Homological stability minicourse

Alexander Kupers

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Abstract

These are the collected lecture notes for eCHT minicourse on homological stability in spring 2021.

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Chapter 1

The homology of symmetric groups

In the first lecture of this minicourse, we introduce the phenomenon of homological stability through the example of the symmetric groups Σ_n. We also explain what one use can these homological stability results for.

1.1 Homology groups of symmetric groups

The *symmetric group* Σ_n is the group of bijections of the finite set $\underline{n} = \{1, \ldots, n\}$, under composition. The classifying space BG of a discrete group G, such as Σ_n, is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It can be constructed by extracting from G the groupoid $* /\!\!/ G$ given by:

- a single object $*$,
- morphisms given by $* \xrightarrow{g} *$ for $g \in G$, and
- composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space $|N(* /\!\!/ G)|$; this is a model for BG. Exercise 1.3.1 proves it indeed has the desired property.

Question 1.1.1. What are the homology groups $H_*(B\Sigma_n; \mathbb{Z})$?

Remark 1.1.2. This is the same as computing the group homology of Σ_n with coefficients in $\mathbb{Z}$ in the sense of [Bro94], see Exercise 1.3.2.

Let us compute these groups and the homology of their classifying spaces for the first few values of n.

Example 1.1.3. For $n = 0, 1$, the group Σ_n is trivial so its classifying space is weakly contractible and hence has trivial homology.

Example 1.1.4. For $n = 2$, Σ_2 is isomorphic to the cyclic abelian group $\mathbb{Z}/2$. Then $B\mathbb{Z}/2$, as constructed above, is homotopy equivalent to $\mathbb{R}P^\infty$. We conclude that

$$H_*(B\mathbb{Z}/2;\mathbb{Z}) = H_*(\mathbb{R}P^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ is odd,} \\ 0 & \text{if } * > 0 \text{ is even.} \end{cases}$$

For an alternative argument, see Exercise 1.3.3.

Example 1.1.5. For $n = 3$, the group Σ_3 is the dihedral group D_3 with 6 elements (i.e. the symmetries of a triangle). A more complicated computation given in Exercise 1.3.5 yields the homology of D_3:

$$H_*(BD_3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ and } * \equiv 1 \pmod 4, \\ \mathbb{Z}/6 & \text{if } * > 0 \text{ and } * \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

As the previous example indicates, direct computation of homology becomes increasingly difficult. However, it is certainly possible by computer.[1] Let's look at Fig. 1.1 to see whether we can discern some patterns:

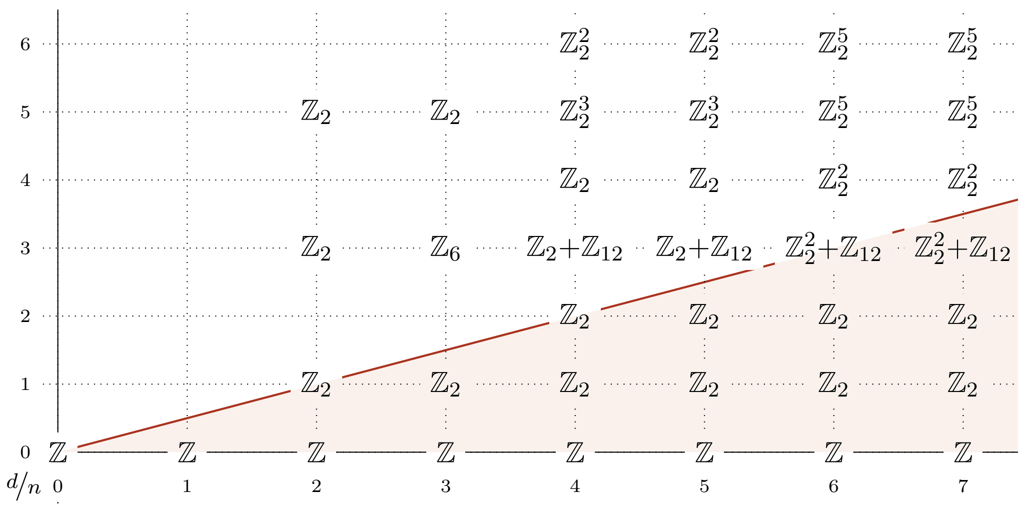


Figure 1.1: The homology groups $H_d(B\Sigma_n;\mathbb{Z})$. To keep this table readable, some compromises had to be made: we wrote $\mathbb{Z}_d^r$ for $(\mathbb{Z}/d)^{\oplus r}$, + for $\oplus$, and combined some summands. The stable range from Theorem 1.1.8 is shaded.

Here some things one might conjecture after looking at these computations:

(1) Each reduced homology group $\widetilde{H}_d(B\Sigma_n;\mathbb{Z})$ is finite and has small exponent.

[1] We in fact know all homology groups of all symmetric groups, in the sense that there is a mechanical procedure for determining them. This can be done combining the work of Nakaoka with that of May [CLM76].

(2) The homology in fixed degree $* = d$ becomes independent of n as $n \to \infty$.

(3) Before becoming independent of n, the homology only increases in size.

(4) The p-power torsion only changes when $p|n$.

If we want to attempt to prove (2)–(4), we need a better way to compare the homology groups for different n than just as abstract abelian groups. This is done by observing that the inclusion $\underline{n} \hookrightarrow \underline{n+1}$ of finite sets gives a homomorphism

$$\sigma\colon \Sigma_n \longrightarrow \Sigma_{n+1},$$

by extending a permutation of $\underline{n}$ by the identity on $n+1 \in \underline{n+1}$ to a permutation of $\underline{n+1}$. Our construction of BG is natural in groups and homomorphisms, so this homomorphism induces a map

$$\sigma\colon B\Sigma_n \longrightarrow B\Sigma_{n+1},$$

which in turn induces a map $\sigma_*\colon H_*(B\Sigma_n;\mathbb{Z}) \to H_*(B\Sigma_{n+1};\mathbb{Z})$ on homology. We can then give sharper formulations of (2)–(4) in terms of these *stabilisation maps*:

(2') The maps σ_* are isomorphisms in a range increasing with n.

(3') The maps σ_* are injective.

(4') The maps σ_* are isomorphisms on p-power torsion unless $p|n+1$.

Property (1) holds for all finite groups, and the result which proves it also implies (4'):

Proposition 1.1.6. *For a finite group G, $\widetilde{H}_*(BG;\mathbb{Z}[1/|G|]) = 0$. More generally, for $H \subset G$ the map $\iota_*\colon H_*(BH;\mathbb{Z}[1/[G:H]]) \to H_*(BG;\mathbb{Z}[1/[G:H]])$ admits a right inverse τ (i.e. $\iota_* \circ \tau = \mathrm{id}$).*

Proof. The first statement follows from the second by taking $H = \{e\}$. The second is proven in Exercise 1.3.4 using transfer maps. ☐

To deduce (4') from Proposition 1.1.6, note that $[\Sigma_{n+1} : \Sigma_n] = n+1$ so by the long exact sequence on homology groups so that $H_*(B\Sigma_n;\mathbb{Z}) \to H_*(B\Sigma_{n+1};\mathbb{Z})$ is surjective after inverting $n+1$. Now set $n+1$ equal to p and invoke (3').

It is phenomenon indicated by (2') that is the subject of this minicourse:

Definition 1.1.7. A sequence $X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\sigma} X_2 \xrightarrow{\sigma} \cdots$ exhibits *homological stability* if the maps $\sigma_*\colon H_*(X_n;\mathbb{Z}) \to H_*(X_{n+1};\mathbb{Z})$ are isomorphisms in a range of degrees $*$ increasing with n.

In the next two lectures we will prove the following result, due to Nakaoka [Nak60] (though he proved much more):

Theorem 1.1.8. *The sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely, the induced map*

$$\sigma_*\colon H_*(B\Sigma_n;\mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1};\mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ and an isomorphism if $* \leq \frac{n-1}{2}$.

Remark 1.1.9. Of course, if we know property (3') holds then the range in the previous theorem in which σ_* is an isomorphism improves to $* \leq \frac{n}{2}$. However, property (3') is rather special—related to the existence of transfer maps—and you should not expect it to hold for general sequences of classifying spaces of groups. We will not comment on it again, but see Exercise 1.3.6.

Remark 1.1.10. The ranges in the previous remark are optimal among those of the form $* \leq an + b$ with $a, b \in \mathbb{Q}$.

1.2 Using homological stability for symmetric groups

Homological stability is a structural property of a sequence of groups, or more generally topological spaces, but it is also useful *tool*. In fact, many homological stability theorems are proven in service of obtaining other mathematical results. To illustrate this, I now want to explain some straightforward applications of Theorem 1.1.8. These concern the transfer of information from low n to high n and vice-versa. They can be obtained by other methods as well, but their generalisations to other sequences of groups often can not.

1.2.1 Alternating groups

Recall that for path-connected X, the Hurewicz map $\pi_1(X) \to H_1(X;\mathbb{Z})$ coincides with abelianisation (we are suppressing the basepoint). In particular, the map $G \to H_1(BG;\mathbb{Z})$ induces an isomorphism $G^{\mathrm{ab}} \to H_1(BG;\mathbb{Z})$ naturally in G. Thus we can understand the abelianisation of Σ_n by computing its first homology group.

The sign homomorphism $\mathrm{sign}\colon \Sigma_n \to \mathbb{Z}/2$ yields a map

$$\mathrm{sign}\colon B\Sigma_n \longrightarrow B\mathbb{Z}/2,$$

which induces a map on homology. This is compatible with stabilisation, in the sense that $\mathrm{sign} \circ \sigma = \mathrm{sign}$, so we get a commutative squares

$$\begin{array}{ccc} H_1(B\Sigma_{n-1};\mathbb{Z}) & \xrightarrow{\sigma_*} & H_1(B\Sigma_n;\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{sign}} & & \downarrow{\scriptstyle \mathrm{sign}} \\ \mathbb{Z}/2 & = & \mathbb{Z}/2. \end{array}$$

The map $H_1(B\Sigma_2;\mathbb{Z}) \to \mathbb{Z}/2$ is an isomorphism because $\mathrm{sign}\colon \Sigma_2 \to \mathbb{Z}/2$ is. By Theorem 1.1.8, in the commutative diagram

$$\begin{array}{ccccccc} H_1(B\Sigma_2;\mathbb{Z}) & \twoheadrightarrow & H_1(B\Sigma_3;\mathbb{Z}) & \xrightarrow{\cong} & H_1(B\Sigma_4;\mathbb{Z}) & \xrightarrow{\cong} & \cdots \\ {\scriptstyle \cong}\downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}/2 & = & \mathbb{Z}/2 & = & \mathbb{Z}/2 & = & \cdots \end{array}$$

the right-most top horizontal map is surjective and the other top horizontal maps are isomorphisms. A single diagram chase then deduces from the fact that the left-most vertical map is an isomorphism that all other vertical maps are.

Thus we have used homological stability to prove that

$$\mathrm{sign}\colon \Sigma_n \longrightarrow \mathbb{Z}/2$$

is the abelianisation for $n \geq 2$, or equivalently that the kernel of the sign homomorphism is exactly the subgroup $[\Sigma_n, \Sigma_n]$ generated by commutators. Recalling that this kernel is exactly the alternating group A_n, we conclude that:

Theorem 1.2.1. $[\Sigma_n, \Sigma_n] = A_n$.

Remark 1.2.2. This is a fact you likely knew already, and elementary group-theoretic arguments exist. We could have used this fact instead to give an elementary proof of Theorem 1.1.8 in degree $* = 1$.

1.2.2 Group completion

Homological stability implies that for in fixed degree $*$, for n sufficienty large the canonical map

$$H_*(B\Sigma_n; \mathbb{Z}) \longrightarrow \operatorname*{colim}_{n\to\infty} H_*(B\Sigma_n; \mathbb{Z})$$

is an isomorphism; the right hand side is known as the *stable homology*. This has two somewhat tautological consequences:

1. We can compute the right side from the left side.
2. We can compute the left side from the right side.

This is particularly interesting because the stable homology on the right side has a more familiar description.

When we constructed the stabilisation map, we used that inclusion $\underline{n} \to \underline{n+1}$ yields a homomorphism $\Sigma_n \to \Sigma_{n+1}$. More generally, disjoint union induces a homomorphism $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, which yields "multiplication" maps

$$B\Sigma_n \times B\Sigma_m \longrightarrow B\Sigma_{n+m},$$

making the space $\bigsqcup_{n\geq 0} B\Sigma_n$ into a unital topological monoid (these are associative but not commutative, and it is probably better to say E_1-space since that is a homotopy-invariant notion).

Example 1.2.3. The topological monoid structure makes $\pi_0 \coloneqq \pi_0(\bigsqcup_{n\geq 0} B\Sigma_n)$ into a unital monoid, and this can be identified with $\mathbb{N}$ with its usual addition. That addition is commutative reflects the fact that $\bigsqcup_{n\geq 0} B\Sigma_n$ is homotopy-commutative.

The stabilisation maps assemble to a map $\sqcup_{n\geq 0} B\Sigma_n \to \sqcup_{n\geq 0} B\Sigma_n$, mapping the nth component to the $(n+1)$st, which can equivalently be described a multiplication by an element of $B\Sigma_1$. This makes $H_*(\sqcup_{n\geq 0} B\Sigma_n; \mathbb{Z})$ into $\mathbb{Z}[\pi_0]$-module. On $\mathrm{colim}_\sigma\, H_*(\sqcup_{n\geq 0} B\Sigma_n; \mathbb{Z})$ multiplication by σ is invertible (this map induces a pro-isomorphism on diagrams), so there is a map

$$H_*(\sqcup_{n\geq 0} B\Sigma_n; \mathbb{Z})[\pi_0^{-1}] \longrightarrow \mathrm{colim}_\sigma\, H_*(\sqcup_{n\geq 0} B\Sigma_n; \mathbb{Z}), \tag{1.1}$$

and it is an algebraic fact about computing localisations that this is an isomorphism.

