INVARIANT ALGEBRAIC SETS OF COMPLEX MULTIJETS, AND THE FOUR OPERATIONS

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ABSTRACT. In the first part of this talk we define the type of sets of multijets which is used to produce the stratification in Goodwillie's proof: *IASCMs*. These will be constructed from a single such set Z^0 by operations A, B, C, and D, which we will discuss in the second part of this talk.

In these notes, we discuss Chapters II.A and II.B of [Goo90].

1. Context

Recall that we are trying to modify a fibered concordance

$$F\colon I \times P \times D^s \longrightarrow I \times N \times D^s$$

through a fibered isotopy of concordances, so as to avoid some submanifold $I \times M$ of $I \times N$ (or at least one in a collection of these). Our strategy is to give, after putting F in general position, a stratification of $P \times D^s$ with good and bad strata such that the bad strata have large codimension and on good strata we can inductively remove the intersections with $I \times M$ by a sunny collapsing procedure.

In the next talk we define this stratification. In this talk we define the sets of jets of locally holomorphic maps $\mathbb{C}^{p+1} \to \mathbb{C}^n$ near r points that describe these strata in (complexified) local coordinates. They will be produced by the four operations A, B, C, and D we have heard so much about already, from a basic set Z^0.

2. Invariant algebraic sets of complex multijets

Let $\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n)$ be the set of equivalence classes z of pairs of a point $sz \in \mathbb{C}^{p+1}$ and a jet of order $\leq m$ of a holomorphic function $\mathbb{C}^{p+1} \subset U \to \mathbb{C}^n$ for U near $sz \in U$. (As the dimensions indicate, we will be looking at the fiberwise jets of the composition $pF\colon I \times P \times D^s \to I \times N \times D^s \to N \times D^s$ eventually.) Think of sz as the *source* of z, and let $tz \coloneqq z(sz) \in \mathbb{C}^n$ be its *target*.

Multijets are then given by jets at a configuration of distinct points:

$${}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n) \coloneqq \{(z_1, \ldots, z_r) \mid sz_i \neq sz_j \text{ if } i \neq j\} \subset \mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n)^r.$$

Forgetting from order m' to m, yields a map

$$p^{m'}_m\colon {}_r\mathcal{J}^{m'}_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n) \longrightarrow {}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n) \quad \text{for } m \leq m',$$

and we let ${}_r\mathcal{J}^\infty(\mathbb{C}^{p+1}, \mathbb{C}^n)$ be the limit over these maps, which comes with induced maps $p^\infty_m\colon {}_r\mathcal{J}^\infty_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n) \to {}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n)$.

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Notation 2.1. We shorten ${}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n)$ to ${}_r\mathcal{J}^m$.

Then ${}_r\mathcal{J}^m$ is a smooth complex algebraic variety (a Zariski open subset of affine space, in fact) defined over $\mathbb{R}$.

Definition 2.2. A subset $Z \subset {}_r\mathcal{J}^\infty$ is *algebraic of level m and rank r* if

$$Z = (p^\infty_m)^{-1}(p^\infty_m(Z))$$

and $p^\infty_m(Z)$ is a closed algebraic subset of ${}_rJ^m$ defined over $\mathbb{R}$.

Remark 2.3. If Z has level m it also has level m' for $m' > m$.

Here are the two most important examples:

Example 2.4. The set $Z^0 \coloneqq \{(z_1, z_2) \in {}_2\mathcal{J}^\infty \mid t(z_1) = t(z_2)\}$ is algebraic of level 0 and rank 2.

Example 2.5. The set $Z^1 \coloneqq \{z \in {}_2\mathcal{J}^\infty \mid \ker(Dz) \neq 0\}$ is algebraic of level 1 and rank 1.

If an algebraic set Z of level m and rank r is invariant under suitable coordinate changes in the domain and target, it can be used to construct sets

$$S(Z, P, N) \subset {}_r\mathcal{J}^\infty_{\mathbb{R}}(\mathbb{R} \times P, N).$$

