Copenhagen lectures on
an operadic setup for embedding calculus and its variants

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Abstract

These are the lectures notes for the Copenhagen workshop on homotopical methods in manifolds theory in 2024.

Several recent applications of homotopy theory to manifold theory rely on Goodwillie—Weiss' embedding calculus, which is a homotopy-theoretic tool to study spaces of smooth embeddings. This series of lectures introduces from scratch a unifying higher-categorical setup for embedding calculus and its variants (such as a version for topological embeddings or Boavida de Brito–Weiss' theory of configuration categories), ready to be used for further applications (as exemplified by the accompanying lectures series). Everything discussed is joint with Manuel Krannich unless mentioned otherwise; this is based on [KK24b, KK24c, KK24d].

Contents

Chapter 1

Calculus of embeddings

1.1 Spaces of embeddings

Let M and N be smooth manifolds of dimension d, for the moment without boundary. A continuous map $M \to N$ is an *embedding* if it is a homeomorphism onto its image and locally, with respect to charts in the domain and target, it is given by the identity map $\mathrm{id}_{\mathbf{R}^d} \colon \mathbf{R}^d \to \mathbf{R}^d$ of Euclidean spaces. The latter implies implies that an embedding it is smooth with invertible derivatives, and conversely, by the inverse function theorem, a smooth map with invertible derivatives is locally of the form $\mathrm{id}_{\mathbf{R}^d} \colon \mathbf{R}^d \to \mathbf{R}^d$.

Definition 1.1.1. $\mathrm{Emb}^o(M, N)$ is the space of smooth embeddings, topologised as a subspace of the space of smooth maps $C^\infty(M, N)$ in the smooth topology.

We can also allow the manifolds to have boundary or corners. Convenient for our purposes is the notion of a *triad*: a manifold M with its boundary ∂M divided into two codimension zero submanifolds $\partial_0 M, \partial_1 M$ meeting at corners $\partial_{01} M = \partial_0 M \cap \partial_1 M$. For an embedding of a triad we then require that:

- $\partial_0 M$ goes into ∂N (locally modelled on the inclusion $\mathbf{R}_{\geqslant 0} \times \mathbf{R}^{d-1} \hookrightarrow \mathbf{R}_{\geqslant 0} \times \mathbf{R}^{d-1}$),
- $\mathrm{int}\, \partial_1 M$ goes into $\mathrm{int}\, N$ (locally modelled on the inclusion $\mathbf{R}_{\geqslant 0} \times \mathbf{R}^{d-1} \hookrightarrow \mathbf{R}^d$), and
- $\partial_{01} M$ necessarily goes into ∂N (locally modelled on the inclusion $\mathbf{R}^2_{\geqslant 0} \times \mathbf{R}^{d-2} \hookrightarrow \mathbf{R}_{\geqslant 0} \times \mathbf{R}^{d-1}$).

In this lecture, for simplicity we will have $\partial_0 M = \varnothing$ unless stated otherwise, but later we will need the above generality.

Example 1.1.2. For $M = N$, the topological group of diffeomorphisms $\mathrm{Diff}(N) \subset \mathrm{Emb}^o(N, N)$ agrees with the subspace of smooth embeddings admitting a strict inverse. If N is closed, or more generally compact and $\partial_0 N = \partial N$, all embeddings do and the inclusion is an equality.

Example 1.1.3. The set of path components of $\mathrm{Emb}^o(S^1 \times D^2, S^3)$ is the set of framed knots up to isotopy. More generally, high-dimensional knot theory is about the set of path components of $\mathrm{Emb}^o(S^d \times D^2, S^{d+2})$.

Example 1.1.4. The inclusions

$$\mathrm{GL}_d(\mathbf{R}) \longrightarrow \mathrm{Diff}(\mathbf{R}^d) \longrightarrow \mathrm{Emb}^o(\mathbf{R}^d, \mathbf{R}^d)$$

are equivalences, with inverse r given by taking the derivative at the origin.

Even if we are only interested in manifolds or moduli spaces thereof, we can not avoid studying embeddings. A manifold can be understood through embeddings into it, and its diffeomorphisms by letting these act on embeddings. The latter frequently uses the *isotopy extension theorem*: if M is compact, and N has no boundary, then the map

$$\mathrm{Diff}(N) \longrightarrow \mathrm{Emb}^o(M, N),$$

given by acting on a fixed embedding $e_0\colon M \to N$, is a Serre fibration. Its fibre over an embedding $e\colon M \to N$ is non-empty if and only if there exists a diffeomorphism φ of N so that $\varphi e_0 = e$. If so, the fibre is given by the group $\mathrm{Diff}_\partial(N - \mathrm{int}\, e(M))$ of diffeomorphisms of the complement of the interior of $e(M)$ that agree with the identity on $\partial e(M)$ (with all derivatives, strictly speaking), and the fibre-inclusion map

$$\mathrm{Diff}_\partial(N - \mathrm{int}\, e(M)) \longrightarrow \mathrm{Diff}(N)$$

is given by extension-by-identity. Thus an informal interpretation of embedding spaces is as "differences between diffeomorphism groups."

Topological variant

The definition of an embedding makes sense for topological manifolds as well, and is historically known as a "locally flat" embedding. We can then define a topological space $\mathrm{Emb}^t(M,N)$ as the subspace of the space of continuous maps $C^0(M,N)$ in the compact-open topology. Topological embeddings are to homeomorphisms as smooth embeddings are to diffeomorphisms, as the isotopy extension theorem holds by [EK71].

Example 1.1.5. Kister's theorem says that the following inclusions are equivalences [Kis64]:

$$\mathrm{Top}(d) \longrightarrow \mathrm{Homeo}(\mathbf{R}^d) \longrightarrow \mathrm{Emb}^t(\mathbf{R}^d, \mathbf{R}^d)$$

where $\mathrm{Top}(d)$ is the space of germs of homeomorphisms of $\mathbf{R}^d$ near the origin. A problem for the third lecture concerns its proof (not Kister's original one but one due to Siebenmann that casts it as a uniqueness result).

Remark 1.1.6. There is of course a continuous map $\mathrm{Emb}^o(M,N) \to \mathrm{Emb}^t(M,N)$, forgetting that a smooth embedding is smooth. Understanding this in homotopy-theoretic terms is part of smoothing theory, which will make an appearance in the third lecture.

1.2 Calculus of embeddings

Goodwillie–Weiss' calculus of embedding attempts—and often succeeds—to give a homotopy-theoretic description of $\mathrm{Emb}^c(M,N)$ for $c \in \{o,t\}$; the original references are [Wei99, BdBW13]. We now give a crash-course on it, only to reinterpret it in the next section.

1.2.1 Setup

As a starting point, we gather all d-dimensional manifolds and embeddings between these into a topologically-enriched 1-category Man^c_d:

- The objects are d-dimensional manifolds M.
- The mapping space $\mathrm{Map}_{\mathsf{Man}^c_d}(M,N)$ is the space of embeddings $\mathrm{Emb}^c(M,N)$.
- The composition is given by composition of embeddings.

Taking coherent nerve, this yields an ∞-category $\mathcal{M}\mathrm{an}^c_d$.

Convention 1.2.1. In this lecture series we will *always* work ∞-categorically unless stated otherwise, as this is the most convenient setting for our higher-algebraic techniques. Thus, *category means ∞-category, operad means ∞-operad, etc.* If you are not familiar with this, not much is lost by pretending ∞-categories are topologically-enriched 1-categories and ∞-operad are topologically-enriched 1-operads.

To a d-dimensional manifold M we can associate a space-valued presheaf

$$\begin{aligned} E_M\colon (\mathcal{M}\mathrm{an}^c_d)^{\mathrm{op}} &\longrightarrow \mathcal{S} \\ M' &\longmapsto \mathrm{Map}_{\mathcal{M}\mathrm{an}^c_d}(M', M) = \mathrm{Emb}^c(M', M), \end{aligned}$$

which tautologically recovers spaces of embeddings into M. This is of course nothing but the image of M under the Yoneda embedding

$$\mathcal{M}\mathrm{an}^c_d \xrightarrow{y} \mathrm{PSh}(\mathcal{M}\mathrm{an}^c_d)$$

and the Yoneda lemma says that the functoriality of y induces an equivalence

$$\mathrm{Emb}^c(M,N) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{PSh}(\mathcal{M}\mathrm{an}^c_d)}(E_M, E_N)$$

with inverse given by evaluation at $\mathrm{id}_M \in E_M(M) = \mathrm{Emb}^c(M,M)$.

Embedding calculus then asks to what extend we can recover E_M and maps into it—which is all there is to know about embeddings into M—from its restriction to finite disjoint unions of open discs. Weiss once called it a "pointillistic" point of view of manifolds, per the following example:

Example 1.2.2. The value of E_M on $S \times \mathbf{R}^d$ for a finite set S is given by the map

$$E_M(S \times \mathbf{R}^d) \simeq \mathrm{Emb}(S \times \mathbf{R}^d, M) \xrightarrow{\simeq} \mathrm{Conf}_S(M) \times_{M^S} \mathrm{Fr}^c(TM)^S$$

taking the value and derivative at $S \times \{0\}$. Here we define $\mathrm{Fr}^c(TM)$ to be the space of germs of embeddings $\mathbf{R}^d \to M$, a convenient model for the frame bundle, and the target is the space of S-labelled configuration of points in M, each labelled by a frame in the corresponding smooth or topological tangent space. In particular, for $S = \varnothing$ we get that $E_M \simeq *$; we say that E_M is a *reduced presheaf.*

More precisely, we let $\mathcal{D}\mathrm{isc}^c_d \subset \mathcal{M}\mathrm{an}^c_d$ be the full subcategory on all objects equivalent to $S \times \mathbf{R}^d$ for a finite set S, and we consider the image of M under the composition of the Yoneda embedding and restriction along $i\colon \mathcal{D}\mathrm{isc}^c_d \to \mathcal{M}\mathrm{an}^c_d$:

$$\mathcal{M}\mathrm{an}^c \xrightarrow{y} \mathrm{PSh}(\mathcal{M}\mathrm{an}^c_d) \xrightarrow{i^*} \mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_d), \tag{1.1}$$

where we can add superscript $(-)^{\mathrm{red}}$ indicating the full subcategories reduced presheaves if we prefer.

Definition 1.2.3. The *embedding calculus approximation* is the map

$$\mathrm{Emb}^c(M,N) = \mathrm{Map}_{\mathcal{M}\mathrm{an}^c_d}(M,N) \longrightarrow T_\infty \mathrm{Emb}^c(M,N) \coloneqq \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_d)}(E_M, E_N)$$

induced by (1.1) on mapping spaces.

Let us now discuss some of the important features of this construction; others appear in the problems.

1.2.2 Universal property

There are many presheaves F in $\mathrm{PSh}(\mathcal{M}\mathrm{an}^c_d)$ that are not of the form E_M, e.g. $P \mapsto \mathrm{Map}(P, M)$, and it is interesting and useful to also consider

$$T_\infty F(M) \coloneqq \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_d)}(E_M, F).$$

Moreover, one may recognise this as the right Kan extension along i,

$$\begin{tikzcd}
\mathcal{D}\mathrm{isc}^c_d \arrow[r, "i^* F"] \arrow[d, "i"'] & \mathcal{S} \\
\mathcal{M}\mathrm{an}^c_d \arrow[ur, dashed, "i_* i^* F \simeq T_\infty F"']
\end{tikzcd}$$

and the natural transformation $F \to i_* i^* F$ as the unit of the adjunction $i^* \dashv i_*$. This is the first characterisations of T_∞, the second one identifies it with $\mathcal{J}_\infty$-sheafification [BdBW13].

1.2.3 Convergence

The theorem that makes considering (1.1) worthwhile, is the following improvement of the celebrated convergence result of Goodwillie–Klein–Weiss [GW99, GK15]. They proved the result when M has handle dimension $\leqslant d-3$, and the improvement is obtained by handle arguments using the isotopy extension for embedding calculus [KK24a].

Definition 1.2.4. We say that a smooth embedding $P \to Q$ is *an equivalence on tangential k-types* if there is a factorisation $Q \to B \to \mathrm{BO}$ or BTop of the stable tangent classifier so that $Q \to B$ and $P \to B$ are both k-connected.

Remark 1.2.5. A factorisation $Q \to B \to \mathrm{BO}$ so that $Q \to B$ is k-connected always exists, by taking a Moore–Postnikov factorisation.

Example 1.2.6. If Q is 1-connected spin, then $P \to Q$ is an equivalence on tangential 2-types if and only if P is also 1-connected spin.

Theorem 1.2.7 (Goodwillie–Klein–Weiss, Krannich–K., improved smooth convergence)**.** *If $d \geqslant 5$, M is compact, and $\partial M \to M$ is an equivalence on tangential 2-types, then the map*

$$\mathrm{Emb}^o(M,N) \longrightarrow T_\infty \mathrm{Emb}^o(M,N)$$

is equivalence. We say that embedding calculus converges*.*

The problems explain how to reduce the original Goodwillie–Klein–Weiss result to this.

Example 1.2.8. $\mathbf{C}P^{2n+1} - \mathrm{int}\, D^{4n+2}$ satisfies the hypothesis for the theorem but has handle dimension $4n$ so does not satisfy the hypothesis for the original Goodwillie–Klein–Weiss convergence result.

Remark 1.2.9. In low dimensions the state of the art is as follows: embedding calculus always converges in dimension $d \leqslant 2$ [KK21], and in dimension $d = 3, 4$ if the handle dimension of M is $\leqslant d-3$, the Goodwillie–Klein–Weiss result.

The same result is true for topological embeddings as long as the target admits a smooth structure (this hypothesis is conjectured to be unnecessary), as we will explain in the third lecture.

