Lectures on algebraic topology

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Abstract

These are the collected lecture notes for Math 231a. They are based on Haynes Miller's notes [Hatb]. Other good references are [Hat02], [Dol95], and [tD08].

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Chapter 1

Singular homology

1.1 Quantifying shapes

1.1.1 Geometry or analysis of position

This is a first course in algebraic topology, a mathematical subject originating in Poincaré's work on "analysis situs." As early as in the 17th century, natural philosophers desired a mathematical theory of shape which ignored distances and angles and only cared about shapes up to deformation. In fact, the first concrete reference I know is Leibniz musing about "geometria situs" (the geometry of position), which concerns the position of objects *ignoring distances* [Lei50]:[1]

> I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitude.

Poincaré was the first to make a serious attempt to make these ambitions reality. His work contains some errors, imprecisions, and infelicitous choices, but what we will learn in this course is in essence the mathematics that he developed [Poi10]:[2] homology theory for topological spaces. In his introduction he echoes Leibniz:

> We know how useful geometric figures are in the theory of imaginary functions and integrals evaluated between imaginary limits, and how much we desire their assistance when we want to study, for example, functions of two complex variables.
>
> If we try to account for the nature of this assistance, figures first of all make up for the infirmity of our intellect by calling on the aid of our senses; but not only this. It is worthy repeating that geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset.

[1] From a 1679 letter to Christian Huygens.

[2] You can find it at https://www.maths.ed.ac.uk/~v1ranick/papers/poincare2009.pdf.

> The use of figures is, above all, then, for the purpose of making known certain relations between the objects that we study, and these relations are those which occupy the branch of geometry that we have called Analysis situs, and which describes the relative situation of points and lines on surfaces, without consideration of their magnitude.

That is, the objects of interests are topological spaces X, without any conditions on their topology. By mapping test spaces—points, intervals, triangles, etc.—into X, we will extract abelian groups $H_n(X)$ whose structure—rank, torsion, etc.—is the sought-after quantification of the shape of X. After defining these *homology groups* of X, we will develop a suite of tools to aid in their computation.

1.1.2 Homology theory is intended to be applied.

Not only will we develop this theory, but we are also interested in applying it to topological spaces X of geometric interest. Such spaces include:

- the *Euclidean spaces* $\mathbb{R}^n$,
- the *spheres* $S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\}$,
- the *real projective spaces* $\mathbb{R}P^{n-1} = S^{n-1}/\{\pm 1\}$ of lines in $\mathbb{R}^n$,
- the *surfaces* Σ_g of genus g.

What they have in common is that they are manifolds. Usually we care about these with their smooth structure (which tells us what it means for a real-valued function on it to be a smooth function), but in this course it will suffice to consider them as *topological manifolds*: second countable Hausdorff topological spaces which are locally homeomorphic to $\mathbb{R}^n$ for some n. The first highlight of this course is that the homology groups of topological manifolds have a symmetry, *Poincaré duality*. Poincaré writes it as[3]

> For a closed manifold the Betti numbers equally distant from the ends of the sequence are equal.

In fact, surgery theory in differential topology tell us that a high-dimensional topological manifold is roughly the same as a topological space with Poincaré duality and a tangent bundle [LÖ2].

Furthermore, we are interested in applying our techniques to other parts of mathematics. When you learn abstract tools like homological algebra or simplicial methods, you should keep in mind that the same techniques are foundational to modern results in algebraic geometry, number theory, symplectic geometry, logic, etc.

1.2 Singular chains

Our goal for the remainder of this chapter is to construct the homology groups of a topological space. So, let us fix a topological space X for the remainder of this section. As mentioned above, we seek to understand X by probing it with certain test spaces.

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[3] For him, the Betti number $\beta_i(M)$ is the rank of the free part of $H_i(M)$. His statement is incorrect, as M needs to be orientable for this to be true.

Definition 1.2.1. The *n-simplex* Δ^n is the topological space given by

$$\Delta^n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{k+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for } i \geq 0 \right\}.$$

Example 1.2.2. For small n, Δ^n is a familiar topological space:

- $n = 0$: a point,
- $n = 1$: an interval,
- $n = 2$: a solid triangle,
- $n = 3$: a solid tetrahedron.

Definition 1.2.3. The *set of n-simplices in X* is

$$\mathrm{Sin}_n(X) := \{\sigma \colon \Delta^n \to X \mid \sigma \text{ continuous}\}.$$

The set $\mathrm{Sin}_n(X)$ is quite large, but we can extract geometric information from it by understanding which collections of simplices are boundaries and which bound. This uses an observation which you may have already made: the boundary of Δ^n is a union of $n+1$ copies of Δ^{n-1}.

Example 1.2.4. The boundary $\partial\Delta^n$ of Δ^n for low k decomposes as follows:

- $n = 0$: it is empty,
- $n = 1$: it is two points,
- $n = 2$: it is three intervals (which overlap in points),
- $n = 3$: it is four solid triangles (which overlap in intervals).

More precisely, for each $0 \leq i \leq n$ the point $(t_0, \dots, t_n)$ with $t_i = 1$ and $t_j = 0$ for $i \neq j$ (this is forced by the conditions on the t_i's) is the *ith vertex of* Δ^n. The map

$$\begin{aligned} \delta_i \colon \Delta^{n-1} &\longrightarrow \Delta^n \\ (t_0, \dots, t_{n-1}) &\longmapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) \end{aligned} \tag{1.1}$$

which skips the ith entry, is the inclusion of the face opposite the ith vertex; the ith face. In this notation, $\partial\Delta^n$ is $\bigcup_{i=0}^{n} \delta_i(\Delta^{n-1})$.

By precomposing with these maps, we can define what it means to take the ith face of a simplex in X: for $\sigma \in \mathrm{Sin}_n(X)$ and $0 \leq i \leq n$,

$$d_i(\sigma) := \sigma \circ \delta_i.$$

Example 1.2.5. We can extract the set $\pi_0(X)$ of path components of X from $\mathrm{Sin}_n(X)$ for $n = 0, 1$. The set $\mathrm{Sin}_0(X)$ may be identified with the set of points of X, and set $\mathrm{Sin}_1(X)$ with the set of continuous paths $[0,1] \to X$. Two points $x_0, x_1 \in \mathrm{Sin}_0(X)$ are in the same path component if and only if they can be connected by a path, that is, if there is a $\sigma \in \mathrm{Sin}_1(X)$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. We conclude that

$$\pi_0(X) = \mathrm{Sin}_0(X)/{\sim}$$

with $x_0 \sim x_1$ if there is a $\sigma \in \mathrm{Sin}_1(X)$ such that

$$d_0(\sigma) = d_1(\sigma). \tag{1.2}$$

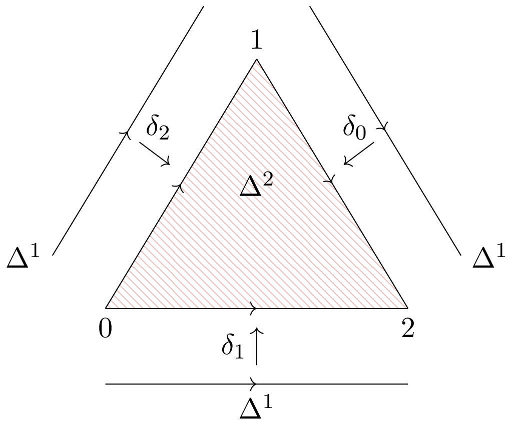


Figure 1.1 The standard 2-simplex Δ^2 and its three faces.

To extend this example to higher dimensions, we rewrite (1.2) as $d_0(\sigma) - d_1(\sigma) = 0$. This only makes sense in the free abelian group on $\mathrm{Sin}_0(X)$:

Definition 1.2.6. The abelian group $S_n(X)$ *of singular n-chains in X* is the abelian group given by

$$S_n(X) \coloneqq \mathbb{Z}[\mathrm{Sin}_n(X)].$$

That is, elements of $S_n(X)$ are finite linear combinations of n-simplices in X with integer coefficients, e.g. $3 \cdot \sigma + 17 \cdot \sigma' - 7346 \cdot \sigma''$. By convention, we set $S_n(X) \coloneqq 0$ for $n < 0$. We can extend $d_i \colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ linearly to $d_i \colon S_n(X) \to S_{n-1}(X)$, e.g. $d_i(3 \cdot \sigma + 17 \cdot \sigma' - 7346 \cdot \sigma'') = 3 \cdot d_i(\sigma) + 17 \cdot d_i(\sigma') - 7346 \cdot d_i(\sigma'')$. The boundary of a n-simplex in X is given by taking its faces, counted with a sign:

Definition 1.2.7. For $\sigma \in S_n(X)$, we define

$$d(\sigma) \coloneqq \sum_{i=0}^{n} (-1)^i d_i(\sigma) \in S_{n-1}(X).$$

Remark 1.2.8. The sign is $+1$ if prepending to the standard orientation $d_i(\sigma)$ an outward-pointing normal vector recovers the standard orientation of σ, and -1 otherwise.

Definition 1.2.9. An *n-cycle* is an n-chain σ such that $d(\sigma) = 0$. The abelian group $Z_n(X)$ *of n-cycles* is given by

$$Z_n(X) \coloneqq \ker[d \colon S_n(X) \to S_{n-1}(X)].$$

In Problem 1.4.1, you will check that $d(d(a)) = 0$ for any $a \in S_n(X)$. Thus we can cheaply construct many cycles by applying d.

Definition 1.2.10. An *n-boundary* is an n-chain a such that $a = d(b)$ for some $\tau \in S_{n+1}(X)$. The abelian group $B_n(X)$ *of n-boundaries* is given by

$$B_n(X) \coloneqq \mathrm{im}[d \colon S_{n+1}(X) \to S_n(X)].$$

Since $d^2 = 0$, $B_n(X)$ is contained in $Z_n(X)$. We now define the homology groups most of this course will be about the group of n-cycles up to n-boundaries (the evident and hence boring n-cycles):

Definition 1.2.11. The *nth homology group* $H_n(X)$ *of* X is given by

$$H_n(x) := \frac{Z_n(X)}{B_n(X)}.$$

Some comments on this definition:

- $H_n(X)$ is a subquotient of $S_n(X)$. Hence for negative n, $H_n(X) = 0$ as a consequence of the convention that $S_n(X) = 0$.
- It is convenient to collect all $H_n(X)$ into a single object: the *graded abelian group* $H_*(X) = \{H_n(X)\}_{n\in\mathbb{Z}}$ given in degree $n \in \mathbb{Z}$ by $H_n(X)$.
- Even though both $Z_n(X)$ and $B_n(X)$ are free abelian groups, as subgroups of a free abelian group, taking the quotient can create torsion in $H_n(X)$. For example, $H_1(\mathbb{R}P^2) = \mathbb{Z}/2$ and in fact any abelian group occurs as a homology group of some topological space X.
- Even though both $Z_n(X)$ and $B_n(X)$ are usually abelian groups with enormous sets of generators, for reasonable topological spaces X the homology groups $H_n(X)$ tend to be finitely-generated.
- If you want to visualize an element of say $H_2(X)$, imagine it as some collection of oriented triangles in X whose edges "cancel out" and which does not "bound" a collection of tetrahedra in X.