The right hand side of (1.1) is nothing but a $\mathbb{Z}$-indexed direct sum of copies of $\operatorname{colim}_n H_*(B\Sigma_n;\mathbb{Z})$. Thus the stable homology is equal to one of these summands of $H_*(\sqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})[\pi_0^{-1}]$. The McDuff–Segal group completion theorem tells us when we can equivalently make the path-components into a group *before* passing to homology [MS76, Proposition 1] (see also [ERW19, Section 6]):

Theorem 1.2.4 (McDuff–Segal)**.** *If M is a homotopy-commutative unital associative topological monoid, then $H_*(M;\mathbb{Z})[\pi_0^{-1}] \cong H_*(\Omega BM;\mathbb{Z})$.*

Here Ω denote the based loop space construction, and BM is the bar construction of M obtained as $|N(* /\!\!/ M)|$, where $* /\!\!/ M$ is the category enriched in topological spaces given by:

- a single object $*$,
- space of morphisms given by M,
- composition given by multiplication.

(Strictly speaking, for this to have the correct homotopy type, the inclusion $\{\mathrm{id}\} \hookrightarrow M$ needs to be a cofibration, a condition satisfied in particular when $M = \bigsqcup_{n\geq 0} BG_n$ with each G_n discrete.) The bar construction BM has a canonical basepoint provided by the object $*$, and that is where our loops are based. There are many techniques, known as *infinite loop space machines* [MT78], to compute the homotopy type of BM. In particular, these apply to $M = \bigsqcup_{n\geq 0} B\Sigma_n$ [BP72, Seg74]:

Theorem 1.2.5 (Barratt–Priddy–Quillen–Segal)**.** *There is a homotopy equivalence $\Omega B(\bigsqcup_{n\geq 0} B\Sigma_n) \simeq \Omega^\infty \mathbb{S}$.*

Here $\mathbb{S}$ is the *sphere spectrum*, described by saying its nth stage is $\mathbb{S}_n = S^n$ and the map $\Sigma\mathbb{S}_n \to \mathbb{S}_{n+1}$ is the usual identification of ΣS^n with S^{n+1}. In particular $\pi_i(\mathbb{S})$ is the ith *stable homotopy group of spheres* given explicitly as $\operatorname{colim}_{k\to\infty}\pi_{i+k}(S^k)$.

Corollary 1.2.6. *Let $\Omega^\infty_0\mathbb{S} \subset \Omega^\infty\mathbb{S}$ denote the basepoint component, then*

$$\operatorname*{colim}_{n\to\infty} H_*(B\Sigma_n;\mathbb{Z}) \cong H_*(\Omega^\infty_0\mathbb{S};\mathbb{Z}).$$

Remark 1.2.7. A fancier way of stating the Barratt–Priddy–Quillen–Segal theorem is that the sphere spectrum $\mathbb{S}$ is the algebraic K-theory spectrum of the category of finite sets. An even fancier way of stating this would involve the "field with one element".

1.2.3 Serre's finiteness theorem and variations

Let us now use Corollary 1.2.6. By (1) the groups $H_*(B\Sigma_n;\mathbb{Z})$ are finite for $* > 0$. By Theorem 1.1.8 the same is true for the stable homology as long as restrict to degrees $* \leq \frac{n}{2}$. Since n is arbitrary, the stable homology is finite in all positive degrees. This has the following consequence:

Theorem 1.2.8 (Serre)**.** *$\pi_*(\mathbb{S})$ is finite for all $* > 0$.*

Proof. By construction, the path-component $\Omega^\infty_0\mathbb{S} \subset \Omega^\infty\mathbb{S}$ corresponding to $0 \in \pi_0(\mathbb{S}) \cong \mathbb{Z}$ (all are homotopy-equivalent) is an infinite loop space and hence a so-called *simple space* (i.e. π_1 is abelian and acts trivially on the higher homotopy groups). Thus we can apply the Hurewicz theorem modulo the Serre class $\mathcal{C}$ of finite abelian groups. This in particular says that $\pi_*(\Omega^\infty_0\mathbb{S})$ is finite for $* \leq d$ if and only if $\widetilde{H}_*(\Omega^\infty_0\mathbb{S};\mathbb{Z})$ is finite for $* \leq d$. Now use that Corollary 1.2.6 identifies $H_*(\Omega^\infty_0\mathbb{S};\mathbb{Z})$ with the stable homology of symmetric groups. □

We can say something similar about torsion: by (4') the groups $H_*(B\Sigma_n;\mathbb{Z})$ contain no p-torsion for when $n < p$. By Theorem 1.1.8 we conclude that same is true for the stable homology as long as we restrict to degrees $* \leq \frac{p-1}{2}$. This has the following consequence, working modulo the Serre class $\mathcal{C}$ of finite abelian groups which only have ℓ-torsion for primes $\ell \neq p$:

Proposition 1.2.9. *$\pi_*(\mathbb{S})$ has no p-torsion for $* \leq \frac{p-1}{2}$.*

In fact, the following better result is known (and this range is optimal):

Theorem 1.2.10 (Serre)**.** *$\pi_*(\mathbb{S})$ has no p-torsion for $* < 2p-3$.*

Arguing in the other direction, we obtain that the stable homology of symmetric groups has no p-torsion for $* < 2p-3$ and conclude using Corollary 1.2.6 and property (3') that:

Proposition 1.2.11. *$H_*(B\Sigma_n;\mathbb{Z})$ has no p-torsion for $* < 2p-3$.*

Proof. If there were p-torsion for $* < 2p-3$ in $H_*(B\Sigma_n;\mathbb{Z})$, by (3') the same would be true for the stable homology. But by a Sere class argument this contradicts Theorem 1.2.10. □

1.3 Exercises

Exercise 1.3.1 (Recognizing BG)**.** One way to recognize that a topological space is weakly homotopy equivalent to BG is to exhibit it as a quotient of a contractible topological space by G acting freely and properly discontinuously. In this exercise we use this to prove $|N(* /\!\!/ G)|$ arises this way.

(i) Consider the groupoid $G /\!\!/ G$ given by:
   - objects given by G,
   - morphisms given by $h \xrightarrow{g} gh$ for $g \in G$,
   - composition given by multiplication.

   Prove that $|N_*(G /\!\!/ G)|$ is contractible by using the fact that if $\eta\colon F \Rightarrow F'$ is a natural transformation, then $|NF|$ and $|NF'|$ are homotopic. (Hint: construct a natural transformation from the identity functor on $G /\!\!/ G$ to a constant functor.)

(ii) Multiplication on the right gives an action of G on $G /\!\!/ G$ and hence on $|N(G /\!\!/ G)|$. Prove that this action is free, properly discontinuous, and that $|N(G /\!\!/ G)|/G \cong |N(* /\!\!/ G)|$.

Exercise 1.3.2 (Group homology)**.** The definition of group homology of G with coefficients in a $\mathbb{Z}[G]$-module M is $H_*(G;M) \coloneqq \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z},M)$. That is, it is computed by the homology of $P_* \otimes_{\mathbb{Z}[G]} M$ where $P_* \to \mathbb{Z}$ is a projective $\mathbb{Z}[G]$-module resolution.

(i) As the geometric realisation of a simplicial set, $|N(G /\!\!/ G)|$ has a canonical CW structure with a single cell for each non-degenerate simplex. Using the cellular chains on $|N(G/G)|$ to construct a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$.

(ii) Prove that $H_*(BG;\mathbb{Z}) \cong \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z},\mathbb{Z})$ and conclude that the homology of BG is the group homology of G with coefficients in $\mathbb{Z}$.

(iii) How do you obtain $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z},M)$ in terms of BG?

Exercise 1.3.3 (Homology of $B\mathbb{Z}/2$)**.** Use cellular homology of $|N(* /\!\!/ \mathbb{Z}/2)|$ to compute $H_*(B\mathbb{Z}/2;\mathbb{Z})$.

Exercise 1.3.4 (Transfer maps)**.**

(i) Prove that $p\colon X \to B$ is a k-fold cover, there is a map $\tau\colon \widetilde{H}_*(B;\mathbb{Z}) \to \widetilde{H}_*(X;\mathbb{Z})$ so that $p_* \circ \tau = k \cdot \mathrm{id}$. (Hint: send a singular simplex $\Delta^n \to B$ to its k distinct lifts.)

(ii) Prove that if G is a discrete group and $H \subset G$ is a subgroup of finite index, there is a model of BH which is a $[G:H]$-fold cover of BG.

(iii) Conclude that Proposition 1.1.6 holds.

Exercise 1.3.5 (Homology of BD_3)**.**

(i) Use the arguments of Exercise 1.3.4 to deduce that $\widetilde{H}_*(BD_3;\mathbb{Z})$ is annihilated by multiplication with 6.

(ii) Use universal coefficient theorems to show that Example 1.1.5 follows once one proves that

$$H^*(BD_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[x] \qquad \text{and} \qquad H^*(BD_3;\mathbb{Z}/3) \cong \mathbb{Z}/3[y]$$

with $|x| = 1$ and $|y| = 2$.

(iii) Prove (ii) using the cohomological Serre spectral sequence for the fibration sequence $B\mathbb{Z}/3 \longrightarrow BD_3 \longrightarrow B\mathbb{Z}/2$ associated to the extension

$$1 \longrightarrow \mathbb{Z}/3 \longrightarrow D_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

with coefficients in $\mathbb{Z}/2$ and $\mathbb{Z}/3$. (Hint: this extension is a semi-direct product with the non-trivial element of $\mathbb{Z}/2$ acting on $\mathbb{Z}/3$ by multiplication with -1.)

Exercise 1.3.6 (Dold's lemma)**.** We will prove [Dol62, Lemma 2].

(i) Suppose we have a sequence of abelian groups and homomorphisms

$$0 \xrightarrow{\sigma_0} A_0 \xrightarrow{\sigma_1} A_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} A_n$$

and homomorphisms $\tau_{k,m}\colon A_m \to A_k$ for $k \leq m \leq n$ such that

(a) $\tau_{k,k} = \mathrm{id}$,

(b) $\tau_{k,m} \circ \sigma_m = \tau_{k,m-1} \pmod{\mathrm{im}(\sigma_k)}$ for $k < m$.

Prove by induction over m that the map

$$T_m \colon A_m \longrightarrow \bigoplus_{k \leq m} A_k/\mathrm{im}(\sigma_k)$$

with kth component given by $\mathrm{proj}\circ\tau_{m,k}$, is an isomorphism and σ_m has a left inverse. (Hint: the left inverse to σ_m will be $T_{m-1}^{-1} \circ p \circ T_m \circ \sigma_m$ where $p\colon \bigoplus_{k \leq m} A_k/\mathrm{im}(\sigma_k) \to \bigoplus_{k \leq m-1} A_k/\mathrm{im}(\sigma_k)$ is the projection onto the first $m-1$ summands.)

(ii) In the sequence

$$0 \xrightarrow{\sigma_0} H_d(B\Sigma_0;\mathbb{Z}) \xrightarrow{\sigma_1} H_d(B\Sigma_1;\mathbb{Z}) \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} H_d(B\Sigma_n;\mathbb{Z}),$$

use transfers to construct maps $\tau_{k,m}\colon H_d(B\Sigma_m;\mathbb{Z}) \to H_d(B\Sigma_k;\mathbb{Z})$ satisfying (a) and (b). Conclude that σ is always injective.

Exercise 1.3.7 (Some stable homotopy groups of spheres)**.**

(i) Use the results in Section 1.2.1 to prove that $\pi_1(\mathbb{S}) = \mathbb{Z}/2$.

(ii) Can you also compute $\pi_2(\mathbb{S})$? (Hint: [Arl90].)

Exercise 1.3.8 (Using Serre's finiteness theorem)**.** Serre proved that $\pi_*(\mathbb{S})$ is finite for $* > 0$. Combine this with Corollary 1.2.6 and Exercise 1.3.6 to prove that the sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. (Hint: you will not be able to give an explicit range.)

Remark 1.3.9. See [McD75] for a similar qualitative argument for configuration spaces of manifolds.

Chapter 2

Homological stability for symmetric groups

In the second lecture of this minicourse, we prove homological stability for the symmetric groups Σ_n.

2.1 The strategy

In this lecture we will prove, following the strategy in [RWW17, Section 3] which originally goes back to Quillen:[1]

Theorem 2.1.1 ([Nak60])**.** *The sequence* $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ *exhibits homological stability. More precisely,*

$$\sigma_* \colon H_*(B\Sigma_n; \mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1}; \mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ *and an isomorphism if* $* \leq \frac{n-1}{2}$*.*

Recall that we constructed BG as the geometric realisation of the nerve of a category $* /\!\!/ G$. As the notation suggests, this can be interpreted as a quotient, or more precisely a *homotopy quotient*. One can construct the homotopy quotient $X /\!\!/ G$ of any space X with G-action by a group G, and here we just take $X = *$. By abuse of notation $* /\!\!/ G = |N(* /\!\!/ G)|$.[2] A reference for its construction and properties is [Rie14], but we will only need the following facts:

1. *Homotopy quotients are natural.* If $X \to Y$ is an equivariant map between G-spaces then there is an induced map $X /\!\!/ G \to Y /\!\!/ G$.
2. *Homotopy quotients preserve homological connectivity.* If $X \to Y$ is an equivariant map between G-spaces which is homologically d-connected then $X /\!\!/ G \to Y /\!\!/ G$ is also homologically d-connected. (Recall that a map is *homologically d-connected* if it is an isomorphism on H_i for $i < d$ and surjection on H_d.)

[1] Quillen used it to study the homology of general linear groups over finite fields. His argument was not published by Quillen, but appears in his notebooks [Qui74]. Unfortunately, the first few pages were left in the sun and were bleached. The argument was reconstructed and generalised in [SW20].

