maps $\mathbb{S} \to \mathbb{S}^{tC_2}$ and the one corresponding to $\mathbb{S} \to \mathbb{S}^{hC_2} \to \mathbb{S}^{tC_2}$ is the *Tate diagonal*; it is a natural map $X \to (X \otimes X)^{tC_2}$ that serves as a spectral substitute for a diagonal map. It is in fact lax symmetric monoidal.

Example 2.4. In general, the Tate construction is not connective. For example, for an ordinary module M thought of as an Eilenberg–Mac Lane spectrum HM we have that

$$\pi_*(HM)^{tC_2} = \begin{cases} H_{*+1}(C_2; M) & \text{if } * > 1, \\ \ker(M_{C_2} \to M^{C_2}) & \text{if } * = 1, \\ \operatorname{coker}(M_{C_2} \to M^{C_2}) & \text{if } * = 0, \\ H^{-*}(C_2; M) & \text{if } * < 0. \end{cases}$$

On the other hand, the now-proven Segal conjecture implies that $\mathbb{S}^{tC_2}$ is the 2-completion of $\mathbb{S}$.

3. Poincaré structures on categories of modules

3.1. **Ring spectra and categories of modules.** The ∞-category $\mathcal{S}\mathrm{p}$ of spectra admits a closed symmetric monoidal structure whose unit is the sphere spectrum $\mathbb{S}$. It is produced by constructing a tensor product of presentable stable ∞-categories, and proving that $\mathcal{S}\mathrm{p}$ is its unit; the unit of any symmetric monoidal category is canonically a commutative algebra and a commutative algebra structure on a category is a symmetric monoidal structure. If we prefer to think of a spectrum as a sequence $(X_0, X_1, \cdots)$ of pointed spaces such that $X_0 \to \Omega X_1$ is an equivalence, then the tensor product $X \otimes Y$ can be computed from the smash products $X_k \wedge Y_l$; see Adams' book.

In particular, we can make sense of associative algebras in $\mathcal{S}\mathrm{p}$, known as E_1-ring spectra for historical reasons. For such a E_1-ring spectrum R, we let $\mathrm{Mod}(R)$ denote the ∞-category of (left) R-modules. The subcategory $\mathrm{Mod}^p(R)$ of perfect modules is its smallest stable subcategory containing R and closed under retracts; an object is perfect if and only if it is compact (mapping out of it preserves filtered colimits).

Example 3.1. The Eilenberg–Mac Lane spectrum lifts to a lax-monoidal functor $H\colon (\mathsf{Ab}, \otimes) \to (\mathcal{S}\mathrm{p}, \otimes)$. In particular, it takes ordinary rings to E_1-ring spectra. In this case $\mathrm{Mod}(R)$ is equivalent to the derived ∞-category $\mathcal{D}(R)$, obtained by taking chain complexes of projective R-modules and inverting the quasi-isomorphisms, and a chain perfect is perfect if it is quasi-isomorphism to a bounded chain complex of finitely-generated projective R-modules.

Example 3.2. The topological K-theory spectra KU and KO are commutative ring spectra, known as E_∞-spectra for historical reasons.

3.2. **Morita theory and modules with genuine involution.** Morita theory says that colimit-preserving functors between module categories are classified in terms of certain bimodules. Using this will lead us to the notion of a module with genuine involution, classifying the hermitian structures on $\mathrm{Mod}^p(R)$ along the way.

Theorem 3.3 (Morita theory)**.** *There is an equivalence of categories*

$$\begin{aligned} \mathrm{Fun}^{\mathrm{L}}(\mathrm{Mod}(R), \mathrm{Mod}(R)) &\longrightarrow \mathrm{Mod}(R^{\mathrm{op}} \otimes S) \\ F &\longmapsto F(R), \end{aligned}$$

where $\mathrm{Fun}^{\mathrm{L}}(-,-)$ *denotes the colimit-preserving functor.*

Proof sketch. We can an inverse by sending $R^{\mathrm{op}} \otimes S$-module P to the functor $M \mapsto P \otimes_R M$. □

This allows us translate the data (B, Λ, α) of a hermitian structure in more concrete language:

- We start with the linear part given by an exact functor $\Lambda\colon \mathrm{Mod}^p(R)^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$. Using the duality equivalence $\mathrm{Map}_R(-,R)\colon \mathrm{Mod}^p(R)^{\mathrm{op}} \to \mathrm{Mod}^p(R^{\mathrm{op}})$ we can interpret this as an exact functor $\mathrm{Mod}^p(R^{\mathrm{op}}) \to \mathcal{S}\mathrm{p}$ and $\mathrm{Mod}^p(R^{\mathrm{op}}) \to \mathrm{Mod}(R^{\mathrm{op}})$ exhibits the latter as the ind-completion, this is equivalent to a colimit-preserving functor $\mathrm{Mod}(R^{\mathrm{op}}) \to \mathcal{S}\mathrm{p}$, so by Morita theory corresponds is given by tensoring with a unique R-module N. Composing all maps we see that
$$\Lambda(X) \simeq \mathrm{Map}_R(X,R) \otimes_R N \simeq \mathrm{Map}_R(X,N),$$
the latter equivalence following because it is true on R and both are exact.
- Similarly, one deduces that the bilinear part $B\colon \mathrm{Mod}^p(R)^{\mathrm{op}} \times \mathrm{Mod}^p(R)^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is of the form
$$B(X,Y) \simeq \mathrm{Map}_{R\otimes R}(X \otimes Y, M)$$
for a unique $R \otimes R$-bimodule M. That it is symmetric means that $M \in \mathrm{Mod}(R \times R)^{hC_2}$ where the action of C_2 flips the two copies of R.
- It remains to understand the gluing map α. As for Λ, R-module corresponding to the exact functor $X \mapsto B(X,X)^{tC_2}$ is given by M^{tC_2}, made an R-module via the Tate diagonal $R \mapsto (R \otimes R)^{tC_2}$. Thus through the Morita theory, the gluing map α is encoded by an R-module map $N \to M^{tC_2}$.

Definition 3.4. A module with genuine involution is a triple (M,N,α) of $M \in \mathrm{Mod}(R \otimes R)^{hC_2}$, $N \in \mathrm{Mod}(N)$ and $\alpha\colon N \to M^{tC_2}$.

Thus concretely, not only can we define a hermitian structure on $\mathrm{Mod}^p(R)$ by taking the pullback
$$\begin{array}{ccc} ϙ^\alpha(X) & \longrightarrow & \mathrm{Map}_{R\otimes R}(X \otimes X, M)^{hC_2} \\ \downarrow & & \downarrow \\ \mathrm{Map}_R(X,N) & \longrightarrow & \mathrm{Map}_{R\otimes R}(X \otimes X, M)^{tC_2} \simeq \mathrm{Map}_R(X, M^{tC_2}), \end{array}$$
but all hermitian structures are of this form.

When is this a Poincaré structure? The bilinear part of $ϙ^\alpha$ is by construction $\mathrm{Map}_{R\otimes R}(X \otimes Y, M)$ and the equivalence $\mathrm{Map}_{R\otimes R}(X \otimes Y, M) \simeq \mathrm{Map}_R(X, \mathrm{Map}_R(Y,M))$ shows that $ϙ^\alpha$ is Poincaré if (1) $Y \mapsto \mathrm{Map}_R(Y,M)\colon \mathrm{Mod}(R)^{\mathrm{op}} \to \mathrm{Mod}(R)$ restricts to a functor $\mathrm{Mod}^p(R)^{\mathrm{op}} \to \mathrm{Mod}^p(R)$ and (2) the evaluation map is an equivalence. Item (1) is the case if M is perfect and item (2) holds if and only if it holds for R, i.e. $R \to \mathrm{Map}_R(\mathrm{Map}_R(R,M),M) \simeq \mathrm{Map}_R(M,M)$ is an equivalence. If this is the case, we say that M is *invertible.*

Theorem 3.5. *Hermitian structures on* $\mathrm{Mod}^p(R)$ *are classified by invertible modules with genuine involution.*

3.3. **Examples.**

Example 3.6 (Modules with involution)*.* Let us start with an $M \in \mathrm{Mod}(R \otimes R)^{hC_2}$. Using this we can define symmetric and quadratic hermitian structures
$$ϙ^q_M(X) := \mathrm{Map}_{R\otimes R}(X \otimes X, M)_{hC_2} \qquad \text{and} \qquad ϙ^s_M(X) := \mathrm{Map}_{R\otimes R}(X \otimes X, M)^{hC_2}.$$
These are given by taking the genuine modules with involution $(M,0,0)$ and $(M, M^{tC_2}, \mathrm{id})$ respectively.