This is a particular case of a general construction. A *chain complex* C_* is a sequence of abelian groups C_n for $n \in \mathbb{Z}$ with homomorphisms $d\colon C_n \to C_{n-1}$ satisfying $d^2 = 0$. Given a chain complex, one can define its homology group $H_n(C_*)$ of C_* in the same way as above: the n-cycles $Z_n(C_*)$ are $\ker[d\colon C_n \to C_{n-1}]$ and the n-boundaries $B_n(C_*)$ are $\operatorname{im}[d\colon C_{n+1} \to C_n]$. The equation $d^2 = 0$ implies $B_n(C_*) \subset Z_n(C_*)$, so we can define homology of C_* as

$$H_n(C_*) = \frac{Z_n(C_*)}{B_n(Z_*)}.$$

The homology groups of X are the homology groups of the *singular chain complex* $S_*(X)$. We will revisit this in the next chapter.

1.3 First examples

Eventually we will be able to do computations such as

$$H_*(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}^{2g} & \text{if } * = 1, \\ \mathbb{Z} & \text{if } * = 2, \\ 0 & \text{if } * > 2. \end{cases} \tag{1.3}$$

This groups exhibit the symmetry discovered by Poincaré, and reflect the geometric properties of the surface Σ_g:

- $H_0(\Sigma_g) = \mathbb{Z}$: it is path-connected,
- $H_1(\Sigma_g) = \mathbb{Z}^{2g}$: it has g handles,
- $H_2(\Sigma_g) = \mathbb{Z}$: it is orientable,
- $H_*(\Sigma_g) = 0$ for $* > 2$: it is 2-dimensional.

It will take some time before we prove this. Instead, now we will only do the simplest of computations. These are simple enough that they can be done using only the definitions.

1.3.1 H_0 of a topological space

We motivated the definition of the homology groups $H_n(X)$ through the relationship of the case $n = 0$ to path components. The following lemma makes this precise:

Lemma 1.3.1. *$H_0(X) \cong \mathbb{Z}[\{$path components of $X\}]$.*

Proof. There is a map of sets

$$\mathrm{Sin}_0(X) \longrightarrow \{\text{path components of } X\}$$

sending each point $x \in \mathrm{Sin}_0(X)$ to the path component which contains it. This extends linearly to a unique homomorphism

$$\lambda\colon S_0(X) = \mathbb{Z}[\mathrm{Sin}_0(X)] \longrightarrow \mathbb{Z}[\{\text{path components of } X\}].$$

Since $S_{-1}(X) = 0$, $Z_0(X) = C_0(X)$, that is, all 0-chains are 0-cycles. To get an induced homomorphism $H_0(X) \to \mathbb{Z}[\{\text{path components of } X\}]$, it suffices to prove that $\lambda(B_0(X)) = 0$. This amounts to proving that λ sends to 0 the generators $d(\sigma)$ for $\sigma \in \mathrm{Sin}_1(X)$. The element σ is a continuous path $[0, 1] \to X$ and $d(\sigma) = \sigma(0) - \sigma(1)$. As both $\sigma(0)$ and $\sigma(1)$ necessarily lie in the same path component, the homomorphism λ sends $d(\sigma)$ to path component $-$ same path component $= 0$.

Now that we have a well-defined homomorphism $\lambda\colon H_0(X) \to \mathbb{Z}[\{\text{path components of } X\}]$, we show it is surjective and injective. It is surjective because each of the generators of $\mathbb{Z}[\{\text{path components of } X\}]$ is in the image of λ: choose once and for all a point $x^{(i)}$ in each of the path components X_i of X, and interpret $x^{(i)}$ as an element of $C_0(X)$.

To see it is injective, we prove that its kernel is exactly $B_0(X)$. To prepare, we observe that in $H_0(X)$ an element $x_0 \in Z_0(X) = S_0(X)$ is equivalent to x_1 if and only if they can be connected by a continuous path τ in X. Indeed, then $x_0 - x_1 = d(\tau)$. Thus an arbitrarily element $a \in \ker(\lambda) \subset H_0(X)$ can be represented by a linear combinition $\sum_i a_i x^{(i)}$ of our chosen points, with finitely many non-zero coefficients. The homomorphism λ sends this to $\sum_i a_i X_i$, and we see this is 0 if and only if $a_i = 0$ for all i. $\square$

Corollary 1.3.2. *If X is path-connected then $H_0(X) = \mathbb{Z}$.*

1.3.2 H_* of a point

The input to further computations is the computation of the homology of a point.

Remark 1.3.3. I prefer to use $*$ for a point, but too many asterisks would be confusing here.

From above we know that $H_0(\mathrm{pt}) = \mathbb{Z}$, but what happens in higher degrees?

Lemma 1.3.4. *We have that*

$$H_*(\mathrm{pt}) = \begin{cases} \mathbb{Z} & \textit{if } * = 0, \\ 0 & \textit{if } * > 0. \end{cases}$$

Proof. There is a single map $\Delta^n \to \mathrm{pt}$, the constant one. This means that each of the sets Sin_n contains a single element const_n and thus

$$S_n(\mathrm{pt}) = \mathbb{Z}[\mathrm{const}_n].$$

What is the differential $d\colon S_n(\mathrm{pt}) \to S_{n-1}(\mathrm{pt})$? Precomposition of the constant map const_n by any δ_i gives the constant map const_{n-1}. Thus the differential either vanishes or gives const_{n-1}: the first occurs when n is even (so has an odd number of faces, all but one of which cancel) and the second occurring when n is odd (so has an even number of faces, all of which cancel).

That is, $S_*(X)$ is the chain complex given by

$$\cdots \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow \cdots,$$

the first copy of $\mathbb{Z}$ in the degree 0 (degree increasing by 1 when we move rightwards). When we take kernel of d modulo its image, we see that only the copy of $\mathbb{Z}$ in degree 0 survives. ☐

1.4 Problems

Problem 1.4.1 (A verification)**.** Recall that the differential $d\colon S_n(X) \to S_{n-1}(X)$ is given by

$$d(a) = \sum_{i=0}^{n} (-1)^i d_i(a),$$

with $d_i\colon S_n(X) \to S_{n-1}(X)$ obtained by linearly extending the function $\sigma \mapsto d_i(\sigma) = \sigma \circ \delta_i$ on n-simplices in X. Verify that $d^2 = 0$.

Chapter 2

Homology as a functor

In this chapter we will discuss the first of several important properties of the homology groups $H_*(X)$: its dependence on the topological space X. This is best phrased in the language of category theory, which serves as the foundation of much of modern mathematics. See [Rie16] for an introduction to this topic.

2.1 Homology is natural

Recall we defined the graded abelian group $H_*(X)$ as the homology groups

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\ker[d\colon S_n(X) \to S_{n-1}(X)]}{\operatorname{im}[d\colon S_{n+1}(X) \to S_n(X)]}$$

of the chain complex

$$\begin{array}{c} \vdots \\ \downarrow \\ S_n(X) = \mathbb{Z}[\mathrm{Sin}_n(X)] = \mathbb{Z}[\{\sigma\colon \Delta^n \to X\}] \\ \downarrow \\ S_{n-1}(X) = \mathbb{Z}[\mathrm{Sin}_{n-1}(X)] = \mathbb{Z}[\{\sigma'\colon \Delta^{n-1} \to X\}] \\ \downarrow \\ \vdots \end{array}$$

Every continuous map $f\colon X \to Y$ induces homomorphisms $f_*\colon H_n(X) \to H_n(Y)$, constructed as follows:

(1) The continuous map f induces functions

$$\begin{aligned} \mathrm{Sin}_n(f)\colon \mathrm{Sin}_n(X) &\longrightarrow \mathrm{Sin}_n(Y) \\ (\sigma\colon \Delta^n \to X) &\longmapsto (f \circ \sigma\colon \Delta^n \to X \to Y). \end{aligned}$$

These satisfy

$$\mathrm{Sin}_n(f)(d_i(\sigma)) = f \circ \sigma \circ \delta_i = d_i(\mathrm{Sin}_n(f)(\sigma)). \tag{2.1}$$

(2) Extending linearly we get homomorphisms

$$S_n(f)\colon S_n(X) \longrightarrow S_n(Y).$$

The formula above implies $d \circ S_n(f) = S_n(f) \circ d$. By linearity, it suffices to check this on generators $\sigma \in \mathrm{Sin}_n(f)$:

$$\begin{aligned} S_n(f)(d(\sigma)) &= S_n(f)(\sum_{i=0}^{n}(-1)^i d_i(\sigma)) \\ &= \sum_{i=0}^{n+1}(-1)^i \mathrm{Sin}_n(f)(d_i(\sigma)) \\ &= \sum_{i=0}^{n+1}(-1)^i d_i(\mathrm{Sin}_n(f)(\sigma)) && (2.1) \\ &= d(S_n(f)(\sigma)). \end{aligned}$$

(3) This in turn implies that

$$S_n(f)\,(Z_n(X)) \subset Z_n(Y), \text{ and } S_n(f)\,(B_n(X)) \subset B_n(Y).$$

For the first of these inclusions, suppose $a \in Z_n(X)$, then $d(a) = 0$ so $d(S_n(f)(a)) = S_n(f)(d(a)) = S_n(f)(0) = 0$. We leave the second one to the reader.

(4) As a consequence, there is an induced map of quotient groups

$$\begin{aligned} H_n(f)\colon H_n(X) = \frac{Z_n(X)}{B_n(X)} &\longrightarrow H_n(Y) = \frac{Z_n(Y)}{B_n(Y)} \\ [a] &\longmapsto [S_n(f)(a)]. \end{aligned}$$

We will not work out the following observation in detail, as in the next section we reduce it to checking a number of easier claims:

Lemma 2.1.1.
- *For $\mathrm{id}_X : X \to X$ the identity map, $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$.*
- *For $f\colon X \to Y$ and $g\colon Y \to Z$, $H_n(g \circ f) = H_n(g) \circ H_n(f)$.*

Proof suggestion. The crucial observations are that $\mathrm{Sin}_n(\mathrm{id})$ sends $\sigma\colon \Delta^n \to X$ to $\mathrm{id}_X \circ \sigma = \sigma$, and both $\mathrm{Sin}_n(g \circ f)$ and $\mathrm{Sin}_n(g) \circ \mathrm{Sin}_n(f)$ send it to $g \circ f \circ \sigma$. □

2.2 Categories, functors, and natural transformations

2.2.1 General discussion

Category theory is a general framework for discussion constructions which are natural in the above sense. It has hard theorems, but at the moment it will serve to us as a language. See [Rie16] for background reading.

Categories

Definition 2.2.1. A *category* C consists of the following data

- a class $\mathrm{ob}(\mathsf{C})$ of *objects*,
- for each $X, Y \in \mathrm{ob}(\mathsf{C})$ a set $\mathsf{C}(X, Y)$ of *morphisms*,
- for every object $X \in \mathrm{ob}(\mathsf{C})$ an *identity morphism* $\mathrm{id}_X \in \mathsf{C}(X, Y)$,
- a *composition law* $\circ\colon \mathsf{C}(X, Y) \times \mathsf{C}(Y, Z) \to \mathsf{C}(X, Z)$.

These should satisfy:

- for $f \in \mathsf{C}(X, Y)$, $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$,
- for $f \in \mathsf{C}(X, Y)$, $g \in \mathsf{C}(Y, Z)$, $h \in \mathsf{C}(Z, W)$,

$$(h \circ g) \circ f = h \circ (g \circ f).$$

The last property says that composition is associative.

