HERMITIAN K-THEORY LEARNING SEMINAR: POINCARÉ CATEGORIES

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Notes from my talk on Poincaré categories. Covers a user's guide to stable ∞-categories, bilinear and quadratic functors and hermtiian and Poincaré categories and finally a classification of these. All categories will be $(\infty, 1)$-categories (i.e quasi-categories), and all constructions are homotopy invariant notions (i.e. derived) unless otherwise mentioned. Refer to [Lur09, Chapter 1] and [Lur17, Chapter 1] for rigorous explanations of stable ∞-categories.

1. A User's Guide to Stable ∞-categories

This section is meant to guide you towards comfortably manipulating and working with stable categories. It is not meant as an explanation of what these mathematical structures *are*, but rather what they *do*.

Definition 1.1. A category $\mathcal{C}$ is called pointed if it contains an object $0 \in \mathcal{C}$ which is both initial and terminal which we call the *zero object*.

Definition 1.2. A pointed category $\mathcal{C}$ is stable if the following two conditions are satisfied:

(1) Every morphism in $\mathcal{C}$ admits a fiber or cofiber, i.e. the two diagrams always exist

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & Y, \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f). \end{array}$$

(2) A sequence $X \to Y \to Z$ is a fiber sequence if and only if it is a cofiber sequence. We call it an *exact sequence* if either (hence both) of these conditions are satisfied.

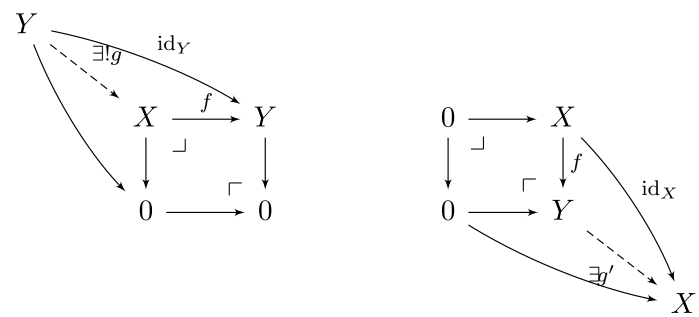
The main examples are $\mathcal{C} = \mathrm{Sp}$ the category of spectra and $\mathcal{D}(R)$ which is the derived category of R. We will return to both of these examples in more detail later on.

There are two important functors defined on any stable category. They are the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ and loop functor $\Omega : \mathcal{C} \to \mathcal{C}$. They are constructed through the fiber and cofiber sequnces

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X, \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array}$$

By property (2) in the definition of stability, they are also respectively pushouts and pullback squares. Hence we conclude that $\Sigma\Omega X \simeq X$ and $\Omega\Sigma X \simeq X$, i.e. loop and suspension are inverse equivalences.

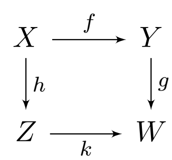
We can also detect whether a morphism $f : X \to Y$ is an equivalence on (co)fibers. This is so, if and only if $\mathrm{cofib}(f) \simeq 0$ if and only if $\mathrm{fib}(f) \simeq 0$. This last if and only if uses $\Sigma\,\mathrm{fib}(f) \simeq \mathrm{cofib}(f)$, and the first follows from considering the (co)fibers of f



The maps g and g' are constructed using the universal properties, and provide a two sided inverse to f.

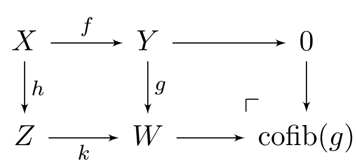
We can also detect whether squares are pushout or pullback sqaures, in fact these turn out to be equivalent, and we therefore call a square that is either (and hence both) an *exact square.*

Proposition 1.3. *A square*



is exact $\iff \mathrm{cofib}(f) \simeq \mathrm{cofib}(k) \iff \mathrm{cofib}(h) \simeq \mathrm{cofib}(g) \iff \mathrm{fib}(f) \simeq \mathrm{fib}(k) \iff \mathrm{fib}(h) \simeq \mathrm{fib}(g)$ $\iff \mathrm{totfib} \simeq 0$.

