RINGS WHOSE GENERAL LINEAR GROUPS DO NOT EXHIBIT HOMOLOGICAL STABILITY

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ABSTRACT. This note gives examples of rings with the property that the first homology of their general linear groups stabilises late, or never at all.

Much has been written about homological stability for general linear groups of a ring R. The purpose of this note is to describe a ring R whose general linear groups do *not* exhibit homological stability. Though this was surely known to experts, we do not know a reference in the literature.

For a topological space X, let $\mathbb{C}^X$ denote the ring of continuous function $f\colon X \to \mathbb{C}$ with pointwise addition and multiplication.

Theorem 0.1. *For odd d, the stabilisation map*

$$H_1(\mathrm{GL}_n(\mathbb{C}^{S^d});\mathbb{Z}) \longrightarrow H_1(\mathrm{GL}_\infty(\mathbb{C}^{S^d});\mathbb{Z})$$

is not surjective for $n < d$.

Proof. There is an isomorphism of groups

$$\mathrm{GL}_n(\mathbb{C}^{S^d}) \cong \mathrm{GL}_n(\mathbb{C})^{S^d},$$

where the right side is the group of continuous maps $S^d \to \mathrm{GL}_n(\mathbb{C})$. Taking the homotopy class of such a map and using that the inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is a homotopy equivalence by Gram–Schmidt, we obtain a homomorphism

$$\mathrm{GL}_n(\mathbb{R})^{S^d} \longrightarrow [S^d, \mathrm{U}(n)],$$

and since $\mathrm{U}(n)$ is a path-connected Lie group we may identify the right side with $\pi_d(\mathrm{U}(n))$. Since this is abelian and $H_1(G;\mathbb{Z}) \cong G^{\mathrm{ab}}$, the result is a homomorphism

$$H_1(\mathrm{GL}_n(\mathbb{C}^{S^d});\mathbb{Z}) \longrightarrow \pi_d(\mathrm{U}(n))$$

which is surjective, by considering a representative of the right side as an element of $\mathrm{U}(n)^{S^d} \subset \mathrm{GL}_n(\mathbb{C})^{S^d} \cong \mathrm{GL}_n(\mathbb{C}^{S^d})$, and compatible with stabilisation. As a consequence we have a commutative diagram with vertical maps surjections

$$\begin{array}{ccc} H_1(\mathrm{GL}_n(\mathbb{C}^{S^d});\mathbb{Z}) & \longrightarrow & H_1(\mathrm{GL}_\infty(\mathbb{C}^{S^d});\mathbb{Z}) \\ \downarrow & & \downarrow \\ \pi_d(\mathrm{U}(n)) & \longrightarrow & \pi_d(\mathrm{U}). \end{array}$$

Since d is odd the target is $\mathbb{Z}$ by Bott periodicity, and by induction over the usual fibration sequences

$$\mathrm{U}(m) \longrightarrow \mathrm{U}(m+1) \longrightarrow S^{2m+1}$$

one proves that the bottom map can not be surjective until $n = d$. This implies that for $n < d$ the top map can not be surjective. □

Remark 0.2. By [Mil71] there is an isomorphism $H_1(\mathrm{GL}_\infty(\mathbb{C}^{S^d});\mathbb{Z}) \cong K_1(\mathbb{C}^{S^d}) \xrightarrow{\sim} (\mathbb{C}^\times)^{S^d} \times \pi_d(U)$ given by the determinant and the construction in the proof of the theorem.

Corollary 0.3. *For* $R = \mathbb{C}^{\sqcup_{d\ odd} S^d}$ *the stabilisation map*

$$H_1(\mathrm{GL}_n(R);\mathbb{Z}) \longrightarrow H_1(\mathrm{GL}_\infty(R);\mathbb{Z})$$

is never surjective.

Proof. Suppose for contradiction that this map is surjective for some n, which we may assume without loss generality is an odd number d. Then because the homomorphism $R \to \mathbb{C}^{S^d}$ is surjective, so are the vertical maps in the commutative diagram

$$\begin{array}{ccc} H_1(\mathrm{GL}_n(R);\mathbb{Z}) & \longrightarrow & H_1(\mathrm{GL}_\infty(R);\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_1(\mathrm{GL}_n(\mathbb{C}^{S^d});\mathbb{Z}) & \longrightarrow & H_1(\mathrm{GL}_\infty(\mathbb{C}^{S^d});\mathbb{Z}) \end{array}$$

and it follows from the assumption that the top horizontal map is surjective that the bottom horizontal is also. This contradicts the previous theorem. □

By [vdK80] the general linear groups over a ring with finite Bass stable rank have homological stability, so R as in the corollary has infinite Bass stable rank. This also follows from [Vas71, Theorem 5].

Remark 0.4. That R has finite Bass stable rank is not necessary for $H_1(\mathrm{GL}_n(R);\mathbb{Z})$ to stabilise. On the one hand, by [Vas71, Theorem 8] the ring $\mathbb{R}[x_1, x_2 \ldots, x_n]$ has stable rank $n+1$ so $\mathbb{R}[x_1, x_2, \ldots]$ has infinite stable rank. On the other hand, by [Sus77, Theorem 6.3] and homotopy invariance of algebraic K-theory the map

$$H_1(\mathrm{GL}_n(\mathbb{R});\mathbb{Z}) \longrightarrow H_1(\mathrm{GL}_n(\mathbb{R}[x_1, x_2 \ldots, x_n]);\mathbb{Z})$$

is an isomorphism for $n \geq 3$, and taking colimits the same is true when we replace the target with $H_1(\mathrm{GL}_n(\mathbb{R}[x_1, x_2 \ldots]);\mathbb{Z})$. Since the groups $H_1(\mathrm{GL}_n(\mathbb{R});\mathbb{Z})$ stabilise so do the groups $H_1(\mathrm{GL}_n(\mathbb{R}[x_1, x_2 \ldots]);\mathbb{Z})$.

References

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