[2] This reflects that in modern homotopy theory, one does not really make a distinction between a category, its nerve, and the geometric realisation of its nerve.

3. *Homotopy quotients commute with geometric realisation.* If $X_\bullet$ is a semi-simplicial G-space, then $||X_\bullet|| /\!\!/ G \simeq ||X_\bullet /\!\!/ G||$. (We will explain the terminology and notation later.)
4. *Homotopy quotients of transitive G-sets.* If S is a transitive G-set, then $S /\!\!/ G \simeq B\mathrm{Stab}_G(s)$ for any $s \in S$.

When proving Theorem 2.1.1 we are only interested in the homology of $B\Sigma_n$ in a range, so by (1) and (2) we may replace $*$ with the trivial Σ_n-action by a different Σ_n-space X as long as X is homologically highly-connected. As (3) and (4) suggest, our desired X is of the form $||X_\bullet||$ with each X_k a transitive G-set. Why is this a good idea? A geometric realisation $||X_\bullet||$ comes with a canonical filtration, yielding a spectral sequence that will relate the homology of $B\Sigma_n$ to that of the classifying spaces $B\Sigma_{n-p}$ of various stabiliser groups. This will allow for an inductive argument.

2.2 Injective words

Recall that Δ is the category whose objects are non-empty ordered finite sets and whose morphisms are order-preserving maps; this has a skeleton given by the ordered finite sets $[p] = (0 < \ldots < p)$ for $p \geq 0$. The combinatorics of this category encodes various maps between the standard simplices $\Delta^p = \{(t_0, \ldots, t_p) \in [0,1]^n \mid t_0 + \cdots + t_p = 1\}$: we can construct a functor

$$\Delta^\bullet \colon \Delta \longrightarrow \mathsf{Top},$$

which sends the morphism $\delta_i \colon [p-1] \to [p]$ for $0 \leq i \leq p$ which skips the ith element to the affine-linear inclusion $\Delta^{p-1} \to \Delta^p$ opposite the ith vertex, and the morphism $\sigma_j \colon [p] \to [p-1]$ for $0 \leq j \leq p-1$ which doubles the jth element to the affine-linear surjection $\Delta^p \to \Delta^{p-1}$ which collapses to the jth and $(j+1)$st vertices to same point. These morphisms generate all morphisms in the category Δ.

A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$ and a *simplicial space* is a functor $\Delta^{\mathrm{op}} \to \mathsf{Top}$. The value $X_p \coloneqq X([p])$ is called the *space of p-simplices.* The morphism δ_i induces a *face map* $d_i \colon X_p \to X_{p-1}$, and the morphism σ_i a *degeneracy map* $s_j \colon X_{p-1} \to X_p$. The geometric realisation is the coend

$$|X_\bullet| \coloneqq \Delta^\bullet \otimes_\Delta X_\bullet = \left(\bigsqcup_{p \geq 0} \Delta^p \times X_p \right) /\!\sim$$

with $\sim$ the equivalence relation generated by $(\delta_i t, x) \sim (t, d_i x)$ and $(\sigma_i t, x) \sim (t, s_i x)$.

Example 2.2.1. In the previous lecture, we already used the simplicial set $N\mathsf{C}$ for a category C (we took the groupoid $* /\!\!/ G$). Interpreting $[p]$ has a poset, which is a particular type of category, we get a functor $[\bullet] \colon \Delta \to \mathsf{Cat}$. Then $N\mathsf{C}_p = \mathrm{Hom}_{\mathsf{C}}([p], \mathsf{C})$, or more concretely, $N\mathsf{C}_p$ is the set of composable sequences of morphisms

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_p} C_p.$$

The face maps are induced by precomposition, and explicitly given by composing or forgetting morphisms. Thhe degeneracy maps are also induced by precomposition, and explicitly given by inserting identity morphisms.

It will suffice for our purposes to keep track of less structure, and replace Δ by its subcategory Δ_{inj} with the same objects but morphisms only injective order-preserving maps. A *semi-simplicial set* is a functor $\Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathsf{Set}$ and a *semi-simplicial space* is a functor $\Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathsf{Top}$. That, it is the analogue of a simplicial space which only face maps and no degeneracy maps. We can restrict $\Delta^\bullet$ to Δ_{inj} and once more take the coend to get a geometric realisation

$$||X_\bullet|| \coloneqq \Delta^\bullet \otimes_{\Delta_{\mathrm{inj}}} X_\bullet = \left(\bigsqcup_{p \geq 0} \Delta^p \times X_p \right) /\sim$$

with $\sim$ the equivalence relation generated by $(\delta_i t, x) \sim (t, d_i x)$. A reference for semi-simplicial spaces and their properties is [ERW19].

Remark 2.2.2. Under suitable cofibrancy conditions, for a simplicial space $X_\bullet$ we have $|X_\bullet| \simeq \mathrm{hocolim}_{\Delta^{\mathrm{op}}} X_\bullet$. One advantage for semi-simplicial spaces, is that for a semi-simplicial space $X_\bullet$ we *always* have $||X_\bullet|| \simeq \mathrm{hocolim}_{\Delta_{\mathrm{inj}}^{\mathrm{op}}} X_\bullet$. This is a consequence of [ERW19, Theorem 2.2], as this theorem implies cofibrantly replacing $X_\bullet$ does not affect the weak homotopy type of the geometric realisation.

We now define the semi-simpicial set with Σ_n-action which will replace $*$. Let FI be the category whose objects are finite sets and whose morphisms are injections.

Definition 2.2.3. $W_n(\underline{1})_\bullet$ is the semi-simplicial set with p-simplices given by

$$W_n(\underline{1})_\bullet = \mathrm{Hom}_{\mathsf{FI}}([p], \underline{n})$$

and face maps d_i given by precomposition with $\delta_i \colon [p-1] \to [p]$.

That is, $W_n(\underline{1})_p$ has as p-simplices the ordered words $(m_0\, m_1\, \cdots\, m_p)$ of elements of $\underline{n}$ and no letter duplicated. The ith face map forgets the ith letter m_i. This explains why we call this the *semi-simplicial set of injective words*. The notation is rather complicated, but will become clear in the next lecture.

Example 2.2.4. $W_2(\underline{1})_\bullet$ has:

- two 0-simplices given by the words (1) and (2),
- two 1-simplices given by the words (1 2) and (2 1).

Its geometric realisation is a circle.

Example 2.2.5. $W_3(\underline{1})_\bullet$ has:

- three 0-simplices (1), (2), (3),
- six 1-simplices (1 2), (2 1), (1 3), (3 1), (2 3), (3 2),
- six 2-simplices (1 2 3), (2 1 3), (1 3 2), (3 1 2), (2 3 1), (3 2 1).

We will see this is 1-connected. Let us just pick the loop given by the 1-simplices corresponding to (1 2) and (2 1) and see this null-homotopic: it is the boundary obtained when we glue the 2-simplices corresponding to (1 2 3) and (2 1 3) along their common 1-simplices.

The group Σ_n acts on $W_n(\underline{1})_\bullet$ by post-composition, and hence on the geometric realisation. We have that:

Proposition 2.2.6.

(i) $||W_n(\underline{1})_\bullet||$ *is homologically* $\frac{n-1}{2}$*-connected.*

(ii) $W_n(\underline{1})_p$ *is a transitive* Σ_n*-set, and the stabiliser of* $x \in W_n(\underline{1})_p$ *is the group of permutations of* $\underline{n} \setminus \mathrm{im}(x)$*.*

Here ((ii)) is evident, but ((i)) requires a proof which we postpone to the next lecture. The upshot is that $||W_n(\underline{1})_\bullet|| /\!\!/ \Sigma_n$ can serve a replacement for $* /\!\!/ \Sigma_n$ for computing the homology in a range.

2.3 The geometric realisation spectral sequence

A filtration $F_0X \subset F_1X \subset \ldots$ on a space X makes the singular chains $C_*(X)$ into a filtered chain complex by setting $F_rC_*(X) := \mathrm{im}(C_*(F_rX) \to C_*(X))$. Assuming that $F_{r-1}X \to F_rX$ is a cofibration and $F_rX/F_{r-1}X$ is at least $(r-1)$-connected, this gives a strongly-convergent first-quadrant *spectral sequence*

$$E^1_{p,q} = \widetilde{H}_{p+q}(F_pX/F_{p-1}X;\mathbb{Z}) \Longrightarrow H_{p+q}(X;\mathbb{Z})$$

with differentials given by $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$.

Remark 2.3.1. If you are unfamiliar with these notions, I recommend you look at [McC01, Hat]. Roughly, a spectral sequence is an algebraic object that conveniently packages all long exact sequences in homology for the pairs (F_sX, F_rX) with $s \geq r$ with the goal of compute the homology of X.

We can in particular apply this to the geometric realisation $||X_\bullet||$. This has a filtration

$$F_r||X_\bullet|| := \left(\bigsqcup_{0 \leq p \leq r} \Delta^p \times X_p \right) /{\sim}$$

with equivalence relation $\sim$ as before, all of whose maps are cofibrations under a mild condition on $X_\bullet$ that will be satisfied in examples in these notes. The associated graded is given by

$$\frac{F_r||X_\bullet||}{F_{r-1}||X_\bullet||} = \frac{\Delta^r}{\partial\Delta^r} \wedge (X_r)_+ \tag{2.1}$$

so is at least $(r-1)$-connected. Thus we get [Seg68] (see also [ERW19, Section 1.4]):

Theorem 2.3.2. *There is a strongly convergent first quadrant spectral sequence*

$$E^1_{p,q} = H_q(X_p;\mathbb{Z}) \Longrightarrow H_{p+q}(||X_\bullet||;\mathbb{Z})$$

with differentials given by $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$*. Moreover* $d^1\colon E^1_{p,q} \to E^1_{p-1,q}$ *is given by* $\sum_{i=0}^p (-1)^i (d_i)_*$*, and the edge homomorphism* $E^1_{0,q} \to E^\infty_{0,q} \to H_q(X;\mathbb{Z})$ *is equal to the map induced on homology by the inclusion* $X_0 \to ||X_\bullet||$*.*

Proof. The identification of the E^1-page follows from (2.1) and the suspension isomorphism. The description of the abutment and edge homomorphism is as for any spectral sequence of a filtered space. For the identification of the d^1-differential, see the references. ☐

2.4 The proof of Theorem 2.1.1

We have now gathered all ingredients for the proof of Theorem 2.1.1. It is a proof by strong induction, so we assume we have proven the result for $m \leq n$ and we will prove it for $n+1$. There is nothing to prove when $n+1 \leq 2$, so we may assume that $n+1 \geq 3$.

Step 1: Replacing $*$ by $||W_{n+1}(\underline{1})_\bullet||$

By Proposition 2.2.6, $||W_{n+1}(\underline{1})_\bullet||$ is homologically $\frac{n}{2}$-connected so the map $||W_{n+1}(\underline{1})_\bullet|| \to *$ is $(\frac{n}{2}+1)$-connected. Taking homotopy quotients by Σ_{n+1}, (1) yields a map

$$||W_{n+1}(\underline{1})_\bullet|| /\!\!/ \Sigma_{n+1} \longrightarrow * /\!\!/ \Sigma_{n+1}$$

which is homologically $(\frac{n}{2}+1)$-connected by (2). To prove Theorem 2.1.1, we may thus replace $* /\!\!/ \Sigma_n$ by $||W_{n+1}(\underline{1})_\bullet|| /\!\!/ \Sigma_{n+1}$.

Step 2: The E^1-page of the geometric realisation spectral sequence

Next, (3) provides a weak homotopy equivalence

$$||W_{n+1}(\underline{1})_\bullet|| /\!\!/ \Sigma_{n+1} \simeq ||W_{n+1}(\underline{1})_\bullet /\!\!/ \Sigma_{n+1}||.$$

Then Theorem 2.3.2 provides a spectral sequence

$$E^1_{p,q} = H_q(W_{n+1}(\underline{1})_p /\!\!/ \Sigma_{n+1}; \mathbb{Z}) \Longrightarrow H_{p+q}(X; \mathbb{Z}).$$

Let us now identify the E^1-page more explicitly, as well as the d^1-differential. Since $W_{n+1}(\underline{1})_p$ is a transitive Σ_{n+1}, for any injective map $f\colon [p] \to \underline{n+1}$, (4) says that the map

$$* /\!\!/ \mathrm{Stab}_{\Sigma_{n+1}}(f) \longrightarrow W_{n+1}(\underline{1})_p /\!\!/ \Sigma_{n+1}$$

sending $*$ to f is a weak homotopy equivalence. Taking f to be the inclusion ι_p of the last $p+1$ elements, $\mathrm{Stab}_{\Sigma(n+1)}(\iota_p) \hookrightarrow \Sigma_{n+1}$ is the usual inclusion $\Sigma_{n-p} \hookrightarrow \Sigma_{n+1}$ as acting on the first $n-p$ elements. This allows us to make the identification

$$E^1_{p,q} \cong H_q(B\Sigma_{n-p}; \mathbb{Z}), \tag{2.2}$$

but it is important to recall how this identification is made when we next compute the d^1-differential.