Describing this construction will lead us to the correct notion of invariance. Using charts, we cover ${}_r\mathcal{J}^\infty_{\mathbb{R}}(\mathbb{R} \times P, N)$ by ${}_r\mathcal{J}^\infty_{\mathbb{R}}(\mathbb{R} \times U, V)$ with $U \subset P$ and $V \subset N$ open and coming with embeddings $\phi\colon U \hookrightarrow \mathbb{R}^p$ and $\psi\colon V \hookrightarrow \mathbb{R}^n$. Then we say

$$z \in S(Z, P, N) \iff \text{there exists } \tilde{z} \in Z \text{ such that } {}_rj^\infty(\psi)(tz) \circ z = \tilde{z} \circ {}_rj^\infty(\mathrm{id} \times \phi)(sz),$$

where we identify ${}_rJ^m_{\mathbb{R}}(\mathbb{R}^{p+1}, \mathbb{R}^n)$ with a subset of ${}_rJ^m_{\mathbb{C}}(\mathbb{C}^{p+1}, \mathbb{C}^n)$ for $0 \leq m \leq \infty$ by identifying Taylor polynomials with real coefficients as a subset of the Taylor polynomials with complex coefficients. A correct notion of independence is the one that makes the right side independent of U, V.

2.1. **Invariance in the domain.**

We start with invariance in the domain. Suppose we are given a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^{p+1} \supset \Omega_1 & \xrightarrow{\mathcal{U}} & \Omega_2 \subset \mathbb{C}^{p+1} \\ \downarrow p_2 & & \downarrow p_2 \\ \mathbb{C}^p \supset \Omega_3 & \xrightarrow{\mathcal{V}} & \Omega_4 \subset \mathbb{C}^{p+1} \end{array}$$

with horizontal maps complex diffeomorphisms, and p_2 given by the projection $\mathbb{C}^{p+1} = \mathbb{C} \times \mathbb{C}^p \to \mathbb{C}^p$. Then precomposition by $\mathcal{U}$ gives an induced map

$$\mathcal{U}^*\colon {}_r\mathcal{J}^m_{\mathbb{C}}(\Omega_2, \mathbb{C}^n) \longrightarrow {}_r\mathcal{J}^m_{\mathbb{C}}(\Omega_1, \mathbb{C}^n).$$

Definition 2.6. An algebraic set $Z \subset {}_r\mathcal{J}^\infty$ of level m is *domain-invariant* if for each diagram of the above form we have

$$\mathcal{U}^*(p^\infty_m(Z) \cap {}_r\mathcal{J}^m_{\mathbb{C}}(\Omega_2, \mathbb{C}^n)) = p^\infty_m(Z) \cap {}_r\mathcal{J}^m_{\mathbb{C}}(\Omega_1, \mathbb{C}^n).$$

Let us give a more local description of domain-invariance. For any $(x_1,\dots,x_r)\in\Omega_1^{(r)}$, the jets of $\mathcal{U}$ and $\mathcal{V}$ given by

$$u_i \coloneqq j^\infty(\mathcal{U})(x_i)\in\mathcal{J}^\infty(\Omega_1,\Omega_2) \qquad\text{and}\qquad v_i\coloneqq j^\infty(\mathcal{V})(p_2x_i)\in\mathcal{J}^\infty_{\mathbb{C}}(\Omega_3,\Omega_4)$$

have the following properties:

- u_i and v_i are invertible,
- $su_i\neq su_j$ if $i\neq j$,
- $tu_i\neq tu_j$ if $i\neq j$,
- $j^\infty(p_2)(tu_i)\circ u_i = v_i\circ j^\infty(p_2)(su_i)$ for all i,
- $sv_i=sv_j$ if and only if $tv_i=tv_j$ if and only if $v_i=v_j$ for all i,j.

Using the complex version of the inverse function theorem, these conditions on multijets suffice to construct a commutative diagram as above (for very small Ω_i, $i=1,2,3,4$ around the source and target of the multijets). This implies:

Lemma 2.7. *An algebraic set Z is domain-invariant if and only if for all $u_1,\dots,u_r\in\mathcal{J}^\infty_{\mathbb{C}}(\mathbb{C}^{p+1},\mathbb{C}^{p+1})$ and $v_1,\dots,v_r\in\mathcal{J}^\infty_{\mathbb{C}}(\mathbb{C}^p,\mathbb{C}^p)$ with the above properties and $z\in Z$ such that $sz_i=tu_i$, the multijet $z\circ u$ lies in Z.*