Example 2.2.2. The category Set of sets has the collection of all sets as its class of objects, and the functions $X \to Y$ as morphisms from X to Y. Composition is given by composition of functions and the identity is the identity function.

It is here where you see why we allow classes of objects; it allows us to have categories whose objects are sets, possibly with some additional structure. Categories with a set of objects are called *small*. We will usually ignore such set-theoretical issues.

The next examples will be given in less detail:

Example 2.2.3. The category Top of topological spaces has topological spaces as objects, and the continuous maps as morphisms.

Example 2.2.4. The category Grp of groups has groups as objects, and the homomorphisms as morphisms.

Example 2.2.5. The category Ab of abelian groups has abelian groups as objects, and the homomorphisms as morphisms.

Functors

We next define a notion of map from one category to another. A map between vector spaces—called a linear map—is a function which preserves all the structure around; all operations and the equations these operations satisfy. The same will be true for categories:

Definition 2.2.6. A *functor* $F\colon \mathsf{C} \to \mathsf{D}$ is given by

- an assignment $F\colon \mathrm{ob}(\mathsf{C}) \to \mathrm{ob}(\mathsf{D})$,
- for all $X, Y \in \mathrm{ob}(\mathsf{C})$ a function $F\colon \mathsf{C}(X, Y) \to \mathsf{D}(F(X), F(Y))$.

These should satisfy:

- for $X \in \mathrm{ob}(\mathsf{C})$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$,
- for $f \in \mathsf{C}(X, Y)$, $g \in \mathsf{C}(Y, Z)$,

$$F(g \circ f) = F(g) \circ F(f).$$

Example 2.2.7. Every abelian group is a group, so there is an inclusion functor $i\colon \mathsf{Ab} \to \mathsf{Grp}$. This regards an abelian group as a group and a homomorphism of abelian groups as a homomorphism of groups.

The next examples will be given in less detail.

Example 2.2.8. Taking the underlying set of a topological space gives a functor $U\colon \mathsf{Top} \to \mathsf{Set}$.

Example 2.2.9. Considering a set as a discrete topological space gives a functor $L\colon \mathsf{Set} \to \mathsf{Top}$.

Example 2.2.10. Considering a set as an indiscrete topological space gives a functor $R\colon \mathsf{Set} \to \mathsf{Top}$.

2.2.2 Homology as a functor

Of course, we were aiming at describing homology as a functor. This is just a reformulation of Lemma 2.1.1, once we introduce one more category.

Definition 2.2.11. The category GrAb of graded abelian groups has collections $A_* = \{A_n\}_{n\in\mathbb{Z}}$ of abelian groups A_n as objects, and morphisms $A_* \to B_*$ given by collections $\{f_n\}_{n\in\mathbb{Z}}$ of homomorphisms $f_n\colon A_n \to B_n$.

Theorem 2.2.12. *Homology is a functor*

$$H_*\colon \mathsf{Top} \longrightarrow \mathsf{GrAb}$$

given on objects by $X \mapsto H_*(X) = \{H_n(X)\}_{n\in\mathbb{Z}}$ *and on morphisms by* $f \mapsto H_*(f) = \{H_n(f)\}_{n\in\mathbb{Z}}$.

Notation 2.2.13. We will eventually shorten $H_*(f)$ to f_*.

2.2.3 Constructing of homology in terms of functors

To get used to category theory, let me describe the construction of homology as a composition of several functors:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "\mathrm{Sin}_\bullet"'] \arrow[rrr, bend left, "H_*"] & \mathsf{ssSet} \arrow[r, "\mathbb{Z}(-)"'] & \mathsf{Ch}_{\mathbb{Z}} \arrow[r, "H_*"'] & \mathsf{GrAb}. \end{tikzcd}$$

Constructing the singular simplicial set, $\mathrm{Sin}_\bullet$

We start with the category Top of topological spaces and continuous maps. Out of this we construct the sets $\mathrm{Sin}_n(f)$, which have some additional structure coming from restriction to faces of simplices. Let us describe this categorically.

Definition 2.2.14. The category Δ_{inj} has objects given by the finite ordered sets $[n] = \{0 < \ldots < n\}$ and morphisms from $[n]$ to $[m]$ giving by the functions $f\colon [n] \to [m]$ such that $f(i) < f(j)$ for $i < j$.

All morphisms of Δ_{inj} are injective, hence the subscript inj. This category models the combinatorial structure of the inclusions of faces:

Example 2.2.15. There is a functor

$$\begin{aligned} \Delta^\bullet \colon \Delta_{\mathrm{inj}} &\longrightarrow \mathsf{Top} \\ [n] &\longmapsto \Delta^n, \end{aligned}$$

sending an $\alpha\colon [n] \to [m]$ to the continuous injection sending a point $(t_0, \ldots, t_n) \in \Delta^n$ to the point in Δ^m giving by putting t_i in the $f(i)$th place and making the remaining coordinates 0. For example, the inclusion $\delta_i \colon [n-1] \to [n]$ skipping the ith entry gives rise to the face inclusion $\delta_i \colon \Delta^{n-1} \hookrightarrow \Delta^n$ of (1.1).

We obtained the sets $\mathrm{Sin}_n(X)$ by mapping Δ^n into X, and the functions $d_i \colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ by precomposing with δ_i. Note that this changes directions: a composition

$$[n-2] \xrightarrow{\delta_i} [n-1] \xrightarrow{\delta_j} [n]$$

induces inclusions of simplices, along which we restrict to get a composition

$$\mathrm{Sin}_n(X) \xrightarrow{d_j} \mathrm{Sin}_{n-1}(X) \xrightarrow{d_i} \mathrm{Sin}_{n-2}(X).$$

The order of composition is reversed. This "contravariance" is encoded by modifying Δ_{inj}:

Definition 2.2.16. Let C be a category, the *opposite category* C^{op} is the category with the same class of objects, but morphisms from X to Y given by the set $\mathsf{C}(Y, X)$.

Sending C to C^{op} is a construction on categories; I will leave it to the reader to define $(-)^{\mathrm{op}}$ on functors.

Example 2.2.17. From any object $X \in \mathsf{ob}(C)$, we obtain a functor $h_X \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ by $Y \mapsto \mathsf{C}(Y, X)$. This is known as the *Yoneda functor*.

The construction of the sets $\mathrm{Sin}_n(X)$ and the maps between them is nothing but the composition of functors

$$\mathrm{Sin}_\bullet(X) = h_X \circ (\Delta^\bullet)^{\mathrm{op}} \colon \Delta_{\mathrm{inj}}^{\mathrm{op}} \longrightarrow \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{Set}.$$

Such functors have a name:

Definition 2.2.18. A *semisimplicial set* is a functor $\Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathsf{Set}$.

Let us spell out the details: a semisimplicial set $X_\bullet \colon \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathsf{Set}$ consists of a collection of sets $\{X_n\}_{n \geq 0}$ of *n-simplices* obtained by evaluating $X_\bullet$ on $[n]$. Each strictly order-preserving injection $[n] \to [m]$ induces a function $X_n \to X_m$. Each morphism of Δ_{inj} is a composition of injections of the form $\delta_i \colon [n-1] \to [n]$, so it is enough to record what function the morphism δ_i induces: the *ith face map* $d_i \colon X_n \to X_{n-1}$.

Example 2.2.19. There is a semisimplicial set $S^1_\bullet$ gives as follows: the set of n-simplices S^1_n is empty for $n > 1$, and both S^1_0 and S^1_1 consists of a single element. This determines the face maps uniquely. It is a combinatorial model for the circle.

More generally, any triangulation of a space gives rise to a semisimplicial set (essentially by definition if I had defined triangulation). This is a reason why piecewise linear topology, which concerns topological manifolds with triangulations, is also called *combinatorial topology*.

Semisimplicial sets are the objects of a category. It remains to define the morphisms of this category, which are given as follows:

Definition 2.2.20. A morphism $f_\bullet \colon X_\bullet \to Y_\bullet$ of semisimplicial sets is collection of functions $f_n \colon X_n \to Y_n$ satisfying $d_i \circ f_n = f_{n-1} \circ d_i$ for all $n \geq 1$ and $0 \leq i \leq n$.

Remark 2.2.21. In fact, these are just natural transformations: a *natural transformation* $\eta \colon F \to G$ between functors $\mathsf{C} \to \mathsf{D}$ is a collection of morphisms $\eta_X \colon F(X) \to G(X)$ such that for each morphism $f \colon X \to Y$ we have $G(f) \circ \eta_X = \eta_Y \circ F(f)$.

Thus, there is a category ssSet which has semisimplicial sets as objects (which are just functors) and morphisms as above (which are just natural transformations). We can now phrase the construction of the sets of simplices categorically: there is a *singular semisimplicial set* functor

$$\begin{aligned} \mathrm{Sin}_\bullet \colon \mathsf{Top} &\longrightarrow \mathsf{ssSet} \\ X &\longmapsto \mathrm{Sin}_\bullet(X). \end{aligned}$$

Of course it remains to verify that this satisfies $\mathrm{Sin}_\bullet(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Sin}_\bullet(X)}$ and $\mathrm{Sin}_\bullet(g \circ f) = \mathrm{Sin}_\bullet(g) \circ \mathrm{Sin}_\bullet(f)$. This can be done by hand without much effort, but one can also do work more abstractly:

Remark 2.2.22. There is a category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ with functors as objects and natural transformations as morphisms. The assignment $X \mapsto h_X$ gives a functor $h \colon \mathsf{C} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$. Given a functor $F \colon \mathsf{D} \to \mathsf{C}$, precomposition with F gives a functor $F^* = \mathsf{Fun}(\mathsf{C}, \mathsf{Set}) \to \mathsf{Fun}(\mathsf{D}, \mathsf{Set})$. Thus taking $\mathsf{C} = \mathsf{Top}$, $\mathsf{D} = \Delta^{\mathrm{op}}_{\mathrm{inj}}$ and $F = \Delta^\bullet$, this exhibits $\mathrm{Sin}_\bullet$ as the composition $F^* \circ h$.

Remark 2.2.23. If you familiar with algebraic topology, you may have seem the category Δ: it has the same objects but morphisms from $[n]$ to $[m]$ given by the functions $f \colon [n] \to [m]$ such that $f(i) \leq f(j)$ for $i \leq j$. There is a similar *singular simplicial set functor*

$$\begin{aligned} \mathrm{Sing}_\bullet \colon \mathsf{Top} &\longrightarrow \mathsf{sSet} \\ X &\longmapsto \mathrm{Sing}_\bullet(X), \end{aligned}$$

which contains a bit more information that the singular semisimplicial set. This information is not necessary to define homology, but it does lead to better formal properties.

Extracting a chain complex

Next in our definition of homology, we extract the chain complex $S_*(X)$ from $\mathrm{Sin}_\bullet(X)$. Recall that chain complex C_* is a collection $\{C_n\}_{n\in\mathbb{Z}}$ of abelian groups with homomorphisms $d\colon C_n \to C_{n-1}$ such that $d^2 = 0$. A morphism $f_*\colon C_* \to D_*$ of chain complexes should induce maps on homology groups. This is the case when we have a collection of homomorphisms $\{f_n\colon C_n \to D_n\}_{n\in\mathbb{Z}}$ satisfying $d \circ f_n = f_{n-1} \circ d$; we call this a *chain map.*

Definition 2.2.24. The category $\mathsf{Ch}_{\mathbb{Z}}$ of *chain complexes* has chain complexes as objects, and the chain maps as morphisms.