1.2.4 Tower and layers

Given the above convergence result and its announced topological counterpart, to understand $\mathrm{Emb}^c(M,N)$ one can compute $T_\infty \mathrm{Emb}^c(M,N)$ instead. To do so, we uses that as the subscript ∞ indicates, $T_\infty \mathrm{Emb}^c(M,N)$ is a limit of finite approximations. Let $\mathcal{D}\mathrm{isc}^c_{\leqslant k} \subset \mathcal{D}\mathrm{isc}^c$ be the full subcategory on objects equivalent to $S \times \mathbf{R}^d$ for a finite set of cardinality $\leqslant k$. Then we have a sequence of subcategory inclusions

$$\mathcal{D}\mathrm{isc}^c_{\leqslant 1} \subset \mathcal{D}\mathrm{isc}^c_{\leqslant 2} \subset \cdots \subset \mathcal{D}\mathrm{isc}^c_{\leqslant \infty} = \mathcal{D}\mathrm{isc}^c$$

exhibiting $\mathcal{D}\mathrm{isc}^c$ as the colimit of the $\mathcal{D}\mathrm{isc}^c_{\leqslant k}$ for finite k. Hence there is a corresponding tower of restriction functors between presheaf categories

$$\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant 1}) \longleftarrow \mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant 2}) \longleftarrow \cdots \longleftarrow \mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant \infty}) = \mathrm{PSh}(\mathcal{D}\mathrm{isc}^c),$$

exhibiting $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c)$ as the limit of $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant k})$ for finite k. On mapping spaces we get a tower of finite approximations

$$\begin{tikzcd}
 & T_\infty\mathrm{Emb}^c(M,N) := \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c)}(E_M,E_N) \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_2\mathrm{Emb}^c(M,N) := \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant 2})}(E_M,E_N) \arrow[d] \\
\mathrm{Emb}^c(M,N) \arrow[r] \arrow[ur] \arrow[uuur] & T_1\mathrm{Emb}^c(M,N) := \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant 1})}(E_M,E_N)
\end{tikzcd}$$

satisfying

$$T_\infty\mathrm{Emb}^c(M,N) \xrightarrow{\simeq} \lim_{k\to\infty} T_k\mathrm{Emb}^c(M,M).$$

These satisfy the analogous universal properties: the natural transformation $F \to T_kF = \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant k})}(E_M,F)$ is the unit of the adjunction $(i_k)^* \dashv (i_k)_*$ for the inclusion $i_k\colon \mathcal{D}\mathrm{isc}^c_{\leqslant k} \to \mathcal{M}\mathrm{an}^c$, and can be interpreted as $\mathcal{J}_k$-sheafification.

This focuses our attention on the first stage $T_1\mathrm{Emb}^c(M,M)$ and the layers

$$\mathrm{fib}_x\big[T_k\mathrm{Emb}^c(M,N) \to T_{k-1}\mathrm{Emb}^c(M,N)\big] \qquad \text{for } x \in T_{k-1}\mathrm{Emb}^c(M,N).$$

If we could understand these, we could inductively compute $T_k\mathrm{Emb}^c(M,N)$ and take a limit to compute $T_\infty\mathrm{Emb}^c(M,N)$. This is indeed possible: the higher layers are covered in the third lecture, but the first stage we can do now.

Right Kan extension identifies $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{=1})$, where $\mathcal{D}\mathrm{isc}^c_{=1} \subset \mathcal{D}\mathrm{isc}$ is the full subcategory on objects equivalent to $\mathbf{R}^d$, with $\mathrm{PSh}^{\mathrm{red}}(\mathcal{D}\mathrm{isc}^c_{\leqslant 1})$. Since E_M is always reduced, we get that

$$\mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{=1})}(E_M,E_N) \simeq \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant 1})}(E_M,E_N).$$

The category $\mathcal{D}\mathrm{isc}^c_{=1}$ is equivalent to the groupoid with a single object and morphisms given by either $\mathrm{GL}_d(\mathbf{R}) \simeq \mathrm{O}(d)$ or $\mathrm{Top}(d)$, and by unstraightening the presheaf categories are equivalent to the categories $\mathcal{S}_{/\mathrm{BO}(d)}$ or $\mathcal{S}_{/\mathrm{BTop}(d)}$ of spaces over $\mathrm{BO}(d)$ or $\mathrm{BTop}(d)$. That is,

$$\mathrm{PSh}^{\mathrm{red}}(\mathcal{D}\mathrm{isc}^c_{\leqslant 1}) \simeq \begin{cases} \mathcal{S}_{/\mathrm{BO}(d)} & \text{if } c = o, \\ \mathcal{S}_{/\mathrm{BTop}(d)} & \text{if } c = t. \end{cases}$$

Under this equivalence, E_M corresponds to M with its tangent classifier and the first stage can be identified with

$$T_1\mathrm{Emb}^c(M,N) \xrightarrow{\simeq} \begin{cases} \mathrm{Map}^{/\mathrm{BO}(d)}(M,N) & \text{if } c = o, \\ \mathrm{Map}^{/\mathrm{BTop}(d)}(M,N) & \text{if } c = t, \end{cases}$$

and equivalently spaces of bundle maps.

1.3 Operadic calculus

Having gotten a first look at embedding calculus, we will reinterpret it as a construction associated to the framed little d-discs operad.

1.3.1 Operads and envelopes

Operads encode certain types of algebraic structures. A prototypical example is the operad E_d^o, unfortunately known as the *(topologically) framed little d-discs operad.* Working in the simplest model of (single-coloured) 1-operads, it has for each $k \geqslant 0$ a *space of k-ary operations* given by

$$E_d^c(k) := \mathrm{Emb}^c(\sqcup_k \mathbf{R}^d, \mathbf{R}^d).$$

Permuting the input induces a Σ_k-action, composition of embeddings induces composition maps

$$E_d^c(k) \times \prod_{i=1}^{k} E_d^c(\ell_i) \longrightarrow E_d^c(\ell_1 + \cdots + \ell_k)$$

that are suitably equivariant, and $\mathrm{id}_{\mathbf{R}^d} \in E_d^o(1)$ acts a unit for these. By taking Lurie's operadic nerve, one extracts from this an operad $\mathcal{E}_d^c$.

Remark 1.3.1. E_d^c is the endomorphism operad of $\mathbf{R}^d$ in $\mathcal{M}\mathrm{an}_d^c$.

Example 1.3.2. $E_d^o(k)$ is equivalent to $\mathrm{Conf}_k(\mathbf{R}^d) \times \mathrm{O}(d)^k$ and $E_d^t(k)$ is equivalent to $\mathrm{Conf}_k(\mathbf{R}^d) \times \mathrm{Top}(d)^k$, and during operadic composition the $\mathrm{O}(d)$ and $\mathrm{Top}(d)$ act on the spaces of configurations. We will see momentarily that they are in a precise sense "semi-direct product" operads.

In general, an operad $\mathcal{O}$ can have more "colours" to serve as inputs and outputs: for all collections of colours $c_1, \ldots, c_k, c$, one has a space of multi-operations $\mathrm{Mul}^{\mathcal{O}}((c_1, \ldots, c_k); c)$ and composition is only possible if the colours line up. This generality is necessary if we want to interpret a symmetric monoidal category $(\mathcal{C}, \otimes)$ as an operad: the colours are the objects of $\mathcal{C}$ and multi-operations are $\mathrm{Mul}^{\mathcal{C}}((x_1, \ldots, x_k); x) = \mathrm{Map}_{\mathcal{C}}(x_1 \otimes \cdots \otimes x_k, x)$.

This gives an inclusion

$$\mathrm{CMon}(\mathcal{C}\mathrm{at}) \longrightarrow \mathcal{O}\mathrm{p}$$

of symmetric monoidal categories into operads, which admits a left adjoint $\mathrm{Env}\colon \mathcal{O}\mathrm{p} \to \mathrm{CMon}(\mathcal{C}\mathrm{at})$, known as the *symmetric monoidal envelope.* Informally, the objects of $\mathrm{Env}(\mathcal{O})$ are pairs $(c_s)_{s \in S}$ of a finite set S and a colour c_s for each elements $s \in S$ and the mapping spaces decompose as

$$\mathrm{Map}_{\mathrm{Env}(\mathcal{O})}((c_s)_{s \in S}, (c_t)_{t \in T}) \simeq \bigsqcup_{\phi\colon S \to T} \prod_{t \in T} \mathrm{Mul}^{\mathcal{O}}((c_s)_{s \in \phi^{-1}(t)}, c_t),$$

with composition induced by the operad composition. The symmetric monoidal structure is given on object by formal disjoint union and on morphisms by taking products. The following is the crucial observation:

Proposition 1.3.3. *There is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Env}(\mathcal{E}_d^c) \xrightarrow{\simeq} \mathcal{D}\mathrm{isc}^c.$$

Proof. Recognising $\mathcal{E}_d^c$ as the endomorphism operad of $\mathbf{R}^d$ in the symmetric monoidal category $\mathcal{D}\mathrm{isc}^c$, where the monoidal structure is given by disjoint union, the universal property of the symmetric monoidal envelope induces a symmetric monoidal functor $\mathrm{Env}(\mathcal{E}_d^c) \to \mathcal{D}\mathrm{isc}^c$. This is essentially surjective since both have objects corresponding by finite sets and fully faithful because the map

$$\bigsqcup_{\phi\colon S \to T} \prod_{t \in T} \mathrm{Emb}(\sqcup_{\phi^{-1}(t)} \mathbf{R}^d, \mathbf{R}^d) \longrightarrow \mathrm{Emb}(\sqcup_S \mathbf{R}^d, \sqcup_T \mathbf{R}^d)$$

is an equivalence, dividing the right into terms corresponding to the underlying map of finite sets of path components. ☐

1.3.2 Operadic calculus

The previous observation leads us to consider for an operad $\mathcal{O}$ the category of "right modules"

$$\mathrm{RMod}(\mathcal{O}) \coloneqq \mathrm{PSh}(\mathrm{Env}(\mathcal{O})),$$

and develop a "calculus" satisfying the following desiderata:

(A) A tower.
(B) A description of the first stage.
(C) A description of the layers.
(D) All the above natural as natural as possible.

Desideratum (A) is straightforward to satisfy: the commutative operad $\mathcal{C}\mathrm{om}$ has a single colour and all multi-operations given by a point, and hence is the terminal operad. It satisfies $\mathrm{Env}(\mathcal{C}\mathrm{om}) \simeq \mathrm{Fin}$, the category of finite sets and all maps between these, so there is a canonical symmetric monoidal functor $\mathrm{Env}(\mathcal{O}) \to \mathrm{Fin}$. Letting $\mathrm{Fin}_{\leqslant k} \subset \mathrm{Fin}$ be the full subcategory of finite sets of cardinality $\leqslant k$, we can form

$$\mathrm{Env}(\mathcal{O})_{\leqslant k} \coloneqq \mathrm{Env}(\mathcal{O}) \times_{\mathrm{Fin}} \mathrm{Fin}_{\leqslant k} \qquad \text{and} \qquad \mathrm{RMod}_k(\mathcal{O}) \coloneqq \mathrm{PSh}(\mathrm{Env}(\mathcal{O})_{\leqslant k}).$$

The inclusions $\mathrm{Fin}_{\leqslant 1} \subset \mathrm{Fin}_{\leqslant 2} \subset \cdots \subset \mathrm{Fin}_{\leqslant \infty} = \mathrm{Fin}$ then induce a tower

$$\mathrm{RMod}_1(\mathcal{O}) \longleftarrow \mathrm{RMod}_2(\mathcal{O}) \longleftarrow \cdots \longleftarrow \mathrm{RMod}(\mathcal{O}).$$

Similarly, Desideratum (B) is straightforward to satisfy if $\mathcal{O}$ is groupoid-coloured (that is, spaces of 0-ary operations are contractible and all 1-ary operations invertible) and we restrict to reduced right modules $\mathrm{RMod}_1^{\mathrm{red}}(\mathcal{O}) \coloneqq \mathrm{PSh}^{\mathrm{red}}(\mathrm{Env}(\mathcal{O})_{\leqslant 1})$, as the latter can be identified with $\mathrm{PSh}(\mathcal{O}^{\mathrm{col}})$ where $\mathcal{O}^{\mathrm{col}}$ is the groupoid of colours of $\mathcal{O}$, and using unstraightening we get

$$\mathrm{RMod}_1^{\mathrm{red}}(\mathcal{O}) \simeq \mathcal{S}_{/\mathcal{O}^{\mathrm{col}}}.$$

1.3.3 More operads

The operadic approach will be powerful for two reasons. Firstly, phrasing everything more generally in terms of operads and right modules is clarifying. Secondly, there are many operads related to $\mathcal{E}_d^c$ that lack a geometric incarnation. We will describe two constructions of such.

You may be more familiar with $\mathcal{E}_d$, the little d-discs operad, than $\mathcal{E}_d^c$; here we only allow embeddings that preserve a framing or equivalently are rectilinear. There is a general theory of tangential structures for reduced operads $\mathcal{O}$, i.e. those where the spaces of 0-ary and 1-ary multioperations are contractible: for any map $\theta\colon B \to \mathrm{BAut}_{\mathcal{O}\mathrm{p}}(\mathcal{O})$ (we will suppress the subscript $\mathcal{O}\mathrm{p}$ from now on if it is clear from the context we consider automorphisms of an operad) one can form the colimit

$$\mathcal{O}^\theta \coloneqq \operatorname*{colim}_B \mathcal{O} \in \mathcal{O}\mathrm{p},$$

which one should interpret as a semi-direct product $\Omega B \ltimes \mathcal{O}$, if B is connected, because $\mathrm{Mul}^{\mathcal{O}^\theta}(k) \simeq \mathcal{O}(k) \times \Omega B^k$.