Indeed, $d^1 = \sum (-1)^i (d_i)_*$ and even though the diagram

$$\begin{array}{ccc} * /\!\!/ \mathrm{Stab}_{\Sigma_{n+1}}(f) & \xrightarrow{\simeq} & W_{n+1}(\underline{1})_p /\!\!/ \Sigma_{n+1} \\ \downarrow & & \downarrow {\scriptstyle d_i /\!\!/ \Sigma_{n+1}} \\ * /\!\!/ \mathrm{Stab}_{\Sigma_{n+1}}(d_i f) & \xrightarrow{\simeq} & W_{n+1}(\underline{1})_{p-1} /\!\!/ \Sigma_{n+1} \end{array}$$

commutes it is not true in general that $d_i\iota_p = \iota_{p-1}$. Rather, we have $h_i d_i \iota_p = \iota_{p-1}$ where h_i is an element of Σ_{n+1} that sends $d_i\iota_p$ to ι_{p-1} (there is more than one such element). The correct commuting diagram involving only standard inclusions is

$$\begin{array}{ccc}
* /\!\!/ \mathrm{Stab}_{\Sigma_{n+1}}(\iota_p) & \xrightarrow{\simeq} & W_{n+1}(\underline{1})_p /\!\!/ \Sigma_{n+1} \\
\big\downarrow & & \big\downarrow {\scriptstyle d_i /\!\!/ \Sigma_{n+1}} \\
* /\!\!/ \mathrm{Stab}_{\Sigma_{n+1}}(d_i\iota_p) & \xrightarrow{\simeq} & W_{n+1}(\underline{1})_{p-1} /\!\!/ \Sigma_{n+1} \\
{\scriptstyle c_{h_i}} \big\downarrow & & {\scriptstyle \simeq} \big\downarrow {\scriptstyle c'_{h_i}} \\
* /\!\!/ \mathrm{Stab}_{\Sigma_{n+1}}(\iota_{p-1}) & \xrightarrow{\simeq} & W_{n+1}(\underline{1})_{p-1} /\!\!/ \Sigma_{n+1}
\end{array}$$

where $c_{h_i}\colon \mathrm{Stab}_{\Sigma_{n+1}}(d_i\iota_p) \to \mathrm{Stab}_{\Sigma_{n+1}}(\iota_{p-1})$ is induced by conjugation in Σ_{n+1} with h_i, and c'_{h_i} is induced by the map of Σ_{n+1}-sets $W_{n+1}(\underline{1})_{p-1} \to W_{n+1}(\underline{1})_{p-1}$ sending given by multiplication with h_i (one also needs to then twist the action by conjugation by h_i). The map c'_{h_i} is homotopic to the identity, so the upshot is that the following diagram commutes

$$\begin{array}{ccc}
H_*(B\Sigma_{n-p};\mathbb{Z}) & \xrightarrow[\cong]{\iota_p} & H_*(W_{n+1}(\underline{1})_p /\!\!/ \Sigma_{n+1};\mathbb{Z}) \\
{\scriptstyle (c_{h_i}\circ \mathrm{inc})_*} \big\downarrow & & \big\downarrow {\scriptstyle (d_i)_*} \\
H_*(B\Sigma_{n-p+1};\mathbb{Z}) & \xrightarrow[\cong]{\iota_{p+1}} & H_*(W_{n+1}(\underline{1})_{p-1} /\!\!/ \Sigma_{n+1};\mathbb{Z}).
\end{array}$$

We are free to choose h_i, and we shall make a fortunate choice: we take it to be the transposition swapping $n-p+1$ and $n-p+i$. This has the advantage of commuting with the image of Σ_{n-p}, so that the left map is equal to just σ_*. The upshot is that under the identification of (2.2), we can identify the d^1-differential as

$$d^1 = \sum_{i=0}^{p} (-1)^i \sigma_* = \begin{cases} \sigma_* & \text{if } p > 0 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

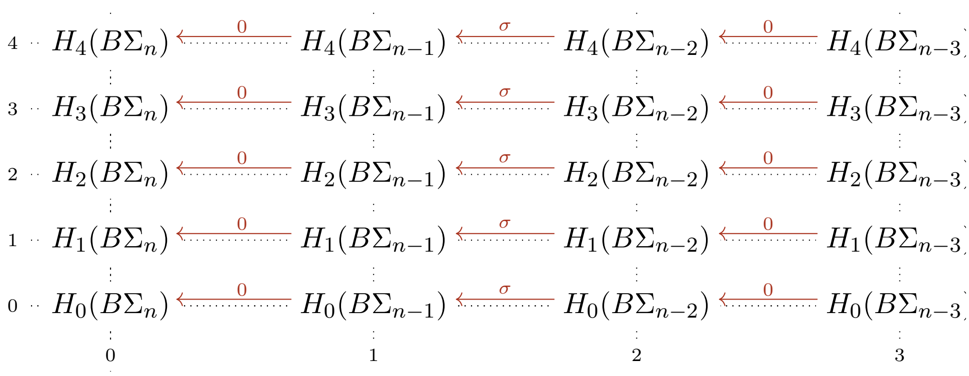


Figure 2.1: The E^1-page entries $E^1_{p,q}$ and the d^1-differentials (we drop the coefficients from homology for the sake of readability).

Step 3: A spectral sequence argument

We want to show that the edge homomorphism

$$E^1_{0,q} = H_q(B\Sigma_n;\mathbb{Z}) \longrightarrow H_q(||W_{n+1}(\underline{1})_\bullet /\!\!/ \Sigma_{n+1}||;\mathbb{Z}) \xrightarrow{\cong} H_q(B\Sigma_{n+1};\mathbb{Z})$$

is a surjection for $* \leq \frac{n}{2}$ and an isomorphism for $* \leq \frac{n-1}{2}$. Indeed, in this range the right map is an isomorphism (as indicated) and it is easy to identify the composition with the stabilisation map σ.

We will do so by studying the geometric realisation spectral sequence, whose E^1-page looks like Figure 2.1. By the inductive hypothesis, whenever we see a d^1-differential equal to σ it is surjective or even an isomorphism in a range.

Putting in the explicit ranges, we get that the E^2-page vanishes in a range for $p > 0$. There are two cases, (a) $n = 2k + 1$ odd and (b) $n = 2k + 1$ even:

(a) The precise vanishing range is that for $r \geq 0$, $E^2_{2r+1,q} = E^2_{2r+2,q} = 0$ for $q \leq k-r-1$. For example, if $n = 9$ then $E^1_{2,q} = H_q(B\Sigma_7;\mathbb{Z}) \to H_q(B\Sigma_8;\mathbb{Z})$ is a surjection for $* \leq \frac{7}{2}$ and an isomorphism for $* \leq \frac{6}{2}$. That is, two adjacent columns connected by a σ vanish in a range, the first pair from the left vanishing for $q \leq k-1$ and this range going down one degree whenever we move two columns to the right. See Fig. 2.2 for $2k = 9$.

Furthermore, the higher differentials are $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ (so upwards and to the left). Thus no differential with non-zero domain can enter the entries $E^2_{0,q} = H_q(B\Sigma_{2k};\mathbb{Z})$ for $q \leq k$, and that no further non-zero groups contribute to $E^\infty_{0,q}$. This implies that $H_q(B\Sigma_{2k+1};\mathbb{Z}) \to H_q(B\Sigma_{2k+2};\mathbb{Z})$ is an isomorphism for $q \leq k = \lfloor \frac{2k+1-1}{2} \rfloor = \lfloor \frac{2k+1}{2} \rfloor$; this what we needed to prove.

(b) The precise vanishing range is that for $r \geq 0$, $E^2_{2r+1,q} = 0$ and $q \leq k-r-1$ and $E^2_{2r+2,q} = 0$ for $q - r - 2$. This is the same pattern as in case (a), but in the right column in each pair the range is one lower. See Fig. 2.2 for $2k = 8$. The result is that no differential with non-zero domain can enter the entries $E^2_{0,q} = H_q(B\Sigma_{2k};\mathbb{Z})$ for $q \leq k-1$, and that no further non-zero groups contribute to $E^\infty_{0,q}$. Furthermore, a single d^2-differential with non-zero domain can enter the entry $E^2_{0,k} = H_k(B\Sigma_{2k};\mathbb{Z})$ and again no further non-zero groups can contribute. This implies that $H_q(B\Sigma_{2k};\mathbb{Z}) \to H_q(B\Sigma_{2k+1};\mathbb{Z})$ is an isomorphism for $q \leq k-1 = \lfloor \frac{2k-1}{2} \rfloor$ and a surjection for $q = \lfloor \frac{2k}{2} \rfloor = k$; this what we needed to prove.

2.5 Exercises

Exercise 2.5.1 (Simplicial and semi-simplicial spaces)**.**

(i) Let sTop and ssTop denote the categories of simplicial, resp. semi-simplicial, spaces. Prove that the forgetful functor $U\colon \mathsf{sTop} \to \mathsf{ssTop}$ has a left-adjoint F. (Hint: it is given by “freely adjoining degeneracies”.)

(ii) Prove that $|F(X_\bullet)| \cong ||X_\bullet||$ for $X_\bullet \in \mathsf{ssTop}$.

Exercise 2.5.2 (The fundamental group of $W_n(\underline{1})_\bullet$)**.** Use Seifert–van Kampen to prove that $||W_n(\underline{1})_\bullet||$ is 1-connected for $n \geq 3$.

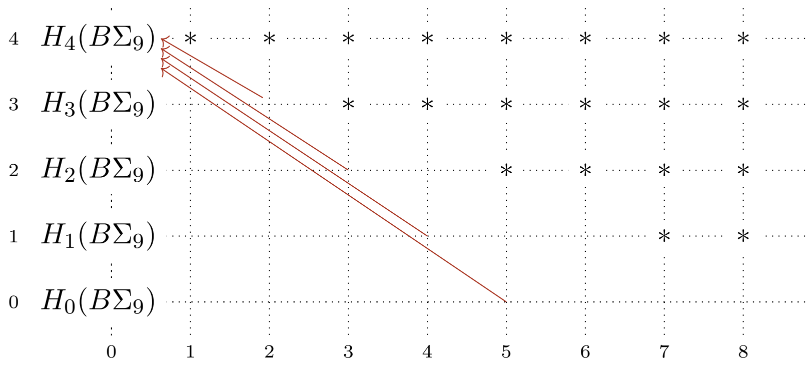


Figure 2.2: The E^2-page entries $E^2_{p,q}$ for $n = 2k+1 = 9$. The empty entries are 0, the $*$ means unknown groups. We have drawn the higher differentials into the entry $E^2_{0,4} = H_4(B\Sigma_9; \mathbb{Z})$.

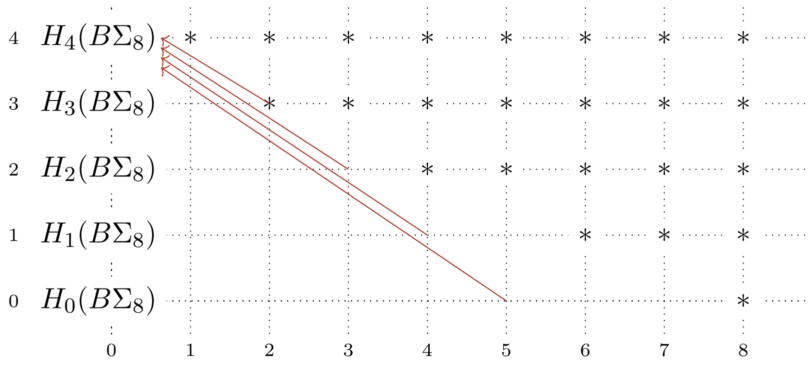


Figure 2.3: The E^2-page entries $E^2_{p,q}$ for $n = 2k = 8$. The empty entries are 0, the $*$ means unknown groups. We have drawn the higher differentials into the entry $E^2_{0,4} = H_4(B\Sigma_8; \mathbb{Z})$.

Exercise 2.5.3 (Homotopy quotients preserve connectivity)**.** There is a fibration sequence natural in G-spaces X

$$X \longrightarrow X /\!\!/ G \longrightarrow * /\!\!/ G.$$

Use the associated Serre spectral sequence to prove property (2) of homotopy quotients.

Exercise 2.5.4 (Homotopy quotients of G-sets)**.** If X is a set with G-action, then the homotopy quotient $X /\!\!/ G$ has a model as the geometric realisation of the nerve of the category with objects elements $x \in X$ and a unique morphism from x to gx. We also denote this category by $X /\!\!/ G$.

(i) For $x \in X$ construct a functor $\iota_x \colon * /\!\!/ \mathrm{Stab}_G(x) \to X /\!\!/ G$ sending $*$ to x.

(ii) Prove that ι_x is an equivalence of categories if X is a transitive G-set and conclude that the map induced by ι_x on geometric realisation of nerves is a homotopy equivalence. This is property (4) of homotopy quotients.

(iii) For $h \in G$ construct a functor $c_h \colon X /\!\!/ G \to X /\!\!/ G$ sending x to hx.

(iv) Prove that there is a natural transformation $\mathrm{id} \Rightarrow c_h$ and conclude that the map induced by c_h on geometric realisation of nerves is homotopic to the identity. This justifies a claim in Step 2 of Section 2.4.

(v) Can the homotopy in (iv) be taken to be based?

Exercise 2.5.5 (Homological stability with constant coefficients)**.** Prove that Theorem 2.3.2 goes through when we replace the coefficients $\mathbb{Z}$ with another abelian group A.

Exercise 2.5.6 (Homological stability with abelian coefficients)**.** If M be a $\mathbb{Z}/2$-module, then we can use the sign homomorphism $\mathrm{sign}\colon \Sigma_n \to \mathbb{Z}/2$ to make it into a local coefficient system M_n on $B\Sigma_n$ and define $H_*(B\Sigma_n; M_n)$.

(i) Prove that the pullback of M_{n+1} along $\sigma\colon B\Sigma_n \to B\Sigma_{n+1}$ is canonically isomorphic to M_n. Construct a stabilisation map

$$\sigma_*\colon H_*(B\Sigma_n; M_n) \longrightarrow H_*(B\Sigma_{n+1}; M_{n+1}).$$

(ii) Modify the argument in this lecture to prove that that $\sigma_*\colon H_*(B\Sigma_n; M_n) \to H_*(B\Sigma_{n+1}; M_{n+1})$ is a surjection if $* \leq \frac{n-1}{3}$ and an isomorphism if $* \leq \frac{n-3}{3}$. (Hint: $(c_{h_i} \circ \mathrm{inc})_*$ is no longer equal to σ_*, but it is so on the image of $H_*(B\Sigma_{n-p-1}; M_{n-p-1})$ in $H_*(B\Sigma_{n-p}; M_{n-p})$.)