Our construction of a chain complex from $\mathrm{Sin}_\bullet(X)$ works for any semisimplicial set $X_\bullet$: we define $\mathbb{Z}(X_\bullet)_n := \mathbb{Z}[X_n]$, the free abelian group on X_n, and give a differential in terms of the face maps by the same formula:

$$\begin{aligned} d\colon \mathbb{Z}(X_\bullet)_n = \mathbb{Z}[X_n] &\longrightarrow \mathbb{Z}(X_\bullet)_{n-1} = \mathbb{Z}[X_{n-1}] \\ x &\longmapsto \sum_{i=0}^{n} (-1)^i d_i(x). \end{aligned}$$

For a morphism $f_\bullet\colon X_\bullet \to Y_\bullet$ of semisimplicial sets, $\mathbb{Z}(f_\bullet)_n\colon \mathbb{Z}(X_\bullet)_n \to \mathbb{Z}(Y_\bullet)_n$ is given by sending a generator $x \in X_n$ of $\mathbb{Z}(X_\bullet)_n = \mathbb{Z}[X_n]$ to $f_n(x) \in \mathbb{Z}(Y_\bullet)_n = \mathbb{Z}[Y_n]$. This is a chain map, as you will verify in Problem 2.4.1.

After checking that $\mathbb{Z}(-)$ preserves identities and composition, we obtain a functor

$$\begin{aligned} \mathbb{Z}(-)\colon \mathsf{ssSet} &\longrightarrow \mathsf{Ch}_{\mathbb{Z}} \\ X_\bullet &\longmapsto \mathbb{Z}(X_\bullet). \end{aligned}$$

Taking homology

It remains to extract the homology from a chain complex. This is done by defining cycles and boundaries in the same way as for $S_*(X)$ and taking their quotient:

$$H_*(C_*) = \frac{\ker[d\colon C_n \to C_{n-1}]}{\mathrm{im}[d\colon C_{n+1} \to C_n]}.$$

The definition of a chain map $f_*\colon C_* \to D_*$ was such that $[a] \mapsto [f(a)]$ gives a well-defined map on homology. After checking that this preserves identities and composition, we obtain a functor

$$\begin{aligned} H_*\colon \mathsf{Ch}_{\mathbb{Z}} &\longrightarrow \mathsf{GrAb} \\ C_* &\longmapsto H_*(C_*). \end{aligned}$$

2.3 First applications of naturality

We refer to the fact that homology as a functor by saying it is “natural.” Let us this chapter by giving two applications that use this fact.

2.3.1 Homology of disjoint unions

The inclusion $X_i \hookrightarrow \bigsqcup_i X_n$ induce homomorphisms $H_*(X_i) \to H_*(\bigsqcup_i X_i)$ which we can sum together into

$$\bigoplus_i H_*(X_i) \longrightarrow H_*\left(\bigsqcup_i X_i\right).$$

Lemma 2.3.1. *This map is an isomorphism.*

Proof. The singular chain complex $S_*(\bigsqcup_i X_i)$ is isomorphic to the direct sum $\bigoplus_i S_*(X_i)$, and taking homology commutes with direct sums. □

2.3.2 Reduced homology

Next, we discuss a structural property of the homology of a *pointed* space X. Every space X has a unique map to the point pt, and X being pointed means that we are also given a map $\mathrm{pt} \to X$. This is of course the same as picking some $x_0 \in X$, so we denote this map x_0 as well. Since the composition is the identity in

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "x_0"'] \arrow[rr, bend left, "\mathrm{id}"] & X \arrow[r] & \mathrm{pt} \end{tikzcd},$$

so is the induced map on homology

$$\begin{tikzcd} H_*(\mathrm{pt}) \arrow[r, "H_*(x_0)"'] \arrow[rr, bend left, "\mathrm{id}"] & H_*(X) \arrow[r] & H_*(\mathrm{pt}) \end{tikzcd}. \tag{2.2}$$

Definition 2.3.2. The *reduced homology groups* of the topological space X are given by

$$\widetilde{H}_*(X) := \ker[H_*(X) \to H_*(\mathrm{pt})].$$

The diagram (2.2) provides an identification

$$\widetilde{H}_*(X) \cong \frac{H_*(X)}{\mathrm{im}[H_*(\mathrm{pt}) \to H_*(X)]},$$

as well as a splitting of $H_*(X)$ as $H_*(X) \cong \widetilde{H}_*(X) \oplus H_*(\mathrm{pt})$.

2.4 Problems

Problem 2.4.1 (Another verification)**.** Let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a morphism of semisimplicial sets. Verify that the map $\mathbb{Z}(X_\bullet)_* \to \mathbb{Z}(Y_\bullet)_*$ determined by $\mathbb{Z}[X_n] \in x \mapsto f_n(x) \in \mathbb{Z}[Y_n]$ is a chain map.

Problem 2.4.2 (Homology of retracts)**.** We say $A \subset X$ is a *retract* if there is a continuous map $r \colon X \to A$ such that $r \circ i = \mathrm{id}_A$, with $i \colon A \to X$ the inclusion. Prove that if A is a retract of X, then $H_*(A)$ is a direct summand of $H_*(X)$.

Some category theory

Definition 2.4.3. A morphism $f\colon X \to Y$ in a category C is an *isomorphism* if there is a morphism $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In this case we call g the *inverse* of f.

Problem 2.4.4.

(i) Prove that every morphism can have at most one inverse.

(ii) Prove that functors preserve isomorphisms, i.e. if f is an isomorphism then so is $F(f)$.

Definition 2.4.5. A natural transformation $\eta\colon F \to G$ of functors $\mathsf{C} \to \mathsf{D}$ is a *natural isomorphism* if for each object X of C the component $\eta_X\colon F(X) \to G(X)$ is an isomorphism.

Example 2.4.6. Let $\mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fd}}$ be the category with finite-dimensional $\mathbb{Q}$-vector spaces as objects, and the linear maps as morphisms.

Problem 2.4.7 (Duals)**.** For $V, W \in \mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fd}}$, let $\mathrm{Hom}(V, W)$ denote the vector space of linear maps between from V to W.

(i) Explain how to make the assignment

$$\begin{aligned} \mathrm{Hom}(-, \mathbb{Q})\colon (\mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fd}})^{\mathrm{op}} &\longrightarrow \mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fd}} \\ V &\mapsto \mathrm{Hom}(V, \mathbb{Q}), \end{aligned}$$

into a functor. That is, define its value on morphisms.

(ii) Prove that $\mathrm{Hom}(-, \mathbb{Q}) \circ \mathrm{Hom}(-, \mathbb{Q})\colon \mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fd}} \to \mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fd}}$ is naturally isomorphic to the identity functor on $\mathsf{Vect}_{\mathbb{Q}}^{\mathrm{fd}}$.

Geometric realization

Definition 2.4.8. The geometric realization of a semisimplicial set $X_\bullet$ is the quotient space

$$||X_\bullet|| := \left(\bigsqcup_{n \geq 0} \Delta^n \times X_n \right) /{\sim}$$

where $\sim$ is the equivalence relation generated by $(\delta_i(t_0, \ldots, t_{n-1}), x) \sim ((t_0, \ldots, t_{n-1}), d_i(x))$.

Problem 2.4.9 (An example)**.** Draw the geometric realization of the semisimplicial set given by

$$\begin{aligned} \Delta_{\mathrm{inj}}^{\mathrm{op}} &\longrightarrow \mathsf{ssSet} \\ [n] &\longmapsto \Delta_{\mathrm{inj}}([n], [2]). \end{aligned}$$

For each $\sigma \in X_n$ there is a *characteristic map* $c_\sigma\colon \Delta^n \to ||X_\bullet||$ given by composing the inclusion of $\Delta^n \cong \Delta^n \times \{\sigma\} \hookrightarrow \bigsqcup_{n \geq 0} \Delta^n \times X_n$ with the quotient map $\bigsqcup_{n \geq 0} \Delta^n \times X_n \to ||X_\bullet||$.

Problem 2.4.10 (The geometric realization of a semisimplicial set)**.**

(i) Show that given a morphism $f_\bullet \colon X_\bullet \to Y_\bullet$ of semisimplicial sets, there is a unique continuous map
$$||f_\bullet|| \colon ||X_\bullet|| \longrightarrow ||Y_\bullet||$$
such that $||f_\bullet|| \circ c_\sigma = c_{f_n(\sigma)}$ for $\sigma \in X_n$.

(ii) Give a natural transformation η from $||\mathrm{Sin}_\bullet(-)|| \colon \mathsf{Top} \to \mathsf{Top}$ to the identity functor on Top which has the property that
$$(f_\bullet \colon X_\bullet \to \mathrm{Sin}_\bullet(Z)) \longmapsto (\eta_Z \circ ||f_\bullet|| \colon ||X_\bullet|| \to Z)$$
gives a bijection between morphisms $X_\bullet \to \mathrm{Sin}_\bullet(Z)$ of semisimplicial sets and continuous maps $||X_\bullet|| \to Z$. (In particular, you must prove it has this property.)

(iii) Show that the map of chain complexes $S_*(\eta_X) \colon S_*(||\mathrm{Sin}_\bullet(X)||) \to S_*(X)$ admits a splitting and conclude that $H_*(X)$ is a summand of $H_*(||\mathrm{Sin}_\bullet(X)||)$.

Remark 2.4.11*.* In fact, the map $\eta_X \colon ||\mathrm{Sin}_\bullet(X)|| \to X$ induces an isomorphism on homology as well as fundamental groups at any basepoint. Moreover, if X has the homotopy type of a CW-complex it is a homotopy equivalence. Thus from the point of view of homotopy theory, X can be recovered from $\mathrm{Sin}_\bullet(X)$.

Chapter 3

Homotopy invariance

In this chapter we prove that homology only depends on the homotopy type of a topological space.

3.1 Homotopies

In your previous point-set or differential topology course, you should have seen the following equivalence relation on continuous maps $X \to Y$:

Definition 3.1.1. Two continuous maps $f, g\colon X \to Y$ are *homotopic* if there is a continuous map $H\colon X \times [0,1] \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$.

We call H a *homotopy*. Every continuous map is homotopic to itself by the constant homotopy, and by reversing or concatenating homotopies we prove that homotopy is an equivalence relation on the set of continuous maps $X \to Y$. An equivalence class is called a *homotopy class*, and the set of homotopy classes is denoted $[X, Y]$.

Example 3.1.2. If $X = S^1$, $[S^1, Y]$ is the set of free homotopy classes of circles in Y. If Y is path-connected, this is the set of conjugacy classes in the fundamental group $\pi_1(Y)$ (for any choice of basepoint).

Definition 3.1.3. The *homotopy category of topological spaces* HoTop has as objects topological spaces, and as morphisms homotopy classes of continuous maps.

Implicit in this definition is the claim that composition of homotopy classes is well-defined, i.e. independent of choice of representatives. An isomorphism in this category is a *homotopy equivalence*, a continuous map with an inverse up to homotopy, and isomorphic objects are homotopy equivalent.