Proof. Here the total fiber is any of the equivalent combinations of taking fibers and then cofibers of horizontal or vertical maps. The total fiber then vanishes precisely when any of the (co)fibers of the various maps are equivalences. To connect exactness with any of the other properties, consider e.g. taking the cofiber of g, giving the diagram



By the pasting lemma, the inner left square is exact if and only if the outer square is exact. But the outer squrare computes the cofiber of h, hence exactness is equivalent to $\mathrm{cofib}(h) \simeq \mathrm{cofib}(g)$ by uniqueness of (co)limits. □

There is also a splitting lemma, namely given an exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

it is split $Y \simeq X \oplus Z$ if and only if there is a map $r : Z \to Y$ with $gr = \mathrm{id}_Z$ if and only if $t : Y \to X$ with $tf = \mathrm{id}_X$. This is occasionally useful.

Any exact sequence also extends infinitely in both directions into exact sequences, meaning each three consecutive terms form an exact sequence,

$$\cdots \to \Omega Y \to \Omega Z \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X \to \Sigma Y \to \dots$$

One sees this by patching together diagrams of succesive cofiber sequences, and applying the pasting lemma succesively

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z & \longrightarrow & \Sigma X & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \Sigma Y & \longrightarrow & \dots \end{array}$$

This is where the various long exact sequences in (co)homology and stable homotopy come from! Finally, let us state some equivalent characterisations of stable categories.

Proposition 1.4. *Let $\mathcal{C}$ be pointed. The following are equivalent:*

(1) $\mathcal{C}$ is stable.
(2) Loop and suspension functors are inverse equivalences.
(3) Finite (co)limits exist and squares are pushouts if and only if they are pullbacks.
(4) finite (co)limits exist and commute.

1.1. **Examples:** Sp **and** $\mathcal{D}(R)$**.** The prototypical example of a stable category is the category of spectra Sp. Let me give a high-tech construction of this category.

Consider the category of based spaces Spc_*. Taking loop spaces defines a functor $\Omega : \mathrm{Spc}_* \to \mathrm{Spc}_*$ succesively applying the functor yields a diagram in Cat_∞ the category of ∞-categories and we consider the limit in Cat_∞

$$\lim(\dots \xrightarrow{\Omega} \mathrm{Spc}_* \xrightarrow{\Omega} \mathrm{Spc}_*) =: \mathrm{Sp}(\mathrm{Spc}_*)$$

This is a general construction for categories $\mathcal{C}$ that admit finite limits, and the output of the above process is always a stable category called the *spectrification* of $\mathcal{C}$. Then the category of spectra is $\mathrm{Sp} := \mathrm{Sp}(\mathrm{Spc}_*)$.

A 0-simplex of Sp then consists of a sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$ along with homotopy equivalences $\Omega X_{n+1} \simeq X_n$. Some might recognize these under the name of Ω-spectra. The category of spectra plays the same role for stable categories that Spc does for categories. That means that every stable category is enriched over Sp in the sense that there is a spectrum $\mathrm{map}_{\mathcal{C}}(x, y)$ such that

$$\Omega^\infty \mathrm{map}_{\mathcal{C}}(x, y) \simeq \mathrm{Map}_{\mathcal{C}}(x, y)$$

for every pair of objects $x, y \in \mathcal{C}$.

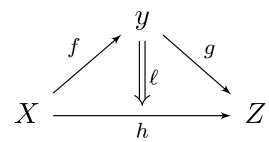
The second (related of course) example is the derived category of a ring $\mathcal{D}(R)$. One way of constructing categories is starting with a (Kan) simplicially enriched 1-category and applying the coherent nerve construction. If one instead starts with a differential graded (dg) 1-category, for instance chain complexes of an additive category $\mathcal{A}$, there is also a dg nerve N_{dg} whose output is a category. This will be how we construct the derived category.

Definition 1.5. Let $\mathcal{A}$ be an abelian 1-category with enough projectives. Let Ch^- denote the chain complexes bounded below and $\mathcal{A}_{proj}$ denotes the 1-category of projective objects. Define

$$\mathcal{D}(\mathcal{A}) := N_{dg}(Ch^-(\mathcal{A}_{proj})),$$

This is also equivalent to the localization at the quasi-isomorphisms $\mathcal{D}(\mathcal{A}) \simeq Ch(\mathcal{A})[q.i.^{-1}]$.

We can describe somewhat what this category looks like: A 0-simplex is an element $X \in Ch^-(\mathcal{A}_{proj})$, a 1-simplex is a chain map $X \to Y$, and a 2-simplex is a triangle



where f, g, h are chain maps and ℓ is a chain homotopy with $d(\ell) = (gf) - h$, and so on. It is a non-trivial fact that in this case the output $\mathcal{D}(\mathcal{A})$ is a stable category.