Exercise 2.5.7 (Homological stability for alternating groups)**.** By picking M appropriately and invoking Shapiro's lemma, use Exercise 2.5.6 to deduce homological stability for alternating groups.

Remark 2.5.8. Homological stability for alternating groups goes back to Mann [Man85].

Chapter 3

Homological stability for automorphism groups

In the third lecture of this minicourse, we finish the proof of homological stability for symmetric groups by proving that the semi-simplicial set of injective words is highly-connected. We then explain the general framework for homological stability due to Randal-Williams and Wahl, with an application to a result of Dwyer.

3.1 The semi-simplicial set of injective words is highly-connected

Last lecture, we proved that the sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability: more precisely,

$$\sigma_*\colon H_*(B\Sigma_n;\mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1};\mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ and an isomorphism if $* \leq \frac{n-1}{2}$. We gave a complete proof, with the exception of using the following result as input:

Proposition 3.1.1. $||W_n(\underline{1})_\bullet||$ *is homologically* $\frac{n-1}{2}$*-connected.*

Remark 3.1.2. It is in fact known to be $(n-2)$-connected (e.g. [Far79, Ker05, RW13a, Gan17]), but we will not use this. See Exercise 3.4.2 for a proof which uses a technique which is useful for proving other semi-simplicial sets are highly-connected.

I want to prove this, after recalling the definition of $W_n(\underline{1})_\bullet$, because we will soon see that the connectivity of semi-simplicial sets like $W_n(\underline{1})$ is often the crux for proving homological stability results. Recall that FI is the category of finite sets and injections.

Definition 3.1.3. $W_n(\underline{1})_\bullet$ is the semi-simplicial set with p-simplices given by

$$W_n(\underline{1})_\bullet = \mathrm{Hom}_{\mathsf{FI}}([p], \underline{n})$$

and face maps d_i given by precomposition with $\delta_i\colon [p-1] \to [p]$.

Proof of Proposition 3.1.1. We will give a proof due to Randal-Williams [RW]. The proof is by strong induction over n; we did the cases $n \leq 2$ before and the case $n = 3$ was an exercise. So we may assume $n \geq 4$, and suppose the cases $< n$ to be known. In our induction we will use different subsets of $\underline{n}$, so it is convenient to temporarily write $W(S)_\bullet$ for the semi-simplicial set obtained by replacing $\underline{n}$ with S.

The inclusions $\underline{n} \setminus \{i\} =: S_i \hookrightarrow \underline{n}$ for $1 \leq i \leq n$ of all $(n-1)$-element subsets give a cover by subcomplexes of the $(n-2)$-skeleton of $||W(\underline{n})_\bullet||$ with its canonical CW-structure. Thus the map

$$\cup_{i=1}^n ||W(S_i)_\bullet|| \longrightarrow ||W(\underline{n})_\bullet||$$

induces an surjection on H_* for $* \leq n-2$, and $n-2 \geq \frac{n-1}{2}$ when $n \geq 4$.

We will now prove that $\bigoplus_{i=1}^n H_*(||W(S_i)_\bullet)||;\mathbb{Z}) \to H_*(\cup_{i=1}^n ||W(S_i)_\bullet||;\mathbb{Z})$ is surjective for $* \leq \frac{n-1}{2}$. To do so, we use the Mayer–Vietoris spectral sequence for this cover:

$$E^1_{pq} = \bigoplus_{1 \leq i_0 < \cdots < i_p \leq n} H_q(\cap_{j=1}^p ||W(S_{i_j})_\bullet||;\mathbb{Z}) \Longrightarrow H_{p+q}(\cup_{i=1}^n ||W(S_i)_\bullet||;\mathbb{Z}).$$

Since $\cap_{j=0}^p ||W(S_{i_j})_\bullet|| \cong ||W(\cap_{0=1}^p S_{i_j})_\bullet||$ is isomorphic to $||W(\underline{n-p-1})_\bullet||$, the entries $E^1_{p,q}$ vanish for $0 < q \leq \frac{n-p-2}{2}$. Furthermore, the chain complex $(E^1_{p,0}, d^1)$ is the cellular chain complex of $\partial\Delta^{n-1}$ and hence on the E^2-page, we get that the bottom row $E^2_{p,0}$ vanishes for $0 < p < n-2$. Since $n \geq 4$, this implies that the edge homomorphism is surjective for $* \leq \frac{n-1}{2}$, which proves the desired statement.

Now we observe that composition

$$E^1_{0,q} \longrightarrow H_q(\cup_{i=1}^n ||W(S_i)_\bullet||;\mathbb{Z}) \xrightarrow{\cong} H_q(||W(\underline{n})_\bullet||;\mathbb{Z})$$

is induced by the inclusions $||W(S_i)_\bullet|| \to ||W(\underline{n})_\bullet||$, which are null-homotopic by "coning off" using the element $i \in \underline{n}$ (see Exercise 3.4.1). Thus in degrees $0 < * \leq \frac{n-1}{2}$

$$H_*(\cup_{i=1}^n ||W(S_i)_\bullet||;\mathbb{Z}) \longrightarrow H_*(||W(\underline{n})_\bullet||;\mathbb{Z})$$

is both zero and surjective, and hence the target vanishes. This completes the proof of the induction step. □

Remark 3.1.4. From an exercise in the previous lecture, we know that $||W_n(\underline{1})_\bullet||$ is also simply-connected for $n \geq 3$. Hence in that case, it is not just homologically $\frac{n-1}{2}$-connected but actually $\frac{n-1}{2}$-connected.

3.2 The framework of Randal–Williams and Wahl

We did not use much about symmetric groups in our arguments and it is possible to completely formalise the properties that we did use; doing so yields [RWW17].

A *monoidal category* C is a category with a functor

$$\oplus \colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$

and an object $\mathbb{1} \in \mathsf{C}$ serving as a unit. There are associativity and unitality isomorphisms relating these, see [ML98]. It is a *symmetric monoidal* if we are additionally given compatible natural isomorphisms $\beta_{X,Y} \colon X \oplus Y \to Y \oplus X$ so that $\beta_{X,Y} \circ \beta_{Y,X} = \mathrm{id}$.

When G is a symmetric monoidal groupoid, then given objects $A, X \in \mathsf{C}$ we can form the automorphism groups

$$G_n := \mathrm{Aut}_{\mathsf{C}}(A \oplus X^{\oplus n}),$$

and the functor $- \oplus X$ induces a stabilisation map $\sigma \colon G_n \to G_{n+1}$ between these.

Question 3.2.1. When does the sequence $BG_0 \xrightarrow{\sigma} BG_1 \xrightarrow{\sigma} BG_2 \xrightarrow{\sigma} \cdots$ exhibit homological stability?

Theorem 3.2.2 ([RWW17]). *Suppose that a symmetric monoidal groupoid* G *has the following properties:*

(i) the monoid of isomorphism classes of objects of G *has cancellation,*

(ii) $\mathrm{Aut}(B) \to \mathrm{Aut}(B \oplus X)$ *is injective for all* B*.*

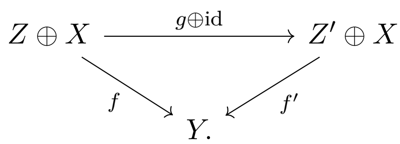
Then we will momentarily describe semi-simplicial sets $W_n(A, X)_\bullet$ *with the following property: if there is a* $k \geq 2$ *with* $||W_n(A, X)_\bullet||$ *is homologically* $\frac{n-2}{k}$*-connected for all* $n \geq 2$*, then*

$$\sigma_* \colon H_*(BG_n; \mathbb{Z}) \longrightarrow H_*(BG_{n+1}; \mathbb{Z})$$

is a surjection for $* \leq \frac{n}{k}$ *and an isomorphism for* $* \leq \frac{n-1}{k}$*.*

Remark 3.2.3. More generally, [RWW17] allows braided monoidal groupoids as inputs. Moreover, the classifying space of a symmetric monoidal groupoid is an E_2-algebra which is a disjoint union of Eilenberg–MacLane spaces. In [Kra19], Krannich proved the analogous result for general E_2-algebras, allowing one to drop the cancellation and injectivity assumptions. This for example allows one to study diffeomorphism groups of high-dimensional manifolds. The reader may noticed that the assumptions in [RWW17] are slightly different than those of Theorem 3.2.2. This is explained in [Kra19, Section 7.3].

The construction of $W_n(A, X)_\bullet$ uses a construction due to Quillen, which we denote $\mathcal{U}\mathsf{G}$ following [RWW17]: this is the category with the same objects as G but morphisms from X to Y given by an equivalence class of pairs (Z, f) with $f \colon Z \oplus X \to Y$ a morphism in G. Two of these, (Z, f) and (Z', f'), are equivalent if there is a morphism $g \colon Z \to Z'$ in G such that the following diagram commutes:



This inherits a symmetric monoidal. Then $W_n(X)_p = \mathrm{Hom}_{\mathcal{U}\mathsf{G}}(X^{\oplus [p]}, A \oplus X^{\oplus n})$ and the ith face map is induced by precomposition.

Example 3.2.4. Suppose G is the groupoid FB of finite sets and bijections, on which disjoint union gives a symmetric monoidal structure. Then $\mathcal{U}\mathsf{G}$ is the category FI of finite sets and injections. Taking $A = \varnothing$ and $X = \underline{1}$ we get that $G_n = \Sigma_n$ and $W_n(\varnothing, X)_\bullet = W_n(\underline{1})_\bullet$ as above. Thus Theorem 3.2.2 recovers Nakaoka's theorem.

Example 3.2.5. Suppose that G is the groupoid of finitely-generated abelian groups and their automorphisms, on which direct sum gives a symmetric monoidal structure. Then taking $A = \mathbb{Z}$, $X = \mathbb{Z}$ we get that $G_n = \mathrm{GL}_{n+1}(\mathbb{Z})$. Properties (i) and (ii) hold by the classification of finitely-generated abelian groups. The semi-simplicial sets $W_n(\mathbb{Z}, \mathbb{Z})_\bullet$ were essentially studied by Charney [Cha84]; her proof can be adapted to prove that these are $\frac{n-2}{2}$-connected (this is the hardest step in the argument). We conclude that the stabilisation map

$$\sigma\colon B\mathrm{GL}_{n+1}(\mathbb{Z}) \longrightarrow B\mathrm{GL}_{n+2}(\mathbb{Z})$$

induced by the inclusion $\mathrm{GL}_{n+1}(\mathbb{Z}) \to \mathrm{GL}_{n+2}(\mathbb{Z})$ in the top-left corner, induces a surjection on H_* for $* \leq \frac{n}{2}$ and an isomorphism for $* \leq \frac{n-1}{2}$.

Example 3.2.6. Here is an incomplete list of further sequences of groups whose classifying spaces exhibit homological stability and equally incomplete references (see also Section 5 of [RWW17]):

- *General linear groups of rings of finite stable rank* (this includes any ring you're likely to think of) [vdK80, Cha80, Dwy80, Cha84, Bet86b, NS89, Gui89, Bet92, HT10, GKRW18, SW20, GKRW20].
- *Unitary groups of rings of finite unitary stable rank* (again this includes any ring you're likely to think of) [Fri76, Pan87, MvdK02, Mir05, Col11, Ess13, SW20]; this includes *symplectic groups* [vdKL11] and *orthogonal groups* [Vog79, Vog81, Vog82, Bet86a, Cha87, Bet87, Bet90, Cat07].
- *Automorphism groups of free groups* [Hat95, HV98a, HV98b, RW18] and related groups [HV04, HW05, HVW06, HW08, Zar14]
- *Diffeomorphism groups* or *mapping class groups of surfaces* [Har85, Har90, Iva93, Wah08, Bol12, Wah13, RW14, RW16, HV17, GKRW19] and *3-manifolds* [HW10, Lam15, Kup20]; this includes *braid groups* [Arn69, Fuk70, CLM76, Vai78]. The latter is an example of a configuration space, which we will discuss in more detail in the next lecture.
- *Diffeomorphism groups of high-dimensional manifolds* [Per16a, Per16b, GRW17, Per18, GRW18] and related groups [CM11, RW13b, BM13, Kup15, Til16, Nar17, Gre19, Kra20].
- Various other groups: *automorphism groups of right-angled Artin groups* [GW16], *Artin monoids* [Boy20], *Houghton groups* [PW16], *Higman–Thompson groups* [SW19], *free nilpotent groups* [Szy14], and Coxeter groups [Hep16].

A somewhat outdated but still interesting survey of stability phenomena is [Coh09].

3.2.1 When should you expect homological stability?

I have three reasons to introduce Theorem 3.2.2:

I. It abstracts those properties of symmetric groups that we use.

II. It allows me to state a generalisation with local coefficients in the next section.

III. It leads to a heuristic for when one expects a sequence of classifying spaces of groups to exhibit homological stability.

It is this last point I want to make more explicit by stating Theorem 3.2.2 informally. Suppose that if you have a natural way to "summing" old objects together to make new ones (a monoidal structure), which is sufficiently symmetric (this monoidal structure is symmetric or more generally braided) and somewhat reasonable (properties (i) and

(ii) in Theorem 3.2.2). Then you should expect homological stability to hold for the automorphism groups of $A \oplus X^{\oplus n}$ exactly when the "space of ways of removing copies of X from $A \oplus X^{\oplus n}$" increases in connectivity with n. As this description suggests, the last property is the hardest one to verify and doing so is more of an art than a science. It requires new ideas specific to each situation.[1]

3.3 Local coefficients and a theorem of Dwyer

3.3.1 Homological stability with polynomial coefficients

In an exercise for the previous lecture, we considered replacing the constant coefficients $\mathbb{Z}$ by certain local coefficients. In the general framework of Theorem 3.2.2, these are as follows:

Definition 3.3.1. Let $\mathcal{U}\mathsf{G}_{A,X}$ be the full subcategory of $\mathcal{U}\mathsf{G}$ on objects $A \oplus X^{\oplus n}$. Then a *coefficient system* is a functor $F \colon \mathcal{U}\mathsf{G}_{A,X} \to \mathsf{Ab}$.