Example 3.1.4. The following letters, as subspaces of $\mathbb{R}^2$, are homotopy equivalent: $W, E, T, Y, U, I, S, F, H, J, K, L, Z, X, C, V, N, M$. Similarly, the following letters are homotopy equivalent: Q, R, O, P, A, D. Finally, B is by itself. You can distinguish these using the fundamental group, or eventually H_1.

Recall that the homology groups are "natural" in their input: each continuous map $f\colon X \to Y$ induces a morphism of graded abelian groups $H_*(f)\colon H_*(X) \to H_*(Y)$ (to save notation, we will write f_* for $H_*(f)$), which satisfies $\mathrm{id}_* = \mathrm{id}$ and $(g \circ f))_* = g_* \circ f_*$. We rephrased this as homology providing a functor

$$H_* \colon \mathsf{Top} \longrightarrow \mathsf{GrAb}.$$

There is a canonical functor

$$\gamma \colon \mathsf{Top} \longrightarrow \mathsf{HoTop},$$

which is the identity on objects and sends a continuous map to its homotopy classes. In this chapter and the next we will prove that the functor $H_* \colon \mathsf{Top} \to \mathsf{GrAb}$ factors over γ:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "H_*"] \arrow[d, "\gamma"] & \mathsf{GrAb} \\ \mathsf{HoTop}. \arrow[ur] & \end{tikzcd}$$

The following theorem is a more down-to-earth statement of the same result:

Theorem 3.1.5 (Homotopy invariance of homology)**.** *If $f, g \colon X \to Y$ are homotopic then the induced maps $f_*, g_* \colon H_*(X) \to H_*(Y)$ are equal.*

Since functors take isomorphisms to isomorphisms by Problem 2.4.4, we get:

Corollary 3.1.6. *If $f \colon X \to Y$ is a homotopy equivalence then $f_* \colon H_*(X) \to H_*(Y)$ is an isomorphism. Thus homotopy equivalent topological spaces have the same homology.*

Since we computed the homology of a point, and topological spaces homotopy equivalent to a point are called *contractible*, we get:

Corollary 3.1.7. *If X is contractible then*

$$H_*(X) = \begin{cases} \mathbb{Z} & \textit{if } * = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Example 3.1.8. It may seem easy to "see" that the homology of a contractible spaces is equal to that of a point (or equivalently, the reduced homology vanishes). The *Whitehead manifold* is a contractible 3-dimensional manifold for which I think this is not so clear: `https://en.wikipedia.org/wiki/Whitehead_manifold`.

3.2 Chain homotopies

Recall that H_* is the composition of functors

$$\mathsf{Top} \xrightarrow{\mathrm{Sin}_\bullet} \mathsf{ssSet} \xrightarrow{\mathbb{Z}[-]} \mathsf{Ch}_\mathbb{Z} \xrightarrow{H_*} \mathsf{GrAb}.$$

Hence, to prove Theorem 3.1.5 it is helpful to know when two chain maps $f_*, g_* \colon C_* \to D_*$ induce the same maps on homology. We will eventually apply this to $S_*(f)$ and $S_*(g)$. To do so, we will study the analogue of a homotopy for chain complexes.

Definition 3.2.1. Two chain maps $f_*, g_* \colon C_* \to D_*$ are *chain homotopic* if there are homomorphism $h_n \colon C_n \to D_{n+1}$ for $n \in \mathbb{Z}$ such that

$$d \circ h_n + h_{n-1} \circ d = f_n - g_n.$$

We leave it to you to verify this is an equivalence relation. We call the collection $\{h_n\}_{n \in \mathbb{Z}}$ a *chain homotopy*, and the hardest step is checking that you can "concatenate" chain homotopies.

Lemma 3.2.2. *If $f_*, g_* \colon C_* \to D_*$ are chain homotopic then they induce the same map on homology.*

Proof. It suffices to prove that for each n-cycle $a \in C_n$, the elements $f_n(a)$, $g_n(a)$ differ by an n-boundary, an element in the image of d. This follows from

$$f_n(a) - g_n(a) = d \circ h_n(a) + h_{n-1} \circ d(a) = d(h_n(a)),$$

using the fact that $d(a) = 0$ since a was assumed to be an n-cycle. □

We can now define a homotopy category of chain complexes $\mathsf{HoCh}_{\mathbb{Z}}$. This has the same objects, but morphisms are the chain homotopy classes of chain maps. Just as for the homotopy category of spaces, we need to check that composition is well-defined. Let's prove a part of this, as an example of the type of arguments involved:

Lemma 3.2.3. *If $f_*, g_* \colon C_* \to D_*$ are chain-homotopic and $k_* \colon D_* \to E_*$ is a chain map, then $k_* \circ f_*, k_* \circ g_* \colon C_* \to E_*$ are chain homotopic.*

Proof. Let h be the chain homotopy from f_* to g_*, i.e. $d \circ h_n + h_{n-1} \circ d = f_n - g_n$. We claim that the maps $k_{n+1} \circ h_n$ assemble to a chain homotopy from $k_* \circ f_*$ to $k_* \circ g_*$. As k_* is a chain map and hence commutes with d, we get

$$\begin{aligned} d \circ (k_{n+1} \circ h_n) + (k_n \circ h_{n-1}) \circ d &= k_n \circ (d \circ h_n + h_{n-1} \circ d) \\ &= k_n \circ (f_n - g_n) \\ &= k_n \circ f_n - k_n \circ g_n. \end{aligned}$$

□

The isomorphisms in $\mathsf{HoCh}_{\mathbb{Z}}$ are those chain maps with an inverse up to chain homotopy, the *chain homotopy equivalences.* If two chain complexes are isomorphic in $\mathsf{HoCh}_{\mathbb{Z}}$ we say they are *chain homotopy equivalent.*

If two chain maps are chain homotopic they induce the same map on homology, so chain homotopy equivalences induce isomorphisms on homology and chain homotopy equivalent chain complexes have the same homology. Hence there is also a factorization

$$\begin{tikzcd} \mathsf{Ch}_{\mathbb{Z}} \arrow[r, "H_*"] \arrow[d, "\gamma"] & \mathsf{GrAb} \\ \mathsf{HoCh}_{\mathbb{Z}}. \arrow[ur] & \end{tikzcd}$$

To prove Theorem 3.1.5, we will prove:

Lemma 3.2.4. *If $f, g\colon X \to Y$ are homotopic, then $f_*, g_*\colon S_*(X) \to S_*(Y)$ are chain-homotopic.*

Remark 3.2.5. Though chain homotopy equivalences induce isomorphisms on homology, the converse is not true: a chain which induces an isomorphism on homology need not be a chain homotopy equivalence. The former are called *quasi-isomorphisms* and it is more common to build a homotopy category of chain complexes where quasi-isomorphisms play the role of homotopy equivalences, instead of using chain homotopy equivalences. Quillen was the first to develop a general theory of categories with homotopies, *model category theory* [Qui67, Hov99]. Nowadays, these are considered as presentations for ∞-categories.

3.3 Star-shaped domains

We will first prove Corollary 3.1.7 for a particular class of contractible topological spaces. This includes the products of n-simplices, and that particular case will go into the proof of Lemma 3.2.4.

Definition 3.3.1. A subset $X \subset \mathbb{R}^n$ is *star-shaped* if there is a point $x_0 \in X$ if for all $x \in X$ the line segment $\{tx_0 + (1-t)x \mid t \in [0,1]\}$ is contained in X.

Example 3.3.2. Non-empty convex subsets are star-shaped.

Example 3.3.3. The union of the coordinate axes is star-shaped, but not convex.

Star-shaped domains are always contractible. In fact, X *deformation retracts* onto x_0: if we use the notation $r\colon X \to \{x_0\}$ for the unique map and $i\colon \{x_0\} \hookrightarrow X$ for the inclusion, there is a homotopy H from id_X to $i \circ r$ such that $H(x_0, t) = x_0$ for all $t \in [0,1]$. For a shar-shaped domain H is given by

$$H(x,t) = tx_0 + (1-t)x.$$

We can think of $\mathbb{Z}$ as a chain complex concentrated in degree 0 with trivial differentials. Then there is a chain map

$$\epsilon\colon S_*(X) \longrightarrow \mathbb{Z},$$

determined uniquely by sending each generator of $S_0(X) = \mathbb{Z}[\mathrm{Sin}_0(X)]$ to 1.

Proposition 3.3.4. *If X is star-shaped then ϵ is a chain homotopy equivalence.*

Proof. We first need to give a chain map $\eta\colon \mathbb{Z} \to S_*(X)$. This can be given by picking an element of $\mathrm{Sin}_0(X)$. We pick x_0. So η is given in degree 0 by sending 1 to x_0, and zero in all other degrees.

We will prove that η and ϵ are inverse up to chain homotopy. This is easy for $\epsilon \circ \eta$, as this is equal to the identity on $\mathbb{Z}$. However, the composition $\eta \circ \epsilon$ is far from the identity on $S_*(X)$, as its image has rank 1 in degree 0 and has 0 in all other degrees. This means the chain homotopy h between $\eta \circ \epsilon$ and $\mathrm{id}_{S_*(X)}$ must be non-trivial.

Intuitively, $\eta \circ \epsilon$ concentrates everything at x_0, and our chain homotopy h will mimic the deformation retraction H doing so. A generator of $S_n(X)$ is a continuous map $\sigma\colon \Delta^n \to X$. Out of this we can construct a new continuous map

$$\begin{aligned} h_n(\sigma)\colon \Delta^{n+1} &\longrightarrow X \\ (t_0, \dots, t_{n+1}) &\longmapsto \begin{cases} t_0 x_0 + (1-t_0)\sigma\left(\frac{(t_1,\dots,t_{n+1})}{1-t_0}\right) & \text{if } t_0 \neq 1 \\ x_0 & \text{if } t_0 = 1. \end{cases} \end{aligned}$$

Composing with δ_0 (restriction opposite the 0th vertex) means setting t_0 to 0, and recovers σ. Composing with δ_i for $i > 0$ gives the same construction, just applied to $\sigma \circ \delta_{i-1} = d_i(\sigma)$. This can not be exactly right when $n = 0$, since we have not defined our construction for $n = -1$. We will deal with this later and assume $n \geq 1$ for now.

