POSETS OF DECOMPOSITIONS

ALEXANDER KUPERS

Abstract. We give a short proof that the poset of unordered proper direct sum decompositions of an n-dimensional vector space is homotopy equivalent a wedge of $(n-2)$-spheres.

1. Introduction

In a recent MathOverflow post, Inna Zakharevich asked a question about the following poset of direct sum decompositions. For a finite-dimensional vector space V over a field $\mathbb{F}$, let $\mathsf{D}(V)$ be the poset of unordered collections $\overline{W} = \{W_0, \ldots, W_p\}$ of proper subspaces of V such that the natural map $W_0 \oplus \cdots \oplus W_p \to V$ is an isomorphism, ordered by

$$\overline{W} = \{W_0, \ldots, W_p\} \leq \overline{W}' = \{W'_0, \ldots, W'_{p'}\} \qquad \text{if each } W'_i \text{ is contained in some } W_j.$$

In this case we say $\overline{W}'$ is a refinement of $\overline{W}$ and in particular $p' \geqslant p$. In other words, maximal elements are decompositions into lines and minimal elements are decompositions into two proper subspaces.

Question. What is the homotopy type of the (nerve of) the poset $\mathsf{D}(V)$?

The answer is:

Theorem A. $\mathsf{D}(V)$ *is homotopy equivalent to a wedge of* $(\dim(V)-2)$*-spheres.*

Welker proved this for finite fields [Wel95], and Randal-Williams explained how the techniques in [GKRW18] can combined with a result of Charney [Cha80, Theorem 1.1] to yield the same result for general field [RW22]. Here we give a more elementary argument. We also give some further applications of the techniques and explain an interpretation in terms of E_1- and E_∞-homology.

2. Proof of Theorem A

In analogy with $\mathsf{D}(V)$, we define a poset $\mathsf{S}(V)$ of *ordered* direct sum decompositions. Its objects are ordered collections $\underline{U} = (U_0, \ldots, U_p)$ of proper subspaces of V such that the natural map $U_0 \oplus \cdots \oplus U_p \to V$ is an isomorphism, ordered by

$$\underline{U} = \{U_0, \ldots, U_p\} \leq \underline{U}' = \{U'_0, \ldots, U'_{p'}\} \qquad \begin{array}{l}\text{if each } U'_i \text{ is contained in some}\\ U_j \text{ so that } i < i' \text{ implies } j < j'\end{array}.$$

Once more we say that $\underline{U}'$ is a refinement of $\underline{U}$, equivalently $\underline{U}$ is obtained by summing together some adjacent terms in $\underline{U}'$.

Theorem 2.1. *The posets* $\mathsf{S}(V)$ *are all* $(\dim(V)-2)$*-spherical if and only if the posets* $\mathsf{D}(V)$ *are all* $(\dim(V)-2)$*-spherical.*

Date: February 22, 2023.

Theorem A follows from this, since Charney proved that $\mathsf{S}(V)$ is $(\dim(V)-2)$-spherical [Cha80, Theorem 1.1] (our $\mathsf{S}(V)$ is isomorphic to her $S_\Lambda(V)$ with $\Lambda = \mathbb{F}$). The technical input is the first part of [vdKL11, Theorem 2.3]. A poset is *bounded* if all chains have finite length; this is true in all of our applications.

Lemma 2.2 (Looijenga–van der Kallen)**.** *Suppose that* A *and* B *are bounded posets and we have a map of posets*

$$F\colon \mathsf{A} \longrightarrow \{\textit{downward closed subposets of } \mathsf{B}\textit{, ordered by inclusion}\}^{\mathrm{op}}.$$

Suppose we have $n \in \mathbb{Z}$ *and functions* $t_\mathsf{A}\colon \mathsf{A} \to \mathbb{Z}$ *and* $t_\mathsf{B}\colon \mathsf{B} \to \mathbb{Z}$ *so that the following hold:*

(i) $\mathsf{A}_{<a} = \{a' \in \mathsf{A} \mid a' < a\}$ *is* $(t_\mathcal{A}(a)-2)$*-connected and* $F(a)$ *is* $(n-t_\mathsf{A}(a)-1)$*-connected,*
(ii) $\mathsf{B}_{<b} = \{b' \in \mathsf{B} \mid b' < b\}$ *is* $(t_\mathsf{B}(b)-2)$*-connected and* $\mathsf{A}_b = \{a \in \mathsf{A} \mid b \in F(a)\}$ *is* $(n-t_\mathsf{B}(b)-1)$*-connected.*

Then A *is* $(n-1)$*-connected if and only if* B *is* $(n-1)$*-connected.*

Proof of Theorem 2.1. Note that both $\mathsf{S}(V)$ and $\mathsf{D}(V)$ are $(\dim(V)-2)$-dimensional, so they are $(\dim(V)-2)$-spherical if and only if they are $(\dim(V)-3)$-connected.