More concretely, this provides a collection of $F_n := F(A \oplus X^{\oplus n})$ of abelian groups with G_n-action together with G_n-equivariant morphisms $F_n \to F_{n+1}$. In particular, we can ask whether the map

$$\sigma_* \colon H_*(BG_n; F_n) \longrightarrow H_*(BG_{n+1}; F_{n+1})$$

is an isomorphism in a range. That is, does the sequence $BG_0 \xrightarrow{\sigma} BG_1 \xrightarrow{\sigma} \cdots$ exhibit homological stability with coefficients in F? This is the case for those coefficient system where he can reduce to trivial coefficients.

There is a functor $\Sigma_X \colon \mathcal{U}\mathsf{G}_{A,X} \to \mathcal{U}\mathsf{G}_{A,X}$ given on objects by sending $A \oplus X^{\oplus n}$ to $A \oplus X^{\oplus n+1}$ and morphism f to $(\beta_{A,X}^{-1} \oplus \mathrm{id}) \circ (X \oplus f) \circ (\beta_{A,X} \oplus \mathrm{id})$. There is a natural transformation $\sigma_X \colon \mathrm{id} \to \Sigma_X$.

Definition 3.3.2.

- $\mathrm{coker}(F) := \mathrm{coker}(F \to \Sigma_X F)$ and $\ker(F) := \ker(F \to \Sigma_X F)$.
- F is *polynomial of degree* -1 if $F(X \oplus A^{\oplus n}) = 0$ for sufficiently large n.
- F is *polynomial of degree* r if $\mathrm{coker}(F)$ is polynomial of degree $r-1$ and $\ker(F)$ is polynomial of degree -1.

Example 3.3.3. F is polynomial of degree 0 if it is eventually constant.

By reduction to the constant coefficients, we can deduce a homological stability with coefficients in a polynomial functor of finite degree.

Theorem 3.3.4 ([RWW17])**.** *Under the assumptions of Theorem 3.2.2, if F is polynomial of degree r, there exists an $N \geq 0$ such when $n \geq N$, the map*

$$\sigma_* \colon H_*(BG_n; F_n) \longrightarrow H_*(BG_{n+1}; F_{n+1})$$

is a surjection for $* \leq \frac{n}{k} - r$ and an isomorphism for $* \leq \frac{n}{k} - r - 1$.

[1] This is one of the reasons for including the long list of references above; if you want to prove a connectivity result for a semi-simplicial set, look at such a result in a related situation.

3.3.2 Dwyer's finiteness theorem

Let us give an application of this, due to Dwyer. Recall homological stability for symmetric groups proved that $\pi_*(\mathbb{S})$ is finite for $* > 0$. We can prove a similar result for $K(\mathbb{S})$, the algebraic K-theory spectrum of the sphere spectrum (also known as $A(*)$):

Theorem 3.3.5 ([Dwy80])**.** *$\pi_*(K(\mathbb{S}))$ is finitely generated for $* > 0$.*

By Waldhausen's work [Wal85], the infinite loop space $\Omega^\infty K(\mathbb{S})$ can be obtained as the group completion of the topological monoid

$$\bigsqcup_{n\geq 0} \operatorname*{hocolim}_{d\to\infty} B\mathrm{hAut}_*(\vee_n S^d),$$

with map $\mathrm{hAut}_*(\vee_n S^d) \to \mathrm{hAut}_*(\vee_n S^{d+1})$ induced by suspension and multiplication map induced by wedging. For the sake of brevity, we abbreviate $\mathrm{hocolim}_{d\to\infty} B\mathrm{hAut}_*(\vee_n S^d)$ to $B\mathrm{hAut}_*(\vee_n \mathbb{S})$.

We may as well assume that $d \geq 2$. The action on H_d then induces an isomorphism $\pi_0(\mathrm{hAut}_*(\vee_n S^d)) \cong \mathrm{GL}_n(\mathbb{Z})$. We can further understand the higher homotopy groups with their $\mathrm{GL}_n(\mathbb{Z})$-action using the Freudenthal suspension theorem and Hilton–Milnor theorem: for $* \leq d$, the suspension map $\pi_*(\mathrm{hAut}_*(\vee_n S^d)) \to \pi_*(\mathrm{hAut}_*(\vee_n S^{d+1}))$ is an isomorphism, and as a $\mathbb{Z}[\mathrm{GL}_n(\mathbb{Z})]$-module we have

$$\pi_*(\mathrm{hAut}_*(\vee_n S^d)) \cong \pi_*(\mathbb{S}) \otimes_{\mathbb{Z}} \mathrm{Ad}_n \qquad \text{for } 0 < * \leq d,$$

where Ad_n is given by $\mathrm{GL}_n(\mathbb{Z})$ acting on $(n \times n)$-matrices with integral entries by conjugation and the $\mathrm{GL}_n(\mathbb{Z})$-action on $\pi_*(\mathbb{S})$ is trivial. These descriptions are compatible with stabilisation.

As a consequence, the group $\pi_*(B\mathrm{hAut}_*(\vee_n \mathbb{S}))$ for $* = 1$ is $\mathrm{GL}_n(\mathbb{Z})$ and the coefficient system $\pi_*(B\mathrm{hAut}_*(\vee_{(-)} \mathbb{S}))$ for $* > 1$ lies in the class $\mathcal{P}$ of polynomial functors which are of finite degree and objectwise finitely-generated. This class is quite well-behaved:

Lemma 3.3.6.

(i) $\mathcal{P}$ is closed under passing to subobjects, quotients, and extensions: in an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow Q \longrightarrow 0$$

of coefficients systems, $F, Q \in \mathcal{P}$ if only if $G \in \mathcal{P}$.

(ii) $\mathcal{P}$ is closed under tensor products and Tor*: if $F, G \in \mathcal{P}$ then $F \otimes_{\mathbb{Z}} G$ and $\mathrm{Tor}^1_{\mathbb{Z}}(F, G)$ are in $\mathcal{P}$.*

(iii) $\mathcal{P}$ is closed under taking homology of Eilenberg–Mac Lane spaces: if $F, G \in \mathcal{P}$ then $H_q(K(F, n); G) \in \mathcal{P}$ for all $q, n > 0$.

The proof of this lemma is quite involved, and will appear in a forthcoming joint paper with Manuel Krannich. Dwyer worked with a more restricted class sufficient for the purpose of proving Theorem 3.3.5.

We will use Lemma 3.3.6 to prove two qualitative statements about the homology of $B\mathrm{hAut}_*(\vee_n\mathbb{S})$. Let $\mathrm{hAut}^{\mathrm{id}}_*(\vee_n S^d) \subset \mathrm{hAut}_*(\vee_n S^d)$ be the path component containing the identity map. For all $n \geq 0$, there is then a fibration sequence

$$\operatorname*{hocolim}_{d\to\infty} B\mathrm{hAut}^{\mathrm{id}}_*(\vee_n S^d) \longrightarrow \operatorname*{hocolim}_{d\to\infty} B\mathrm{hAut}_*(\vee_n S^d) \longrightarrow B\mathrm{GL}_n(\mathbb{Z}),$$

compatible with stabilisation and suspension. Letting $d \to \infty$ we thus get Serre spectral sequences

$$H_p(B\mathrm{GL}_n(\mathbb{Z}); H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_n\mathbb{S}); \mathbb{Z})) \Longrightarrow H_{p+q}(B\mathrm{hAut}_*(\vee_n\mathbb{S}); \mathbb{Z})$$

for all $n \geq 0$, connected by stabilisation maps.

A standard Serre class argument over the Postnikov tower of $B\mathrm{hAut}^{\mathrm{id}}_*(\vee_n\mathbb{S})$ using Lemma 3.3.6 implies the following, see Exercise 3.4.7:

Lemma 3.3.7. *For all $q \geq 0$, the coefficient system $H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_{(-)}\mathbb{S}); \mathbb{Z})$ lies in $\mathcal{P}$.*

Proposition 3.3.8. *The sequence $B\mathrm{hAut}_*(\vee_0\mathbb{S}) \xrightarrow{\sigma} B\mathrm{hAut}_*(\vee_1\mathbb{S}) \xrightarrow{\sigma} \cdots$ exhibits homological stability.*

Proof. Applying spectral sequence comparison to the maps of spectral sequences

$$\begin{array}{c} H_p(B\mathrm{GL}_n(\mathbb{Z}); H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_n\mathbb{S}); \mathbb{Z})) \Longrightarrow H_{p+q}(B\mathrm{hAut}_*(\vee_n\mathbb{S}); \mathbb{Z}) \\ \big\downarrow \\ H_p(B\mathrm{GL}_{n+1}(\mathbb{Z}); H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_{n+1}\mathbb{S}); \mathbb{Z})) \Longrightarrow H_{p+q}(B\mathrm{hAut}_*(\vee_{n+1}\mathbb{S}); \mathbb{Z}), \end{array}$$

we see it suffices to prove that for all $q \geq 0$ the maps

$$\sigma_*\colon H_*(B\mathrm{GL}_n; H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_n\mathbb{S}); \mathbb{Z})) \longrightarrow H_*(B\mathrm{GL}_{n+1}; H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_{n+1}\mathbb{S}); \mathbb{Z}))$$

are isomorphisms in a range of degrees $*$ tending to ∞ with n. This follows by combining Theorem 3.2.2 with Example 3.2.5 and Lemma 3.3.7. ☐

The previous argument only uses that the coefficient systems $H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_{(-)}\mathbb{S}); \mathbb{Z})$ are polynomial of finite degree, not that they are objectwise finitely generated. This instead is used to prove:

Proposition 3.3.9. *Fixing n, $H_*(B\mathrm{hAut}_*(\vee_n\mathbb{S}); \mathbb{Z})$ is finitely-generated for each $* \geq 0$.*

Proof. This uses once more the spectral sequence

$$H_p(B\mathrm{GL}_n(\mathbb{Z}); H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_n\mathbb{S}); \mathbb{Z})) \Longrightarrow H_{p+q}(B\mathrm{hAut}_*(\vee_n\mathbb{S}); \mathbb{Z}).$$

Recall that if X is a CW-complex with finitely many cells in each dimension, then for any local coefficient system $\mathcal{A}$ which is finitely generated as an abelian group, the homology groups $H_*(X; \mathcal{A})$ are finitely generated in each degree. Since each group $H_q(B\mathrm{hAut}^{\mathrm{id}}_*(\vee_n\mathbb{S}); \mathbb{Z})$ is finitely generated, it suffices to prove that $B\mathrm{GL}_n(\mathbb{Z})$ has the homotopy type of a CW-complex with finitely many cells in each dimension. This is a result of Borel–Serre [BS73, §11.1]. ☐

Proof of Theorem 3.3.5. The space $\Omega_0^\infty K(\mathbb{S})$ is simple, so by a Serre class argument its homotopy groups are finitely generated if and only if its homology groups are. By Waldhausen's work and the group completion theorem, the homology of $\Omega_0^\infty K(\mathbb{S})$ is equal to the stable homology of $B\mathrm{hAut}_*(\vee_n \mathbb{S})$. By Proposition 3.3.8 the map

$$H_*(B\mathrm{hAut}_*(\vee_n \mathbb{S}); \mathbb{Z}) \longrightarrow \operatorname*{colim}_{n \to \infty} H_*(B\mathrm{hAut}_*(\vee_n \mathbb{S}); \mathbb{Z})$$

is an isomorphism in a range tending to ∞ with n. That the right side is finitely generated in all degrees thus follows from Proposition 3.3.9. □

3.4 Exercises

Exercise 3.4.1 (Coning off)**.**

(i) Let $X_\bullet$ be a semi-simplicial set and $\mathsf{simp}(X_\bullet)$ be the poset with objects given by the simplices $X_\bullet$ and $\sigma \leq \tau$ if σ can be obtained from τ be applying face maps. Prove that $||X_\bullet||$ is homeomorphic to $|N(\mathsf{simp}(X_\bullet)|$. (Hint: Barycentric subdivision.)

(ii) Identify $\mathsf{simp}(W_n(\underline{1})_\bullet)$ with the poset $I(\underline{n})$ of ordered non-empty subsets of $\underline{n}$, ordered by order-preserving inclusions.

(iii) For $S_i = \underline{n} \backslash \{i\}$, construct a zigzag of natural transformations between the inclusion $I(S_i) \to I(\underline{n})$ and the constant map $I(S_i) \to I(\underline{n})$ with value (i).

(iv) Conclude that $||W(S_i)_\bullet|| \to ||W(\underline{n})_\bullet||$ is null-homotopic, as claimed in the proof of Proposition 2.2.6.

Exercise 3.4.2 (Simplicial complexes)**.** A *simplicial complex* X is a set V (called *vertices*) and a collection S of unordered finite subsets of V (called *simplices*) satisfying (a) $\{v\} \in S$ for all $v \in V$, (b) if $\sigma \in S$ then any subset of X is in S as well. If $X \in S$ in S has $p+1$ element we call it a *p-simplex*. The geometric realisation $|X|$ is defined taking

$$|X| = \left(\bigsqcup_k \Delta^k \times \{k\text{-simplices of } X\} \right) /\sim$$

where the equivalence relation $\sim$ is similar to that for the geometric realisation of a semi-simplicial space (we leave the details for the reader).

(i) From a simplicial complex X one can extract a semi-simplicial set $X_\bullet^{\mathrm{ord}}$ by taking its p-simplices to be ordered $(p+1)$-element subsets of V whose underlying unordered set is a p-simplex in X. Describe the face maps and verify this is indeed a semi-simplicial set.