There is a close relationship between Sp and $\mathcal{D}(\mathbb{Z})$. There is a t-structure on Sp consisting of two subcategories $\mathrm{Sp}^{\geq 0}$ and $\mathrm{Sp}^{\leq 0}$ consisting of connective and coconnective spectra, respectively. The intersection of these two subcategories is called the *heart*, and is always a 1-category. For Sp, the result is

$$\mathrm{Sp}^{\heartsuit} = Ab$$

The resulting functor from the heart $Ab \to \mathrm{Sp}$ is the *Eilenberg-MacLane* functor sneding an abelian group A to the Eilenberg-MacLane spectrum HA.

Furthermore, both of these categories have symmetric monoidal structures $(Ab, \otimes)$ and $(\mathrm{Sp}, \otimes)$ where $\otimes$ on Sp is also known as the smash product (and is highly non-trivial to construct!). With respect to these monoidal structure the Eilenberg-MacLane functor is oplax monoidal in the sense that there is a map

$$H(A \otimes B) \to HA \otimes HB.$$

2. Bilinear and Quadratic Functors

Let us now dive into the main topic of this talk, namely Poincaré categories. This will require a slew of definitions, but let us first try to motivate what we are trying to do. From now on every category will be a stable category unless otherwise indicated.

Motivation. We are interested in various flavours of forms on a ring R. We take our inspiration for their higher generalizations from two observations. The first is the obvious correspondence

$$\{\text{Symmetric forms on } R\} \longleftrightarrow \mathrm{Hom}_R(X \otimes_R X, R)^{C_2}$$

where C_2 acts on by flipping the input variables, and we are considering the fixed points. Likewise (though less obvious) we have a correspondence

$$\begin{aligned} \{\text{Quadratic forms on } R\} &\longleftrightarrow \mathrm{Hom}_R(X \otimes_R X, R)_{C_2} \\ q_b(x) = b(x, x) &\longleftarrow [b] \end{aligned}$$

where we now consider the orbits or coinvariants. Figuring this out is left as an exercise for the reader.

The most important Poincaré structures are homotopical counterparts of the above, where we consider elements of $\mathcal{D}^p(R)$ along with homotopy fixed points and homotopy orbits instead

$$\Qoppa^s_R(X) := \mathrm{Hom}_R(X \otimes_R X, R)^{hC_2}, \quad \Qoppa^q_R(X) := \mathrm{Hom}_R(X \otimes_R X, R)_{hC_2}.$$

A functor $f : \mathcal{C} \to \mathcal{D}$ is *reduced* if $f(0) = 0$. If it preserves exact squares as well it is *exact.* However, if it only preservers exact square it is 1-*excisive.*

Definition 2.1. A 3-cube $\rho : (\Delta^1)^3 \to \mathcal{C}$ is called *exact* if $\rho(0,0,0)$ is the limit of ρ restricted to the subsimplicial set $(\Delta^1)^3 \setminus \{0,0,0\}$. A 3-cube is called *strongly exact* if in addition each face is exact (and strongly exact implies exact). A functor $f : \mathcal{C} \to \mathcal{D}$ is 2-*excisive* if it maps strongly exact cubes to exact cubes.

Quadratic functors will be reduced 2-excisive functors, but we will develop criteria for verifying the above condition that is easier in practice. First we need a digression on bireduced functors.

Given a reduced functor $B : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ (i.e. $B(0,0) \simeq 0$) there is a retract diagram

$$B(x,0) \oplus B(0,y) \to B(x,y) \to B(x,0) \oplus B(0,y)$$

whose composite is the identity, where the various maps are induced by the unique maps $0 \to x \to 0$ and $0 \to y \to 0$. By the splitting lemma, we obtain an equivalence

$$B(x,y) \simeq B^{red}(x,y) \oplus B(x,0) \oplus B(0,y)$$

where $B^{red}(x,y)$ is either the fiber or cofiber of the maps in the above retract diagram. It might be useful to think of it in a slightly silly form as

$$B^{red}(x,y) \simeq \frac{B(x,y)}{B(x,0) \oplus B(0,y)}$$

This new functor is new *bireduced* in the sense that $B^{red}(x,0) \simeq B^{red}(0,y) \simeq 0$.