We will do an induction over $\dim(V)$, observing there is nothing to prove in the initial case $\dim(V)=0$. For the induction step we will apply Lemma 2.2 to $\mathsf{A}=\mathsf{S}(V)$, $\mathsf{B}=\mathsf{D}(V)$, $n=\dim(V)-2$, and the map

$$\begin{aligned} F\colon \mathsf{S}(V) &\longrightarrow \{\text{downward closed subposets of } \mathcal{D}(V)^{\mathrm{op}}\}^{\mathrm{op}} \\ \underline{U} = (U_0,\ldots,U_k) &\longmapsto \{\overline{V} = \{V_0,\ldots,V_l\} \text{ such that } \overline{V} \text{ is a refinement of } \overline{U}\}, \end{aligned}$$

where $\overline{U} = \{U_0,\ldots,U_k\}$, i.e. we forget the ordering. The values of the map F are downwards closed as any refinement $\underline{V}$ is also a refinement of $\underline{U}$ (remember they are subposets of $\mathsf{D}(V)^{\mathrm{op}}$ rather than $\mathsf{D}(V)$). The map F is a map of posets because if $\underline{U}' \leq \underline{U}$ in $\mathsf{S}(V)$, i.e. $\underline{U}$ is a refinement of $\underline{U}'$, then $F(\underline{U}) \subseteq F(\underline{U}')$ because all refinements of $\overline{U}$ are in particular refinements of $\overline{U}'$.

For (i), we take $t_{\mathsf{S}(V)}=0$ and then there is nothing to prove about $\mathsf{S}(V)_{<\underline{U}}$. This works because $F(\underline{U})$ is always contractible, as $\overline{U}$ is terminal in $F(\underline{U}) \subset \mathsf{D}(V)^{\mathrm{op}}$.

For (ii) we take $t_\mathsf{D}(\overline{W}) = \dim(V)-k-1$ if $\overline{W} = \{W_0,\ldots,W_k\}$. Then $(\mathsf{D}(V)^{\mathrm{op}})_{<\overline{W}}$ is the opposite of $\mathsf{D}(V)_{>\overline{W}}$, given by refinements of $\overline{W}$. Since a refinement of $\overline{W}$ is given by refining at least any of the W_i, this is the join $\mathsf{D}(W_0) * \cdots * \mathsf{D}(W_k)$. As $\dim(W_i) < \dim(V)$, by induction this is

$$-1 + \sum_{i=0}^{k} (\dim(W_i)-2+1) = (\dim(V)-k-2)\text{-spherical}$$

so $(\dim(V)-k-3) = (t_{\mathsf{D}(V)}(\overline{W})-2)$-connected. Finally $\mathsf{S}(V)_{\overline{W}}$ consists of those ordered splittings $\underline{U} = (U_0,\ldots,U_l)$ such that $\overline{W}$ is a refinement of $\overline{U} = \{U_0,\ldots,U_l\}$. That is, each U_i is obtained by combining some W_j's. Thus this poset is isomorphic to the poset of ordered partitions of the set $\{0,\ldots,k\}$ into at two least two non-empty subsets, ordered by refinement. This is the boundary of the permutahedron of order $k+1$, which is a $(k-1)$-sphere so

$$(k-2) = ((\dim(V)-2)-(\dim(V)-k-1)-1) = (n-t_\mathcal{D}(\overline{W})-1)\text{-connected}.$$

It now follows from Lemma 2.2 that $\mathcal{S}(V)$ is $(\dim(V)-3) = (n-1)$-connected if and only if $\mathcal{P}(V)$ is. ∎

