INTRODUCTION TO HERMITIAN K-THEORY

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1. K_0-, GW_0-, and W_0-groups

1.1. K_0. To understand a commutative ring R or situations in mathematics where it plays a role, one wants to classify R-modules. Of particular interest are the "small" ones: finitely generated projective R-modules. If R is commutative, one can interpret these as vector bundles over $\operatorname{Spec}(R)$.

Example 1.1. Every finitely-generated projective $\mathbb{F}$-module for a field $\mathbb{F}$ is free.

Example 1.2. Every finitely-generated projective $\mathbb{Z}$-module is free.

Example 1.3 (Serre–Swan). For a compact Hausdorff space X, there is an equivalence of categories between finite-dimensional vector bundles over X and finitely generated projective $C(X,\mathbb{R})$-modules.

Direct sum makes the set

$$\operatorname{Proj}(R) := \frac{\{\text{finitely generated projective } R\text{-modules}\}}{\text{isomorphism}}$$

into an abelian monoid. These are hard to understand and we can formally adjoin inverses by "group completing". Group completion is the left adjoint to the inclusion $\mathsf{AbGrp} \to \mathsf{AbMon}$, in the sense that it is the initial map $M \to M^{\mathrm{gc}}$ of into an abelian group. In other words, every map $M \to A$ with A an abelian group factors uniquely over $M \to M^{\mathrm{gc}}$

$$\begin{array}{ccc} M & \longrightarrow & A \\ \downarrow & \overset{\exists!}{\nearrow} & \\ M^{\mathrm{gc}}. & & \end{array}$$

Explicitly, we can construct it as

$$M^{\mathrm{gc}} := \frac{\mathbb{Z}[M]}{\{[a]+[b]=[ab]\}}.$$

Definition 1.4. The *zeroth K-group of R* is given by

$$K_0(R) := \operatorname{Proj}(R)^{\mathrm{gc}}.$$

Example 1.5. Using that every finitely-generated projective $\mathbb{F}$-module for a field $\mathbb{F}$ is free, we get an isomorphism

$$\dim\colon K_0(\mathbb{F}) \xrightarrow{\cong} \mathbb{Z}.$$

Example 1.6. Similarly as for fields, we get an isomorphism

$$\operatorname{rk}\colon K_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}.$$

More generally, if $\mathcal{O}$ is a Dedekind domain then the classification of finitely-generated $\mathcal{O}$-modules yields $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \operatorname{Cl}(\mathcal{O})$, where the second term is the class group of fractional ideals modulo principal ideals.

Example 1.7. For compact Hausdorff X, the Serre–Swan correspondence gives an isomorphism $K_0(C(X,\mathbb{R})) \cong K^0(X)$, with the latter the (real) topological K-theory of X defined as the group completion $\operatorname{Vect}(X)^{\mathrm{gc}}$ of the abelian monoid of isomorphism classes of finite-dimensional real vector bundles over X under direct sum.

1.2. GW_0. Often we are interested in modules with additional structure. Drawing inspiration from Poincaré duality, a common one is a quadratic form.

Definition 1.8. A *quadratic form* on a finitely-generated projective R-module P is a map $q\colon P \to R$ satisfying:

(1) $q(rx) = r^2 q(x)$,
(2) $b(x,y) := q(x+y) - q(x) - q(y)$ is a symmetric bilinear form.

It is *unimodular* if the induced map $b_\sharp\colon P \to P^*$ is an isomorphism.

Remark 1.9. If 2 is invertible in R then q can be recovered from b by $q(x) = \frac{1}{2}b(x,x)$. This shows a general feature of this learning seminar: a lot of the subtleties arise from dealing with the case that 2 is not invertible.

Orthogonal direct sum makes the set

$$\mathrm{Unimod}(R) := \frac{\{\text{finitely-generated projective } R\text{-modules with unimoudlar quadratic form}\}}{\text{isomorphism}}$$

into an abelian monoid.

Definition 1.10. The *zeroth Grothendieck–Witt group* of R is given by

$$GW_0(R) := \mathrm{Unimod}(R)^{\mathrm{gc}}.$$

Example 1.11 (Sylvester). A quadratic form over $\mathbb{R}$ is determined uniquely up to isomorphism by its dimension and its signature, inducing an isomorphism

$$(\dim, \sigma)\colon GW_0(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}.$$