Definition 2.2. Given a reduced functor $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$, let $B_{\Omega} := \Omega((-)\oplus(-))^{red} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ denote the *cross effect*. This gives a functor $B_{(-)} : \mathrm{Fun}_*(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \to BiFun(\mathcal{C})$

Definition 2.3. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be stable and $b : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is *bilinear* if it is exact in each variable separately. It is *symmetric* if it is an element of $\mathrm{Fun}^s(\mathcal{C}) = [\mathrm{Fun}^b(\mathcal{C})]^{hC_2}$ under the flip action.

This might all seem very abstract and confusing, but we are trying to caputure the following elementary situation: If we have a quadratic function, the simplest one being $q(x) = x^2$ we can extract a symmetric and a bilinear part using $q(x+y) = (x+y)^2 = x^2 + y^2 + 2xy$. Here the symmetric part is $x^2 + y^2$ and the bilinear part is $q(x+y) - q(x) - q(y) = 2xy$. The bilinear part is exactly what the cross effect is picking up

$$B_{\Omega}(X,Y) = \Omega((X) \oplus (Y))^{red} \simeq \frac{\Omega(X \oplus Y)}{\Omega(X) \oplus \Omega(Y)}$$

Example 2.4. If $(\mathcal{C}, \otimes)$ is a monoidal category, we obtain a bilinear functor $B_a(x,y) = \hom_{\mathcal{C}}(x \otimes y, a)$. If $\otimes$ refines to a symmetric monoidal structure B_a is a symmetric bilinear functor.

Proposition 2.5. *Let $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be reduced functor. The following are equivalent:*

(1) Ω is 2-excisive.
(2) B_{Ω} is bilinear and $\mathrm{fib}(\Omega(x) \to B_{\Omega}(x,x)^{hC_2})$ is exact in x.
(3) B_{Ω} is bilinear and $\mathrm{cofib}(B_{\Omega}(x,x)_{hC_2} \to \Omega(x))$ is exact in x.

Definition 2.6. Any reduced $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ satisfying any of the conditions of Proposition 2.5 is called *quadratic*.

Example 2.7. Let us return to our central example. Let $B \in \mathrm{Fun}^s(\mathcal{C})$. Consider $\Omega^s_B(X) := B(X,X)^{hC_2}$ and $\Omega^q_B(X) := B(X,X)_{hC_2}$. We clam that these are both quadratic functors. This follows by calculating their symmetric bilinear parts[1]

[1]In the calculation we use the fact that if we have a finite group G then the spectrum $\oplus_G X$ with G-action permuting the entries (the (co)induced action) has homotopy fixed points $(\oplus_{g \in G} X)^{hG} \simeq X$. This follows from

$$\begin{aligned} B_{\Qoppa^s_B}(x,y) := \Qoppa^s_B(x \oplus y)^{red} &\simeq \frac{\Qoppa^s_B(x \oplus y)}{\Qoppa^s_B(x) \oplus \Qoppa^s_B(y)} \\ &\simeq \frac{B(x \oplus y, x \oplus y)^{hC_2}}{B(x,x)^{hC_2} \oplus B(y,y)^{hC_2}} \\ &\simeq \frac{(B(x,x) \oplus B(x,y) \oplus B(y,x) \oplus B(y,y))^{hC_2}}{B(x,x)^{hC_2} \oplus B(y,y)^{hC_2}} \\ &\simeq \frac{\left(B(x,x) \oplus \mathrm{Ind}_e^{C_2}(B(x,y)) \oplus B(y,y)\right)^{hC_2}}{B(x,x)^{hC_2} \oplus B(y,y)^{hC_2}} \\ &\simeq \frac{B(x,x)^{hC_2} \oplus B(x,y) \oplus B(y,y)^{hC_2}}{B(x,x)^{hC_2} \oplus B(y,y)^{hC_2}} \\ &\simeq B(x,y). \end{aligned}$$

This is certainly bilinear and conditions 2 and 3 are satisfied for $\Qoppa^s_B$ and $\Qoppa^q_B$ respectively, as the 0 functor is exact.

The superscripts $(-)^q$ and $(-)^s$ remind of classical quadratic and symmetric forms. For the perfect derived category

$$\mathcal{D}^p(R) = [\text{bounded, levelwise f.g. projective complexes}][q.i.^{-1}]$$

we have the bilinear functor $B_R(X,Y) := \mathrm{Hom}(X \otimes_R Y, R)$. Points in $\Omega^\infty B_R(X,Y)$ are then bilinear forms $b : X \otimes Y \to R$ if X, Y are projective and concentrated in a single degree. Similarly, we get the motivating identifications of $\pi_0(B_R(X,X))^{C_2}$ with symmetric forms on X and $\pi_0(B_R(X,X))_{C_2}$ with quadratic forms on X.