Example 1.3.4. We have

$$\begin{aligned} \mathcal{E}_d^o &\simeq (\mathcal{E}_d)^o && \text{for } o\colon \mathrm{BO}(d) \to \mathrm{BAut}(\mathcal{E}_d), \\ \mathcal{E}_d^t &\simeq (\mathcal{E}_d)^t && \text{for } t\colon \mathrm{BTop}(d) \to \mathrm{BAut}(\mathcal{E}_d) \end{aligned}$$

for maps o, t constructed as follows, doing only the latter for brevity. If we take the definition of $\mathcal{E}_d^t$ but no longer require the discs to be disjointly embedded, we get an operad $\mathcal{E}_d^{t,\sqcup}$

with $\mathcal{E}_d^{t,\sqcup}(k) = \prod_k \mathrm{Emb}^t(\mathbf{R}^d, \mathbf{R}^d)$, which is equivalent to $\prod_k \mathrm{Top}(d)$. There is inclusion of operads

$$\mathcal{E}_d^t \longrightarrow \mathcal{E}_d^{t,\sqcup}$$

and taking fibres over the identity elements we recover an operad equivalent to $\mathcal{E}_d$. Now we observe that this map is equivariant for the action of $\mathrm{Top}(d)$ on $\mathcal{E}_d^t$ and $\mathcal{E}_d^{t\sqcup}$ by conjugation, given by sending $e \in \mathcal{E}_d^t(k) = \mathrm{Emb}^t(\sqcup_k \mathbf{R}^d, \mathbf{R}^d)$ to $\phi \circ e \circ (\sqcup_k \phi^{-1})$ and similarly for $\mathcal{E}_d^{t,\sqcup}(k)$. This fixes the points $(\mathrm{id}_{\mathbf{R}^d}, \dots, \mathrm{id}_{\mathbf{R}^d})$ so we get an induced action $\mathrm{Top}(d)$ on $\mathcal{E}_d$.

However, in the accompanying lecture we will need the follow novel variant:

$$\mathcal{E}_d^p \simeq (\mathcal{E}_d)^{\mathrm{id}} \qquad \text{for } p := \mathrm{id}\colon \mathrm{BAut}(\mathcal{E}_d) \to \mathrm{BAut}(\mathcal{E}_d).$$

Here the superscript p stands for "particle" because of its close relationship to configuration categories as in [BdBW18], to be explained in [KK24c].

Remark 1.3.5. An operad is unital if its spaces of 0-ary multioperations are contractible. Lurie's work of disintegration and assembly allows one to prove that any unital operad $\mathcal{O}$ whose category of colours $\mathcal{O}^{\mathrm{col}}$ (given by taking the objects and 1-ary operations between these) is an ∞-groupoid arises by adding a tangential structure to a reduced operad. More precisely, there is an equivalence

$$\mathcal{O}\mathrm{p}^{\mathrm{gc}} \simeq \int_{\mathcal{S}} \mathrm{Fun}(-, \mathcal{O}\mathrm{p}^{\mathrm{red}}),$$

where $(-)^{\mathrm{gc}}$ denotes "groupoid-coloured" operads (unital so that $\mathcal{O}^{\mathrm{col}}$ is a groupoid) and $(-)^{\mathrm{red}}$ reduced operads.

Secondly, we can k-truncate operads by discarding multi-operations of arity exceeding k. When we restrict to unital operads, i.e. those where the spaces of 0-ary operations are contractible, there are functors

$$\mathcal{O}\mathrm{p}^{\mathrm{un}} = \mathcal{O}\mathrm{p}^{\mathrm{un}}_{\leqslant\infty} \longrightarrow \cdots \longrightarrow \mathcal{O}\mathrm{p}^{\mathrm{un}}_{\leqslant 2} \longrightarrow \mathcal{O}\mathrm{p}^{\mathrm{un}}_{\leqslant 1},$$

which admit both left and right adjoints. By truncating and applying the right adjoint we can form the reduced operads $(\mathcal{E}_d)_{\leqslant k}$ and then we can modify these by taking tangential structures. The operads obtained this way do not necessarily arise by taking a tangential structure on $\mathcal{E}_d$ and truncating, e.g. when we taking the tangential structure $\mathrm{id}\colon \mathrm{BAut}((\mathcal{E}_d)_{\leqslant k}) \to \mathrm{BAut}((\mathcal{E}_d)_{\leqslant k})$.

1.4 Problems

The following problems are intended to illustrate or extend the material of the lectures. Parts labelled by $(*)$ require concepts not explained in the lectures or the problems.

Problem 1 (Isotopy equivalence)**.** We say that M and M' are *isotopy-equivalent* if there are embeddings $M \hookrightarrow M'$ and $M' \hookrightarrow M$ that are inverse up to isotopy (i.e. M and M' are isomorphic in $\mathcal{M}\mathrm{an}_d^c$).

(a) Use a collar to prove that every compact manifold M is isotopy equivalent to its interior.

(b) Prove that if M and M' are isotopy equivalent then

$$\mathrm{Emb}^c(M, N) \simeq \mathrm{Emb}^c(M', N).$$

Problem 2 (Smooth Kister's theorem)**.** Prove Example 1.1.4.

Problem 3 (The micro-extension trick)**.** There is of course no issue with defining embeddings between manifolds that are not of the same dimension. However, considering only codimension zero embeddings comes at little loss of generality by the following arguments:

(a) Using the tubular neighbourhood theorem, prove for every smooth embedding $e\colon M \to N$ between manifolds of dimension d and d' respectively, there is a unique vector bundle ξ over M so that e extends to an embedding $E\colon D(\xi) \to N$ of the total space of a $(d'-d)$-disc bundle in ξ.

(b) Let $\mathrm{Bun}(-,-)$ denote the space of bundle monomorphisms. Prove there is a pullback square

$$\begin{tikzcd}
\mathrm{Emb}(D(\xi), N) \arrow[r] \arrow[d] & \mathrm{Bun}(TM \oplus \xi, TN) \arrow[d] \\
\mathrm{Emb}(M, N) \arrow[r] & \mathrm{Bun}(TM, TN).
\end{tikzcd}$$

(c) $(*)$ What happens in the topological setting?

Problem 4.

(a) Defining $\mathrm{Fr}^c(TM)$ as the space of germs near the origin of embeddings $\mathbf{R}^d \to M$ as in Example 1.2.2, use the isotopy extension theorem to establish a fibre sequence

$$M \longleftarrow \mathrm{Fr}^c(TM) \longleftarrow \begin{cases} \mathrm{O}(d) & \text{if } c = o, \\ \mathrm{Top}(d) & \text{if } c = t. \end{cases}$$

(b) Extend this to $k > 1$.

Problem 5 ($\mathcal{J}_k$-covers and $\mathcal{J}_k$-descent)**.** Fix $1 \leqslant k \leqslant \infty$. An open cover $\mathcal{U}$ of a topological space X is a $\mathcal{J}_k$-cover if every finite subset of X of cardinality $\leqslant k$ is contained in some $U \in \mathcal{U}$.

(a) Prove that an open cover $\mathcal{U}$ is a $\mathcal{J}_k$-cover if and only if for all $j \leqslant k$ we have that $\{\mathrm{Conf}_j(U) \mid U \in \mathcal{U}\}$ is an open cover of $\mathrm{Conf}_j(X)$.

(b) $(*)$ Prove that every smooth manifold admits a $\mathcal{J}_\infty$-cover so that all finite intersections are finite disjoint unions of open discs. Is the same true topologically?[1]

Let $N\mathcal{U}$ denote the nerve of the open cover $\mathcal{U}$; the poset of ordered finite collections $(U_1, \ldots, U_n)$ of elements of $\mathcal{U}$, ordered by inclusion. Dugger–Isaksen proved that the map

$$\operatorname*{colim}_{(U_1,\ldots,U_n) \in N\mathcal{U}} (U_1 \cap \cdots \cap U_n) \longrightarrow X$$

is an equivalence (here we use an ∞-categorical colimit, i.e. a homotopy colimit).

(c) Prove that for a $\mathcal{J}_k$-cover $\mathcal{U}$ of a d-dimensional manifold M, we have an equivalence in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_{\leqslant k})$

$$\operatorname*{colim}_{(U_1,\ldots,U_n) \in N\mathcal{U}} E_{U_1 \cap \cdots \cap U_n} \longrightarrow E_M.$$

(d) Prove that if $\mathcal{U}$ is a $\mathcal{J}_k$-cover of M then for any $F \in \mathrm{PSh}(\mathcal{M}\mathrm{an}^c)$ the map

$$T_k F(M) \longrightarrow \lim_{(U_1,\ldots,U_n) \in N\mathcal{U}} T_k F(U_1 \cap \cdots \cap U_n)$$

is an equivalence.

(e) $(*)$ Extend the above to hypercovers.

Part (d) says that $T_k F$ satisfies a sheaf property for $\mathcal{J}_k$-property. [BdBW13] proves is that it is the initial $\mathcal{J}_k$-sheaf, i.e. $F \to T_k F$ is $\mathcal{J}_k$-sheafification; we suggest the interested reader peruse the paper, as it is instructive.

Problem 6 (Examples of $\mathcal{J}_\infty$-covers)**.**

(a) Prove that $\{D^k \backslash \{x\}\}_{x \in \mathrm{int}\, D^k}$ is a $\mathcal{J}_\infty$-cover of D^k.

---

[1] This seems to be an open problem.

(b) Prove that if $\{U_i\}_{i \in I}$ is a $\mathcal{J}_\infty$-cover of X then $\{U_i \times Y\}_{i \in I}$ is a $\mathcal{J}_\infty$-cover of $X \times Y$.

(c) Deduce that $\{D^{d-k} \times (D^k \backslash \{x\})\}_{x \in \mathrm{int}\, D^k}$ is a $\mathcal{J}_\infty$-cover of $D^{d-k} \times D^k$.

(d) Use (d) to prove that if M is a d-dimensional manifold with r handles of top index $k \geqslant 1$, then it admits a $\mathcal{J}_\infty$-cover with $< r$-handles of top index k.

Problem 7 (Equivalences on tangential k-type)**.**

(a) Prove that if an embedding $P \to Q$ is k-connected then it is an equivalence on tangential k-types.

(b) Prove that if an embedding $P \to Q$ is an equivalence on tangential k-types then it is $(k-1)$-connected.

(c) Prove that an embedding $D^d \hookrightarrow Q$ is an equivalence on tangential 2-types if and only if Q is 1-connected spin.

(d) Justify Example 1.2.6.

Problem 8 (Handle dimension)**.**

(a) Suppose that a d-dimensional manifold M admits a handle decomposition with only $\leqslant d-k-1$-handles. Prove that $\partial M \to M$ is k-connected.

(b) Conclude that if M has handle dimension $\leqslant d-3$ then $\partial M \to M$ is an equivalence on tangential 2-types.

Problem 9 (Smale–Hirsch–Lees)**.** The immersion $\mathrm{Imm}^c(M,N)$ differ from the embeddings $\mathrm{Emb}^c(M,N)$ by still requiring that maps are locally given by $\mathrm{id}_{\mathbf{R}^d} \colon \mathbf{R}^d \to \mathbf{R}^d$ but no longer that it is a homeomorphism onto its image.

(a) Prove that $\mathrm{Emb}^c(\mathbf{R}^d, N) \to \mathrm{Imm}^c(\mathbf{R}^d, N)$ is an equivalence.

The technical core of the Smale–Hirsch–Lees theorem is that the presheaf $\mathrm{Imm}^c(-,N) \in \mathrm{PSh}(\mathcal{M}\mathrm{an}^c_d)$ satisfies descent for $\mathcal{J}_1$-covers of manifolds that have no closed components.

(b) Use this to prove that the maps $T_1\mathrm{Emb}^c(M,N) \to T_1\mathrm{Imm}^c(M,N) \leftarrow \mathrm{Imm}^c(M,N)$ are equivalences if M has no closed components.

Chapter 2

Boundary conditions and bordism categories

In many applications we want to restrict our attention to those embeddings agree with a fixed embedding on the boundary. In this lecture we explain how to adapt embedding calculus as explained in the previous lecture to such restricted embeddings, using the language of algebras and modules.

2.1 Day convolution of presheaves

Recall that we advocated a perspective on embedding calculus as being about the category

$$\mathrm{RMod}(\mathcal{E}_d^\theta) \coloneqq \mathrm{PSh}(\mathrm{Env}(\mathcal{E}_d^\theta))$$

for a tangential structure $\theta\colon B \to \mathrm{BAut}(\mathcal{E}_d)$, e.g. take $\theta = o, t, p$ to get the (smooth, topological, "particle") framed little d-discs operads, and where $\mathrm{Env}(\mathcal{E}_d^c)$ is the symmetric monoidal envelope. In fact, more generally one may as well consider $\mathrm{RMod}(\mathcal{O}) \coloneqq \mathrm{PSh}(\mathrm{Env}(\mathcal{O}))$ for any operad $\mathcal{O}$.

We also explained that the existence of a functor $\mathrm{Env}(\mathcal{E}_d^\theta) \to \mathrm{Fin}$ allows one construct a tower of truncated right module categories $\mathrm{RMod}_k(\mathcal{E}_d^\theta)$, but we did not yet use that symmetric monoidal envelopes are symmetric monoidal (as the name and definition make clear): this will make $\mathrm{RMod}(\mathcal{E}_d^\theta)$ symmetric monoidal as well, which can be used to model gluing of manifolds in embedding calculus.