However, over $\mathbb{C}$ (or any algebraically closed field) we have an isomorphism

$$\dim\colon GW_0(\mathbb{C}) \xrightarrow{\cong} \mathbb{Z}.$$

1.3. W_0. The computation of Grothendieck–Witt groups can be approached through the computation of so-called Witt groups. Their definition uses that for any finitely-generated projective R-module P, not only is its dual P^* is also a finitely-generated projective R-module, but $P \oplus P^*$ has a canonical unimodular quadratic form given by $q_{\mathrm{hyp}}(x,\alpha) = \alpha(x)$, referred to as a hyperbolic form. Indeed, we have

(1) $q_{\mathrm{hyp}}(r(x,\alpha)) = (r\alpha)(rx) = r^2\alpha(x)$,
(2) $q_{\mathrm{hyp}}((x,\alpha)+(y,\beta)) - q_{\mathrm{hyp}}(x,\alpha) - q_{\mathrm{hyp}}(y,\beta) = \alpha(y) + \beta(x)$.

This gives a map

$$\begin{aligned} K_0(R) &\longrightarrow GW_0(R) \\ P &\longmapsto (P \oplus P^*, q_{\mathrm{hyp}}). \end{aligned}$$

This sends P and P^* to isomorphic quadratic modules, so factors over the coinvariants $K_0(R)_{C_2}$ for the involution on $K_0(R)$ given by $P \mapsto P^*$. For some rings this action is trivial—when P is always isomorphic to its own dual—and we have that $K_0(R) \to K_0(R)_{C_2}$ is an isomorphism. At any rate, the result is a map

$$\mathrm{hyp}\colon K_0(R)_{C_2} \longrightarrow GW_0(R).$$

Definition 1.12. The *zeroth Witt group of* R is given by

$$W_0(R) := \mathrm{coker}(\mathrm{hyp}\colon K_0(R)_{C_2} \to GW_0(R)).$$

That is, the Witt group is obtained by taking the quotient of the Grothendieck–Witt group by the subgroup generated by hyperbolic forms. There is a tautological exact sequence

$$K_0(R)_{C_2} \longrightarrow GW_0(R) \longrightarrow W_0(R) \longrightarrow 0. \tag{1}$$

Example 1.13. Since hyperbolic forms are always even-dimensional, over $\mathbb{C}$ the exact sequence (1) is given by

$$0 \longrightarrow K_0(\mathbb{C})_{C_2} \cong \mathbb{Z} \xrightarrow{2\cdot -} GW_0(\mathbb{C}) \cong \mathbb{Z} \longrightarrow W_0(\mathbb{C}) \cong \mathbb{Z}/2 \longrightarrow 0.$$

Example 1.14. The classification of unimodular quadratic forms over $\mathbb{Z}$ is the same as the classification of even unimodular symmetric forms. The latter is hard in general, but it is easier once we allow them to be replaced by indefinite ones by direct sum with hyperbolic forms. It is a result of Serre that even indefinite unimodular symmetric forms over $\mathbb{Z}$ are all isomorphic to direct sums of $H = (\mathbb{Z}^2, [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}])$ and E_8. We conclude that (1) is given by

$$0 \longrightarrow K_0(\mathbb{Z})_{C_2} \cong \mathbb{Z} \xrightarrow{\text{hyp}} GW_0(\mathbb{Z}) \to W_0(\mathbb{Z}) \xrightarrow[\sigma/8]{\cong} \mathbb{Z} \longrightarrow 0$$

where injectivity of the left map follows by considering the dimension and the existence of the E_8-form given a splitting of the right map, yielding $GW_0(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Question 1.15. Can we extend (1) to a long exact sequence?

1.4. **More generality.** There are several generalisations that a useful theory of modules with forms should encompass:

- *Allow involutions.* Over $\mathbb{C}$, we tend not to care about symmetric bilinear forms but hermitian ones, i.e. those that are $\mathbb{C}$-linear in the second entry and satisfying $b(x,y) = \overline{b(x,y)}$. We would hence like to allow rings with involutions.
- *More general rings.* For a group ring $R = \mathbb{Z}[G]$, the Witt group $W_0(\mathbb{Z}[G])$ (involving the involution $\overline{a_1 g_1 + \cdots + a_n g_n} = a_1 g_1^{-1} + \cdots + a_n g_n^{-1}$) is the same as Wall's quadratic L-group $L_0^q(\mathbb{Z}[G])$ from surgery theory. We would hence like to allow non-commutative rings.
- *More general forms.* In manifold theory, Poincaré duality endows the middle-dimensional homology group of an oriented closed manifold of dimension $2n$ into a bilinear form, which is symmetric if n is even but anti-symmetric if n is odd. We would hence like to allow other types of "quadratic" forms.

Question 1.16. What is a convenient framework allowing all these generalisations?

2. K-, GW-, and L-spectra

The above two questions go together, as we will see by first considering a related situation in algebraic K-theory.

