HOMOTOPY EXCISION FOR $(a+1)$-ADS

ABSTRACT. This a summary of "*Preliminaries B. $(a+1)$-ads and connectivity*" in T. Goodwillie's thesis.

1. Warm-up: triads

A triad $\mathbf{X} = (X; X_1, X_2)$ is $\mathbf{k} = (k_1, k_2, k_{12})$-connected if the pairs $(X_1, X_1 \cap X_2), (X_2, X_1 \cap X_2), (X, X_1 \cup X_2)$ are k_1, k_2 and k_{12}-connected respectively, with $k_{12} \leq k_1 + k_2$.

Theorem 1.1. *If a CW triad $\boldsymbol{X}$ is $\boldsymbol{k}$-connected, then $\pi_s(\boldsymbol{X}) = 0$ for all $2 \leq s \leq k_{12}$.*

Remark. The difference between this theorem and Blakers–Massey's is that the in the latter it is assumed that $X = X_1 \cup X_2$, in which case one takes $k_{12} = k_1 + k_2$.

Proof. Consider the following subsets of $I = \{1, 2\}$:

$$T_1 = I, \quad T_2 = \{1\}, \quad T_3 = \{2\}, \quad T_4 = \emptyset.$$

Set

$$\mathcal{S}^i = \{T_1, \ldots, T_i\} \subset 2^I, \quad \text{and} \quad Y^i = \mathcal{S}^i_X$$

Furthermore define the following triads

$$\mathbf{Y}^i = (Y^i; Y^i \cap X_1, Y^i \cap X_2).$$

Explicitly, these are

$$\begin{aligned} \mathbf{Y}^1 &= (X_1 \cap X_2; X_1 \cap X_2, X_1 \cap X_2) \\ \mathbf{Y}^2 &= (X_1; X_1, X_1 \cap X_2) \\ \mathbf{Y}^3 &= (X_1 \cup X_2; X_1, X_2) \\ \mathbf{Y}^4 &= (X; X_1, X_2) \end{aligned}$$

Observe that by Blakers-Massey $\pi_s(\mathbf{Y}^3) = 0$ for all $2 \leq s \leq k_{12}$. We will use this in order to show that the homotopy groups of $\mathbf{Y}^4$ vanish in the same range of degrees. Define the following triad

$$\mathbf{Z} = (X; X_1, X_1 \cup X_2).$$

It turns out that there is a long exact sequence

$$\cdots \to \pi_s(\mathbf{Y}^3) \to \pi_s(\mathbf{Y}^4) \to \pi_s(\mathbf{Z}) \to \cdots \to \pi_2(\mathbf{Y}^3) \to \pi_2(\mathbf{Y}^4) \to \pi_2(\mathbf{Z})$$

We want to show that $\pi_s(\mathbf{Z}) = 0$ for all $s \leq k_{12}$. For this we define a dyad

$$\mathbf{W} = (X, X_1 \cup X_2)$$

and note that since $X_1 \subset X_1 \cup X_2$, we can drop the space X_1 from $\mathbf{Z}$ after taking homotopy groups, that is $\pi_s(\mathbf{W}) \cong \pi_s(\mathbf{Z})$ for $s \geq 2$. Now by assumption the homotopy groups of this dyad vanish up to degrees k_{12}. This completes the proof. □

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2. The general case: homotopy excision for topological $(a+1)$-ads

A *topological* $(a+1)$ is a space X together with a subspaces $X_1, \ldots, X_a$. We typically denote it by $\mathbf{X} = (X; X_1, \ldots, X_a)$. If every X_i is open in X, the $\mathbf{X}$ is called an open $(a+1)$. If X is a CW-complex and each X_i is a subcomplex, then $\mathbf{X}$ is called a CW-$(a+1)$-ad. If a basepoint $* \in \bigcap_{i=1}^{a} X_i$ has been chosen, then $\mathbf{X}$ is called a pointed $(a+1)$-ad. A map $f : \mathbf{X} \to \mathbf{Y}$ between $(a+1)$-ads is a map $f : X \to Y$ such that $f(X_i) \subset Y_i$.