2.1.1 Day convolution

Lurie proved that if $\mathcal{C}$ is a symmetric monoidal category then $\mathrm{PSh}(\mathcal{C})$ is also, with a symmetric monoidal structure known as *Day convolution.*

Its universal property—that it the corepresents $\mathrm{Map}_{\mathrm{CMon}^{\mathrm{lax}}(\mathcal{C}\mathrm{at})}(- \times \mathcal{C}^{\mathrm{op}}, \mathcal{S})$—is useful for establishing formal properties but not so much for computations. Fortunately, Lurie also gave a formula for the Day convolution monoidal product in terms of left Kan extension

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \arrow[r, "F \times G"] \arrow[d, "\otimes"'] & \mathcal{S} \\
\mathcal{C}^{\mathrm{op}}. \arrow[ru, dashed, "F \otimes G"']
\end{tikzcd}$$

For the symmetric monoidal envelope $\mathrm{Env}(\mathcal{O})$ you can simplify this formula using a cofinality argument: the result is that for $F, G \in \mathrm{RMod}(\mathcal{O})$ (assuming for simplicity there is only one colour for brevity) one has

$$(F \otimes G)(S) \coloneqq \bigsqcup_{S = S_1 \sqcup S_2} F(S_1) \times G(S_2).$$

One feature of Day convolution, visible from the formula, is that it commutes with small colimits in each entry separately, so in particular with geometric realisations; we say the symmetric monoidal structure is *compatible with geometric realisations.*

This formula make sense also for $\mathrm{RMod}_k(\mathcal{O}) = \mathrm{PSh}(\mathrm{Env}(\mathcal{O})_{\leqslant k})$, but this does *not* fit into the above framework because $\mathrm{Env}(\mathcal{O})_{\leqslant k}$ is *not* symmetric monoidal (we can't disjoint union of discs whose total number exceeds k). The solution is to recognise the formula as $j^*(j_* F \otimes j_* G)$ for $j\colon \mathrm{Env}(\mathcal{O})_{\leqslant k} \to \mathrm{Env}(\mathcal{O})$ the inclusion, where j^* is the restriction functor and j_* its fully faithful right adjoint given by right Kan extension. and think of $\mathrm{RMod}_k(\mathcal{O})$ as a localisation of $\mathrm{RMod}(\mathcal{O})$ and its symmetric monoidal structure as the localisation of the Day convolution symmetric monoidal structure on $\mathrm{RMod}(\mathcal{O})$. With this construction, the restriction functors $\mathrm{RMod}(\mathcal{O}) \to \mathrm{RMod}_k(\mathcal{O})$ and $\mathrm{RMod}_k(\mathcal{O}) \to \mathrm{RMod}_{k-1}(\mathcal{O})$ are symmetric monoidal; they also preserve colimits and in particular geometric realisations because they admit a right adjoint; we say they are *compatible with geometric realisations.*

We can further restrict Day convolution to reduced presheaves, motived by the observation that E_M is always reduced. This is interesting especially in the case $k = 1$, as then the formula reduces to a coproduct

$$(F \otimes G)(*) \simeq F(*) \times G(\varnothing) \sqcup F(\varnothing) \times G(*) \simeq F(*) \sqcup G(*),$$

and we get an equivalence $(\mathrm{RMod}_1^{\mathrm{red}}(\mathcal{O}), \mathrm{day}) \simeq (\mathrm{PSh}(\mathcal{O}^{\mathrm{col}}), \sqcup)$ of symmetric monoidal categories.

The upshot of this discussion is that:

Proposition 2.1.1. *The tower of (truncated) module categories lifts a tower*

$$(\mathrm{RMod}(\mathcal{O}), \mathrm{day}) \longrightarrow \cdots \longrightarrow (\mathrm{RMod}_2(\mathcal{O}), \mathrm{day}) \longrightarrow (\mathrm{RMod}_1(\mathcal{O}), \mathrm{day})$$

of symmetric monoidal categories and symmetric monoidal functors compatible with geometric realisations, and similarly for reduced right modules.

2.1.2 Presheaf of a disjoint union

To see how the Day convolution symmetric monoidal structures on $\mathrm{RMod}(\mathcal{E}_d^c)$ for $c \in \{o, t\}$ relate to the geometry of manifolds, we consider a d-dimensional manifold of the form $M \sqcup M'$ and the associated presheaf $E_{M \sqcup M'} \in \mathrm{RMod}(\mathcal{E}_d^c)$. We can compute directly its values

$$E_{M \sqcup M'}(S \times \mathbf{R}^d) \simeq \mathrm{Emb}(S \times \mathbf{R}^d, M \sqcup M')$$

by observing the term on the right decomposes as a disjoint union over decompositions $S = S_1 \sqcup S_2$,

$$\mathrm{Emb}(S \times \mathbf{R}^d, M \sqcup M') \cong \bigsqcup_{S = S_1 \sqcup S_2} \mathrm{Emb}(S_1 \times \mathbf{R}^d, M) \times \mathrm{Emb}(S_2 \times \mathbf{R}^d, M')$$

because an S-labelled collection of open discs in $M \sqcup M'$ decomposes uniquely as a pair of of a collection of open discs in M and a collection of open discs in M'. This is exactly the formulate for Day convolution, and indeed E is the composition of symmetric monoidal functors

$$(\mathcal{M}\mathrm{an}^c, \sqcup) \longrightarrow (\mathrm{PSh}(\mathcal{M}\mathrm{an}^c), \mathrm{day}) \longrightarrow (\mathrm{PSh}(\mathcal{D}\mathrm{isc}^c), \mathrm{day}).$$

2.2 Calculus of embeddings with boundary

Recall from the first lecture that it is convenient to consider a manifold triad M, its boundary decomposed as $\partial_0 M \cup_{\partial_{01} M} \partial_1 M$.

Definition 2.2.1. Suppose M be a triad and we are given an embedding $e_\partial\colon \partial_0 M \to \partial N$. Then $\mathrm{Emb}^c_{\partial_0}(M, N) \subset \mathrm{Emb}^c(M, N)$ denotes the embeddings agreeing with e_∂ on $\partial_0 M$.

On the one hand, this has the advantage that we can define extension-by-identity maps (strictly speaking, in the smooth case this requires that the embedding agrees with a fixed collar near the boundary, but imposing this condition does not change the homotopy type). On the other hand, this is closely related to embedding spaces considered previously, as it fits in a fibre sequence

$$\mathrm{Emb}^{\mathrm{c}}_{\partial_0}(M,N) \longrightarrow \mathrm{Emb}^c(M,N) \longrightarrow \mathrm{Emb}^c(\partial_0 M, \partial N)$$

where in the middle term we require, per our convention for triads, that $\partial_0 M$ is mapped into ∂N. We now explain how to adapt embedding calculus to this setting.

2.2.1 Algebras and modules

In any symmetric monoidal category $\mathcal{C}$ one can define (associative) algebras as a map of operads $\mathcal{E}_1 \to \mathcal{C}$ where $\mathcal{E}_1$ is the little 1-disc operad:

$$\mathrm{Alg}(\mathcal{C}) = \mathrm{Fun}_{\mathcal{O}\mathrm{p}}(\mathcal{E}_1, \mathcal{C}).$$

Remark 2.2.2. To justify calling these associative algebras, note passing to path components induces an equivalence

$$\mathcal{E}_1 \xrightarrow{\simeq} \mathrm{Ass}$$

to the associative operad, so one can think of $\mathcal{E}_1$ as a "thickening" of Ass more suitable for interacting with manifolds.

Informally, the value on the unique colour of $\mathcal{E}_1$ specifies the underlying object A of the algebra, the value on the unique 0-ary operation a unit map $\mathbb{1} \to A$, and the value on the 2-ary operations specifies a multiplication map

$$A \otimes A \longrightarrow A.$$

The k-ary operations are obtained from this by iterated composition and the relations between these enforce associativity and unitality.

Similarly, one can define pairs of an algebra and left module over it as maps out of the variant $\mathcal{L}\mathcal{E}_1$ of the $\mathcal{E}_1$-operad. This variant has two colours, $I = (0,1)$ and $L = (0,1]$, and multioperations with target L given by $\mathrm{Emb}^{\mathrm{rec}}_{\partial L}(S \times I \sqcup L, L)$ (note we always require a single L be present). Informally, a map out of $\mathcal{L}\mathcal{E}_1$ is a map out of $\mathcal{E}_1$ specifying an algebra and the following additional data: the value on the colour L specifies the underlying object M of the module, and the 2-ary operations into L specify an action map

$$A \otimes M \longrightarrow M$$

satisfying associativity and unitality. The inclusion $\mathcal{E}_1 \to \mathcal{L}\mathcal{E}_1$ induces a restriction map which one may use to fix the algebra and define an ∞-category of A-modules:

$$\mathcal{M}\mathrm{od}_A(\mathcal{C}) = \mathrm{fib}_A\big[\mathrm{Fun}_{\mathcal{O}\mathrm{p}}(\mathcal{L}\mathcal{E}_1, \mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}\mathrm{p}}(\mathcal{E}_1, \mathcal{C})\big].$$

A symmetric monoidal functor $\mathcal{C} \to \mathcal{D}$ is in particular a map of operads so in particular takes algebras, resp. left modules, in $\mathcal{C}$ to such objects in $\mathcal{D}$.

2.2.2 Algebras and modules in manifolds

Let us consider the case $\mathcal{C} = \mathcal{M}\mathrm{an}^c$, whose symmetric monoidal structure has monoidal product given by disjoint union and monoidal unit given by the empty manifold. Any triad gives rise to an algebra and module over it as follows. We start by observing that the open interval $I = (0,1)$ is canonically an $\mathcal{E}_1$-algebra in $\mathcal{M}\mathrm{an}^c$ for $d = 1$: the k-ary operations in $\mathcal{E}_1(k)$ where by definition given by embeddings $\sqcup_k \to I$ that on each term are a composition of scaling and translation. Taking the product with $\partial_0 M$, we see that $\partial_0 M \times I$ is also an $\mathcal{E}_1$-algebra in $\mathcal{M}\mathrm{an}^c$. E.g. for $\partial_0 M = \partial M = S^1$, the multiplication on $S^1 \times I$ is represented by the embedding

picture

Pick a collar $\partial_0 M \times [0,1] \hookrightarrow M$ on $\partial_0 M$ in M—the space of collars is contractible by [Con71]—we can make M into a left $\partial_0 M \times [0,1]$-module. E.g. for $\partial_0 M = \partial M = S^1$ and $M = D^2$, the action of $S^2 \times I$ on $D^2 \times I$ is represented by the embedding

picture

2.2.3 Boundary conditions in embedding calculus

Let us collect what we have done above:

(1) A triad M yields a pair of an algebra $\partial_0 M \times I$ and a left module M over it in $(\mathcal{M}\mathrm{an}^c, \sqcup)$.

(2) The embedding calculus construction

$$(\mathcal{M}\mathrm{an}^c, \sqcup) \xrightarrow{y} (\mathrm{PSh}(\mathcal{M}\mathrm{an}^c), \mathrm{day}) \xrightarrow{i^*} (\mathrm{RMod}(\mathcal{E}^c_d), \mathrm{day})$$

is symmetric monoidal.

(3) Symmetric monoidal functors preserve algebras and left modules.

Combining these, we find that a triad M yields a pair of an algebra $E_{\partial_0 M \times I}$ and a left module E_M over it in $(\mathrm{RMod}(E^c_d), \mathrm{day})$. The analogous statement goes through in the truncated setting, as the restriction $(\mathrm{RMod}(E^c_d), \mathrm{day}) \to (\mathrm{RMod}_k(E^c_d), \mathrm{day})$ is symmetric monoidal.

A self-embedding $e\colon M \to M$ that fixes not only $\partial_0 M$ but also the collar $\partial_0 M \times [0,1]$ pointwise, preserves the $\partial_0 M \times I$-module structure described above. Thus the map

$$e \circ (-)\colon E_M \longrightarrow E_M$$

of presheaves it induces is one of $E_{\partial_0 M \times I}$-modules in $\mathrm{RMod}(\mathcal{E}^c_d)$. However, by contractibility of space of collars, the hypothesis that it fixes the collar is homotopically vacuous. More generally, if we have a manifold N and an embedding $e_\partial\colon \partial_0 M \to \partial N$, we can identify its image with $\partial_0 M$ to N into a triad with $\partial_0 N = \partial_0 M$ and hence into a $\partial_0 M \times I$-module. Doing so, any embedding $M \to N$ agreeing with e_∂ on $\partial_0 M$ induces a map $E_M \to E_N$ of $E_{\partial_0 M \times I}$-modules in $\mathrm{RMod}(\mathcal{E}^c_d)$.

The conclusion is that if we let $\mathcal{M}\mathrm{an}^c_P$ denote the category of d-dimensional triads M with $\partial_0 M$ identified with P and whose mapping spaces are embeddings fixing P pointwise, then there are functors

$$\mathcal{M}\mathrm{an}^c_P \longrightarrow \mathcal{M}\mathrm{od}_{P \times I}(\mathcal{M}\mathrm{an}^c) \longrightarrow \mathcal{M}\mathrm{od}_{E_{P \times I}}(\mathrm{RMod}(E^c_d)).$$

Definition 2.2.3. Suppose M and N are triads with $\partial_0 M = P = \partial_0 N$. Then the *embedding calculus approximation with boundary* P is the map

$$\mathrm{Emb}^c_{\partial_0}(M,N) \longrightarrow T_\infty \mathrm{Emb}^c_{\partial_0}(M,N) := \mathrm{Map}_{\mathcal{M}\mathrm{od}_{E_{P \times I}}(\mathrm{RMod}(E^c_d))}(E_M, E_N).$$

One can similarly define $T_k \mathrm{Emb}^c_{\partial_0}(M,N)$ by replacing $\mathrm{RMod}(E^c_d)$ with $\mathrm{RMod}_k(E^c_d)$, using that restriction is symmetric monoidal. In fact, such a treatment boundary conditions in a tower makes sense for any operad $\mathcal{O}$ by replacing $\mathrm{RMod}(E^c_d)$ or $\mathrm{RMod}_k(E^c_d)$ with $\mathrm{RMod}(\mathcal{O})$ or $\mathrm{RMod}_k(\mathcal{O})$, again with Day convolution symmetric monoidal structure.

The first piece of evidence that this is a good way to treat boundary conditions is the following convergence result.

Theorem 2.2.4 (Goodwillie–Klein–Weiss, Krannich–K., improved smooth convergence)**.** *If $d \geqslant 5$, M is compact, and $\partial_1 M \to M$ is an equivalence on tangential 2-types, then the map*

$$\mathrm{Emb}^o_{\partial_0}(M,N) \longrightarrow T_\infty \mathrm{Emb}^o_{\partial_0}(M,N)$$

is equivalence.

Remark 2.2.5. As the case without boundary, we will eventually establish a topological variant as long as N is smoothable.

2.2.4 Gluing of manifolds

If we have a left module M and a right module N over an algebra A in a symmetric monoidal category that has geometric realisations, we can form the relative tensor product

$$M \otimes_A N = |\mathrm{Bar}_\bullet(M,A,N)|.$$

The relationship of this relative tensor product to gluing of manifolds is the second piece of evidence that we found a good way to treat boundary conditions. Two triads M and N with $\partial_0 M = P = \partial_1 M$ can be glued along P to form a manifold $M \cup_P N$ and the following is left for the problem session:

Proposition 2.2.6. *There is an equivalence in* $\mathrm{RMod}(E^c_d)$

$$E_{M\cup_P N} \simeq E_M \otimes_{E_{P\times I}} E_N.$$

2.3 Bordism categories

These gluing maps are part of a larger coherent structure. Indeed, a usable form of embedding calculus should have compatible and coherent analogues of geometric constructions on embeddings:

1. Composition of embeddings.
2. Gluing of embeddings.
3. Disjoint unions of embeddings.

How does one encode all this data conveniently? For embeddings, this may be done by constructing a *symmetric monoidal non-compact bordism double ∞-category* $\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d$, as was done in [KK22]. This is an object in $\mathrm{CMon}(\mathcal{C}\mathrm{at}(\mathcal{C}\mathrm{at}))$, an definition I will unwind now, illustrating it with $\mathrm{nc}\mathcal{B}\mathrm{ord}^c$.