(ii) For a finite set V, let Δ^V be the simplicial complex where each finite subset of V is a simplex. Prove that $|\Delta^V|$ is homeomorphic to $\Delta^{\#V-1}$ and $\mathrm{ord}_\bullet(\Delta^V) = W_{\#V}(\underline{1})_\bullet$.

(iv) A *link* of a simplex σ of a simplicial complex X consists of all simplices τ such that $\sigma \cap \tau = \varnothing$ and $\sigma \cup \tau$ is a simplex. Describe how this can be made into a simplicial complex $\mathrm{link}_X(\sigma)$ and show that $\mathrm{link}_{\Delta^V}(Y)$ for a subset $Y \subset V$ is isomorphic to $\Delta^{V \backslash W}$.

(v) A simplicial complex X is said to be *weakly Cohen–Macaulay of dimension d* if (a) $|X|$ is $(d-1)$-connected and (b) for all k-simplices σ and $k \geq 0$, $|\mathrm{link}_X(\sigma)|$ is $(d-k-2)$-connected. Show that Δ^V is weakly Cohen–Macaulay of dimension $\#V - 1$.

(vi) By [RWW17, Proposition 2.14], if X is weakly Cohen–Macaulay of dimension d then $||X^{\mathrm{ord}}_\bullet||$ is $(d-1)$-connected. Use this to prove that $||W_n(\underline{1})_\bullet||$ is $(n-2)$-connected.

Exercise 3.4.3 (An example)**.** Prove that $||W_2(\mathbb{Z}, \mathbb{Z})_\bullet||$ from Example 3.2.5 is path-connected.

Exercise 3.4.4 (Homological stability for hyperoctahedral groups)**.** Let G be the symmetric monoidal groupoid of finite free $\mathbb{Z}/2$-sets, with monoidal structure given by disjoint union.

(i) For $X = \mathbb{Z}/2$, prove that $\mathrm{Aut}(X^{\oplus n}) = \mathbb{Z}/2 \wr \Sigma_n$, the *$n$th hyperoctahedral group*.

(ii) Describe $W_n(\varnothing, X)_\bullet$ and prove that it is homologically $\frac{n-1}{2}$-connected along the lines of Proposition 2.2.6.

(iii) Deduce a homological stability result for hyperoctahedral groups.

Exercise 3.4.5 (A non-injective stabilisation map)**.** Theorem 3.2.2 also generalises to give homological stability with abelian coefficients. You may assume this (with an unspecified range) in this exercise.

(i) Use this to prove that the sequence $B\mathrm{SL}_0(\mathbb{Z}) \xrightarrow{\sigma} B\mathrm{SL}_1(\mathbb{Z}) \xrightarrow{\sigma} \cdots$ exhibits homological stability.

(ii) Use the facts that (a) $\pm\mathrm{id} \subset \mathrm{SL}_2(\mathbb{Z})$ is the center and (b) $\mathrm{SL}_2(\mathbb{Z})/\{\pm\mathrm{id}\} \cong \mathbb{Z}/2 * \mathbb{Z}/3$, to prove that $H_1(B\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/12$.

(iii) Use the fact that $\mathrm{SL}_n(\mathbb{Z})$ is perfect for $n \geq 3$ to prove that the stabilisation map $\sigma_* \colon H_1(B\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z}) \to H_1(B\mathrm{SL}_3(\mathbb{Z}); \mathbb{Z})$ is not injective.

Exercise 3.4.6 (Algebraic K-theory of the integers)**.** Instead of asking about the algebraic K-theory of $\mathbb{S}$, one could ask about that of$\mathbb{Z}$. These are the homotopy groups of a spectrum $K(\mathbb{Z})$ so that $\Omega^\infty K(\mathbb{Z})$ is the group completion of the topological monoid $\bigsqcup_{n \geq 0} B\mathrm{GL}_n(\mathbb{Z})$.

(i) Explain that $\pi_*(K(\mathbb{Z}))$ is finitely generated for $* > 0$ if and only if $H_*(\Omega^\infty_0 K(\mathbb{Z}); \mathbb{Z})$ is finitely generated for $* > 0$.

(ii) Explain that there is an isomorphism $H_*(\Omega^\infty_0 K(\mathbb{Z}); \mathbb{Z}) \cong \mathrm{colim}_{n \to \infty} H_*(B\mathrm{GL}_n(\mathbb{Z}); \mathbb{Z})$.

(iii) Prove that right term in (ii) is finitely generated for all $* > 0$ using Theorem 3.2.2, Example 3.2.5, and the result of Borel–Serre.

Exercise 3.4.7 (Some polynomial coefficients systems)**.** Use Lemma 3.3.6 to prove Lemma 3.3.7.

Chapter 4

More subtle stability phenomena

In this last lecture, we look past homological stability to a pair of more subtle stability phenomena: representation stability and higher-order homological stability. In either case we will do so through an illustrative example, and will not attempt to explain the general theory.

Remark. Other topics of recent interest that we will not discuss is stability phenomena near the cohomological dimension, e.g. [CFP14], and stable stability, e.g. [GRW17].

4.1 Representation stability

Close cousins of the symmetric groups are the configuration spaces of unordered points in a Euclidean space $\mathbb{R}^d$. We shall give the definition for a general topological space X:

Definition 4.1.1. The *configuration space of n unordered points in X* is given by

$$C_n(X) \coloneqq \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}/\Sigma_n,$$

where the elements of Σ_n act by permuting the particles.

By construction this is the quotient by the symmetric group Σ_n of the *configuration space of n ordered points in X* given by

$$\mathrm{Conf}_n(X) \coloneqq \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Lemma 4.1.2.

(i) $\mathrm{Conf}_n(\mathbb{R}^d)$ *is $(d-2)$-connected.*

(ii) There is a $(d-1)$-connected map $C_n(\mathbb{R}^d) \to B\Sigma_n$.

Proof. Part (ii) follows from part (i) by observing that the action of Σ_n on $\mathrm{Conf}_n(\mathbb{R}^d)$ is free and proper, so the quotient by Σ_n is a homotopy quotient (see Exercise 4.3.3 (i)). To prove (i) we prove more generally that $\mathrm{Conf}_n(\mathbb{R}^d \setminus \{k \text{ points}\})$ is $(d-2)$-connected, by induction over n. For $n = 1$, we have $\mathrm{Conf}_1(\mathbb{R}^d \setminus \{k \text{ points}\}) = \mathbb{R}^d \setminus \{k \text{ points}\} \simeq \vee_k S^{d-1}$. The induction step uses the fibration sequences

$$\mathrm{Conf}_{n-1}(\mathbb{R}^d \setminus \{k+1 \text{ points}\}) \longrightarrow \mathrm{Conf}_n(\mathbb{R}^d \setminus \{k \text{ points}\}) \longrightarrow \mathbb{R}^d \setminus \{k \text{ points}\}$$

for $k \geq 1$, with right map obtained by remembering only the location of the first point [FN62]. □

There are stabilisation maps $\sigma\colon C_n(\mathbb{R}^d) \to C_{n+1}(\mathbb{R}^d)$ given informally by adding a new particle far away in the e_1-direction (see Exercise 4.3.3 (ii)), fitting into a homotopy-commutative diagram

$$\begin{array}{ccc} C_n(\mathbb{R}^d) & \xrightarrow{\sigma} & C_{n+1}(\mathbb{R}^d) \\ \downarrow & & \downarrow \\ B\Sigma_n & \xrightarrow{\sigma} & B\Sigma_{n+1}. \end{array}$$

As a consequence, the homological stability result for symmetric groups implies one for the spaces $C_n(\mathbb{R}^d)$ in degrees $* < d-1$. In fact, it is true in all degrees:

Theorem 4.1.3. *The sequence of spaces $C_0(\mathbb{R}^d) \xrightarrow{\sigma} C_1(\mathbb{R}^d) \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely, the map*

$$\sigma_*\colon H_*(C_n(\mathbb{R}^d);\mathbb{Z}) \longrightarrow H_*(C_{n+1}(\mathbb{R}^d);\mathbb{Z})$$

is a surjection for $* \leq \frac{n}{2}$ and an isomorphism for $* \leq \frac{n-1}{2}$.

Remark 4.1.4. The stabilisation maps are always injective, see Exercise 4.3.5.

Remark 4.1.5. Homological stability for unordered configuration spaces and their variants is a well-studied topic. Here is an incomplete list of references: [McD75, Seg79, LS01, Ker05, Chu12, RW13a, BM14, KM15, CP15, KMT16, KM16, EVW16, Knu17, Pal18]. In many cases it is possible to compute the homology of configuration spaces outright [CLM76, Nap03, FT05, Pet17, BG18, Sch19, Pet20, Pag20].

Is something similar true for the ordered configuration spaces $\mathrm{Conf}_n(\mathbb{R}^d)$? The answer is no, because in the lowest degree where the reduced homology can be non-trivial by Lemma 4.1.2 (i), we have

$$H_{d-1}(\mathrm{Conf}_n(\mathbb{R}^d);\mathbb{Q}) \cong \mathbb{Q}^{n-1}.$$

This computation is more easily interpreted when we recall that $\mathrm{Conf}_n(\mathbb{R}^d)$ is a topological space with Σ_n-action and we describe this homology group as a rational Σ_n-representation: it is the kernel of the augmentation $\epsilon\colon \mathbb{Q}[\underline{n}] \to \mathbb{Q}$ (i.e. the *reduced regular representation*). Thus, when we take into account the naturally present group actions, the homology *does* admit a uniform description for n sufficiently large (in fact for all n in this particular case). This notion of stability is known as *representation stability* [CF13, Far14].

Let us now give a precise statement for the rational cohomology of ordered configuration spaces of a manifold M. Doing so uses that the rational representations of Σ_n are classified by partitions of n (or equivalently Young diagrams with n boxes) [FH91, Chapter 4], which we list as decreasing sequences $(i_1, \ldots, i_k)$ of positive integers with $\sum i_k = n$. For example, the trivial representation is (n) and the reduced regular representation is $(n-1,1)$. We can stabilise partitions by adding 1 to the first entry, and can think of this as an operation taking isomorphism classes of rational Σ_n-representations to rational Σ_{n+1}-representations. It may be extended to all isomorphism classes of rational Σ_n-representations by first decomposing these into irreducibles. We say that a sequence $\{V_n\}_{n\geq 0}$ of representations of symmetric groups exhibits *multiplicity stability* if for n sufficiently larger V_{n+1} can be obtained from V_n by this operation. Church proved the cohomology of ordered configuration spaces of manifolds has this property (though he proved much more):

Theorem 4.1.6 ([Chu12]). *Suppose M is a finite type manifold of dimension ≥ 2. Then for all $d \geq 0$ the representations $\{H^d(\mathrm{Conf}_n(M);\mathbb{Q})\}_{n\geq 0}$ exhibit multiplicity stability.*

Remark 4.1.7. By the universal coefficient theorem and self-duality of rational representation of symmetric groups, we could also have phrased this in terms of homology. Here is a reason to prefer cohomology: since M may be closed there are no stabilisation maps adding a point far away, like we used for $\mathbb{R}^d$. Rather, in contrast with the case of unordered configuration spaces there are now maps which forget a point. On cohomology these induce maps $H^*(\mathrm{Conf}_{n-1}(M);\mathbb{Q}) \to H^*(\mathrm{Conf}_n(M);\mathbb{Q})$ resembling stabilisation maps.

In general, representation stability is more subtle because for other sequences of groups or more complicated coefficients it is not possible to classify all representations and give them consistent names. This occurs even in such a simple case as the homology of configuration spaces with *integer coefficients*. In general, one needs to study the representation theory of certain categories which combine the groups acting and (de)stabilisation maps, and phrase the representation stability in terms of finite generation or presentation. For configuration spaces, the relevant category is FI (objects are non-empty finite sets and morphisms are injective functions) and the relevant objects are functors $\mathsf{FI} \to \mathsf{Ab}$. These are called FI-*modules* and, assuming that all homology groups are finitely generated, the analogue of exhibiting homological stability is being finitely-generated as a FI-module. The collection $\{H^d(\mathrm{Conf}_n(M);\mathbb{Z})\}_{n\geq 0}$ forms an FI-module using the forgetful maps discussed in the previous remark. That this is a finitely-presented FI-module for each $d \geq 0$ is proven in [CEF15]. For a proof closer to the ones in the previous lectures, see [MW19, MW20].

Remark 4.1.8. You can interpret ordinary homological stability as representation stability with trivial group action. In this case, the categorical representation theory is that of the poset $(\mathbb{N}, \leq)$. The category of functors $(\mathbb{N}, \leq) \to \mathsf{Ab}$ is equivalent to the category of graded $\mathbb{Z}[x]$-modules, and a degreewise finitely-generated graded $\mathbb{Z}[x]$-module is finitely generated if and only if it is finitely presented (because $\mathbb{Z}[x]$ is a noetherian ring) if and only if it is eventually constant.

Example 4.1.9. There is a lot of literature on representation stability for ordered configuration spaces [AAB15, HR17, KM18, FW18, MW19, MW20, Ram20] and their variants [JRW19, Gad17, Bib18]. Here is an incomplete list of classifying spaces of groups which exhibit representation stability and equally incomplete references:

- Groups related to mapping class groups [JR11, JR15, JRMD18, JR19] and Torelli groups [BHD12, Pat18, MPW19].
- Groups related to linear groups [Put15, PS17, MPW19].
- Groups related to automorphism groups of free groups [DP17].