2.1. **Notation.** For every subset $S \subset I = \{1, \ldots, a\}$ we define

$$X_S = \begin{cases} \bigcap_{j \in S} X_j & \text{if } S \neq \emptyset \\ X & \text{if } S = \emptyset \end{cases}$$

and

$$X^S = \bigcup_{j \in S} X_j.$$

For Ω a subset of 2^I (the power set of I) we let

$$\Omega_X = \bigcup_{S \in \Omega} X_S.$$

2.2. **Homotopy groups.** In order to define the homotopy groups of an $(a+1)$-ad $\mathbf{X}$ we consider the following $(a+2)$-ad

$$\mathbf{I}^a = \left(I^a; F_1^1, \ldots, F_a^1, \bigcup_{i=1}^{a} F_i^0 \right)$$

where $\epsilon = 0, 1$ and F_j^ϵ denotes the face of the cube consisting of points whose j-th coordinate equals ϵ.

Let $\mathbf{X} = (X; X_1, \ldots, X_a, *)$ be a pointed $(a+1)$-ad. The homotopy groups/sets of $\mathbf{X}$ are defined as

$$\pi_s(\mathbf{X}) := \pi_{s-a}(\mathrm{Map}(\mathbf{I}^a, \mathbf{X}), \mathrm{const}).$$

This makes sense only for $s \geq a$ and they are sets when $s = a$, groups when $s = a+1$ and abelian groups when $s \geq a+2$. Here are some useful remarks:

- Note that the homotopy groups of a 2-ad $(X; X_1)$ as defined above coincide with those of the pair (X, X_1).
- "absolute compression": if $X_i = X$ for some i, then $\pi_s(\mathbf{X}) = 0$ for all $s \geq a$.
- "relative compression": if $X_j \subset X_i$ for some $i \neq j$ then $\pi_s(\mathbf{X}) = \pi_s(X; X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_a)$.
- Define
$$D_j^1(\mathbf{X}) = (X; X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_a)$$
and
$$D_j^0(\mathbf{X}) = (X_j; X_1 \cap X_j, \ldots, X_{j-1} \cap X_j, X_{j+1} \cap X_j, \ldots, X_a \cap X_j)$$
then there is a long exact sequence
$$\cdots \to \pi_{s+1}(\mathbf{X}) \to \pi_s(D_j^0(\mathbf{X})) \to \pi_s(D_j^1(\mathbf{X})) \to \pi_s(\mathbf{X}) \to \cdots$$
- Let $I = \{1, \ldots, a\}$. If we define $H_*(\mathbf{X}) = H_*(X, X^I)$ then there is an analog long exact sequence for homology groups which can be easily derived from the LES of the triple $(X, X^I, X^{I \setminus \{j\}})$ and excision.

2.3. **Connectivity.** Let $a \geq 1$ and $\mathbf{k} = \{k_T\}_{T \subset I} \subset \mathbb{N} \cup \{0\}$, such that if $T \subset \bigcup_i T_i$ then $k_T \leq \sum_i k_i$. An $(a+1)$-ad $\mathbf{X}$ is $\mathbf{k}$-connected if for every nonempty subset $T \subset I$ the pair

$$(X_{I-T}, X_{I-T} \cap X^T)$$

is k_T-connected.

2.4. **Homotopy excision for $(a+1)$-ads.** Here is the main theorem.

Theorem 2.1 (Homotopy excision)**.** *If an open or CW $(a+1)$-ad $\boldsymbol{X}$ is $\boldsymbol{k}$-connected and $k_T \geq 2$ for all nonempty $T \subset I$, then $\pi_s(\boldsymbol{X}) = 0$ for all $a \leq s \leq k_I$ and for all basepoints in X_I.*

Remark 2.2.