The starting point is a *double ∞-category*, an object of $\mathcal{C}\mathrm{at}(\mathcal{C}\mathrm{at})$ so by definition a category object in $\mathcal{C}\mathrm{at}$. This is a functor $\mathcal{D}\colon \Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ satisfying a Segal condition

$$\mathcal{D}_{[n]} \xrightarrow{\simeq} \underbrace{\mathcal{D}_{[1]} \times_{\mathcal{D}_{[0]}} \cdots \times_{\mathcal{D}[0]} \mathcal{D}_{[1]}}_{n}.$$

Informally, it has a category of objects $\mathcal{D}_{[0]}$, a category of morphisms $\mathcal{D}_{[1]}$, and a composition functor $\mathcal{D}_{[1]} \times_{\mathcal{D}_{[0]}} \mathcal{D}_{[1]} \to \mathcal{D}_{[1]}$. The Segal condition says $\mathcal{D}_{[n]}$ is a category of strings of n composable morphisms. For the non-compact bordism category we have:

- $(\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)_{[0]}$ has objects (possibly non-compact) $(d-1)$-dimensional manifolds P and mapping spaces given by embeddings between these.
- $(\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)_{[1]}$ has objects (possibly non-compact) bordisms and mapping spaces given by embeddings between these.
- $(\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)_{[n]}$ has as objects (possibly non-compact) bordisms written as a composition of n bordisms and mapping spaces given by embeddings between these preserving the decomposition.

- The composition in $(\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)^c$ is given by gluing of bordisms.

In particular, the mapping category

$$\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d(P,Q) \coloneqq \mathrm{fib}_{(P,Q)}\big[(\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)_{[1]} \to (\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)_{[0]} \times (\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)_{[0]}\big]$$

has objects given by bordisms from P to Q and mapping spaces given by embeddings between these fixing P and Q pointwise. In terms of these, the composition functor

$$\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d(P,Q) \times \mathrm{nc}\mathcal{B}\mathrm{ord}^c_d(Q,R) \longrightarrow \mathrm{nc}\mathcal{B}\mathrm{ord}^c_d(P,R)$$

is giving by gluing of bordisms and embeddings.

Example 2.3.1. Picking objects W and W' of $\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d(P,Q)$ and $\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d(Q,R)$ and setting $\partial_0 W = P \sqcup Q$, $\partial_0 W' = Q \sqcup R$, the induced map on endomorphism spaces is the gluing map

$$\mathrm{Emb}^c_{\partial_0}(W,W) \times \mathrm{Emb}^c_{\partial_0}(W',W') \longrightarrow \mathrm{Emb}^c_{\partial_0}(W \cup_Q W', W \cup_Q W')$$

where $\partial_0(W \cup_Q W') = P \sqcup R$. Restricting to the identity in $\mathrm{Emb}^c_{\partial_0}(W',W')$ gives an extension-by-identity map

$$\mathrm{Emb}^c_{\partial_0}(W,W) \longrightarrow \mathrm{Emb}^c_{\partial_0}(W \cup_Q W', W \cup_Q W').$$

Remark 2.3.2. Closed $(d-1)$-manifolds and compact d-dimensional bordisms gives a subcategory $\mathcal{B}\mathrm{ord}^c_d \subset \mathrm{nc}\mathcal{B}\mathrm{ord}^c_d$. It has the feature that the mapping categories in the object and morphisms categories are groupoids because embeddings between closed manifolds, or embeddings between compact manifold relative the boundary, are diffeomorphisms or homeomorphisms.

Just like symmetric monoidal categories are commutative monoid objects in $\mathcal{C}\mathrm{at}$, *symmetric monoidal double categories* are commutative monoid objects in $\mathcal{C}\mathrm{at}(\mathcal{C}\mathrm{at})$. This amounts to giving symmetric monoidal structures on all $(\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d)_{[n]}$ so that all functors between them are symmetric monoidal. In the case of $\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d$, this is given by disjoint unions.

To reflect this structure on side of embedding calculus, we need a symmetric monoidal double ∞-category of algebras and bimodules. Bimodules appear here rather than modules because bordism have they boundary divided into three pieces: an incoming boundary, an outgoing boundary, and a free part. To recover triads take null-bordisms, where the outgoing boundary is empty.

Such a construction was provided by Haugseng [Hau17], in the form of a symmetric monoidal double ∞-category *Morita category* $\mathrm{ALG}(\mathcal{C})$, defined for any symmetric monoidal ∞-category compatible with geometric realisations.

- $\mathrm{ALG}(\mathcal{C})_{[0]}$ has objects given by algebras in $\mathcal{C}$ and mapping spaces given by maps of algebras.
- $\mathrm{ALG}(\mathcal{C})_{[1]}$ has objects triples (A,M,B) of two algebras A,B and an (A,B)-bimodule and mapping spaces given by maps of such.
- $\mathrm{ALG}(\mathcal{C})_{[n]}$ has as objects given by string of $(n+1)$-algebras and n bimodules and mapping spaces given by maps of such.
- The composition in $\mathrm{ALG}(\mathcal{C})$ is given by relative tensor products.
- The symmetric monoidal structure is given by external tensor product of algebras and bimodules.

This construction in particular applies to $\mathrm{RMod}(\mathcal{E}^c_d)$ or $\mathrm{RMod}_k(\mathcal{E}^c_d)$

A strong statement of the naturality of embedding calculus is as follows (proven for $k = \infty$ in the smooth case in [KK22] and in general in [KK24b]), which informally says that (truncated) embedding calculus with boundary conditions has composition, gluing, and disjoint unions operations, as compatible as one could hope for.

Theorem 2.3.3 (Krannich–K., naturality of embedding calculus)**.** *There are maps*

$$\mathrm{nc}\mathcal{B}\mathrm{ord}^c_d \longrightarrow \mathrm{ALG}(\mathrm{RMod}(\mathcal{E}^c_d)) \longrightarrow \cdots \longrightarrow \mathrm{ALG}(\mathrm{RMod}_2(\mathcal{E}^c_d)) \longrightarrow \mathrm{ALG}(\mathrm{RMod}_1(\mathcal{E}^c_d))$$

of symmetric monoidal double ∞-categories, and similarly for reduced modules.

Remark 2.3.4. There is a stronger statement that is provable with our techniques: taking an extended non-compact bordism category as the domain, there ought to be a map of symmetric monoidal $(d+1)$-fold ∞-categories. We did not write down the details because we do not know of a use for this.

Example 2.3.5. As in Example 2.3.1, the existence of the Morita category can be used to produce extension-by-identity map

$$T_\infty \mathrm{Emb}^c_{\partial_0}(W, W) \longrightarrow T_\infty \mathrm{Emb}^c_{\partial_0}(W \cup_Q W', W \cup_Q W'),$$

which by previous theorem fit into a commutative square

$$\begin{tikzcd}
\mathrm{Emb}^c_{\partial_0}(W, W) \arrow[r] \arrow[d] & \mathrm{Emb}^c_{\partial_0}(W \cup_Q W', W \cup_Q W') \arrow[d] \\
T_\infty \mathrm{Emb}^c_{\partial_0}(W, W) \arrow[r] & T_\infty \mathrm{Emb}^c_{\partial_0}(W \cup_Q W', W \cup_Q W').
\end{tikzcd}$$

Example 2.3.6. It is a result of Haugseng that $\mathrm{ALG}(\mathcal{C})$ where $\mathcal{C}$ has a cocartesian symmetric monoidal structure is given by a cospan symmetric monoidal double ∞-category $\mathrm{COSPAN}(\mathcal{C})$. Thus we get

$$\mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}_1(\mathcal{E}^c_d)) \simeq \begin{cases} \mathrm{COSPAN}(\mathcal{S}_{/\mathrm{BO}(d)}) & \text{if } c = o, \\ \mathrm{COSPAN}(\mathcal{S}_{/\mathrm{BTop}(d)}) & \text{if } c = t. \end{cases}$$

2.4 Problems

Problem 10 (Gluing as relative tensor product)**.** The relative tensor product $E_M \otimes_{E_{P\times I}} E_N$ can computed by the following semisimplicial object in $\mathrm{RMod}(\mathcal{E}^c_d)$: using collars we can write $M \cup_P N \cong M \cup_P P \times [0,1] \cup_P N$, and then $X_\bullet$ is given

$$[p] \longmapsto \bigsqcup_{0<t_0<\cdots<t_p<1} E_{M\cup_P P\times([0,1]\setminus\{t_0,\ldots,t_p\})\cup_P N}.$$

(a) Prove that $\{P \times (I\setminus\{x\})\}_{x\in I}$ is a Weiss $\mathcal{J}_\infty$-cover.

(b) Give an augmentation $X_\bullet \to E_{M\cup_P N \cong M\cup_P P\times[0,1]\cup_P N}$ and combine (a) with a problem from the last lecture to prove it realises to an equivalence.

Problem 11 (Ad-hoc boundary conditions)**.** One can also treat boundary conditions as follows. Fixing a $(d-1)$-dimensional manifold P, recall that $\mathcal{M}\mathrm{an}^c_P$ is the category of d-dimensional manifolds ∂M with inclusion $P \hookrightarrow \partial M$ and mapping spaces given by embeddings that are the identity on P. Let $\mathcal{D}\mathrm{isc}^c_P \subset \mathcal{M}\mathrm{an}^c_P$ be the full subcategory on objects equivalent to $P \times [0,1) \sqcup S \times \mathbf{R}^d$ and embedding calculus with boundary P through the functor

$$\mathcal{M}\mathrm{an}^c_P \xrightarrow{y} \mathrm{PSh}(\mathcal{M}\mathrm{an}^c_P) \xrightarrow{i^*} \mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_P).$$

(a) Construct analogously to the case with empty boundary a tower

$$T^{\mathrm{adhoc}}_\infty \mathrm{Emb}_P(M, N) \longrightarrow \cdots \longrightarrow T^{\mathrm{adhoc}}_2 \mathrm{Emb}_P(M, N) \longrightarrow T^{\mathrm{adhoc}}_1 \mathrm{Emb}_P(M, N)$$

for $M, N \in \mathcal{M}\mathrm{an}^c_P$.

(b) Prove that $\mathcal{D}\mathrm{isc}^c_P$ is equivalent to the full subcategory of $\mathcal{M}\mathrm{od}_{E_{P\times I}}(\mathrm{RMod}(\mathcal{E}^c_d))$ on free $E_{P\times I}$-modules on representables.

(c) Use (b) to construct a functor

$$\mathcal{M}\mathrm{od}_{E_{P\times I}}(\mathrm{RMod}(\mathcal{E}^c_d)) \longrightarrow \mathrm{PSh}(\mathcal{D}\mathrm{isc}^c_P)$$

(d) Extend (b) and (c) to the truncated setting.

(e) ($*$) Prove that (c) and its generalisations in (d) are fully faithful.

Problem 12 (Contractibility of collars)**.** As usual in manifold theory, one proves relative uniqueness and then deduces existence; uniqueness is just foliated uniqueness.

(a) Let U be an open neighbourhood of $M \times \mathbf{R}_{\leqslant 0}$ in $M \times \mathbf{R}$ and $h_0 \colon U \to M \times \mathbf{R}$ be an embedding that is the identity on $M \times \mathbf{R}_{\leqslant 0}$. Prove that there exists an isotopy $h_t \colon U \to M \times \mathbf{R}$ of embeddings fixing $M \times \mathbf{R}_{\leqslant 0}$ so that $h_1 = \mathrm{inc}$. (Hint: conjugate by a “slide.”)

(b) Formulate a relative version of (a) and prove the existence of collars.

(c) Formulate a foliated version of (a) and prove the contractibility of spaces of collars.

Chapter 3

Tools in operadic calculus

In the previous lectures we discussed that to an operad $\mathcal{O}$ one can associate a tower of truncated reduced right $\mathcal{O}$-modules

$$\mathrm{RMod}^{\mathrm{red}}(\mathcal{O}) \longrightarrow \cdots \longrightarrow \mathrm{RMod}^{\mathrm{red}}_2(\mathcal{O}) \longrightarrow \mathrm{RMod}^{\mathrm{red}}_1(\mathcal{O}),$$

consisting of symmetric monoidal categories and symmetric monoidal functors, which for the operads $\mathcal{E}^c_d$ with $c \in \{o,t\}$ yields embedding calculus with boundary conditions through maps of symmetric monoidal double ∞-categories

$$\mathrm{nc}\mathcal{B}\mathrm{ord}^o_d \to \mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}^o_d)) \to \cdots \to \mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}_2(\mathcal{E}^o_d)) \to \begin{array}{c}\mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}_1(\mathcal{E}^o_d)) \\ \simeq \mathrm{COSPAN}(\mathcal{S}_{/\mathrm{BO}(d)})\end{array}$$

and similarly in the topological setting. Today we discuss the two remaining desiderata: (C) a description of the layers, (D) naturality in the operad. As an application we deduce convergence of topological embedding calculus from smooth embedding calculus.

3.1 Layers

The layers of the embedding calculus tower were given by

$$\mathrm{fib}_x\big[T_k\mathrm{Emb}^c(M,N) \to T_{k-1}\mathrm{Emb}^c(M,N)\big] \qquad \text{for } x \in T_{k-1}\mathrm{Emb}^c(M,N)$$

and to describe these we will explain how to obtain for a groupoid-coloured operad $\mathcal{O}$ a pullback square

$$\begin{tikzcd}
\mathrm{RMod}_k(\mathcal{O}) \arrow[r] \arrow[d] & \mathcal{S}^{[2]} \arrow[d, "d^1"] \\
\mathrm{RMod}_{k-1}(\mathcal{O}) \arrow[r] & \mathcal{S}^{[1]}
\end{tikzcd}$$

for concrete horizontal functors; a pullback square of ∞-categories induces pullback squares of mapping spaces and the description of the layers amounts to noting that the vertical fibres on mapping spaces agree.

