MAT1305S: ADDITIONAL HANDOUT
ISOTOPY EXTENSION

ALEXANDER KUPERS

In this note we give a complete proof of the following result:

Theorem A. *Suppose that $M \subset N$ is an inclusion of closed manifolds. Then the map $\mathrm{Diff}(N) \to \mathrm{Emb}(M,N)$ given by $f \mapsto f|_M$ is a Serre fibration.*

Other proofs can be found in [Pal60, Lim64], which in fact prove that the map is a locally trivial fibre bundle. We instead deduce it quickly from a general fact about Serre microfibrations.

1. Serre microfibrations

The following definitions appears in [Wei05], inspired by Gromov's notion of microflexibility from [Gro86].

Definition 1.1. A map $p\colon E \to B$ is a *Serre microfibration* if in any commutative square

$$\begin{array}{ccc} D^i & \xrightarrow{\quad f \quad} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ D^i \times [0,1] & \xrightarrow{\ F\ } & B \end{array}$$

there exists a $\epsilon > 0$ and a lift of $F|_{D^i \times [0,\epsilon]}$ agreeing with f on $D^i \cong D^i \times \{0\}$.

Example 1.2. Every Serre fibration is a Serre microfibration, and famously a Serre microfibration with weakly contractible fibres is a Serre fibration [Wei05].

By induction over cells one proves that Serre microfibrations satisfy the following stronger lifting property:

Lemma 1.3. *If X is a finite CW-complex and A is a subcomplex, then in any commutative square*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\ F\ } & B \end{array}$$

there exists an open neighbourhood U of A in X and a lift of $F|_U$.

The reason for introducing Serre microfibrations is their behaviour under restriction, clear from the definitions:

Lemma 1.4. *If $p\colon E \to B$ is a Serre microfibration and $U \subset E$ is open, then $p|_U\colon U \to B$ is a Serre microfibration.*

Date: February 2, 2023.

Given these facts, we will prove the following proposition:

Proposition 1.5. *Suppose that a topological group G acts on a topological space X si that for all $x_0 \in X$ the map $\alpha_{x_0} \colon G \to X$ given by $g \mapsto g \cdot x_0$ is a Serre microfibration. These maps are also Serre fibrations.*

Proof. Suppose we have a lifting problem

$$\begin{array}{ccc} D^i & \xrightarrow{f} & G \\ \downarrow & & \downarrow \alpha_x \\ D^i \times [0,1] & \xrightarrow{F} & X, \end{array} \quad \text{or equivalently} \quad \begin{array}{ccc} [0,1]^i & \xrightarrow{f} & G \\ \downarrow & & \downarrow \alpha_x \\ [0,1]^{i+1} & \xrightarrow{F} & X. \end{array}$$

For each point $t \in [0,1]^{i+1}$ we then apply the lifting property from Lemma 1.3 in the modified diagram

$$\begin{array}{ccc} \{t\} & \xrightarrow{e} & G \\ \downarrow & & \downarrow \alpha_{F(t)} \\ [0,1]^{i+1} & \xrightarrow{F} & X \end{array}$$

to find an open subset $U_t \subset [0,1]^{i+1}$ around t and a map $\beta_t \colon U(t) \to G$ such that $\alpha_{F(t)} \circ \beta_t = F|_{U_t}$ or in other words $\beta_t \cdot F(t) = F|_{U_t}$ with $\cdot$ denoting the action; we will use the latter from now on. By the Lebesgue number lemma, there is a large integer N such that each of the cubes $I_k := \prod_{j=1}^{i+1} [k_j/N, (k_j+1)/N]$ is contained in some U_t. We pick such an open subset for each $k = (k_1, \ldots, k_{i+1})$, denote it U_k, and observe it comes with corresponding map $\beta_k \colon U_k \to G$.

We now by induction over the dimension i and the last index k_{i+1} construct a lift of F on $[0,1]^i \times [0, k_{i+1}/N]$. For the induction over i, in the initial case $i = -1$ there is nothing to prove, so we may assume that $i \geqslant 0$ and prove the induction step for i given the cases $i' < i$. For the induction over k_{i+1}, the initial case $k_{i+1} = 0$ is provided by f. For the induction step, the induction hypothesis for i implies that we can lift F over the sides $\partial(\prod_{j=1}^{i} [k_j/N, (k_j+1)/N]) \times [k_{i+1}/N, (k_{i+1}+1)/N]$ of I_k, so it remains to extend the lift to the interior of I_k. Reparametrising this cube as $[0,1]^{i+1}$ and the sides over which we already have a lift as $[0,1]^i$, we are in the situation where we have to solve the above lifting problem but have the additional information of a point $t \in [0,1]^{i+1}$ and a map $\beta \colon [0,1]^{i+1} \to G$ such that $\beta \cdot F(t) = F$. Our desired lift is then given by

$$[0,1]^i \times [0,1] \ni (s, t_{i+1}) \longmapsto \beta(s, t_{i+1}) \cdot \beta(s,0)^{-1} \cdot f(s) \in G,$$

as is verified by

$$\begin{aligned} \beta(s, t_{i+1}) \cdot \beta(s,0)^{-1} \cdot f(s) \cdot x_0 &= \beta(s, t_{i+1}) \cdot \beta(s,0)^{-1} \cdot F(s,0). \\ &= \beta(s, t_{i+1}) \cdot F(t) \\ &= F(s, t_{i+1}). \end{aligned}$$

□

2. Isotopy extension

We will first prove the following:

Lemma 2.1. *Suppose that $M \subset N$ is an inclusion of closed manifolds. Then the map $C^\infty(N,N) \to C^\infty(M,N)$ given by $f \mapsto f|_M$ is a Serre microfibration.*

Proof. We first pick an embedding of N into $\mathbb{R}^d$, identify N with its image, and choose a a tubular neighbourhood $\nu_N \cong W \subset \mathbb{R}^d$ for N in $\mathbb{R}^d$ with projection map $\pi\colon W \to N$. We next choose a tubular neighbourhood $\nu_M \cong V \subset N$ for M in N with projection map $\varpi\colon V \to M$. We finally pick a compactly-supported smooth function $\chi\colon V \to [0,1]$ which takes the value 1 on M. Given a commutative square

$$\begin{array}{ccc} D^i & \xrightarrow{f} & C^\infty(N,N) \\ \downarrow & & \downarrow \\ D^i \times [0,1] & \xrightarrow{F} & C^\infty(M,N) \end{array}$$

we then write the following candidate for a lift of F starting at f:

$$\begin{aligned} \overline{F}\colon D^i \times [0,1] &\longrightarrow C^\infty(N,N) \\ (s,t) &\longmapsto \left(p \mapsto \begin{cases} \pi\big[f(s)(p) + \chi(p)(f(s)(\varpi(p)) - F(s,t)(\varpi(p)))\big] & \text{if } p \in V, \\ f(s)(p) & \text{otherwise.} \end{cases}\right) \end{aligned}$$

This is well-defined as long as $f(s)(\varpi(p)) + \chi(p)(f(s)(\varpi(p)) - F(s,t)(\varpi(p))) \in W$ for all $p \in \mathrm{supp}(\chi)$. Since $\mathrm{supp}(\chi)$ is compact and this is true for $t = 0$, it is true for all $t \in [0,\epsilon]$ for some small $\epsilon > 0$. It is easily seen to be continuous in (s,t) and satisfies $\overline{F}(s,0) = f(s)$ since $f(s)(\varpi(p)) = F(s,0)(\varpi(p))$. Thus the formula for the candidate lift provides the desired partial lift when restricted to $D^i \times [0,\epsilon]$. ☐

Proof of Theorem A. Given an embedding $e\colon M \to N$ we may identify M with its image. Then when we restrict the Serre fibration $C^\infty(N,N) \to C^\infty(M,N)$ to the open subset $\mathrm{Diff}(N) \subset C^\infty(N,N)$, we land in $\mathrm{Emb}(M,N)$. Thus applying Lemma 1.4 and Lemma 2.1 we get that $\mathrm{Diff}(N,N) \to \mathrm{Emb}(M,N)$ is a Serre microfibration for any embedding $e\colon M \to N$. Now apply Proposition 1.5. ☐

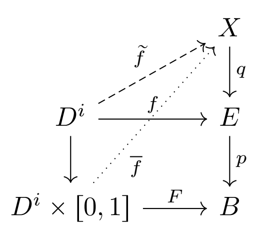
We deduce from the following variant:

Corollary 2.2. *Suppose that $P \subset M \subset N$ are closed submanifolds. Then the map $\mathrm{Emb}(M,N) \to \mathrm{Emb}(P,N)$ given by $e \mapsto e|_M$ is also a Serre fibration.*

Its proof uses the following lemma:

Lemma 2.3. *If $X \xrightarrow{q} E \xrightarrow{p} B$ are such that q and $p \circ q$ are Serre fibrations and q is surjective, then p is also a Serre fibration.*

Proof. Any map $D^i \to B$ admits a lift along the surjective Serre fibration $q\colon X \to B$. This is proven by induction over i; the initial case $i = 0$ is surjectivity and for the induction step write D^i as $D^{i-1} \times [0,1]$. Thus given a lifting problem



we can gift a dashed lift $\widetilde{f}$. Now using the fact that $p \circ q$ is a Serre fibration we get a dotted lift $\overline{f}$. The composition $q \circ \overline{f}$ solves the lifting problem. ☐

Proof. We may restriction our attention to path components $\mathrm{Emb}(M,N)_e$ and $\mathrm{Emb}(P,N)_{e|_P}$ of $e \in \mathrm{Emb}(M,N)$ and $e|_P \in \mathrm{Emb}(P,N)$, as long as we replace $\mathrm{Diff}(N)$ by the stabiliser $\mathrm{Diff}(N)_{\mathrm{stab}(e)}$ of the isotopy class of e. Then we consider

$$\mathrm{Diff}(N)_{\mathrm{stab}(e)} \longrightarrow \mathrm{Emb}(M,N)_e \longrightarrow \mathrm{Emb}(P,N)_{e|_P}$$

where the left map and composition are Serre fibrations by Theorem A. Moreover, by the easy unparametrised version of isotopy extension the left map is surjective. An application of Lemma 2.3 finishes the proof. □

References

[Gro86] Mikhael Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9, Springer-Verlag, Berlin, 1986. MR 864505 (90a:58201) 1

[Lim64] Elon L. Lima, *On the local triviality of the restriction map for embeddings*, Comment. Math. Helv. **38** (1964), 163–164. MR 161343 1

[Pal60] Richard S. Palais, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. **34** (1960), 305–312. MR 123338 1

[Wei05] Michael Weiss, *What does the classifying space of a category classify?*, Homology Homotopy Appl. **7** (2005), no. 1, 185–195. MR 2175298 (2007d:57059) 1