So if we define

$$h_n\colon S_n(X) \longrightarrow S_{n+1}(X)$$

on generators by $\sigma \mapsto h_n(\sigma)$, we compute that

$$d(h_n(\sigma)) = \sigma + \sum_{i=1}^{n+1} (-1)^i h_{n-1}(d_{i-1}(\sigma)) = \sigma - h_{n-1}(d(\sigma)).$$

As $\eta \circ \epsilon$ vanishes in positive degrees, this can be rewritten as

$$d(h_n(\sigma)) + h_{n-1}(d(\sigma)) = \sigma = \mathrm{id}_{S_*(X)}(\sigma) - \eta \circ \epsilon(\sigma).$$

As mentioned before, this formula does not make sense when $n = 0$; in that case precomposing $h_0(\sigma)$ with δ_0 gives us x_0 and we get

$$d(h_0(\sigma)) = \sigma - x_0 = \mathrm{id}_{S_*(X)}(\sigma) - \eta \circ \epsilon(\sigma).$$

We have thus verified the formula

$$d \circ h_n + h_{n-1} \circ d = \mathrm{id}_{S_*(X)} - \epsilon \circ \eta,$$

which finishes the proof. □

3.4 The cross product

Homotopy invariance says that if $f, g\colon X \to Y$ are homotopic, then the induced maps $f_*, g_*\colon H_*(X) \to H_*(Y)$ are equal. As a consequence the homology groups of homotopy equivalent spaces are equal, which we already proved for star-shaped domains $X \subset \mathbb{R}^n$: $H_*(X) = H_*(\mathrm{pt})$. Last time, we reduce this to showing:

Lemma 3.4.1. *If $f, g\colon X \to Y$ are homotopic, then $f_*, g_*\colon S_*(X) \to S_*(Y)$ are chain-homotopic.*

Let us first reduce Lemma 3.4.1 to the "universal case" using naturality. A homotopy H between f and g fits in a commutative diagram (this means that composing any two paths of morphisms with same start and end point gives the same map):

$$\begin{tikzcd}
X \times \{0\} \arrow[dr, "i_0"'] \arrow[drr, bend left, "f"] & & \\
& X \times [0,1] \arrow[r, "H"] & Y \, . \\
X \times \{1\} \arrow[ur, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Since $S_*\colon \mathsf{Top} \to \mathsf{Ch}_{\mathbb{Z}}$ is a functor, on the level of chain complexes this gives equations $S_*(f) = S_*(H) \circ S_*(i_0)$ and $S_*(g) = S_*(H) \circ S_*(i_1)$. Since composing a chain homotopy with a chain map gives a chain homotopy by Lemma 3.2.3, it suffices to prove:

Lemma 3.4.2. $S_*(i_0), S_*(i_1)\colon S_*(X) \to S_*(X \times [0,1])$ *are chain homotopic.*

A chain homotopy g between $S_*(i_0)$ and $S_*(i_1)$ consists of maps $h_n\colon S_n(X) \to S_{n+1}(X \times [0,1])$ for $n \in \mathbb{Z}$, and we want to produce these by taking a "product" of $\sigma \in \mathrm{Sin}_n(X)$ with $\Delta^1 \cong [0,1]$. If we take cartesian products we get $\sigma \times \mathrm{id}\colon \Delta^n \times [0,1] \to X \times [0,1]$. We could then explicitly, and in a manner consistent in n, decompose the prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices to get a number of maps $(\sigma \times \mathrm{id})^{(i)}\colon \Delta^{n+1} \to X \times [0,1]$. This is the *Eilenberg–Zilber* approach to the *cross product*, worked out in Problem 3.5.1. We will instead use more abstract techniques to define it. Our starting point is:

Theorem 3.4.3. *There is a cross product*

$$\times\colon S_p(X) \times S_q(Y) \longrightarrow S_{p+q}(X),$$

satisfying

- naturality*: if $f\colon X \to X'$, $g\colon Y \to Y'$ are continuous maps, $a \in S_p(X)$, $b \in S_q(Y)$, then $(f \times g)_*(a \times b) = f_*(a) \times g_*(b)$,*
- bilinearity*: it is bilinear in both entries,*
- Leibniz rule*: $d(a \times b) = d(a) \times b + (-1)^p a \times d(b)$,*
- normalization*: writing $j_x\colon Y \to X \times Y$ for $y \mapsto (x,y)$ and $i_y\colon X \to X \times Y$ for $x \mapsto (x,y)$, we have $\{x\} \times b = (j_x)_*(b)$ and $a \times \{y\} = (i_y)_*(a)$.*

Proof. The proof is an induction over $m = p+q$. The initial cases $p+q = 0, 1$ are dealt with by normalization. For the induction step, we assume we have constructed the cross product for p- and q-simplices satisfying $p+q = m-1$, and will define it for those satisfying $p+q = m$. We may assume $p+q > 1$.

To ensure bilinearity, we define the cross product on the generators and extend linearly. Let $\iota_p\colon \Delta^p \to \Delta^p$ denote the identity map. Then $\sigma\colon \Delta^p \to X$ is given by $\sigma_*(\iota_p)$. So it we want naturality to hold, we must have

$$\sigma \times \tau \coloneqq (\sigma \times \tau)_*(\iota_p \times \iota_q).$$

In particular, it suffices to define the cross product for the "universal example" $\iota_p \times \iota_q$ and define $\sigma \times \tau$ by the above formula.

Normalization tells us how to define $\iota_p \times \iota_q$ when $p = 0$ or $q = 0$, so we assume both are positive. If we want the Leibniz rule to hold, we must have

$$d(\iota_p \times \iota_q) = d(\iota_p) \times \iota_q + (-1)^p \iota_p \times d(\iota_q).$$

Since $d^2 = 0$, a necessary condition for this is that $d[d(\iota_p) \times \iota_q + (-1)^p \iota_p \times d(\iota_q)]$ vanishes. It does, using that the Leibniz rule for total degree $m-1$ holds by the inductive hypothesis:

$$\begin{aligned}
&d[d(\iota_p) \times \iota_q + (-1)^p \iota_p \times d(\iota_q)] \\
&\quad = d^2(\iota_p) \times \iota_q + (-1)^{p-1} d(\iota_p) \times d(\iota_q) + (-1)^p d(\iota_p) \times d(\iota_q) + \iota_p \times d^2(\iota_q) \\
&\quad = 0.
\end{aligned}$$

Because $\Delta^p \times \Delta^q$ is star-shaped, $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ as $p + q > 1$. Thus the $(p+q-1)$ cycle $d(\iota_p) \times \iota_q + (-1)^p \iota_p \times d(\iota_q)$ is a boundary of some element: we pick such an element and declare it to be $\iota_p \times \iota_q$. By construction the Leibniz rule holds, as do the other three properties. □

Remark 3.4.4. The cross product is not unique, as it uses a choice of certain chains. However, any two choices are unique up to chain homotopy. Similarly, it need not be associative or graded-commutative, but is so up to chain homotopy.

We can now finish the proof:

Proof of Lemma 3.4.2. Pick a 1-simplex $\iota\colon \Delta^1 \to [0,1]$ such that $d_0(\iota) = \{1\}$ and $d_1(\iota) = \{0\}$; then $d(\iota) = \{1\} - \{0\}$ We then define

$$\begin{aligned}
h_n\colon S_n(X) &\longrightarrow S_{n+1}(X \times [0,1]) \\
a &\longmapsto (-1)^n a \times \iota.
\end{aligned}$$

The bilinearity of the cross product tells us this is a homomorphism.

We then need to compute $d \circ h_n + h_{n-1} \circ d$:

$$\begin{aligned}
d \circ h_n(a) + h_{n-1} \circ d(a) &= d((-1)^n a \times \iota) + (-1)^{n-1} d(a) \times \iota \\
&= (-1)^n d(a) \times \iota + a \times d(\iota) - (-1)^n d(a) \times \iota \quad && \text{Leibniz rule} \\
&= a \times \{1\} - a \times \{0\} \\
&= (i_1)_*(a) - (i_0)_*(a). && \text{normalization}
\end{aligned}$$

So indeed h is a chain homotopy between i_0 and i_1. □

3.5 Problems

Problem 3.5.1 (The Eilenberg–Zilber map)**.** In this problem you give an explicit construction of the cross product.

A (p,q)-*shuffle* (a,b) is a partition of the ordered set $\{1,\ldots,p+q\}$ into two disjoint sets $1 \leq a_1 \leq \cdots \leq a_p \leq p+q$ and $1 \leq b_1 \leq \cdots \leq b_q \leq p+q$. Its *sign* $\epsilon(a,b)$ is the sign of the permutation $(a_1, \ldots, a_p, b_1, \ldots, b_q)$. A shuffle (a, b) induces an affine linear map $j^a \colon \Delta^{p+q} \longrightarrow \Delta^p$ uniquely determined by sending the ith vertex of Δ^{p+q} to the jth one of Δ^p if $a_j \leq i \leq a_{j+1}$. It similarly defines an affine linear map $j^b \colon \Delta^{p+q} \to \Delta^q$.

The *Eilenberg–Zilber map* is given by

$$\begin{aligned} \mathrm{EZ}_{p,q} \colon S_p(X) \times S_q(X) &\longrightarrow S_{p+q}(X \times Y) \\ (\sigma, \tau) &\longmapsto \sum_{(p,q)\text{-shuffles}} \epsilon(a, b)(\sigma \circ j^a, \tau \circ j^b). \end{aligned}$$

Prove that this satisfies the properties of Theorem 3.4.3.

Chapter 4

Relative homology

In this chapter we define a refinement of homology, working relative a subspace A of a topological space X. Taking A to be empty recovers homology as defined before.

4.1 Relative homology groups

If $A \subset X$ is a subspace, then we can identify $S_*(A)$ with a subcomplex of $S_*(X)$. That is, for all $n \in \mathbb{Z}$, the group $S_n(A)$ is a subgroup of $S_n(X)$, and the differential on $S_n(X)$ restricts to the differential on $S_n(A)$. The quotient of a chain complex by a subcomplex is again a chain complex:

Lemma 4.1.1. *If $A_* \subset B_*$ is a subcomplex then the quotient groups $C_n \coloneqq B_n/A_n$ with differentials $d\colon C_n \to C_{n-1}$ induced by $d\colon B_n \to B_{n-1}$ form a chain complex. Furthermore, the quotient maps assemble to a chain map $B_* \to C_*$.*

Proof sketch. This amounts to showing that the differential $d\colon C_n \to C_{n-1}$ is well-defined, i.e. independent of choice of representative. But if we represent $[b] \in C_n$ by $b + a$ instead of b, then $d(b + a) = d(b) + d(a)$ and $d(b)$ differ by $d(a) \in A_*$. □

Thus the following definition makes sense:

Definition 4.1.2. The *relative singular chain complex of $A \subset X$* is the chain complex given by

$$S_*(X, A) \coloneqq S_*(X)/S_*(A).$$

Definition 4.1.3. The *relative homology groups $H_*(X, A)$ of $A \subset X$* are the homology groups of $S_*(X, A)$.

Example 4.1.4. $H_*(X, \varnothing) = H_*(X)$ and $H_*(X, X) = 0$.

Example 4.1.5. If we take A to be some basepoint $x_0 \in X$, then $H_*(X, x_0)$ is isomorphic to the reduced homology $\widetilde{H}_*(X)$.

Example 4.1.6. There is a canonical relative n-cycle for $(\Delta^n, \partial\Delta^n)$: the identity map $\iota_n\colon \Delta^n \to \Delta^n$ satisfies $d(\iota_n) \in S_*(\partial\Delta^n)$ so $d(\iota_n) \equiv 0 \in S_*(\Delta^n, \partial\Delta^n)$. We will later see that this generates $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Relative homology is a functor with the following domain:

Definition 4.1.7. The category Top_2 has pairs (X, A) of a topological space X and a subspace $A \subset X$ as objects, and as morphisms $(X, A) \to (Y, B)$ those continuous maps $f : X \to Y$ such that $f(A) \subset B$.

There is a notion of a homotopy between two maps of pairs $f, g : (X, A) \to (Y, B)$: it is map $(X, A) \times [0, 1] = (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ which restricts to f and g at 0 and 1 respectively. In other words, it is a homotopy H between f and g as maps $X \to Y$, satisfying the additional condition that $H(A \times [0, 1]) \subset B$.

Definition 4.1.8. The homotopy category HoTop_2 has pairs of a topological space and a subspace as objects, and the homotopy classes of maps of pairs as morphisms.