2.1. **K-spectra.** Let us start with a digression on a method to compute algebraic K-theory groups. If $s \in R$ is not a zero-divisor we can form the localisation $s^{-1}R$ and localising induces a surjective map $K_0(R) \to K_0(s^{-1}R)$. Let me momentarily ignore that modules needed to be projective; suppose the ring is regular Noetherian will allow you to deal with this. Then in its kernel are finitely-generated modules obtained from R/sR-modules by restriction along $R \to R/sR$. It turns out that

$$K_0(R/sR) \longrightarrow K_0(R) \longrightarrow K_0(s^{-1}R) \longrightarrow 0$$

is exact when R satisfies certain hypotheses. Given an element $A \in \mathrm{GL}_n(s^{-1}(R))$, we can represent it as A/s^n with $A \in \mathrm{M}_n(R)$ a matrix that becomes invertible after inverting s, and then construct a well-defined element of $K_0(R/(s))$ as $[\mathrm{coker}(A)] - n[R/sR]$. One may check that this factors over

$$K_1(s^{-1}R) := \operatorname*{colim}_{n\to\infty} H_1(B\mathrm{GL}_n(s^{-1}R))$$

and extends to an exact sequence

$$K_1(R/sR) \longrightarrow K_1(R) \longrightarrow K_1(s^{-1}R) \longrightarrow K_0(R/sR) \longrightarrow K_0(R) \longrightarrow K_0(s^{-1}R) \longrightarrow 0.$$

This exact sequence strongly resembles part of a long exact sequence of homotopy groups of a fibre sequence, and it indeed is. Stating this requires the construction of algebraic K-theory spectra, and one way to do so is to simply repeat the construction of $K_0(R)$ in a homotopy-theoretic manner.

Our starting point is to replace the set $\mathrm{Proj}(R)$ of isomorphism classes of finitely-generated projective R-modules with the groupoid $\mathsf{Proj}(R)$ whose objects are finitely-generated projective R-modules and whose morphisms are isomorphisms. Direct sum makes this into a symmetric monoidal groupoid, i.e. a commutative algebra object in groupoid. A groupoid G is special case of an ∞-groupoid or equivalently space—the classifying space BG—and a commutative algebra object in ∞-groupoids is an E_∞-space. We say that an E_∞-space is an E_∞-group if its path components form a group. The inclusion $\mathrm{Alg}_{E_\infty}(\mathcal{S})^{\mathrm{grp}} \to \mathrm{Alg}_{E_\infty}(\mathcal{S})$ has an ∞-categorical left adjoint $(-)^{\mathrm{gc}}$, a homotopy-theoretic form of group completion, which can be explicitly modelled by Segal's ΩB-construction. We can then define an *algebraic K-theory space* as

$$\Omega^\infty \mathrm{K}(R) := (B\mathsf{Proj}(R))^{\mathrm{gc}}.$$

As the notation suggests, this is the infinite loop space of an *algebraic K-theory spectrum* $\mathrm{K}(R)$. One way to obtain this is to use that there is an equivalence of ∞-categories $\mathrm{Alg}_{E_\infty}(\mathcal{S})^{\mathrm{grp}} \simeq \mathcal{S}\mathrm{p}^{\mathrm{conn}}$ between E_∞-groups and connective spectra.

Example 2.1. If X is an E_∞-space then $\pi_0(X^{\mathrm{gc}}) \cong (\pi_0(X))^{\mathrm{gc}})$. Thus $\pi_0(\mathrm{K}(R)) \cong K_0(R)$.

Given this construction, we have:

Theorem 2.2 (Localisation)**.** *If R is regular Noetherian and s is not a zero-divisor then there is a fibre sequence*

$$\mathrm{K}(R/sR) \longrightarrow \mathrm{K}(R) \longrightarrow \mathrm{K}(s^{-1}R).$$

Relaxing the conditions on R involves the following observation: the construction of the spectrum $\mathrm{K}(R)$ depends on R through the category $\mathsf{Proj}(R)$. More generally, you can define algebraic K-theory of certain categories, e.g. for abelian categories through Quillen's Q-construction. In fact, the input can be the ∞-categorical analogue of an abelian category, a *stable ∞-category.* We can recover $\mathrm{K}(R)$ in this manner from the derived perfect stable ∞-category $\mathcal{D}^p(R)$. This will be our official definition of algebraic K-theory, because it is the one used to establish most important formal properties.