4.2 Higher-order homological stability

The mapping class group $\Gamma_{g,1}$ of a surface $\Sigma_{g,1}$ with genus g and one boundary component, is the group of isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ fixing the boundary pointwise. This is closely related to algebraic geometry as $\mathcal{M}_{g,1}$, the orbifold moduli space of curves with marked point and non-zero tangent vector, is homotopy equivalent to $B\Gamma_{g,1}$. Taking the boundary connected sum of $\Sigma_{g,1}$ with $\Sigma_{1,1}$ yields a surface diffeomorphic to $\Sigma_{g+1,1}$ and extending a diffeomorphism of $\Sigma_{g,1}$ by the identity to $\Sigma_{1,1}$ thus gives a stabilisation map

$$\sigma\colon B\Gamma_{g,1} \longrightarrow B\Gamma_{g+1,1}.$$

We mentioned in passing in the previous lecture that this sequence of classifying spaces exhibits homological stability: the map

$$\sigma_*\colon H_*(B\Gamma_{g,1};\mathbb{Z}) \longrightarrow H_*(B\Gamma_{g+1,1};\mathbb{Z})$$

is a surjection for $* \leq \frac{2g}{3}$ and an isomorphism for $* \leq \frac{2g-2}{3}$. This is the optimal range, proven in [GKRW19], but a proof of a nearly optimal result is explained in [Wah13]. A similar result with worse range can be obtained from theorem of [RWW17] that we explained in the previous lecture. (In particular, we note for later use that the mapping class groups assemble to a braided monoidal groupoid.)

This result informally says that the homology groups $H_d(B\Gamma_{g,1};\mathbb{Z})$ are independent of g when g is sufficiently large. In analogy with calculus, we could think of this as a function which is eventually constant, or equivalently as a function whose first derivative eventually vanishes. Higher-order homological stability is then analogous to its higher derivatives eventually vanishing.

To convert this analogy into a precise statement, we observe that the role of the first derivative can be played by the relative homology groups

$$H_*(B\Gamma_{g+1,1}, B\Gamma_{g,1};\mathbb{Z}),$$

which of course depend on the maps σ (even though the notation unfortunately does not reflect this). Their eventual vanishing is equivalent to homological stability, as the long exact sequence of a pairs implies that the induced map $\sigma_*\colon H_*(B\Gamma_{g,1};\mathbb{Z}) \to H_*(B\Gamma_{g+1,1};\mathbb{Z})$ is a surjection for $* \leq d$ and an isomorphism for $* < d$ if and only if $H_*(B\Gamma_{g+1,1}, B\Gamma_{g,1};\mathbb{Z})$ for $* \leq d$.

To formulate higher-order homological stability, we need to construct higher-order stabilisation maps. This is not straightforward—in this example they will not be unique(!)—but it turns out that there are maps

$$\varphi_*\colon H_d(B\Gamma_{g,1}, B\Gamma_{g-1,1};\mathbb{Z}) \longrightarrow H_{d+2}(B\Gamma_{g+3,1}, B\Gamma_{g+2,1};\mathbb{Z}).$$

The following is *secondary homological stability* for mapping class groups:

Theorem 4.2.1 ([GKRW19])**.** *The maps φ_* are surjections for $d \leq \frac{3g}{4}$ and isomorphisms for $d \leq \frac{3g-4}{4}$.*

Remark 4.2.2. Sometimes the secondary stabilisation maps are both isomorphisms in a range and zero, and one obtains an improved homological stability range.

Example 4.2.3. Few higher-order stability results have been proven at the time of writing these notes: [GKRW18, GKRW19, MW19, MPP19, GKRW20, Ho20, Him21]. One could ask whether such higher-order stability phenomena exist in any of the examples where homological stability is known.

4.2.1 The strategy for proving Theorem 4.2.1

We gave the "usual" homological stability result for symmetric groups, and isolated the crucial ingredients: given a braided monoidal groupoid G with some mild properties and objects $A, X \in \mathsf{G}$, we extracted a semi-simplicial set $W_n(A, X)_\bullet$ of "destabilisations" and given that these are homologically highly connected, homological stability follows using a spectral sequence argument. Let me now give a brief outline of the proof of Theorem 4.2.1, as it contains some useful ideas.

The input for the argument as in [RWW17] is a symmetric monoidal, or more generally braided monoidal, groupoid G. This structure on G endows $|N\mathsf{G}|$ with the structure of an E_2-algebra. In homotopy theory algebras can be not only coherently associative (E_1) or coherently commutative (E_∞), but there are also intermediate notions of commutativity: for an E_k-algebra the space of multiplications is a $(k-1)$-sphere, see Example 4.2.5. For $k = 1$ this means it is disconnected—there is a left and a right multiplication, and these can be quite different—and for $k = \infty$ this means it is contractible—from a homotopical viewpoint there is just a single multiplication.

More precisely, these are encoded by the operads of little k-discs [May72]. An *operad* $\mathcal{O}$ has Σ_r-spaces $\mathcal{O}(r)$ of r-ary operations, with a unit $1 \in \mathcal{O}(1)$, and compositions $\mathcal{O}(r) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_r) \to \mathcal{O}(k_1 + \cdots + k_r)$. These have two satisfy suitable equivariance, unitality, and associativity axioms. An $\mathcal{O}$-algebra is a space A with maps

$$\mathcal{O}(r) \times A^r \longrightarrow A,$$

which you should think of as defining operations on r-tuples of elements in A, indexed by the points of $\mathcal{O}(r)$. These should similarly satisfy suitable equivariance, unitality, and associativity axioms.

For the little k-discs operad, $E_k(r)$ is the space of k-tuples of rectilinear embeddings $D^k \hookrightarrow D^k$ with disjoint interior, Σ_r permuting the discs in the domain. The element $1 \in E_k(1)$ is the identity embedding, and composition is given by composition of embeddings.

Example 4.2.4. E_1 is homotopy equivalent to the associative operad, whose r-ary operations are the linear orders of $\underline{r}$. Algebras over the latter are associative algebras. E_∞ is homotopy equivalent to commutative operad, whose r-ary operations are contractible. Algebras over the latter are commutative algebras.

Example 4.2.5. The 2-ary operations of the E_k-operad are encoded by the space $E_k(2)$. This is the space of two k-discs in a k-discs, and is homotopy equivalent to S^{k-1}.

Since the mapping class groups assemble to a braided monoidal groupoid G, $|N\mathsf{G}| \simeq \bigsqcup_{g\geq 0} B\Gamma_{g,1}$ admits the structure of an E_2-algebra (see Exercise 4.3.7 for a more geometric approach).

In the homological stability arguments we gave before, we used from this E_2-structure just the multiplication on the right by a point representing $X \in |N\mathsf{G}|$ and some coherence from the braiding. Theorem 4.2.1 will need the full E_2-algebra structure. This is because it will build an approximation $\mathbf{A}$ to the E_2-algebra $\mathbf{R} := |N\mathsf{G}|$ from free E_2-algebras. The free E_2-algebra functor Free^{E_2} is the left adjoint to the forgetful functor from E_2-algebras to spaces, and the underlying space of its values are of the form

$$\mathrm{Free}^{E_2}(X) \simeq \bigsqcup_{r\geq 0} E_2(r) \times_{\Sigma_r} X^r.$$

The E_2-algebra $\mathbf{A}$ will be a good approximation in a sense relevant to Theorem 4.2.1: it captures the homological stability properties in a range. One should compare this to how a CW-approximation of a skeleton of a topological space captures its homology in a range. The proof of Theorem 4.2.1 has the following steps (see [GKRW19] for details):

(1) *Understand how many free E_2-algebras one needs to build* $\mathbf{A}$ *such that it is a good enough approximation.* This requires two inputs: an understanding of $H_d(B\Gamma_{g,1};\mathbb{Z})$ for low d and g, and a connectivity result for certain semi-simplicial set. Unlike the semi-simplicial set $W_n(A,X)_\bullet$, these E_1*-splitting semi-simplicial sets* $S^{E_1}(g)_\bullet$ will be given by decompositions of a surface into boundary connected summands of lower genus.

(2) *Build the small cellular E_2-algebra approximation* $\mathbf{A}$. This uses techniques similar to CW-approximation for spaces, but in the category of E_2-algebras; it will only have three "E_2-cells." These are pushouts of the form

$$\begin{array}{ccc} \mathrm{Free}^{E_2}(S^{k-1}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Free}^{E_2}(D^k) & \longrightarrow & Y \end{array}$$

in the category of E_2-algebras.

(3) *Construct the primary and secondary stabilisation maps.* The primary stabilisation maps will be σ as given above, and come from the E_2-algebra structure. However, the secondary stabilisation maps are constructed by obstruction theory.

(4) *Prove Theorem 4.2.1.* In steps (1) and (2), we made sure that $\mathbf{A}$ has the same homological stability properties as $\mathbf{R}$. In particular, $\mathbf{R}$ has (secondary) homological stability in a range when $\mathbf{A}$ does. This reduces the proof to a computation in $\mathbf{A}$. This is doable because F. Cohen completely computed the homology of free E_2-algebras in terms of certain homology operations [CLM76].

As a slogan, we call this a "multiplicative" approach to homological stability rather than an "additive" one, because it uses the full E_2-algebra structure instead of just the stabilisation maps extracted from it. In [GKRW18, GKRW20] we applied the same techniques to general linear groups.

4.3 Exercises

Exercise 4.3.1 (Interpretations of $C_n(\mathbb{C})$)**.**

(i) Prove that $C_n(\mathbb{C})$ is an Eilenberg–Mac Lane space. Its fundamental group is the *nth braid group* Br_n, so Theorem 4.1.3 says that braid groups exhibit homological stability.

(ii) Prove that $C_n(\mathbb{C})$ is homeomorphic to the space of monic polynomials of degree $\leq n$ with complex coefficients and distinct roots.

Remark 4.3.2. Exercise 4.3.1 (ii) led to an interesting application of the homology of configuration spaces to the complexity of root-finding algorithms [Sma87].

Exercise 4.3.3 (Maps between configuration spaces)**.**

(i) Construct a map $C_n(\mathbb{R}^d) \to C_n(\mathbb{R}^{d+1})$ and prove that $\mathrm{colim}_{d\to\infty} C_n(\mathbb{R}^d) \simeq B\Sigma_n$.

(ii) Construct a map $\sigma\colon C_n(\mathbb{R}^d) \to C_{n+1}(\mathbb{R}^d)$ fitting in a homotopy-commutative diagram

$$\begin{array}{ccc} C_n(\mathbb{R}^d) & \xrightarrow{\sigma} & C_{n+1}(\mathbb{R}^d) \\ \downarrow & & \downarrow \\ B\Sigma_n & \xrightarrow{\sigma} & B\Sigma_{n+1}. \end{array}$$

(iii) What conditions do we need to impose on a manifold M to be able to construct a similar map $\sigma\colon C_n(M) \to C_{n+1}(M)$?

Exercise 4.3.4 (Abelianisation of braid groups)**.** Use Exercise 4.3.1 (i) to compute the abelianisations of the braid groups.

Exercise 4.3.5 (Applying Dold's Lemma to unordered configuration spaces)**.**

(i) Construct "transfer maps" $\tau_{n,k}\colon H_*(C_n(\mathbb{R}^d);\mathbb{Z}) \to H_*(C_{n-k}(\mathbb{R}^d);\mathbb{Z})$ by summing over all ways of deleting k of the points.

(ii) Apply Dold's Lemma (Exercise 1.3.6) to prove that the stabilisation maps $\sigma_*\colon H_*(C_n(\mathbb{R}^d);\mathbb{Z}) \to H_*(C_{n+1}(\mathbb{R}^d);\mathbb{Z})$ are injective.

(iii) Generalise these results to any connected open manifold M replacing $\mathbb{R}^d$.

Exercise 4.3.6 (Betti numbers of configuration spaces of closed manifolds)**.**

(i) Use the fibration sequence

$$\mathrm{Conf}_n(M) \longrightarrow C_n(M) \longrightarrow B\Sigma_n$$

to establish an isomorphism $H^*(C_n(M);\mathbb{Q}) \cong H^*(\mathrm{Conf}_n(M);\mathbb{Q})^{\Sigma_n}$.

(ii) Deduce from Theorem 4.1.6 that for any connected finite type manifold M of dimension ≥ 2, the ith rational Betti number of $C_n(M)$ is independent of n for $n \gg 0$.

Exercise 4.3.7. Let $\mathcal{M}(\Sigma_{g,1})$ be the colimit as $n \to \infty$ of the space of surfaces in $D^2 \times \mathbb{R}^n$ which are diffeomorphic to $\Sigma_{g,1}$ and coincide with $D^2 \times \{0\}$ near $\partial D^2 \times \mathbb{R}^n$. (For the topology on this space, see [GRW10]). It is a fact that

$$B\Gamma_{g,1} \simeq \mathcal{M}(\Sigma_{g,1}).$$

Endow $\bigsqcup_{g\geq 0} \mathcal{M}(\Sigma_{g,1})$ with the structure of an E_2-algebra.

Exercise 4.3.8 (The E_2-operad and braid groups)**.**

(i) Prove that $E_2(r) \simeq \mathrm{Conf}_2(r)$, and hence an Eilenberg–Mac Lane space. Its fundamental group is the *rth pure braid group*, which is the kernel of the permutation homomorphism $\mathrm{Br}_n \to \Sigma_n$ recording how the strands of a braid permute the endpoints.

(ii) Prove that the free E_2-algebra $\mathrm{Free}^{E_2}(*)$ is homotopy equivalent to $\bigsqcup_{r\geq 0} C_n(r)$.

(iii) Compute $H_1(\mathrm{Free}^{E_2}(*); \mathbb{Z})$ using Exercise 4.3.4.

Exercise 4.3.9 (Stabilisation maps which are never isomorphisms)**.** With rational coefficients the ranges in Theorem 4.2.1 can be improved to a surjection for $d \leq \frac{4g+1}{5}$ and an isomorphism for $d \leq \frac{4g-4}{5}$. It is a fact that $H_4(B\Gamma_{6,1}, B\Gamma_{5,1}; \mathbb{Q}) \neq 0$ (using relations in the tautological ring). Prove that $H_{4+2k}(B\Gamma_{6+3k,1}, B\Gamma_{5+3k,1}; \mathbb{Q})$ is non-zero for all $k \geq 0$.

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