Definition 2.8. For quadratic functors $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ define the *linear part* $\Lambda_\Qoppa = \mathrm{cofib}(B_\Qoppa(X,X)_{hC_2} \to \Qoppa(X))$. This is exact by Proposition 2.5 (3).

3. Hermitian and Poincaré Categories

We will begin with the weaker definition of a hermitian category. It is in fact the stronger notion of a Poincaré category which is of primary interest, but many constructions only use the weaker notion of a hermitian category, and can be more transparently done in this setting.

Definition 3.1. A *hermitian category* is a pair $(\mathcal{C}, \Qoppa)$ where $\mathcal{C}$ category and $\Qoppa\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is a quadratic functor. The morphisms between hermitiancategories are *hermitian functors*. These are given by a pair $(f, \eta)\colon (\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa')$ where $f\colon \mathcal{C} \to \mathcal{C}'$ is an exact functor and $\eta\colon \Qoppa \to f^*\Qoppa' := \Qoppa' \circ f^{\mathrm{op}}$ is a natural transformation.

These assemble to an category Cat^h_∞ with objects hermitian categories $(\mathcal{C}, \Qoppa)$ and morphisms the hermitian functors.

Given a hermitian functor $(f, \eta)\colon (\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa')$, we can also relate the bilinear functors $B_\Qoppa$ and $B'_\Qoppa$, as η gives rise to a natural transformation $\beta_\eta : B_\Qoppa \to B_{f^*\Qoppa'} \simeq B_{f^*\Qoppa'}$.

Poincaré categories will be hermitian categories subject to two non-degenracy assumptions.

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the two adjunctions $\mathrm{Sp} \underset{(-)^{hG}}{\overset{triv}{\rightleftarrows}} Sp^{BG} \underset{\oplus_G(-)}{\overset{forget}{\rightleftarrows}} \mathrm{Sp}$, as composing the two left adjoints give the identity, hence the composition $(\oplus_G(-))^{hG}$ is right adjoint to the identity and thus the identity itself.

Definition 3.2. A bilinear functor $B \in \mathrm{Fun}^b(\mathcal{C})$, is called *right non-degenerate* if the presheaf of spectra $c \mapsto B(-, y)$ is representable for every $y \in \mathcal{C}$, i.e.

$$B(-, y) \simeq \hom_{\mathcal{C}}(-, D_B y)$$

where $D_B : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is the duality associated to B. There is a similar notion of *right non-degenerate.* If B is symmetric and satisfies either notion, we say it is *non-degenerate.*

If $(\mathcal{C}, \Omega)$ is a hermitian category, we say that it is non-degenerate if B_Ω is non-degenerate.

The two dualities obtained from a non-degenerate bilinear functor are adjoints, using the equivalences

$$\hom \mathcal{C}(x, D_B y) \simeq B_(x, y) \simeq B_K(y, x) \simeq \hom \mathcal{C}(y, D_B x) \simeq \mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(D_B^{\mathrm{op}} x, y).$$

The unit of the adjunction is $\mathrm{ev} : id \to DD^{\mathrm{op}}$.

Example 3.3. For $B_R(X, Y) = \hom_R(X \otimes Y, R) \simeq \hom_R(X, \hom_R^{cx}(Y, R))$ where we have used the tensor-hom adjunction, we see that $D_R(Y) = \hom_R^{cx}(Y, R)$ is the internal mapping complex. Hence ev really is evaluation, and in fact an equivalence.

Lemma 3.4. *Let $(\mathcal{C}, \Omega)$ and $(\mathcal{C}', tK')$ be two non-degenerate hermitian categories with dualities D_Ω and $D_{\Omega'}$. Then for $f, g : \mathcal{C} \to \mathcal{D}$ exact we have*

$$\mathrm{nat}(B_\Omega, (f \times g)^* B_{\Omega'}) \simeq \mathrm{nat}(f D_\Omega, D_{\Omega'} g^{\mathrm{op}})).$$

Proof. Using that left Kan extensions are adjoint to restriction and that they preserve representable functors and the Yoneda lemma, we have the following equivalences

$$\begin{aligned} \mathrm{nat}(B_\Omega, (f \times g)^* B_{\Omega'}) &\simeq \mathrm{nat}(B_\Omega, (f \times \mathrm{id})^* (\mathrm{id} \times g)^* B_{\Omega'}) \\ \mathrm{nat}((f \times \mathrm{id})_! B_\Omega, (\mathrm{id} \times g)^* B_{\Omega'}) &\simeq \mathrm{nat}(\hom(-, f D_\Omega)), \hom(-, D_{\Omega'} g^{\mathrm{op}})) \\ &\simeq \mathrm{nat}(f D_\Omega, D_{\Omega'} g^{\mathrm{op}})) \end{aligned}$$

□

Given a hermitian functor $(f, \eta) : (\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ denote by $\tau_\eta : f D_\Omega \to D_{\Omega'} f^{\mathrm{op}}$ the natural transformation corresponding to β_η under the above equivalences.