2.2. ***GW*-spectra.** We can similarly define a *classical Grothendieck–Witt spectrum* as the connective spectrum associated to the *classical Grothendieck–Witt space*

$$\Omega^\infty \mathrm{GW}(R)^{\mathrm{cl}} := (B\mathsf{Unimod}(R))^{\mathrm{gc}},$$

where Unimod is the symmetric monoidal groupoid of finitely-generated projective R-modules with unimodular quadratic form under orthogonal sum.

Example 2.3. $\pi_0(\mathrm{GW}(R)^{\mathrm{cl}}) \cong GW_0(R)$.

The reason I am calling this a "classical" Grothendieck–Witt spectrum, is that it is a theorem that it coincides with our official definition of a Grothendieck–Witt spectrum: we will associate a spectrum $\mathrm{GW}(\mathcal{C}, ♀)$ to a stable ∞-category with an additional structure known as a *Poincaré structure.*

A Poincaré structure is akin to a quadratic form on $\mathcal{C}$ and encodes the type of forms we want to consider. More precisely, it is a functor $♀\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ which is quadratic in the sense of Goodwillie calculus (its third cross-effect vanishes) and inducing a duality equivalence $D_♀\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. A *Poincaré object* is then an object $x \in \mathcal{C}$ with an $q \in ♀(x)$ such that $q_\sharp\colon x \to D_♀(x)$ is an equivalence and from we can extract the Grothendieck–Witt space as the group completion of the space of Poincaré objects under orthogonal sum

$$\Omega^\infty \mathrm{GW}(\mathcal{C}, ♀) := \mathrm{Poinc}(\mathcal{C}, ♀)^{\mathrm{gp}}$$

and the spectrum $\mathrm{GW}(\mathcal{C}, \Omega)$ directly via the hermitian Q-construction.

Allowing slightly more general types of forms—(anti-)quadratic or (anti-)symmetric—we have:

Theorem 2.4 (Hebestreit–Steimle)**.** $\mathrm{GW}^{\lambda}(R)^{\mathrm{cl}} \simeq \mathrm{GW}(\mathcal{D}^p(R), \Omega^{g\lambda})$ *for* $\lambda \in \{q, -q, s, -s\}$.

The subtlety here is in the definition of the "genuine" Poincaré structures $\Omega^{g\lambda}$. These are related to more straightforward guesses by natural transformations

$$\Omega^q = \mathrm{Hom}(- \otimes -, R)_{hC_2} \Longrightarrow \Omega^{gq} \Longrightarrow \Omega^{gs} \Longrightarrow \Omega^s = \mathrm{Hom}(- \otimes -, R)^{hC_2}$$

which are equivalences if 2 is invertible in R. This is one of the reasons that the development of hermitian K-theory had to wait until higher category theory was in place.

This will in fact be the last theorem we discuss in this learning seminar. Instead, we will in detail set up the framework of Grothendieck–Witt spectra leading to an fibre sequence inducing (1).

2.3. ***L*-spectra.** At this point we know the spectrum-level analogues of $K_0(R)$ and $GW_0(R)$ in (1) but not yet of $W_0(R)$. This turns out to be an L-theory spectrum $\mathrm{L}(\mathcal{C}, \Omega)$, satisfying not only $\pi_0 \, \mathrm{L}(\mathcal{D}^p(R), \Omega^{gq}) = W_0(R)$ and also featuring in the following theorem:

Theorem 2.5 (Fundamental theorem)**.** *There is a fibre sequence*

$$\mathrm{K}(\mathcal{C})_{hC_2} \longrightarrow \mathrm{GW}(\mathcal{C}, \Omega) \longrightarrow \mathrm{L}(\mathcal{C}, \Omega)$$

which is split after inverting 2.

Idea of the proof. We will establish a universal property of $\mathrm{L}(-)$ and prove that $\mathrm{cofib}[\mathrm{K}(-)_{hC_2} \to \mathrm{GW}(-)]$ has the same universal property. □

This theorem is useful, because it allows one to compute Grothendieck–Witt groups in terms of algebraic K-theory groups. The crucial input for this is that L-theory groups are easy to compute because they are often 4-periodic. For example, that the homotopy groups of $\mathrm{L}(\mathcal{D}^p(R), \Omega^s)$ are 4-periodic is due to Ranicki and the difference between this L-theory spectrum and those for other Poincaré structures can be analysed.

Remark 2.6. There will be a strong analogy between the constructions and proofs in hermitian K-theory and those in study of cobordism categories. This is not coincidental; one of the eventual goals is to study stable moduli spaces of odd-dimensional manifolds. We know these are infinite loop spaces of spectra but do not know what these spectra are. The few results that are known suggest they are similar but not equal to fibres of maps from Thom spectra to L-theory spectra.