Example 3.7 (Interpolating between quadratic and symmetric)*.* We can define further Poincaré structures $ϙ^{\geqslant m}_M(X)$ by replacing $0 \to M^{tC_2}$ and $M^{tC_2} \to M^{tC_2}$ by the connective cover $\tau_{\geqslant n} M^{tC_2} \to M^{tC_2}$ for $n \in \mathbb{Z}$. For $m = \infty$ we get $ϙ^q_M$ and for $m = -\infty$ we get $ϙ^s_M$. Note that the definition of $ϙ$ as a pullback gives fibre sequences
$$ϙ^q_M(X) \longrightarrow ϙ^{\geqslant m}_M(X) \longrightarrow \mathrm{Map}_R(X, \tau_{\geqslant m} M^{tC_2}).$$

$$\Omega_M^{\geqslant m}(X) \longrightarrow \Omega_M^s(X) \longrightarrow \mathrm{Map}_R(X, \tau_{\leqslant m-1}M^{tC_2})$$

allowing us to compare these intermediate hermitian structures to the quadratic and symmetric ones.

Example 3.8. How do you describe $M \in \mathrm{Mod}(R\otimes R)^{hC_2}$ when if R is an ordinary ring and M is an ordinary projective $R\otimes R$-module? In this case being a C_2-fixed point means we have an involution $m \mapsto \overline{m}$ satisfying $\overline{r_1 m r_2} = r_2 \overline{m} r_1$.

What if we want $M = R$, which is the easiest way to get an invertible module with genuine involution? If R admits an anti-homomorphism $\tau\colon R \to R^{\mathrm{op}}$ then we can consider R as an $R\otimes R$-module via $a(r)b = ar\tau(b)$ with involution $\overline{r} = \tau(r)$. Examples include:

- A commutative ring with automorphism of order 2, e.g. $\mathbb{C}$ with complex conjugation.
- A group ring $\mathbb{Z}[G]$ admits an anti-homomorphism determined uniquely by $\tau(g) = g^{-1}$.

More generally, these can be given by specifying an anti-homomorphism $r \mapsto \tau(r)$ and a unit ϵ of R such that $\tau^2(r) = \epsilon^{-1} r \epsilon$ and $\tau(\epsilon) = \epsilon^{-1}$; a *Wall anti-structure*. Then we can make R into a $R\otimes R$-module by $a(b)c = ab\tau(c)$ with involution given by $\overline{b} = \epsilon b$. It turns out all involutions on R are of this form.

Example 3.9. The hermitian structures associated to a module with genuine involution interacts nicely with suspension. First we can do post-compose with Σ^n: Ω is associated to (M, N, α) then $\Sigma^n \Omega(X)$ is associated to $(\Sigma^n M, \Sigma^n N, \Sigma^n\alpha)$.

Second we can combine this with pre-composition with Σ^m: Ω is associated to (M, N, α) then $\Sigma^{n+m}\Omega(\Sigma^n X)$ is associated to $(\Sigma^{m-n\sigma} M, \Sigma^m N, \Sigma^m \alpha)$ where σ is the sign representation of C_2. We say M is $n\sigma$-oriented if $\Sigma^{n\sigma - n} M \simeq M$; for example, if R is a $\mathbb{Z}$-algebra it is 2σ-oriented, by proving this in the universal case $R = \mathbb{Z}$. Thus for such R, we have an equivalence

$$\Omega^4\colon \Sigma^4 \Omega_M^\alpha(X) \xrightarrow{\simeq} \Omega_M^\alpha(X),$$

and this induces the 4-fold periodicity of L-theory groups, if we recall that $L_n(\mathcal{C}, \Omega^\alpha)$ is defined is defined as $\mathrm{coker}(\pi_n \mathrm{Pn}^\partial(\mathcal{C}, \Omega^\alpha) \to \pi_n \mathrm{Pn}(\mathcal{C}, \Omega^\alpha))$.