3.1.1 A recollement theorem

When we have full subcategory inclusion $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ the restriction functor $i^*\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C}_0)$ has both a left and right adjoint, given respectively by left and right Kan extension

$$\begin{tikzcd}
\mathrm{PSh}(\mathcal{C}_0) \arrow[r, shift left=2, "i_!"] \arrow[r, shift right=2, "i_*"'] & \mathrm{PSh}(\mathcal{C}) \arrow[l]
\end{tikzcd} ,$$

with a counit natural transformation ϵ for $i_! \dashv i^*$ and the unit natural transformation $\overline{\eta}$ for $i^* \dashv i^*$ of endofunctors of $\mathrm{PSh}(\mathcal{C})$

$$i_! i^* \xrightarrow{\epsilon} \mathrm{id}_{\mathrm{PSh}(\mathcal{C})} \xrightarrow{\overline{\eta}} i_* i^*. \tag{3.1}$$

Similarly, we have a natural transformation of functors $\mathrm{PSh}(\mathcal{C}_0) \to \mathrm{PSh}(\mathcal{C})$

$$i_! \underset{\simeq}{\xleftarrow{i_! \overline{\eta}}} i_! i^* i_* \xrightarrow{\epsilon i_*} i_*, \tag{3.2}$$

which agrees with the composition of (3.1) upon precomposition with i^*. That is, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{PSh}(\mathcal{C}) \arrow[r, "(3.1)"] \arrow[d] & \mathrm{PSh}(\mathcal{C})^{[2]} \arrow[d, "d^1"] \\
\mathrm{PSh}(\mathcal{C}_0) \arrow[r, "(3.2)"] & \mathrm{PSh}(\mathcal{C})^{[1]}.
\end{tikzcd}$$

Now let $j\colon \mathcal{C}_{\neq 0} \hookrightarrow \mathcal{C}$ be the inclusion of subcategory of objects *not* equivalent to those to $\mathcal{C}_0$ and morphisms that do *factor* through $\mathcal{C}_0$. For this to make sense, the latter need to be closed under composition. Using this we can extend the previous square to

$$\begin{tikzcd}
\mathrm{PSh}(\mathcal{C}) \arrow[r, "(3.1)"] \arrow[d] & \mathrm{PSh}(\mathcal{C})^{[2]} \arrow[r, "j^*"] \arrow[d, "d^1"] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[2]} \arrow[d] \\
\mathrm{PSh}(\mathcal{C}_0) \arrow[r, "(3.2)"] & \mathrm{PSh}(\mathcal{C})^{[1]} \arrow[r, "j^*"] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[1]}.
\end{tikzcd}$$

Several groups independently obtained the following result; Krannich–K., Ayala–Mazel-Gee–Moldstad–Rozenbluym, and Ramzi–Steinebruner.

Theorem 3.1.1 (Recollement for presheaves)**.** *Let $\mathcal{C}_0 \subset \mathcal{C}$ be as above and suppose that for all $c \in \mathcal{C}_{\neq 0}$ the counit map*

$$i_! i^* y_{\mathcal{C}}(c) \longrightarrow y_{\mathcal{C}}(c)$$

is an equivalence onto sub-presheaf of those path components that factor through $\mathcal{C}_0$. Then the following is a pullback square

$$\begin{tikzcd}
\mathrm{PSh}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[2]} \arrow[d] \\
\mathrm{PSh}(\mathcal{C}_0) \arrow[r] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[1]}.
\end{tikzcd}$$

We can simplify this further if $\mathcal{C}_{\neq 0}$ is a groupoid, we can use unstraightening and extend this square by a pullback square

$$\begin{tikzcd}
\mathrm{PSh}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[2]} \simeq (\mathcal{S}_{/\mathcal{C}_{\neq 0}})^{[2]} \simeq (\mathcal{S}^{[2]})_{/\mathrm{const}_{\mathcal{C}_{\neq 0}}} \arrow[r, "c"] \arrow[d] & \mathcal{S}^{[2]} \arrow[d, "d^1"] \\
\mathrm{PSh}(\mathcal{C}_0) \arrow[r] & \mathrm{PSh}(\mathcal{C}_{\neq 0})^{[1]} \simeq (\mathcal{S}_{/\mathcal{C}_{\neq 0}})^{[1]} \simeq (\mathcal{S}^{[1]})_{/\mathrm{const}_{\mathcal{C}_{\neq 0}}} \arrow[r, "c"] & \mathcal{S}^{[1]}
\end{tikzcd}$$

where the horizontal arrows on the right amount to taking a colimit over $\mathcal{C}_{\neq 0}$ (actually lax colimit, but these agree for groupoids).

3.1.2 A recollement theorem for right modules

We will apply this $\mathcal{C}_0 \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{C}_{\neq 0}$ given by

$$\mathrm{Env}(\mathcal{O})_{\leqslant k-1} \longhookrightarrow \mathrm{Env}(\mathcal{O})_{\leqslant k} \longhookleftarrow \mathrm{Env}(\mathcal{O})_{\Sigma_k},$$

where the right term is the subcategory whose objects whose underlying finite set has cardinality k and whose morphisms are bijections on underlying finite sets. The verification of the hypothesis is a computation of a relatively straightforward coend; we will see this explicitly for $\mathcal{E}_d^o$ momentarily. For the addendum, we note that $\mathrm{Env}(\mathcal{O})_{\Sigma_k}$ is a groupoid if and only if the category $\mathcal{O}^{\mathrm{col}}$ of colours is a groupoid. The conclusion is that there is a pullback square

$$\begin{tikzcd}
\mathrm{RMod}_k(\mathcal{O}) \arrow[r] \arrow[d] & \mathcal{S}^{[2]} \arrow[d, "d^1"] \\
\mathrm{RMod}_{k-1}(\mathcal{O}) \arrow[r] & \mathcal{S}^{[1]}
\end{tikzcd}$$

for groupoid-coloured $\mathcal{O}$, so in particular $\mathcal{E}_d^c$ and more generally $\mathcal{E}_d^\theta$ for a tangential structure $\theta\colon B \to \mathrm{BAut}(E_d)$.

3.1.3 The latching and matching spaces for $\mathcal{E}_d^o$

We will now make this as explicit as possible when the operad is $\mathcal{E}_d^o$ and the right module is of the form E_M. Let us start by understanding the middle functor of the top arrow, i.e.

$$\mathrm{RMod}_k(\mathcal{E}_d^o) \ni E_M \longmapsto cj^* E_M \in \mathcal{S}.$$

The computation $\mathrm{Emb}^o(\mathbf{R}^d, \mathbf{R}^d) \simeq \mathrm{O}(d)$ allow us to identify $\mathrm{Env}(\mathcal{O})_{\Sigma k}$ as being equivalent to the category with a single object and automorphisms $\Sigma_k \wr \mathrm{O}(d)$, and under this equivalence $j^* E_M \in \mathrm{PSh}(\mathrm{Env}(\mathcal{E}_d^o)_{\Sigma_k})$ is the $\Sigma_k \wr \mathrm{O}(d)$-space given by the framed configuration space

$$\mathrm{Conf}_k(M) \times_{M^k} \mathrm{Fr}(TM)^k,$$

and applying c takes $\Sigma_k \wr \mathrm{O}(d)$-orbits so we get the unordered configuration space $C_k(M) := \mathrm{Conf}_k(M)/\Sigma_k$. For later use we recall that this admits a compactification $\overline{C}_k(M)$ with the same homotopy type, where points are allowed to be infinitesimally close. Next we need to know the left and right functor of the top arrow, i.e.

$$E_M \ni \mathrm{RMod}_k(\mathcal{E}_d^o) \longmapsto cj^* i_! i^* E_M, \text{ or } cj^* i_* i^* E_M \in \mathcal{S}.$$

Some of the details are given in the problems, but the answer is as follows:

Lemma 3.1.2. *We have that*

$$cj^* i_! i^* E_M \simeq \partial \overline{C}_k(M) \qquad \textit{and} \qquad cj^* i_* i^* E_M \simeq \lim_{I \subsetneq \{1,\ldots,k\}} \overline{C}_I(M).$$

Theorem 3.1.3. *The layers of the embedding calculus tower* $\mathrm{fib}_x\big[T_k \mathrm{Emb}^o(M,N) \to T_{k-1}\mathrm{Emb}^o(M,N)\big]$ *for* $x \in T_{k-1}\mathrm{Emb}^o(M,N)$ *are given by the space of dashed fillers*

$$\begin{tikzcd}
\partial \overline{C}_k(M) \arrow[r] \arrow[d] & \partial \overline{C}_k(N) \arrow[d] \\
\overline{C}_k(N) \arrow[r, dashed] \arrow[d] & \overline{C}_k(N) \arrow[d] \\
\lim\limits_{I \subsetneq \{1,\ldots,k\}} \overline{C}_I(M) \arrow[r] & \lim\limits_{I \subsetneq \{1,\ldots,k\}} \overline{C}_I(N).
\end{tikzcd}$$

Remark 3.1.4. There is a similar description in the topological case, but it is harder to state because there exists no natural compactification of the configuration spaces [Kup20]. Instead of adding a boundary $\partial\overline{C}_k(M)$ to $C_k(M)$ one takes limit over regular open neighbourhoods of the removed fat diagonal (see problems).

Let us return to a promised point: that we verify the hypothesis that for the recollement theorem in the case of $\mathcal{E}_d^o$. What we need to check is that for $\underline{k} = \{1, \ldots, k\}$, the map

$$i_! i^* E_{\underline{k}\times\mathbf{R}^d} \to E_{\underline{k}\times\mathbf{R}^d}$$

hits upon evaluation those components that factors through an inclusion of $< k$ open discs. This is automatic when we evaluate at $S \times \mathbf{R}^d$ for $|S| < k$ (then both sides agree with $\mathrm{Emb}(S \times \mathbf{R}^d, \underline{k} \times \mathbf{R}^d)$. For the case $|S| = k$, we use a step in the proof of the lemma: there is a commutative diagram

$$\begin{tikzcd}
i_! i^* E_M(S \times \mathbf{R}^d) \arrow[r] \arrow[d, "\simeq"] & E_M(S \times \mathbf{R}^d) \arrow[d, "\simeq"] \\
\partial\overline{C}_k(M) \times_{M^k} \mathrm{Fr}^o(TM)^k \arrow[r] & \overline{C}_k(M) \times_{M^k} \mathrm{Fr}^o(TM)^k
\end{tikzcd}$$

Taking $M = \underline{k} \times \mathbf{R}^d$, the bottom map is the inclusion of the boundary into the compactification of framed S-labelled configuration of points in $\underline{k} \times \mathbf{R}^d$. By pushing configurations towards the origins in $\underline{k} \times \mathbf{R}^d$, the components where one of the open disc contains at least two points of the configuration deformation retract onto the boundary.

3.2 Smoothing theory

Let us remark on one crucial feature of the previous result: the groupoid of colours $\mathrm{BO}(d)$ makes no appearance. This is a general result: if $\varphi\colon \mathcal{E}_d^\alpha \to \mathcal{E}_d^\beta$ is a map of operads induced by the map of tangential structures for a reduced operad $\mathcal{E}_d$

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\
& \mathrm{BAut}(\mathcal{E}_d), &
\end{tikzcd}$$

then the induced map on layers for calculus between right $\mathcal{E}_d^\alpha$-modules M, N and right $\mathcal{E}_d^\beta$-modules $\varphi_! M, \varphi_! N$ is an equivalence.

This generalises to say that

$$\begin{tikzcd}
\mathrm{RMod}_k(\mathcal{E}_d^\alpha) \arrow[r, "\varphi_!"] \arrow[d] & \mathrm{RMod}_k(\mathcal{E}_d^\beta) \arrow[d] \\
\mathrm{RMod}_{k-1}(\mathcal{E}_d^\alpha) \arrow[r, "\varphi_!"] & \mathrm{RMod}_{k-1}(\mathcal{E}_d^\beta)
\end{tikzcd}$$

is a pullback, and by induction over k and restricting to reduced modules, we get:

Theorem 3.2.1 (Smoothing theory for right modules)**.** *There is a pullback square*

$$\begin{tikzcd}
\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}_d^\alpha) \arrow[r, "\varphi_!"] \arrow[d] & \mathrm{RMod}^{\mathrm{red}}(\mathcal{E}_d^\beta) \arrow[d] \\
\mathcal{S}_{/A} \arrow[r, "\varphi_!"] & \mathcal{S}_{/B}
\end{tikzcd}$$

The same is true with $\mathcal{E}_d$ replaced by any reduced operad $\mathcal{U}$.

The categories and maps in this square are symmetric monoidal compatible, so this is also a pullback of symmetric monoidal categories. They are moreover with geometric realisation so we can apply the Morita category construction ALG(−), which takes pullbacks in $\mathrm{CMon}(\mathcal{C}\mathrm{at})^{\mathrm{grtp}}$ to pullbacks in $\mathrm{CMon}(\mathcal{C}\mathrm{at}(\mathcal{C}\mathrm{at}))$:

Corollary 3.2.2 (Smoothing theory for Morita categories)**.** *is a pullback square*

$$\begin{tikzcd}
\mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}_d^{\alpha})) \arrow[r, "\varphi_!"] \arrow[d] & \mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}_d^{\beta})) \arrow[d] \\
\mathrm{COSPAN}(\mathcal{S}_{/A}) \arrow[r, "\varphi_!"] & \mathrm{COSPAN}(\mathcal{S}_{/B})
\end{tikzcd}$$

The same is true with $\mathcal{E}_d$ replaced by any reduced operad $\mathcal{U}$.