As a jet in $\mathcal{J}^m_{\mathbb{R}}(\mathbb{R}^{p+1},\mathbb{R}^n)$ is invertible if and only if its image in $\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1},\mathbb{C}^n)$ is, we conclude that given a commutative diagram

$$\begin{array}{ccc}\mathbb{R}^{p+1}\supset\Omega_1 & \xrightarrow{\mathcal{U}} & \Omega_2\subset\mathbb{R}^{p+1}\\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2}\\ \mathbb{R}^p\supset\Omega_3 & \xrightarrow{\mathcal{V}} & \Omega_4\subset\mathbb{R}^{p+1}\end{array}$$

with horizontal maps diffeomorphisms, we have that

$$\mathcal{U}^*(p^\infty_m(Z)\cap{}_r\mathcal{J}^m_{\mathbb{R}}(\Omega_2,\mathbb{R}^n)) = p^\infty_m(Z)\cap{}_r\mathcal{J}^m_{\mathbb{R}}(\Omega_1,\mathbb{R}^n).$$

2.2. **Invariance in the range.** Invariance in the range is even easier. Suppose we have a complex diffeomorphism

$$\mathbb{C}^n\supset\Omega_1\xrightarrow{\mathcal{W}}\Omega_2\subset\mathbb{C}^{p+1},$$

then postcomposition by $\mathcal{W}$ gives an induced map

$$\mathcal{W}_*\colon {}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1},\Omega_1)\longrightarrow{}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1},\Omega_2).$$

Definition 2.8. An algebraic set $Z\subset{}_r\mathcal{J}^\infty$ of level m is *range-invariant* if for each diagram of the above form we have

$$\mathcal{W}_*(p^\infty_m(Z)\cap{}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1},\Omega_1)) = p^\infty_m(Z)\cap{}_r\mathcal{J}^m_{\mathbb{C}}(\mathbb{C}^{p+1},\Omega_2).$$

We can give a similar local characterisation. For any $(y_1,\dots,y_r)\in\Omega_1^{(r)}$, the jets of $\mathcal{W}$ given by

$$w_i\coloneqq j^\infty(\mathcal{W})(y_i)\in\mathcal{J}^\infty(\Omega_1,\Omega_2)$$

have the following properties:

- w_i are invertible,
- $sw_i=sw_j$ if and only if $tw_i=tw_j$ if and only if $w_i=w_j$ for all i,j.

As above, these are sufficient conditions for constructing a complex diffeomorphism on small open neighborhoods of the source and target of the multijet. We thus have a real version as above.

2.3. IASCM's.

Definition 2.9. An *invariant algebraic set of complex multijets* (IASCM) is an algebraic set of complex multijets that is both domain-invariant and range-invariant.

All algebraic sets of multijets that appear in the proof of multiple disjunction are of this type. By construction, for any IASCM Z we may define a set $S(Z,P,N) \subset {}_r\mathcal{J}^\infty(\mathbb{R}\times P,N)$ as those multijets which lie in Z when using charts in the domain and target.

Example 2.10. For the set $Z^0 = \{(z_1,z_2) \in {}_2\mathcal{J}^\infty \mid t(z_1)=t(z_2)\}$ we have

$$S(Z^0,P,N) = \{(z_1,z_2) \mid tz_1 = tz_2\} \subset {}_2\mathcal{J}^\infty(\mathbb{R}\times P,N).$$

For the set $Z^1 = \{z \in {}_1\mathcal{J}^\infty \mid \ker(Dz) \neq 0\}$ we have

$$S(Z^1,P,N) = \{z \mid \ker(Dz)\neq 0\} \subset {}_1\mathcal{J}^\infty(\mathbb{R}\times P,N).$$

The sets $S(Z,P,N)$ are well-behaved. Let me list some of their properties, all of which are proven by straightfoward verifications which do not hold any surprises. For Z of level m we have

(1) $S(Z,P,N) = (p^\infty_m)^{-1}(p^\infty_m(S(Z,P,N)))$ and $p^\infty_m(S(Z,P,N))$ is a closed subset of ${}_r\mathcal{J}^m(\mathbb{R}\times P,N)$.

(2) We will define operation A later, but it produces the "singular subset" $A(Z)\subset Z$, which is again an IASCM. When we set $S^*(Z,P,N) \coloneqq S(Z,P,N)\setminus S(A(Z),P,N)$, this has the following properties: $S^*(Z,P,N) = (p^\infty_m)^{-1}(p^\infty_m(S^*(Z,P,N)))$ and $p^\infty_m(S^*(Z,P,N))$ is a smooth submanifold of ${}_r\mathcal{J}^m(\mathbb{R}\times P,N)$.