Definition 3.5. A hermitian functor (f, η) is *duality preserving* if τ_η is an equivalence

Definition 3.6. A symmetric bilinear functor B is called *perfect* if the evaluation map $ev : \mathrm{id}_{\mathcal{C}} \Rightarrow D_B D_B^{\mathrm{op}}$ is an equivalence. A hermitian structure Ω is called *Poincaré* if the underlying symmetric bilinear form B_Ω is perfect. If this is the case, we declare $(\mathcal{C}, \Omega)$ to be a *Poincaré category.* These assemble into a category Cat_∞^p with morphisms the duality preserving hermitian functors.

Remark 3.7. A perfect symmetric bilinear functor implies that it is in particular non-degenerate, as the unit and counit being equivalences is equivalent to the dualities being equivalences.

Example 3.8. We already verified that $B_R(X, Y)$ on $\mathcal{D}^p(R)$ was perfect. Hence both Ω_R^s and Ω_R^q give rise to Poincaré categories.

4. Classificiation of hermitian and Poincaré Categories

The classification will be in the style of Goodwillie calculus. We will show that quadratic functors can be recovered from a homogeneous and cohomogeneous part.

Lemma 4.1. *Let $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be quadratic. The following are equivalent:*

(1) $B_\Omega(x, x)_{hC_2} \to \Omega(x)$ is an equivalence.
(2) Ω is equivalent to Ω_B^q for some $B \in \mathrm{Fun}^s(\mathcal{C})$.

(3) The spectrum of natural transformation $\mathrm{nat}(\Qoppa, L) \simeq 0$ *for any exact functor* L*.*

Condition (3) is the definition of a functor being homogeneous. Similarly, if $\mathrm{nat}(L, \Qoppa) \simeq 0$ for any exact functor L we say that $\Qoppa$ is cohomogeneous. We can now state the classification of hermitian structures on $\mathcal{C}$.

Proposition 4.2. *The following square is a pullback square*

$$\begin{array}{ccc} \mathrm{Fun}^q(\mathcal{C}) & \xrightarrow{\tau} & \mathrm{Ar}(\mathrm{Fun}^{ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \\ \downarrow B & \lrcorner & \downarrow t \\ \mathrm{Fun}^s(\mathcal{C}) & \xrightarrow{(B^{\Delta})^{tC_2}} & \mathrm{Fun}^{ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}). \end{array}$$

The maps are: $\tau(\Qoppa) = (\Lambda_{\Qoppa} \to (B^{\Delta})^{tC_2})$, t is the target and $(-)^{tC_2}$ denotes the Tate construction, which is the cofiber of the norm map.

A similar classification for Poincaré structures also exists.

Proposition 4.3. *The following square is a pullback square*

$$\begin{array}{ccc} \mathrm{Fun}^p(\mathcal{C}) & \xrightarrow{\tau} & \mathrm{Ar}(\mathrm{Fun}^{ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \\ \downarrow B & \lrcorner & \downarrow t \\ \mathrm{Fun}^{ps}(\mathcal{C}) & \xrightarrow{(B^{\Delta})^{tC_2}} & \mathrm{Fun}^{ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}). \end{array}$$

Here $(-)^{ps}$ denotes perfect symmetric bilinear functors and duality preserving morphisms between them.

Example 4.4. The squares being pullbacks tells us that specifying a hermitian structure is essentially the same as specifying a natural transformation $L \to (B^{\Delta})^{tC_2}$ from an exact functor and some $B \in Fun^s(\mathcal{C})$. There are two obvious choices, namely $id : (B^{\Delta})^{tC_2} \to (B^{\Delta})^{tC_2}$ (which corresponds uniquely to the Poincaré structure $\Qoppa^s_B$) and $0 : 0 \to (B^{\Delta})^{tC_2}$ (which corresponds uniquely to the Poincaré structure $\Qoppa^q_B$).