Example 3.2.3. In terms of mapping categories from $\partial_0 M$ to $\varnothing$ (i.e. triads), this means that there are pullback squares

$$\begin{tikzcd}
\mathcal{M}\mathrm{od}_{E_{\partial_0 M\times I}}(\mathrm{RMod}(\mathcal{E}_d^{\alpha})) \arrow[r] \arrow[d] & \mathcal{M}\mathrm{od}_{E_{\partial_0 M\times I}}(\mathrm{RMod}(\mathcal{E}_d^{\beta})) \arrow[d] \\
(\mathcal{S}_{/A})^{\partial_0 M/} \arrow[r] & (\mathcal{S}_{/B})^{\partial_0 M}
\end{tikzcd}$$

and then on mapping spaces

$$\begin{tikzcd}
T_\infty\mathrm{Emb}_{\partial_0}^{\alpha}(M,N) \arrow[r] \arrow[d] & T_\infty\mathrm{Emb}_{\partial_0}^{\beta}(M,N) \arrow[d] \\
\mathrm{Map}_{\partial_0}^{/A}(M,N) \arrow[r] & \mathrm{Map}_{\partial_0}^{/B}(M,N).
\end{tikzcd}$$

3.3 Application to topological convergence

Taking $A = \mathrm{BO}(d)$ and $B = \mathrm{BTop}(d)$, passing to mapping spaces between E_M, E_N for smooth d-manifold triads M, N the previous example in particular gives a comparison between smooth embedding calculus and topological embedding calculus in the form of a pullback square

$$\begin{tikzcd}
T_\infty\mathrm{Emb}_{\partial_0}^{o}(M,N) \arrow[r] \arrow[d] & T_\infty\mathrm{Emb}_{\partial_0}^{t}(M,N) \arrow[d] \\
\mathrm{Map}_{\partial_0}^{/\mathrm{BO}(d)}(M,N) \arrow[r] & \mathrm{Map}_{\partial_0}^{/\mathrm{BTop}(d)}(M,N).
\end{tikzcd}$$

We can add on top of this the map from embedding spaces to embedding calculus:

$$\begin{tikzcd}
\mathrm{Emb}_{\partial_0}^{o}(M,N) \arrow[r] \arrow[d] & \mathrm{Emb}_{\partial_0}^{t}(M,N) \arrow[d] \\
T_\infty\mathrm{Emb}_{\partial_0}^{o}(M,N) \arrow[r] \arrow[d] & T_\infty\mathrm{Emb}_{\partial_0}^{t}(M,N) \arrow[d] \\
\mathrm{Map}_{\partial_0}^{/\mathrm{BO}(d)}(M,N) \arrow[r] & \mathrm{Map}_{\partial_0}^{/\mathrm{BTop}(d)}(M,N)
\end{tikzcd}$$

and it is a result of Lashof, deduced from smoothing theory for moduli spaces of manifolds, that the outer square is a pullback as long as the dimension $d \geqslant 5$. We conclude from the pullback square

$$\begin{tikzcd}
\mathrm{Emb}^o_{\partial_0}(M,N) \arrow[r] \arrow[d] & \mathrm{Emb}^t_{\partial_0}(M,N) \arrow[d] \\
T_\infty\mathrm{Emb}^o_{\partial_0}(M,N) \arrow[r] & T_\infty\mathrm{Emb}^t_{\partial_0}(M,N)
\end{tikzcd}$$

if the left map is an equivalence, the same is true for all path components in the image of the horizontal maps. The claim is now that by varying the smooth structure of M you can exhaust all path components; a convenient feature of the codimension zero situation. Since smoothing theory is not just true on the level of spaces of automorphisms of manifolds but also for moduli spaces, to give a smooth structure on M amounts to giving a lift of the tangent classifier $M \to \mathrm{BTop}(d)$ to $\mathrm{BO}(d)$, and any component of $T_\infty\mathrm{Emb}^t(M,N)$ provides such a lift through the induced map

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr, dashed] \arrow[ddr] & & N \arrow[dl] \arrow[ddl] \\
& \mathrm{BO}(d) \arrow[d] & \\
& \mathrm{BTop}(d) &
\end{tikzcd}$$

Theorem 3.3.1 (Krannich–K., topological convergence)**.** *If $d \geqslant 5$, M is compact, $\partial_1 M \to M$ is an equivalence on tangential 2-types, and N is smoothable, then the map*

$$\mathrm{Emb}^t_{\partial_0}(M,N) \longrightarrow T_\infty\mathrm{Emb}^t_{\partial_0}(M,N)$$

is equivalence.

Remark 3.3.2. In the next lecture we will prove that the hypothesis is necessary in the case of 1-connected spin manifolds.

In fact, going up two levels we have that in the commutative diagram of double ∞-categories

$$\begin{tikzcd}
\mathcal{B}\mathrm{ord}^o_d \arrow[r] \arrow[d] & \mathrm{nc}\mathcal{B}\mathrm{ord}^o_d \arrow[r] \arrow[d] & \mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}^o_d)) \arrow[r] \arrow[d] & \mathrm{COSPAN}(\mathcal{S}_{/\mathrm{BO}(d)}) \arrow[d] \\
\mathcal{B}\mathrm{ord}^t_d \arrow[r] & \mathrm{nc}\mathcal{B}\mathrm{ord}^t_d \arrow[r] & \mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}^t_d)) \arrow[r] & \mathrm{COSPAN}(\mathcal{S}_{/\mathrm{BTop}(d)}),
\end{tikzcd}$$

the middle square induces a pullback on mapping categories, as do the outer squares since they are even pullbacks of double ∞-categories.

3.4 Problems

Problem 13 (Computation of matching spaces)**.** Let $\mathcal{P}_0(S)$ be the poset of proper subsets of S.

(a) Construct a functor $\mathcal{P}_0(S) \to \mathrm{Env}(\mathcal{E}^c_d)_{\leqslant k-1} \times_{\mathrm{Env}(\mathcal{E}^c_d)_{\leqslant k}} (\mathrm{Env}(\mathcal{E}^c_d)_{\leqslant k})_{/S \times \mathbf{R}^d}$.

(b) Prove it is cofinal.

(c) Deduce that $j^* i_! i^* E_M$ can be computed as a limit over a punctured k-cubical diagram.

Problem 14 (Computation of latching spaces)**.**

(a) Extend $cj^* i_! i^* E_M$ to a functor $cj^* i_! i^* E_{(-)} \colon \mathcal{O}(M) \to \mathcal{S}$, where $\mathcal{O}(M)$ is the poset of open subsets of M, and prove it satisfies codescent for $\mathcal{J}_{k-1}$-covers.

(b) Extend $\partial\overline{C}_k(M)$ to a functor $\partial\overline{C}_k(-)\colon \mathcal{O}(M) \to \mathcal{S}$ and prove it also satisfies codescent for $\mathcal{J}_{k-1}$-covers.

(c) Prove that $cj^*i_!i^*E_{S\times\mathbf{R}^d} \simeq \partial\overline{C}_k(S \times \mathbf{R}^d)$ for $|S| \leqslant k$.

(d) $(*)$ Use (a), (b), (c) to prove that $cj^*i_!i^*E_M \simeq \partial\overline{C}_k(M)$.

Problem 15 (Regular open neighbourhoods)**.** We start with some definitions due to Siebenmann [Sie73, SGH73]. Y be a topological space, $X \subset Y$ be a subspace, and $V \subset U \subset Y$ be neighbourhoods of X. Then V *is I-compressible to* X *in* U (we write $V \searrow_U X$) if for all neighbourhoods $W \subset Y$ of X there exists an isotopy h_t of Y satisfying

1. $h_0 = \mathrm{id}_Y$,
2. $h_1(V) \subset W$, and
3. h_t for $t \in [0,1]$ fixes pointwise $Y\backslash U$ and a neighbourhood of X.

An open neighbourhood $E \subset X$ is *regular* if there is a sequence $E_0 \subset E_1 \subset E_2 \subset \cdots$ of neighbourhoods of X such that $E_i \searrow_{E_{i+1}} X$ and $\bigcup_{i\geqslant 0} E_i = E$.

(a) Prove that if X has a regular open neighbourhood, then any neighbourhood of X contains a regular open neighbourhood.

(b) Prove that if E and E' are regular open neighbourhood of X then there is an isotopy φ_t of embedding $E \to Y$ fixing X pointwise so that $\varphi_0 = \mathrm{inc}_E$ and φ_1 induces a homeomorphism of E onto E'. (Hint: it is a swindle.)

Problem 16 (Kister's theorem)**.**

(a) Prove that if $e\colon \mathbf{R}^d \to \mathbf{R}^d$ is an embedding fixing the origin then $E = e(\mathbf{R}^d)$ is a regular open neighbourhood of the origin.

(b) Use the uniqueness result for regular open neighbourhoods to prove that e is isotopic to a homeomorphism.

(c) $(*)$ Use a foliated variant of regular open neighbourhoods to prove that the inclusion $\mathrm{Homeo}(\mathbf{R}^d) \to \mathrm{Emb}(\mathbf{R}^d, \mathbf{R}^d)$ is an equivalence.

Problem 17 (Layers with boundary conditions)**.** Combine the ad-hoc variant of boundary conditions from the problems of the last lecture with the recollement theorem to describe the layers when there are boundary conditions.

Chapter 4

The $\mathcal{D}$isc-structure space

4.1 Structure spaces and the $\mathcal{D}$isc-structure space

The philosophy of "structure spaces" is that you could understand moduli spaces of manifolds by understanding moduli spaces of manifold structures on objects you understand better.

4.1.1 Structure spaces in surgery theory

The first example is to look at manifold structure on spaces (i.e. homotopy types). We will work with smooth manifolds but the same goes through topologically. Recall that $\mathcal{M}\mathrm{an}_d^o$ is the category whose objects are d-dimensional smooth manifolds and whose mapping spaces are given by spaces of embeddings. Taking the underlying spaces yields a functor $\mathcal{M}\mathrm{an}_d^o \to \mathcal{S}$ to the category of spaces. Passing to groupoid cores and then restricting to the subcategory $\mathcal{M}\mathrm{an}_d^{o,\cong} \subset \mathcal{M}\mathrm{an}_d^{o,\simeq}$ of closed manifolds, so that all self-embeddings are diffeomorphisms, we get a map of spaces

$$\mathcal{M}\mathrm{an}_d^{o,\cong} \longrightarrow \mathcal{S}^{\simeq}. \tag{4.1}$$

Definition 4.1.1. For a space X, the *structure space* is given by

$$S^{\mathcal{S},o}(X) := \mathrm{fib}_X\big[\mathcal{M}\mathrm{an}_d^{o,\cong} \to \mathcal{S}^{\simeq}\big].$$

As the name indicates, this should be interpreted as the space of manifold structures on the space X; the set $\pi_0\, S^{\mathcal{S},o}(X)$ of path components is given by equivalence classes of pairs (M, ϕ) of a closed d-dimensional smooth manifold M and a homotopy equivalence $M \to X$, up to diffeomorphism of M, so is empty unless X is homotopy equivalent to a smooth manifold. So we might as well take $X = M$, and then we further have that the path component of (M, id) is given by the path component of $\mathrm{Aut}(M)/\mathrm{Diff}(M)$ containing id_M.

Classically, one makes two observations. Firstly, closed manifolds have a preferred simple homotopy type and diffeomorphisms are simple homotopy equivalences. Secondly, the map from diffeomorphisms to simple homotopy automorphisms factors over the block diffeomorphisms. That is, we have a factorisation of (4.1) as

$$\mathcal{M}\mathrm{an}_d^{o,\cong} \longrightarrow \widetilde{\mathcal{M}\mathrm{an}}_d^{o,\cong} \longrightarrow \mathcal{S}_s^{\simeq} \longrightarrow \mathcal{S}^{\simeq}.$$

The classical simple structure space of surgery theory is

$$S^{s,o}(X) := \mathrm{fib}_X\big[\widetilde{\mathcal{M}\mathrm{an}}_d^{o,\cong} \to \mathcal{S}_s^{\simeq}\big]$$

and the space-level surgery exact sequence expresses it as the fibre of a map between two infinite loop spaces, namely of normal invariants and L-theory. Pseudoisotopy theory is then concerned with the remaining map

$$\mathrm{fib}_M\big[\mathcal{M}\mathrm{an}_d^{o,\cong} \longrightarrow \widetilde{\mathcal{M}\mathrm{an}}_d^{o,\cong}\big],$$

and work of Igusa and Weiss–Williams says it has an roughly $d/3$-connected map to an infinite loop space, namely of the C_2-orbits of the smooth Whitehead spectrum.

We can add boundary conditions to this story: fix a closed $(d-1)$-dimensional P, let $\mathcal{M}\mathrm{an}_P^{o,\cong}$ denote the subcategory of the groupoid core of $\mathcal{M}\mathrm{an}_P^o$ of compact manifolds with boundary P, and consider the map

$$\mathcal{M}\mathrm{an}_P^{o,\cong} \longrightarrow (\mathcal{S}^{P/})^{\simeq},$$

and the fibres

$$S_\partial^{\mathcal{S}}(X) \coloneqq \mathrm{fib}_X\big[\mathcal{M}\mathrm{an}_P^{o,\cong} \longrightarrow (\mathcal{S}^{P/})^{\simeq}\big].$$

Surgery theory and pseudoisotopy theory generalise to this setting, and taking $P = \varnothing$ we recover the previous story.

4.1.2 The $\mathcal{D}$isc-structure space

The proposal of our paper [KK22] is to rather factor (4.1), or better its enhancement with boundary conditions, as

$$\mathcal{M}\mathrm{an}_P^{o,\cong} \longrightarrow \mathcal{M}\mathrm{od}_{E_{P\times I}}(\mathrm{RMod}(\mathcal{E}_d^o))^{\simeq} \longrightarrow \mathcal{S}_{P/}^{\simeq}$$

and defining the smooth $\mathcal{D}$isc-structure space of a $E_{P\times I}$-module X as

$$S_\partial^{\mathcal{D}\mathrm{isc}}(X) \coloneqq \mathrm{fib}_X\big[\mathcal{M}\mathrm{an}_P^{o,\cong} \to \mathcal{M}\mathrm{od}_{E_{P\times I}}(\mathrm{RMod}(\mathcal{E}_d^o))^{\simeq}\big].$$

It should be interpreted as a space of manifold structure relative to P on the presheaf X; the set $\pi_0\, S^{\mathcal{D}\mathrm{isc}}(X)$ of path components is given by equivalence classes of pairs (M, φ) of a closed d-dimensional smooth manifold M with boundary P and an equivalence $E_M \to X$ of $E_{M\times I}$-modules, up to diffeomorphism of M relative to the boundary, so is empty unless X is equivalent to a presheaf of a smooth manifold. So we might as well take $X = E_M$, and then we further have that the path component of (M, id) is given by the path component of $T_\infty\mathrm{Emb}_\partial^o(M, M)^\times/\mathrm{Diff}_\partial(M)$ containing id_M where we let $T_\infty\mathrm{Emb}_\partial^o(M, M)^\times \subset T_\infty\mathrm{Emb}_\partial^o(M, M)$ denote the set of invertible path components.