(3) $S(Z,P,N)$ is preserved by fibered diffeomorphisms of the domain and target. Let me spell this out for the domain: given a diffeomorphism $\mathcal{U}\colon \mathbb{R}\times P\times D^s \to \mathbb{R}\times P\times D^s$ fibered over D^s, we have

$$\mathcal{U}^*(p^\infty_m(S(Z,P,N))\times D^s) = p^\infty_m(S(Z,P,N))\times D^s.$$

(4) $S(Z,P,N)$ is preserved by a sunny collapsing: if $F^u = (h^u,f^u,p_3)\colon I\times P\times D^s \to I\times N\times D^s$ is the fibered isotopy arising from a sunny collapsing data $\phi^u\colon P\times D^s\to(0,1]$ then we have[1]

$$\begin{aligned}&{}_rj^\infty(f^u)^{-1}(S(Z,P,N))\\ &\quad= \{(t_1,x_1,\ldots,t_r,x_r,y) \mid (t_1\phi^u(x_1,y),x_1,\ldots,y) \in {}_rj^\infty(f)^{-1}(S(Z,P,N))\}.\end{aligned}$$

3. The operations A, B, C, and D

We now define the operations A, B, C, and D, which produce new IASCM's from old ones. We will focus on a few illustrative examples.

[1]Recall that we intend to project the target onto $N\times D^s$ before taking multijets, hence h^u does not appear.

3.1. **Operation A.** We saw this operation before. Its slogan is:

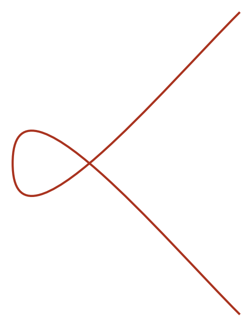
Relegate non-manifold points of strata to deeper strata.

In Goodwillie's paper, an algebraic variety is a quasi-projective variety over $\mathbb{C}$ (i.e. a subset of some complex projective space defined by homogeneous polynomial equations and inequalities). For an algebraic variety X, the tangent space T_xX to $x \in X$ is the linear dual to $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$.

Recall that the dimension of an algebraic variety is the maximal length of a chain $V_0 \subset V_1 \subset \ldots \subset V_d = X$ of distinct non-empty subvarieties.

Definition 3.1. If X is a d-dimensional algebraic variety, then the *singular set* $\Sigma(X)$ is given those $x \in X$ such that $\dim(T_xX) > d$ and those points which lies on components of dimension $< d$.

Example 3.2. The following example illustrates that the dimension of this tangent space can be larger than d:



The singular set $\Sigma(X)$ is once more an algebraic variety, and it is Zariski closed in X. It can in fact be defined for a complex analytic variety and is invariant under complex diffeomorphisms.

Definition 3.3. *Operation A* replaces an IASCM Z of level m and rank k by $(p_m^\infty)^{-1}(\Sigma(p_m^\infty(Z)))$.

Lemma 3.4. *$A(Z)$ is an IASCM.*

Proof. Let us prove it is domain-invariant. Given a complex diffeomorphism $\mathcal{U}\colon \mathbb{R}^{p+1} \supset\colon \Omega_1 \to \Omega_2 \subset \mathbb{R}^{p+1}$, the domain-invariance of Z says that

$$\mathcal{U}^*(p_m^\infty(Z) \cap {}_r\mathcal{J}_{\mathbb{C}}^m(\Omega_2, \mathbb{C}^n)) = p_m^\infty(Z) \cap {}_r\mathcal{J}_{\mathbb{C}}^m(\Omega_1, \mathbb{C}^n).$$

Then we get

$$\begin{aligned}\mathcal{U}^*(p_m^\infty(A(Z)) \cap {}_r\mathcal{J}_{\mathbb{C}}^m(\Omega_2, \mathbb{C}^n)) &= \mathcal{U}^*(\Sigma(p_m^\infty(Z)) \cap {}_r\mathcal{J}_{\mathbb{C}}^m(\Omega_2, \mathbb{C}^n)) \\ &= \Sigma(p_m^\infty(Z) \cap {}_r\mathcal{J}_{\mathbb{C}}^m(\Omega_1, \mathbb{C}^n)) \\ &= p_m^\infty(A(Z)) \cap {}_r\mathcal{J}_{\mathbb{C}}^m(\Omega_1, \mathbb{C}^n))\end{aligned}$$

The important step is the second one: this is the invariance of singular sets under complex diffeomorphisms. The proof that it is range-invariant is similar. □

3.2. **Operation B.** Recall that the strata will be subsets of $P \times D^s$ and are eventually defined by projecting configurations in $I \times P \times D^s$ to $P \times D^s$. The slogan for operation B is:

Relegate intersections of strata to deeper strata.