(1) The statement for open ads follows from the statement for CW-ads using cellular approximation.
(2) If in addition $\mathbf{X}$ is a complete $(a+1)$-ad, that is

$$X = \bigcup_{i=j}^{a} X_{I\setminus\{j\}}$$

then homotopy excision holds. This is a theorem of Barratt-Whitehead[1].
(3) Completeness implies that $X_{I\setminus T} \subset X^T$ for all T with more than element. So in checking that a complete $(a+1)$-ad has certain connectivity, one has to only analyze the pairs of the form $(X_{I\setminus\{j\}}, X_I)$.
(4) It is important to have this general form of homotopy excision for $(a+1)$-ads that are open and not complete. This is because we want to apply this theorem to an $(a+1)$-ad

$$(C(P,N); C(P, N-Q_1), \ldots, C(P, N-Q_a))$$

made out of concordance embedding spaces, with P, and N compact manifolds and $Q_i \subset N$ a compact submanifolds.

Proof of homotopy excision. The proof goes by induction on a. The base case, $a = 1$, is trivial because CW 2-ads and CW pairs are the same thing and the connectivity of a pair is defined in terms of the vanishing of its homotopy groups.

Now assume that Homotopy Excision holds for all $(i+1)$-ads with $1 \leq i < a$ and let $\mathbf{X}$ and $\mathbf{k}$ be as in the statement of the theorem.

Let $T_1, T_2, \ldots, T_{2^a}$ be the collection of all subsets of $I = \{1, \ldots, a\}$, listed in a way such that

$$1 \leq i \leq j \leq 2^a \Rightarrow |T_i| \geq |T_j|.$$

Let $\Omega^i = \{T_1, \ldots, T_i\} \subset 2^I$. With these subsets of the power set of I we define the following $(a+1)$-ads

$$\mathbf{Y}^i = (Y^i; Y^i_1, \ldots, Y^i_a) = (\Omega^i_X; \Omega^i_X \cap X_1, \ldots, \Omega^i_X \cap X_a).$$

Observe that $\mathbf{Y}^{2^a} = \mathbf{X}$, so if we prove that $\pi_s(\mathbf{Y}^i) = 0$ for all $a \leq s \leq k_I$ and for all $a+1 \leq i \leq 2^a$ we will be done. We will do this by induction on i.

The base case is $i = a+1$. By the way we are listing the subsets T_i, we have that T_1 must be all of I, and the subsets $T_2, \ldots, T_{a+1}$ must have $a-1$ elements, so they are of the form $I - \{j\}$ for some $j \in I$. It follows that

$$Y^{a+1} = \bigcup_{j=1}^{a} Y^{a+1}_{I-\{j\}}$$

and so it is a complete $(a+1)$-ad. Furthermore we have that $(Y^{a+1}_{I-\{j\}}, Y^{a+1}_I) = (X_{I-\{j\}}, X)$, so these pairs are k_j-connected by hypothesis. Homotopy excision follows in this case from Barratt-Whitehead's theorem (see the remark above.)

[1]Actually, I think the vanishing of the homotopy groups for complete ads was announced by Toda. The point of Barratt-Whitehead was to compute the first nontrivial homotopy group in terms of the homotopy groups of the other terms.

Let us now assume that $a+1 < i \leq 2^a$ and that $\pi_s(\mathbf{Y}^{i-1}) = 0$ for all $a \leq s \leq k_I$. We want to show that $\pi_s(\mathbf{Y}^i) = 0$ in the same range of degrees. We will show this in two steps:

- **step 1:** we will define an $(a+1)$-ad $\mathbf{Z}$ such that there is a long exact sequence
$$\cdots \to \pi_s(\mathbf{Y}^{i-1}) \to \pi_s(\mathbf{Y}^i) \to \pi_s(\mathbf{Z}) \to \cdots \to \pi_2(\mathbf{Y}^{i-1}) \to \pi_2(\mathbf{Y}^i) \to \pi_2(\mathbf{Z})$$
- **step 2:** we will then show that the homotopy groups of $\mathbf{Z}$ vanish in the desired range. Combined with the induction hypothesis the result will follow.