If $f : (X, A) \to (Y, B)$ is a map of pairs then f sends $S_*(A) \subset S_*(X)$ to $S_*(B) \to S_*(Y)$ so yields a chain map of quotient complexes $S_*(X, A) \to S_*(Y, B)$. This in turn induces a map on relative homology groups. I will leave to you to check that this is compatible with identities and composition, so gives a functor

$$H_* : \mathsf{Top}_2 \longrightarrow \mathsf{GrAb}.$$

Last chapter we proved that homology was homotopy invariant. We did so by constructing a chain homotopy between $S_*(i_0)$ and $S_*(i_1)$ with $i_0 : X \times \{0\} \to X \times [0, 1]$ and $i_1 : X \times \{1\} \to X \times [0, 1]$. This chain homotopy is natural in X, so the one for X restricts to the one for A when $A \subset X$ is a subspace. Thus it induces a chain homotopy between the two chain maps

$$S_*(X \times \{0\})/S_*(A \times \{0\}) \longrightarrow S_*(X \times [0, 1])/S_*(A \times [0, 1])$$

$$S_*(X \times \{1\})/S_*(A \times \{1\}) \longrightarrow S_*(X \times [0, 1])/S_*(A \times [0, 1])$$

This implies that homology is also homotopy invariant on pairs, that is, descends to a functor

$$H_* : \mathsf{HoTop}_2 \longrightarrow \mathsf{GrAb}.$$

4.2 Homology is an excisive invariant

Unlike homotopy groups, homology is supposed to the "excisive." Intuitively this means it is some sense linear in gluing constructions.

Remark 4.2.1. The prototypical "gluings" in topology are the disjoint union and quotients of topological spaces, and in chain complexes they are the direct sum and quotients of chain complexes. Homology does not behave well when we allow arbitrarily pathological gluings; only homotopically well-behaved gluing constructions are allowed. The precise statement is that S_* sends homotopy colimits to homotopy colimits.

It is this linearity that makes homology computable. In particular, intuitively we expect the following properties to hold:

(P1) *long exact sequence of a pair*: $H_*(X, A)$ "is" $H_*(X) - H_*(A)$, (P2) *excision*: if $U \subset A \subset X$, then $H_*(X, A)$ "is" $H_*(X \setminus U, A \setminus A)$,

(P3) *Mayer–Vietoris*: if X is covered by two open subsets U, V, then $H_*(X)$ "is" $H_*(U) + H_*(V) - H_*(U \cap V)$.

We will make these statements precise in this chapter and the next.

4.3 The long exact sequence in homology

Let us start with claim (P1), that $H_*(X, A)$ "is" $H_*(X) - H_*(A)$. This will be a consequence of a general result about chain complexes.

4.3.1 Long exact sequences from short exact sequences

Definition 4.3.1. A sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow A_{i+2} \longrightarrow \cdots$$

is *exact* if the composition of any two adjacent homomorphisms is 0, and $\ker(A_i \to A_{i+1}) = \mathrm{im}(A_{i-1} \to A_i)$ for all i.

In other words, we first require that A_* is a chain complex and next that its homology vanishes. However, the intuition is different: one usually does not intend to take the homology of an exact sequence (the result would be boring), but instead considers it a tool to "compute" A_i from A_j for $j \neq i$.

Definition 4.3.2. A *short exact sequence* of abelian groups is an exact sequence of the form (you can imagine the 0s extending indefinitely in both directions if you prefer)

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Example 4.3.3. Explicitly, a pair of homomorphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

gives a short exact sequence if and only if is i is injective, p is surjective, and $\ker(p) = \mathrm{im}(i)$.

A short exact sequence is said to be *split* if there is a map $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

Example 4.3.4. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_1} \mathbb{Z}^2 \xrightarrow{\pi_2} \mathbb{Z} \longrightarrow 0$$

is split, but the next one is not:

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{q} \mathbb{Z}/p \longrightarrow 0.$$

Definition 4.3.5. A *short exact sequence* of chain complexes is a sequence of maps of chain complexes

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$$

such that for all n we have a short exact sequence of abelian groups

$$0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0.$$

Short exact sequences behave well under taking homology, but not in the way you might expect. It is *not* true that if the sequence of chain complexes

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$$

is exact, that then the sequence

$$0 \longrightarrow H_n(A_*) \longrightarrow H_n(B_*) \longrightarrow H_n(C_*) \longrightarrow 0$$

is also exact. Let us give a counterexample:

Example 4.3.6. Consider the following short exact sequence of chain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Here each chain complex is drawn vertically. The middle chain complex has vanishing homology, while the left and right ones have homology equal to $\mathbb{Z}$ in one degree. That is, the homology groups are given by

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

and we see that the horizontal maps fail to be injective once and surjective once.

Instead there is a long exact sequence, involving connecting homomorphisms which change degree. In the previous example, there is a connecting homomorphism from the right to left $\mathbb{Z}$, which is an isomorphism.

Theorem 4.3.7. *If $0 \to A_* \to B_* \to C_* \to 0$ is a short exact sequence of chain complexes, then there are natural homomorphisms $\partial \colon H_n(C_*) \to H_{n-1}(A_*)$ such that*

$$\cdots \longrightarrow H_n(B_*) \longrightarrow H_n(C_*) \longrightarrow H_{n-1}(A_*) \longrightarrow H_{n-1}(B_*) \longrightarrow \cdots$$

is a long exact sequence. This is natural in short exact sequence of chain complexes.

Proof indication. The proof is a diagram chase best done in privacy, and can be found in any textbook on algebraic topology.

However, let me indicate how to produce ∂. Pick a representative $c \in C_n$ of $[c] \in H_n(C_*)$. Since $B_n \to C_n$ is surjective, we can find a $b \in B_n$ mapping to c. The image of $d(b)$ in C_{n-1} is $d(c) = 0$, so $d(b)$ is in the kernel of $B_{n-1} \to C_{n-1}$. Thus it is the image of a unique $a \in A_{n-1}$. This turns out to be a cycle, so we can $\partial([c]) = [a] \in H_{n-1}(A_*)$. □

4.3.2 The long exact sequence of a pair

Let us draw a conclusion for relative homology groups. This is the precise version of (P1), that $H_*(X, A)$ "is" $H_*(X) - H_*(A)$.

Corollary 4.3.8. *Let (X, A) be a pair, then there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots .$$

This is natural in the pair (X, A).

Proof. Apply Theorem 4.3.7 to the short exact sequence of chain complexes

$$0 \longrightarrow S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X, A) = S_*(X)/S_*(A) \longrightarrow 0.$$ □

We can generalize this to a long exact sequence of a *triple* $B \subset A \subset X$ of topological spaces. Then the kernel of the surjection

$$S_*(X, B) = S_*(X)/S_*(B) \longrightarrow S_*(X, A) = S_*(X)/S_*(A)$$

is exactly $S_*(A, B) = S_*(A)/S_*(B)$. Applying Theorem 4.3.7 again we get a long exact sequence

$$\cdots \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A, B) \longrightarrow H_{n-1}(X, B) \longrightarrow \cdots .$$

The previous result is just the case $B = \varnothing$.

4.3.3 First applications of relative homology

We now give the first two computations of relative homology.

Relative H_0

Let us start with a computation of relative H_0, analogous to the isomorphism $H_0(X) \cong \mathbb{Z}[\{\text{path-components of } X\}]$ of Lemma 1.3.1.

Lemma 4.3.9. $H_0(X, A) \cong \mathbb{Z}[\{\textit{path-components of } X \textit{ not hit by } A\}]$.

Proof. The end of the long exact sequence is given by

$$\begin{tikzcd} H_0(A) \arrow[r] & H_0(X) \arrow[r] & H_0(X,A) \arrow[dll, rounded corners] \\ 0, & & \end{tikzcd}$$

and we know that the homomorphism $H_0(A) \to H_0(X)$ is given on generators by sending a path-component of A to the corresponding path-component of X. Thus its cokernel is isomorphic to the free abelian group generated by all path-components of X that do not contain a point of A. □

Disks relative to their boundary

Let us look at the pair (D^i, S^{i-1}). Since D^i is contractible, by Corollary 3.1.7 we know that its homology vanishes in all degrees except 0, in which case it is given by $\mathbb{Z}$. Thus for $n \geq 2$ we see

$$\begin{tikzcd} \cdots \arrow[r] & H_n(D^i) = 0 \arrow[r] & H_n(D^i, S^{i-1}) \arrow[dll, rounded corners] \\ H_{n-1}(S^{i-1}) \arrow[r] & H_{n-1}(D^n) = 0 \arrow[r] & \cdots, \end{tikzcd}$$

and conclude that the connecting homomorphism has both vanishing kernel and cokernel. Hence it is an isomorphism for $n \geq 2$

$$H_n(D^i, S^{i-1}) \xrightarrow{\cong} H_{n-1}(S^{i-1}).$$

This leaves degrees $n = 0, 1$. For $n = 0$ we can use the previous lemma, and conclude that $H_0(D^i, S^{i-1}) = 0$. For $n = 1$ and $i > 1$, we get

$$\begin{tikzcd} \cdots \arrow[r] & H_1(D^i) = 0 \arrow[r] & H_1(D^i, S^{i-1}) \arrow[dll, rounded corners] \\ H_0(S^{i-1}) = \mathbb{Z} \arrow[r, "\cong"] & H_0(D^n) = \mathbb{Z} \arrow[r] & 0, \end{tikzcd}$$

so $H_1(D^i, S^{i-1}) = 0$.

For $i = 1$ something more subtle happens, as $S^{i-1} = S^0$ has two path-components:

$$\begin{tikzcd} \cdots \arrow[r] & H_1(D^i) = 0 \arrow[r] & H_1(D^i, S^{i-1}) \arrow[dll, rounded corners] \\ H_0(S^{i-1}) = \mathbb{Z}^2 \arrow[r, "\cong"] & H_0(D^n) = \mathbb{Z} \arrow[r] & 0, \end{tikzcd}$$

and $H_1(D^1, S^0) = \mathbb{Z}$.

It is probably most convenient to get rid of this noise in low degrees. This can be done by using reduced homology instead, using Problem 4.4.1. In that case, the clean answer is that for all n and i there is an isomorphism

$$H_n(D^i, S^{i-1}) \xrightarrow{\cong} \widetilde{H}_{n-1}(S^{i-1}).$$

4.4 Problems

Problem 4.4.1 (Long exact sequence of reduced homology)**.** Construct an analogous long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_n(X, A) \arrow[dll, rounded corners] \\ \widetilde{H}_{n-1}(A) \arrow[r] & \widetilde{H}_{n-1}(X) \arrow[r] & \cdots \end{tikzcd}$$

for each pair (X, A).

Problem 4.4.2 (Naturality of the connecting homomorphism)**.** Suppose we are given a commutative diagram of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r] \arrow[d] & B_* \arrow[r] \arrow[d] & C_* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & C'_* \arrow[r] & 0 \end{tikzcd}$$

with both rows short exact sequences of chain complexes. Prove that the following diagram involving the connecting homomorphisms commutes

$$\begin{tikzcd} H_n(C_*) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A_*) \arrow[d] \\ H_n(C'_*) \arrow[r, "\partial"] & H_{n-1}(A'_*). \end{tikzcd}$$

Problem 4.4.3 (Retracts)**.** Recall that $A \subset X$ is a *retract* if there is a continuous map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$, with $i\colon A \to X$ the inclusion. Prove that if $A \subset X$ is a retract, then $H_*(X) = H_*(A) \oplus H_*(X, A)$, a sharpening of Problem 2.4.2.