This is done by defining new strata which encode when the phenomena measured by previous strata coincide vertically in $I \times P$. This is a binary operation: its input is two IASCMs Z and Z' of rank $r(Z)$ and $r(Z')$, two injections

$$\phi\colon \{1,\ldots,r(Z)\} \longrightarrow \{1,\ldots,r\} \qquad \phi'\colon \{1,\ldots,r(Z')\} \longrightarrow \{1,\ldots,r\}$$

such that $\mathrm{im}(\phi) \cup \mathrm{im}(\phi') = \{1,\ldots,r\}$, and a pair of $i \in \{1,\ldots,r(Z)\}$ and $i' \in \{1,\ldots,r(Z')\}$. The latter are the indices of points to vertically coincide in the domain.

Definition 3.5. *Operation B takes this input to*

$$B_{\phi,\phi',i,i'}(Z,Z') = \{z \in {}_rJ^\infty \mid z \circ \phi \in Z,\, z \circ \phi' \in Z',\, p_2 s z_{\phi(i)} = p_2 s z_{\phi'(i')}\}.$$

Lemma 3.6. *This is an IASCM of rank r and level $\max(\mathit{level}(Z), \mathit{level}(Z'))$.*

3.3. **Operation C.** Recall that the construction of sunny collapses runs into difficulty when a certain subset we want to remove has an outwards pointing tangent vector which is vertical (really, only upwards is a problem). The slogan for operation C is:

> *Relegate to deeper strata those points in a stratum where such a tangent vector exists.*

Suppose that Z is an IASCM of level m and rank r.

Definition 3.7. For $z \in p^\infty_{m+1}(Z)$ we say that $v \in T_{sz}((\mathbb{C}^{p+1})^{(r)})$ *is along* Z if for some (or equivalently any) holomorphic map $f\colon \mathbb{C}^{p+1} \supset \Omega \to \mathbb{C}^n$ with $sz_1,\ldots,sz_r \in \Omega$ and ${}_rj^{m+1}(f)(sz) = z$ we have

$$D({}_rj^m(f))(sz) \circ v \in T(p^\infty_m(Z)),$$

the latter being the Zariski tangent space.

Definition 3.8. We define $\widetilde{C}_i(Z) = \{(z,v,c)\} \subset {}_r\mathcal{J}^{m+1} \times (T((\mathbb{C}^{p+1})^{(r)}) \setminus \text{0-section}) \times \mathbb{C}$ satisfying $z \in p^\infty_{m+1}(Z)$, $v \in T_{sz}((\mathbb{C}^{p+1})^{(r)})$ non-zero and along Z, $c \in \mathbb{C}$, and $Dz_i \circ (v_i - c\frac{\partial}{\partial x_1}) = 0$.

Definition 3.9. *Operation C takes Z to the image of $\widetilde{C}_i(Z)$ under the map $(z,v,c) \mapsto z$.*

Lemma 3.10. *This is an IASCM of rank r and level $m+1$.*

3.4. **Operation D.** The final operation we will spend more time on, as it is the most interesting one:

> *Relegate to deeper strata those points obtained by collisions of points in previous strata.*

Suppose that Z is an IASCM of level m and rank r, and there is a surjection

$$\phi\colon \{1,\ldots,r\} \longrightarrow \{1,\ldots,r'\}$$

which is not a bijection. We will produce from this an IASCM $D_\phi(Z)$ of rank r' and level $\max_{i=1}^{r'}((m+1)\#\phi^{-1}(i') - 1)$.

Take K^n to be the space of all complex polynomial maps $\mathbb{C}^{p+1} \to \mathbb{C}^n$ of degrees $< r\binom{p+1+m}{m}$. Then we define

$$X = \{(x,f) \in (\mathbb{C}^{p+1})^{(r)} \times K^n \mid {}_rj^m(f)(x) \in p^\infty_m(Z)\}$$

and letting $\overline{X}$ denote the Zariski closure of X in $(\mathbb{C}^{p+1})^r \times K^n$ (note that we added the fat diagonal back), we define

$$Y = \{(y, f) \in (\mathbb{C}^{p+1})^{(r')} \times K^n \mid (y \circ \phi, f) \in \overline{X}\}.$$

We will need a fact from algebraic geometry: if $f\colon X \to Y$ is a map of algebraic varieties (here the inclusion of $(\mathbb{C}^{p+1})^{(r)} \times K^n$ into $(\mathbb{C}^{p+1})^r \times K^n$, then the Zariski closure of $f(X)$ equals the closure in the usual complex topology.