The first step is not too bad: firstly, since $i \geq a+1$, then $|T_i| \leq |T_{a+2}|$, but T_{a+2} must have $a-2$ elements. So we have
$$0 \leq m := |T_i| \leq a-2.$$

Lemma 2.3. *For all $j > m$,*
$$Y^i_j = Y^i \cap X_j \subset Y^{i-1} \subset Y^i.$$

Proof. This follows directly from the definitions. Only note that since $j > m$, $Y^i_j \subset Y^i \cap X^{I-T_i}$, and that $Y^i = Y^{i-1} \cup X_{T_i}$. □

The desired $(a+1)$-ad is
$$\mathbf{Z} = (Y^i; Y^i_1, \ldots, Y^i_{a-1}, Y^{i-1}),$$
and one can check by hand that the above sequence is exact.

The second step is more difficult. We want to show that the homotopy groups of $\mathbf{Z}$ vanish up to degree k_I. The first observation is that because of the previous Lemma and the relative compression property above, the homotopy groups of $\mathbf{Z}$ are isomorphic to those of the $(m+2)$-ad
$$\mathbf{W} = (W; W_1, \ldots, W_{m+1}) = (Y^i, Y^i_1, \ldots, Y^i_m, Y^{i-1}).$$
So we will show that $\pi_s(\mathbf{W}) = 0$ for all $m+1 \leq s \leq k_I$. Recall that we are still under the induction hypothesis, namely that homotopy excision holds for $(i+1)$-ads with $i < a$. In particular, as $m+1 < a$ we can apply the induction hypothesis to $\mathbf{W}$. So to prove the vanishing of the homotopy groups, it suffices to show that the pairs
$$(W_{I'-T'}, W_{I'-T'} \cap W^{T'})$$
are $k'_{T'}$-connected for $T' \subset \{1, \ldots, m+1\}$, where $k'_{T'} = k_{T'}$ if $T' \subset \{1, \ldots, m\}$ and $k_{T\cup\{m+1\}} = k'_{T\cup\{m+1,\ldots,a\}}$ if $T \subset \{1, \ldots, m\}$.

Let T' be a nonempty subset of $I' = \{1, \ldots, m+1\}$. Then there exists a subset $\Omega \subset 2^I$ such that
$$W_{I'-T'} = \Omega_X$$
(for example $W_j := \{T_1 \cup \{j\}, \ldots, T_i \cup \{j\}\}_X$).

There are two cases:

Case A: $T' \subset \{1, \ldots, m\}$. In this case we have
$$(W_{I'-T'}, W_{I'-T'} \cap W^{T'})(\Omega_X, \Omega_X \cap X^{T'}).$$
This is because $W^{T'} = \Omega^i_X \cap X^{T'} = X^{T'}$.

Case B: $T' = T \cup \{m+1\}$ and $T \subset \{1, \ldots, m\}$. Then
$$(W_{I'-T'}, W_{I'-T'} \cap W^{T'}) = (\Omega_X, \Omega_X \cap X^{T\cup\{m+1,\ldots a\}})$$

This is because

$$\begin{aligned}W^{T'} &= W^T \cup W^{m+1}\\ &= X^T \cup W_{m+1}\\ &= X^T \cup Y^{i-1}\\ &= X^T \cup (Y^i \cap X^{I-T_i})\\ &= X^T \cup (Y^i \cap X^{\{m+1,\dots,a\}})\\ &= X^T \cup (W \cap X^{\{1,\dots,a\}})\\ &= (X^T \cup W) \cap X^{T\cup\{m+1,\dots,a\}}\\ &= X^{T\cup\{m+1,\dots,a\}}.\end{aligned}$$

Cases A and B follow from the following more general lemma which will conclude the proof the homotopy excision theorem..