Remark 2.3. The proof of Lemma 2.2 in [vdKL11] uses that there is a zigzag of maps of posets

$$\mathsf{S}(V) \longleftarrow X \longrightarrow \mathsf{D}(V)$$

where $X \subset \mathsf{S}(V) \times \mathsf{D}(V)$ is the poset of pairs $(\underline{U}, \overline{W})$ with $\overline{W} \in F(\underline{U})$, with induced partial order, and diagonal maps the projections. Their proof then shows that both diagonal maps independently are highly connected. In (i) we have some leeway, as there $n = \infty$ would have worked, and their argument in fact shows that the left diagonal map is a weak homotopy equivalence. Now observe that it admits a section $\underline{U} \mapsto (\underline{U}, \overline{U})$ whose composition with the right diagonal map is the forgetful map $\underline{U} \mapsto \overline{U}$. We conclude that the latter is $(\dim(V) - 2)$-connected and in particular the map

$$\widetilde{H}_{\dim(V)-2}(\mathsf{S}(V); \mathbb{Z}) \longrightarrow \widetilde{H}_{\dim(V)-2}(\mathsf{D}(V); \mathbb{Z})$$

it induces between the only non-zero reduced homology groups, is surjective.

3. Generalisations

Our proof uses little about vector spaces, and applies in many settings to transfer connectivity results from ordered decompositions to unordered ones, and vice versa. We will not attempt to formalise this here, but give two examples.

3.1. **Dedekind domains.** The definitions of $\mathsf{D}(V)$ and $\mathsf{S}(V)$ generalise to direct sum decompositions of finitely-generated projective modules over a ring R with invariant basis property; one only needs to replace "subspace" with "direct summand" and "dimension" with "rank". For R a Dedekind domain and M a finitely-generated projective R-module, Charney proved that $\mathsf{S}(M)$ is $(\mathrm{rk}(M) - 2)$-spherical [Cha80, Theorem 1.1] and hence so is the unordered version $\mathsf{D}(M)$.

3.2. **Free groups.** The definitions of $\mathsf{D}(V)$ and $\mathsf{S}(V)$ also generalise to free product decompositions of free groups. For a free group G, Hatcher and Vogtmann proved that $\mathsf{D}(G)$ is $(\mathrm{rk}(G) - 2)$-spherical [HV98, Theorem 6.1] and hence so is the ordered version $\mathsf{S}(G)$.

4. Relation to E_1- and E_∞-homology

The nerve of the poset $\mathsf{S}(\mathbb{F}^n)$ is isomorphic to the E_1-splitting complex $\mathrm{S}^{E_1}(\mathbb{F}^n)$ of [GKRW18, Section 17.2] for the graded E_∞-algebra $\mathbb{N} \ni n \mapsto B\mathrm{GL}(\mathbb{F}^n)$. As explained in [RW22], the nerve of the poset $\mathsf{D}(\mathbb{F}^n)$ is weakly equivalent to the E_∞-splitting complex $\mathrm{S}^{E_\infty}(\mathbb{F}^n)$ of [GKRW18, Section 17.4]. Thus Theorem 2.1 can be thought of way to a transfer the standard connectivity hypothesis, without using bar spectral sequences as in [GKRW18, Chapter 14]:

Corollary 4.1. *$H^{E_1}_{n,d}(R) = 0$ for $d < n - 1$ if and only $H^{E_\infty}_{n,d}(R) = 0$ for $d < n - 1$.*

Remark 4.2. In this language, Lemma 2.2 says there is a surjection $H^{E_1}_{n,n-1}(R) \to H^{E_\infty}_{n,n-1}(R)$.

References

[Cha80] R. M. Charney, *Homology stability for* GL_n *of a Dedekind domain*, Invent. Math. **56** (1980), no. 1, 1–17. MR 557579 1, 2, 3

[GKRW18] S. Galatius, A. Kupers, and O. Randal-Williams, *Cellular* E_k*-algebras*, 2018, arXiv:1805.07184. 1, 3

[HV98] A. Hatcher and K. Vogtmann, *Cerf theory for graphs*, J. London Math. Soc. (2) **58** (1998), no. 3, 633–655. MR 1678155 3

[RW22] O. Randal-Williams, *What is the homotopy type of the poset of nontrivial decompositions of* $\mathbf{R}^n$*?*, MathOverflow, 2022, https://mathoverflow.net/q/414964. 1, 3

[vdKL11] W. van der Kallen and E. Looijenga, *Spherical complexes attached to symplectic lattices*, Geom. Dedicata **152** (2011), 197–211. MR 2795243 2, 3

[Wel95] V. Welker, *Direct sum decompositions of matroids and exponential structures*, J. Combin. Theory Ser. B **63** (1995), no. 2, 222–244. MR 1320168 1

Department of Computer and Mathematical Sciences, University of Toronto Scarborough, 1265 Military Trail, Toronto, ON M1C 1A4, Canada

Email address: a.kupers@utoronto.ca