The following theorem suggests that this is an interesting object to consider:

Theorem 4.1.2 (Krannich–K.)**.** *In high dimensions d ($d \geqslant 8$ suffices for all), the $\mathcal{D}$isc-structure spaces have the following properties:*

(A) $\mathcal{S}_\partial^{\mathcal{D}\mathrm{isc}}(M)$ is an infinite loop space,

(B) $\mathcal{S}_\partial^{\mathcal{D}\mathrm{isc}}(M)$ only depends on the tangential 2-type of M,

(C) $\mathcal{S}_\partial^{\mathcal{D}\mathrm{isc}}(M)$ does not depend on the smooth structure,

(D) $\mathcal{S}_\partial^{\mathcal{D}\mathrm{isc}}(M)$ is non-trivial if M has a finite cover which is spin.

We think of (A) and (B) as indications that the $\mathcal{D}$isc-structure space $\mathcal{S}_\partial^{\mathcal{D}\mathrm{isc}}(M)$ is closer to homotopical algebra. Firstly, instead of being related to several different infinite loop spaces through fibre sequences and in a range, like $S_\partial(M)$, (A) says it actually *is.* an infinite loop space. Secondly, (B) says it behaves more like L-theory or the fibre of the cyclotomic trace, which also depend on weakly on the input space. The crucial input is that there is a version of the isotopy extension theorem for $T_\infty\mathrm{Emb}^c(M, N)$ as long as embedding calculus converges whenever the domain is $M \sqcup S \times \mathbf{R}^d$ for any finite set S.

In the remainder of this talk, I will explain why (C) and (D) are true using results from the previous and accompanying lectures.

4.2 Independence of smooth structure

By definition, the $\mathcal{D}$isc-structure spaces are fibres of the top horizontal map in the commutative square

$$\begin{tikzcd}
\mathcal{M}\mathrm{an}_P^{o,\cong} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{od}_{E_{P\times I}}(\mathrm{RMod}(\mathcal{E}_d^o))^{\simeq} \arrow[d] \\
\mathcal{M}\mathrm{an}_P^{t,\cong} \arrow[r] & \mathcal{M}\mathrm{od}_{E_{P\times I}}(\mathrm{RMod}(\mathcal{E}_d^t))^{\simeq}.
\end{tikzcd} \tag{4.2}$$

This can be obtained from the commutative diagram of symmetric monoidal double ∞-categories

$$\begin{tikzcd}
\mathcal{B}\mathrm{ord}_d^o \arrow[r] \arrow[d] & \mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}_d^o)) \arrow[d] \\
\mathcal{B}\mathrm{ord}_d^t \arrow[r] & \mathrm{ALG}(\mathrm{RMod}^{\mathrm{red}}(\mathcal{E}_d^t)),
\end{tikzcd}$$

by considering mapping categories from P to $\varnothing$ and taking groupoid cores. We saw before that left square induces pullbacks on mapping categories if $d \neq 4$, and groupoid core is a right adjoint, so (4.2) is a pullback square. Thus the horizontal fibres agree and we conclude:

Theorem 4.2.1. *If $d \neq 4$, then $\mathcal{S}_\partial^{\mathcal{D}\mathrm{isc}}(M)$ does not depend on the smooth structure.*

Remark 4.2.2. It *does* depend on the smooth structure if $d = 4$. This is because smoothing theory for embedding calculus goes through in dimension 4, and if the above theorem held in dimension 4 one could essentially *deduce* smoothing theory for moduli spaces of manifolds 4, which is known to be false.

4.3 Non-triviality

We will now prove that $S_\partial^{\mathcal{D}\mathrm{isc}}(M)$ is non-trivial if M is 1-connected spin of dimension $d \geqslant 6$, even after looping arbitrarily many times. For the more general statement one uses naturality and transfer tricks.

This non-triviality result implies that embedding calculus for such manifolds converges neither smoothly nor topologically, since there is an inclusion of components

$$\begin{tikzcd}
\mathrm{fib}[B\mathrm{Diff}_\partial(M) \to BT_\infty\mathrm{Emb}_\partial^o(M,M)^\times] \simeq \mathrm{fib}[B\mathrm{Homeo}_\partial(M) \to BT_\infty\mathrm{Emb}_\partial^t(M,M)^\times] \arrow[d] \\
S_\partial^{\mathcal{D}\mathrm{isc}}(M).
\end{tikzcd}$$

By the 2-type invariance (A), this follows from the crucial case $M = D^d$, as in the problems we discussed the hypothesis is equivalent to any inclusion $D^d \to M$ being an equivalence on tangential 2-types.

4.3.1 A delooping result

To understand $S_\partial^{\mathcal{D}\mathrm{isc}}(D^d)$ we may as well use topological manifolds and take advantage of some results particular to that setting: using the topological Poincaré conjecture and the Alexander trick, it is given by fibre of

$$* \simeq B\mathrm{Homeo}_\partial(D^d) \longrightarrow BT_\infty\mathrm{Emb}_\partial^t(D^d, D^d)^\times,$$

where as before the superscript $(-)^\times$ indicates we take the invertible components. We conclude that

$$S_\partial^{\mathcal{D}\mathrm{isc}}(D^d) \simeq T_\infty\mathrm{Emb}_\partial^t(D^d, D^d)^\times.$$

The left term being non-trivial is thus equivalent to the Alexander trick failing for topological embedding calculus. To understand it, we can invoke another smoothing theory once more, comparing topological embedding calculus to particle embedding calculus and yielding a pullback square

$$\begin{tikzcd}
T_\infty \mathrm{Emb}^t_\partial(D^d, D^d)^\times \arrow[r] \arrow[d] & T_\infty \mathrm{Emb}^p_\partial(D^d, D^d)^\times \arrow[d] \\
\Omega^{d+1}\mathrm{BTop}(d) \arrow[r] & \Omega^{d+1}\mathrm{BAut}(E_d).
\end{tikzcd}$$

To understand the top-right corner, we use the following "Alexander trick for configuration categories" (by [KK24c], it will be equivalent to theorem of this name by Boavida de Brito–Weiss [BdBW18]):

Lemma 4.3.1. *$T_\infty \mathrm{Emb}^p_\partial(D^d, D^d)^\times$ is contractible.*

Proof. We need understand

$$T_\infty \mathrm{Emb}^p_\partial(D^d, D^d)^\times = \mathrm{Aut}_{\mathcal{M}\mathrm{od}_{E_{\partial D^d \times I}}(\mathrm{RMod}(\mathcal{E}^p_d))}(E_{D^d}, E_{D^d}).$$

This can be done by a concatenation of results of Lurie, which for a general tangential structure $\theta \colon B \to \mathrm{BAut}(E_d)$ yield that

$$\mathrm{Aut}_{\mathcal{M}\mathrm{od}_{E_{\partial D^d \times I}}(\mathrm{RMod}(\mathcal{E}^\theta_d))}(E_{D^d}, E_{D^d}) \simeq \Omega^{d+1}\mathrm{fib}(B \to \mathrm{BAut}(E_d)).$$

This makes clear why one might care about particle embedding calculus: then we take the fibre of the identity map and this is of course contractible. ☐

We conclude the following, variants of which were proven by Boavida de Brito–Weiss [**?**] and Ducolombier–Turchin [DT22]:

Theorem 4.3.2. *For $d \neq 4$, $S^{\mathcal{D}\mathrm{isc}}_\partial(D^d) \simeq \Omega^{d+1}\mathrm{Aut}(E_d)/\mathrm{Top}(d)$ where $\mathrm{Aut}(E_d)/\mathrm{Top}(d) := \mathrm{fib}(\mathrm{BTop}(d) \to \mathrm{BAut}(E_d))$.*

Remark 4.3.3. This is true even for $d = 4$ for the topological variant of the $\mathcal{D}$isc-structure space, which in this dimension does not need agree with the smooth variant.

4.3.2 The question of Dwyer–Hess

This relates the non-triviality of the $\mathcal{D}$isc-structure space for D^d (and by some naturality and transfer tricks, of a manifold with a finite cover which is cover) to the following:

Question 4.3.4 (Dwyer–Hess). Is the map $\mathrm{Top}(d) \to \mathrm{Aut}(E_d)$ an equivalence?

Remark 4.3.5. This is true for $d \leqslant 2$ by work of Horel.

Theorem 4.3.6 (Krannich–K.)**.** *This is false for $d \geqslant 3$.*

We will now give a proof that applies for $d \geqslant 8$ and show that $\mathrm{Aut}(E_d)/\mathrm{Top}(d)$ remains non-trivial after looping arbitrarily many times. For the contradiction we assume that map $\mathrm{BTop}(d) \to \mathrm{BAut}(E_d)$ becomes an equivalence after looping n times. The idea is to consider the commutative diagram

$$\begin{tikzcd}
\mathrm{BSTop}(d-2) \arrow[r] \arrow[d, "s_{\mathrm{Top}}"] & \mathrm{BSAut}(E_{d-2}) \arrow[r] \arrow[d, "s_{\mathrm{Aut}}"] & \mathrm{BSAut}(E^{\mathbf{Q}}_{d-2}) \arrow[d, "s_{\mathrm{Aut}^{\mathbf{Q}}}"] \\
\mathrm{BSTop}(d) \arrow[r] & \mathrm{BSAut}(E_d) \arrow[r, "r"] & \mathrm{BSAut}(E^{\mathbf{Q}}_d),
\end{tikzcd}$$

where $\mathrm{STop}(d) \subset \mathrm{Top}(d)$ denotes the orientation-preserving homeomorphisms and $\mathrm{SAut}(E_d) \subset \mathrm{Aut}(E_d)$ denotes the components that preserve the orientation of $\mathcal{E}_d(2) \simeq S^{d-1}$, and similarly rationally. Let us first assume that the bottom map r is a rationalisations (this is not clear, because you can not in general commute taking automorphisms and rationalising). Then on the one hand, the accompanying lectures proved that the map $s_{\mathrm{Aut}^{\mathbf{Q}}}$ is null-homotopic and hence so is

$$s_{\mathrm{Top}} \colon \mathrm{BSTop}(d-2) \longrightarrow \mathrm{BSTop}(d)$$

after rationalising and looping n times. But on the other hand, in those lectures we proved that the composition

$$\mathrm{BSTop}(d-2) \longrightarrow \mathrm{BSTop}(d) \longrightarrow \mathrm{BSTop} \simeq_{\mathbf{Q}} \prod_{i \geqslant 1} K(\mathbf{Q}, 4i),$$

is a surjection on rational homotopy groups and hence is non-trivial after looping n times. This is a clear contradiction.

Let us now look at the assumption that the map r is a rationalisations. It is possible to give conditions under which this is true: in the accompanying lectures we say that $\mathrm{BAut}(E_d) \simeq \lim_{k \to \infty} \mathrm{BAut}((E_d)_{\leqslant k})$ is a limit of nilpotent spaces with countable homotopy groups. The homotopy groups are then computed by a Milnor exact sequence

$$0 \longrightarrow \lim_{k \to \infty}{}^1 \pi_{*+1} \mathrm{BAut}((E_d)_{\leqslant k}) \longrightarrow \pi_* \mathrm{BAut}(E_d) \longrightarrow \lim_{k \to \infty} \pi_* \mathrm{BAut}((E_d)_{\leqslant k}) \longrightarrow 0.$$

There is a dichotomy. On the one hand, some homotopy group of $\mathrm{BAut}(E_d)$ could be uncountable. Then $\mathrm{BTop}(d) \to \mathrm{BAut}(E_d)$ can not be an equivalence for easy reasons: $\mathrm{Top}(d)$ is a second-countable locally contractible space so has countable homotopy groups. On the other hand, all homotopy groups of $\mathrm{BAut}(E_d)$ could be countable. Then the $\lim^1$-terms are non-trivial, as non-trivial $\lim^1$'s of inverse systems of countable groups are, and by an induction over the layers of the tower for mapping spaces of operads explained in the previous lecture all homotopy groups of $\mathrm{Aut}((E_d)_{\leqslant k})$ are countable. This in turns applies the inverse systems are Mittag–Leffler with countable limit, hence pro-constant, and thus the limit commutes with rationalisation, implying that r is a rationalisation. This argument is clearly quite robust, and also applies after looping n times.

Remark 4.3.7. For $3 \leqslant d \leqslant 7$ one needs to argue differently, using that if $\mathrm{BSAut}(E_d) \to \mathrm{BSAut}(E_d^{\mathbf{Q}})$ were a rationalisation, we would know the rational homotopy groups by [FW20].

4.4 Problems

Problem 18. Use smoothing theory for embedding calculus to deduce that for $d \neq 4$ we have

$$T_\infty \mathrm{Emb}^o_\partial(D^d, D^d)^\times \simeq \Omega^{d+1} \mathrm{Aut}(E_d)/\mathrm{O}(d).$$

Problem 19 (A weaker version of Horel's result)**.**

(a) Prove that $\mathrm{Conf}_k(D^2)$ is a $K(\pi, 1)$.

(b) Prove $\overline{C}_2(D^2)$ can be obtained from the subspace given by the union of $\partial\overline{C}_2(D^2)$ and those configurations where at least one point lies in ∂D^2 by only attaching $\geqslant$ 2-cells. For the remainder of this problem you may assume the same is true for $k \geqslant 2$ replacing 2.

(c) Prove that $T_1 \mathrm{Emb}^o_\partial(D^2, D^2)$ is contractible.

(d) Prove that the layers of the embedding calculus tower for $T_k \mathrm{Emb}^o_\partial(D^2, D^2)$ are contractible.

(e) Prove that $\Omega^2 \mathrm{Aut}(E_2)/\mathrm{O}(2) \simeq *$

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