Lemma 2.4. *Let $\boldsymbol{X}$ and $\boldsymbol{k}$ be as in the statement of the homotopy excision theorem. Then for all $\Omega \subset 2^I$ and nonempty $T \subset I$ the pair $(\Omega_X, \Omega_X \cap X^T)$ is k_T-connected.*

Proof idea: Without lost of generality one assumes that X is 1-connected. Otherwise one passes to universal covers of the components and take preimages of the subspaces.

The proof is by induction on Ω_X with respect to inclusion. The base case $\Omega_X = X_I$ is trivial as X_I is minimal.

Suppose now that for all $\Omega'_X \subsetneq \Omega_X$, the pair $(\Omega'_X, \Omega'_X \cap X^T)$ is k_T-connected.

There are two cases:

Case α: We can write Ω_X as $\Omega_X = \Omega 1_X \cup \Omega 2_X$ with Ωi_X proper subspaces $(i = 1, 2)$. Let $\Omega 3_X = \Omega 1_X \cap \Omega 2_X$, and define the 4-ad:

$$\mathbf{K} = (\Omega_X; \Omega 1_X, \Omega 2_X, \Omega_X \cap X^T).$$

It is easy to see using excision that

$$H_*(D^0_3(\mathbf{K}) = 0 = H_*(D^1_3(\mathbf{K})$$

which implies $H_*(\mathbf{K}) = 0$ (recall the notation and LES from Section 2.2 above).

On the other hand, by induction on Ω_X we have that the pairs $(\Omega i_X, \Omega i_X \cap X^T)$ $(i = 1, 2, 3)$ are k_T-connected, so their homology vanishes up to that degree. Putting all this together and using Mayer-Vietoris for the pair $(\Omega_X, \Omega_X \cap X^T)$ we obtain that $H_s(\Omega_X, \Omega_X \cap X^T) = 0$ for $s \leq k_T$. Thus, by the relative Hurewicz theorem, the proof in this case is completed if we show that the pair $(\Omega_X, \Omega_X \cap X^T)$ is 1-connected. This is a consequence of the following claim:

Claim: For any nonempty subsets $\Omega, \Omega' \subset 2^I$ such that $\Omega'_X \subset \Omega_X$, the pair (Ω_X, Ω'_X) is 2-connected.

To prove this we use induction with respect inclusion again. The base case is trivial again. Assume the claim for smaller subsets and pairs. Take Ω'_X maximal in Ω_X. Recall that we are in Case α, where $\Omega_X = \Omega 1_X \cup \Omega 2_X$. By maximality of Ω'_X we have that $\Omega_X = \Omega'_X \cup \Omega i_X$ for some $i = 1, 2$. Then

$$(\Omega_X, \Omega'_X) = (\Omega'_X \cup \Omega i_X, \Omega'_X)$$

The latter is 2-connected if $(\Omega'_X, \Omega'_X \cap \Omega i_X)$ is 2-connected ("connectivity of pairs is preserved by pushout"), which is true by the induction hypothesis. If we don't have maximality, ie $\Omega'_X \subsetneq \Omega''_X \subset \Omega_X$, with Ω''_X maximal, we have that the first pair is 2-connected by induction and the second too by the previous argument.

Case β: Ω_X cannot be written as a union of smaller subsets. Then $\Omega_X = X_S$ for some $S \subset I$, and

$$(\Omega_X, \Omega_X \cap X^T) = (X_S, X_S \cap X^T).$$

If $S \cap T \neq \emptyset$ then $X_S \cap X^T = X_S$ and there is nothing to show.

If $S \cap T = \emptyset$ then we have two inclusions

$$X_S \cap X^T \subset X_S \cap X^{I-S} \subset X_S.$$

The first inclusion gives a k_T-connected pair (by induction), and the second inclusion gives us, by hypothesis, a k_{I-S}-connected pair (and so k_T-connected because $T \subset I - S$). This concludes the proof of the Claim and of the Lemma... □

and of the homotopy excision theorem. □