Definition 3.11. *Operation D takes the above data to $D_\phi(Z) \subset {}_{r'}J^\infty$ given by those* $(z_1, \ldots, z_{r'})$ such that there exists a $(y, f) \in Y$ satisfying the equation $p^\infty_{(m+1)\#\phi^{-1}(i')-1}(z_{i'}) = j^{(m+1)\#\phi^{-1}(i')-1}(f)(y_{i'})$.

Proving that this is an algebraic set is one of the places where it is useful to work over the complex numbers. But instead of giving this proof, let me state the result which should help you understand this operation. It is consequence of the results discussed in Nils' lecture:

Lemma 3.12. *Let Z and ϕ be as above, $U \subset \mathbb{R}^{n+1}$ be open, $\{f^\nu\}$ a convergent sequence in $C^\infty(U, \mathbb{R}^n)$ with limit f and $\{x^\nu\}$ a sequence in $U^{(r)}$ which converges to $x = y \circ \phi$ in U^r for $y \in U^{(r')}$. Then we have*

$${}_r j^\infty(f^\nu)(x^\nu) \in Z \Longrightarrow {}_{r'} j^\infty(f)(y) \in D_\phi(Z).$$

Now, rather than proving $D_\phi(Z)$ is an IASCM (this amounts to proving that Y is an algebraic vector bundle over $D_\phi(Z)$), we will look at an example.

Example 3.13. Let us take $Z = Z^0 = \{(z_1, z_2) \in {}_2\mathcal{J}^\infty \mid tz_1 = tz_2\}$ and $\phi\colon \{1, 2\} \to \{1\}$ the unique map. We have $r = 2$, $m = 0$, $r' = 1$, and so we are taking polynomials of degree < 2. Thus $f \in K^n$ can be concretely described as

$$f = A + \sum_{j=1}^{p+1} B_j X_j \qquad A, B_j \in \mathbb{C}^n$$

for variables X_j. Then we have

$$X = \{(x_1, x_2, A + \sum_{j=1}^{p+1} B_j X_j) \mid A + \sum_{j=1}^{p+1} B_j x_{1,j} = A + \sum_{j=1}^{p+1} B_j x_{2,j}\} \subset (\mathbb{C}^{p+1})^{(2)} \times K^n.$$

Then $(y, A + \sum_{j=1}^{p+1} B_j X_j) \in Y$ if and only if $(x, x, A + \sum_{j=1}^{p+1} B_j X_j) \in \overline{X}$. We claim this happens if and only if there is a non-zero vector $v \in \mathbb{C}^{p+1}$ such that $\sum_j B_j v_j = 0$. The direction $\Leftarrow$ is easy: given such a v, the sequence $(y, y + v/n, A + \sum_{j=1}^{p+1} B_j X_j)$ will lie in X and converge to $(y, y, A + \sum_{j=1}^{p+1} B_j X_j)$.

For the converse, suppose that $(x_1^\nu, x_2^\nu, A^\nu + \sum_{j=1}^{p+1} B_j^\nu X_j) \in X$ converges to $(y, y, A + \sum_{j=1}^{p+1} B_j X_j)$. Writing $x_2^\nu = x_1^\nu + v^\nu$, we get that $v^\nu \to 0$ and $\sum_j B_j^\nu = 0$ for all ν. Passing to a subsequence, we may assume that $\hat{v}^\nu = v^\nu / ||v^\nu||$ converges to some non-zero vector $\hat{v}$ which satisfies $\sum_j B_j \hat{v}_j = 0$.

Thus we have that

$$Y = \{(y, A + \sum_J B_j X_j) \in \mathbb{C}^{p+1} \times K^n \mid \exists v \neq 0 \text{ such that } \sum_j B_j v_j = 0\}.$$

Now $(m+1)\phi^{-1}(1)-1=1$ and thus we get that $D_\phi(Z^0)$ is given by those $z \in {}_1\mathcal{J}^\infty$ such that $p_1^\infty(z) = j^1(A+\sum_j B_j X_j)(y)$ for $(y, A+\sum_j B_j X_j) \in Y$. That is, $D_\phi(Z^0)$ consists of those z with $\ker(Dz) \neq 0$. We conclude that

$$D_\phi(Z^0) = S(Z^1, P, N).$$

References

[Goo90] T. G. Goodwillie, *A multiple disjunction lemma for smooth concordance embeddings*, Mem. Amer. Math. Soc. **86** (1990), no. 431, viii+317. 1