4. Genuine Poincaré structures

The genuine Poincaré structures on $\mathrm{Mod}^p(R)$ are given by

$$\Omega_M^{gq}(X) := \Omega_M^{\geqslant 2}(X), \quad \Omega_M^{ge}(X) := \Omega_M^{\geqslant 1}(X) \qquad \text{and} \qquad \Omega_M^{gs}(X) := \Omega_M^{\geqslant 0}(X).$$

In this section, we will outline why these particular cases are special.

The starting point is with the ordinary additive category $\mathrm{Proj}(R)$ of finitely-generated projective R-modules. A reduced functor $\mathrm{Proj}(R) \to \mathcal{C}$ is *2-polynomial* if its cross effect is additive in each entry separately.

Theorem 4.1. *The inclusion* $\mathrm{Proj}(R) \to \mathrm{Mod}^p(R)$ *induces an equivalence*

$$\mathrm{Map}^{\text{2-exc}}(\mathrm{Mod}^p(R)^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \longrightarrow \mathrm{Map}^{\text{2-poly}}(\mathrm{Proj}(R)^{\mathrm{op}}, \mathcal{S}\mathrm{p}).$$

Proof sketch. The inverse is given by first extending to $\mathrm{Mod}^p(R)_{\geqslant 0}$ through the Dold–Kan equivalence $\mathrm{Mod}^p(R)_{\geqslant 0} \simeq \mathrm{Proj}(R)^{\Delta^{\mathrm{op}}}$ and applying the functor levelwise, and extending it to $\mathrm{Mod}^p(R)$ by right Kan extension. □

Consider now the 2-polynomial functors $\mathrm{Proj}(R)^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ given by

$$F_M^{gq}(P) := \mathrm{Hom}_{R\otimes R}(P \otimes P, M)_{C_2},$$
$$F_M^{qs}(P) := \mathrm{Hom}_{R\otimes R}(P \otimes P, M)^{C_2},$$
$$F_M^{ge}(P) := \mathrm{im}(\mathrm{Nm}\colon F^{gp}(P) \to F^{gs}(P)).$$

The first of these are nothing but classical quadratic and symmetric forms, an easy exercise you should do. The previous theorem extracts "non-abelian derived" Hermitian structures from these:

Theorem 4.2. *The 2-excisive reduced functors associated to F_M^{gq}, F_M^{ge}, F_M^{gs} are Ω_M^{gq}, Ω_M^{ge}, Ω_M^{gs}.*

Proof sketch. Let us focus on the symmetric case. It suffices to verify the functor associated to F_M^{gs} agrees with Ω_M^{gs} on finitely-generated projective R-modules P. We have that $\mathrm{Map}_{R\otimes R}(P \otimes P, M)$ is concentrated in degree zero, so $\Omega_M^q(P) = \mathrm{Map}_{R\otimes R}(P \otimes P, M)^{hC_2}$ is concentrated in non-positive degrees and agrees with F_M^{qs} in degree 0. Then the fibre sequence

$$\Omega_M^{\geqslant 0}(P) \longrightarrow \Omega_M^s(P) \longrightarrow \mathrm{Map}_R(P, \tau_{\leqslant -1} M^{tC_2})$$

kills off what remains of the linear part. □

The reason we can about these is the following theorem:

Theorem 4.3 (Hebestreit–Steimle)**.** $\mathrm{GW}^\lambda(R)^{\mathrm{cl}} \simeq \mathrm{GW}(\mathrm{Mod}^p(R), \Omega^{g\lambda})$ *for* $\lambda \in \{q, e, s\}$*, where the superscript* $(-)^{\mathrm{cl}}$ *denotes we take the group-completion of the symmetric monoidal category of unimodular forms.*