Problem 4.4.4 (Five lemma)**.** Suppose we are given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

with exact rows.

(i) Prove that γ is surjective if β and δ surjective and ϵ is injective.

(ii) Give conditions on $\alpha, \beta, \delta, \epsilon$ analogous to those in (i) under which γ injective. You do not need to provide a proof.

(iii) Combine (i) and (ii) to give conditions on $\alpha, \beta, \delta, \epsilon$ under which is γ an isomorphism.

Problem 4.4.5 (Isomorphisms on relative homology)**.**

(i) Use the five-lemma and the long exact sequence of a pair to prove that if we are given a map of pairs $(X, A) \to (X', A')$ such that $X \to X'$ and $A \to A'$ are homotopy equivalences, then $H_*(X, A) \to H_*(X', A')$ is an isomorphism.

(ii) Show that the map of pairs $(D^n, \partial D^n) \to (D^n, D^n \setminus \{0\})$ is not a homotopy equivalence of pairs but still induces an isomorphism on relative homology.

Problem 4.4.6 (Alternative definition of relative homology)**.** Define subsets of $S_n(X)$ by

$$\begin{aligned} Z_n(X, A) &:= \{a \in S_n(X) \mid d(a) \in S_{n-1}(A)\} \\ B_n(X, A) &:= \{a \in S_n(X) \mid \exists b \in S_n(A) \text{ such that } a - b \in B_n(X)\}. \end{aligned}$$

(i) Prove that these are subgroups of $S_n(X)$ and that $B_n(X, A) \subset Z_n(X, A)$.

(ii) Prove that $H_n(X, A)$ is naturally isomorphic to $Z_n(X, A)/B_n(X, A)$.

Problem 4.4.7 (Normalized singular chains)**.** A weakly order-preserving map $f\colon \{0, \ldots, n\} \to \{0, \ldots, n-1\}$ determines an affine-linear map $f_*\colon \Delta^n \to \Delta^{n-1}$ by declaring the ith vertex of Δ^n is sent to the $\tau(i)$th vertex of Δ^{n-1}. A simplex $\sigma\colon \Delta^n \to X$ is said to be *degenerate* if it factors as $\sigma = \sigma' \circ f_*$ for some $\sigma'\colon \Delta^{n-1} \to X$ and f as above.

Let $S_n^{\mathrm{deg}}(X) \subset S_n(X)$ be spanned by the degenerate simplices.

(i) Prove that $S_*^{\mathrm{deg}}(X)$ is a subcomplex.

(ii) Prove that the homology of $S_*^{\mathrm{deg}}(X)$ vanishes.

(iii) Conclude we can compute the homology of X using the smaller chain complex $S_*(X)/S_*^{\mathrm{deg}}(X)$.

The quotient chain complex in part (iii) is usually denoted $N_*(X)$ and referred to as the *normalized singular chain complex.*

Chapter 5

Excision and the Eilenberg–Steenrod axioms

Last chapter we proved the existence of a long exact sequence of a pair, making precise (P1), that $H_*(X, A)$ "is" $H_*(X) - H_*(A)$. Today we state excision, (P2) that $H_*(X, A)$ "is" $H_*(X \setminus U, A \setminus A)$.

5.1 Excision

5.1.1 The statement of excision

In the precise statement (P2), the word "is" will be replaced by an isomorphism. However, a minor point-set condition need to satisfied.

Definition 5.1.1. A triple $U \subset A \subset X$ is *excisive* when $\mathrm{cl}(U) \subset \mathrm{int}(A)$.

Theorem 5.1.2 (Excision)**.** *If $U \subset A \subset X$ is an excisive triple then the inclusion of pairs $(X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism*

$$H_*(X \setminus U, A \setminus U) \xrightarrow{\cong} H_*(X, A).$$

That is, you can remove points of your topological space X which lie in the subspace A, as long as there is a little padding. We get the same result when we collapse the subspace to a point, regardless of whether we removed some subset first. Indeed, there is a map of pairs $(X, A) \to (X/A, *)$ which also induces an isomorphism on homology, under minor conditions.

Remark 5.1.3. This requires a bit of care when dealing with empty sets: X/A is by definition the results of collapsing to a point the subsets $A_+ \subset X_+$, where $(-)_+$ means the addition of a disjoint basepoint. So if A is empty we add a disjoint basepoint!

Corollary 5.1.4. *Suppose that there exists a subset $B \subset X$ such that $A \subset B \subset X$ is an excisive triple and B deformation retracts onto A, then the map $(X, A) \to (X/A, *)$ induces an isomorphism*

$$H_*(X, A) \xrightarrow{\cong} H_*(X/A, *) = \tilde{H}_*(X/A).$$

Proof. Consider the commutative diagram of maps of pairs

$$\begin{tikzcd}
(X,A) \arrow[r, "i"] \arrow[d] & (X,B) \arrow[d] & (X\setminus A, B\setminus A) \arrow[l, "j"'] \arrow[d, "k"] \\
(X/A,*) \arrow[r, "i'"] & (X/A,B/A) & (X/A\setminus *, B/A\setminus *). \arrow[l, "j'"']
\end{tikzcd}$$

We want to prove that the left vertical map induces an isomorphism on relative homology. We will do so by proving that the top horizontal two maps, right vertical map, and bottom horizontal two maps do.

The map i induces an isomorphism on relative homology because $A \to B$ is homology equivalence by Problem 4.4.5. The map i' similarly induces an isomorphism on relative homology; the deformation retraction of B onto A induces a deformation retraction of B/A into $* = A/A$ (this uses that $- \times [0,1]$ commutes with quotients since $[0,1]$ is a compact Hausdorff space, Problem 5.3.1).

The map k is a homeomorphism so induces an isomorphism on relative homology. Finally, the maps j and j' induce an isomorphism on relative homology by excision: by assumption $A \subset B \subset X$ is an excisive triple, and the argument given above for i' implies that $* \subset B/A \subset X/A$ is again an excisive triple. $\square$

Example 5.1.5. If we take two points in the 2-sphere, $S^0 \subset S^2$, we can consider $H_*(S^2, S^0)$. A moments reflection shows that the pair (S^2, S^0) is excisive, so by Corollary 5.1.4 $H_*(S^2,S^0) \cong \widetilde{H}_*(S^2/S^0)$. But the pair $(S^2/S^0,*)$ is homotopy equivalent to $(S^2 \vee S^1, *)$, so we obtain that

$$H_*(S^2,S^0) \cong \widetilde{H}_*(S^2 \vee S^1).$$

5.1.2 The homology of a sphere

Since the pair (D^n, S^{n-1}) is excisive and collapsing the boundary of a disk gives a sphere, $H_*(D^n,S^{n-1}) \cong \widetilde{H}_*(D^n/S^{n-1}) = \widetilde{H}_*(S^n)$. Using the isomorphism $H_*(D^n,S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ from the last lecture, we see that

$$\widetilde{H}_*(S^n) \cong \widetilde{H}_{*-1}(S^{n-1}).$$

Since S^0 consists two points, its reduced homology is given by $\mathbb{Z}$ in degree 0 and zero in all other degrees. This is the initial case of an inductive argument computing $\widetilde{H}_*(S^n)$ from $\widetilde{H}_{*-1}(S^{n-1})$ by the previous isomorphism. Passing back from reduced to unreduced homology adds a $\mathbb{Z}$ in degree 0, and thus we conclude:

Theorem 5.1.6. *The homology of the n-sphere is as follows. For $n \geq 1$, we have*

$$H_*(S^n) = \begin{cases} \mathbb{Z} & \text{if } * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 0$, we have

$$H_*(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is *the* fundamental computation of singular homology. In a sense all of our further results are an elaboration of it.

Example 5.1.7. One can explicitly identify a generator of $\widetilde{H}_n(S^n)$ when interpreting this group as $H_n(\Delta^n, \partial\Delta^{n-1})$: it is the homology class associated to the relative cycle $\mathrm{id}_{\Delta^n} \colon \Delta^n \to \Delta^n$. By the construction of the connecting homomorphism, it is equivalent to prove that $d\mathrm{id}_{\Delta^n}$, a signed sum of the $n+1$ faces of Δ^n, is a generator of $\widetilde{H}_{n-1}(\partial\Delta^{n-1})$.

This is proven by induction. The second statement is clear for $n = 1$, and for the induction step we recast the above computation in terms of simplices. To do so, let $x_i \in \partial\Delta^n$ denote the ith vertex, and $\Lambda^n_i \subset \partial\Delta^n$ denote the ith *horn* obtained by deleting the interior of the ith face (opposite the ith vertex). Then excision applies to the map of pairs $(\partial\Delta^n \setminus \{x_0\}, \Lambda^n_0 \setminus \{x_0\}) \to (\partial\Delta^n, \Lambda^{n-1}_0)$ and the former deformation retracts onto $(\Delta^{n-1}, \partial\Delta^{n-1})$ (lying in the 0th face). We thus have isomorphisms

$$\begin{aligned} H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda^n_0) &\xleftarrow{\cong} H_{n-1}(\partial\Delta^n \setminus \{x_0\}, \Lambda^n \setminus \{x_0\}) \\ &\xleftarrow{\cong} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}). \end{aligned}$$

The relative cycle id_{Δ^n} in the left-most term mapped to that of $d\mathrm{id}_{\Delta^n}$, which is equal to relative cycle $\sigma_0 \colon \Delta^{n-1} \to \partial\Delta^n$ given by the inclusion of the 0th face. But this also in the image of the relative cycle $\mathrm{id}_{\Delta^{n-1}} \colon \Delta^{n-1} \to \Delta^{n-1}$ in the right-most term.

5.1.3 Invariance of domain and the Brouwer fixed point

The first application of Theorem 5.1.6 is a pair of geometric results. The first proves that dimension of Euclidean space is well-defined:

Theorem 5.1.8 (Invariance of domain)**.** *If $m \neq n$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.*

Proof. For contradiction, suppose such a homeomorphism f is given. Pick some $x_0 \in \mathbb{R}^n$, then f induces a homeomorphism

$$\mathbb{R}^n \setminus \{x_0\} \xrightarrow{\cong} \mathbb{R}^m \setminus \{f(x_0)\},$$

and thus both sides must have the same homology. However, the left hand side is homotopy equivalent to S^{n-1} and the right hand side to S^{m-1}. These have different homology by Theorem 5.1.6 if $m \neq n$. □

The second proves that every continuous map $f \colon D^n \to D^n$ which is the identity on ∂D^n must have a fixed point. Let us first prove a related proposition:

Proposition 5.1.9. *There is no continuous retraction $r \colon D^n \to \partial D^n$.*

Proof. This evident when $n = 0$, since $\partial D^0 = \varnothing$, so let's assume $n \geq 1$. We will give a proof by contradiction. By definition of a retraction, this would satisfy $r \circ i = \mathrm{id}_{\partial D^n}$ for $i \colon \partial D^n \to D^n$ the inclusion. Hence, the composition

$$H_{n-1}(\partial D^n) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(\partial D^n)$$

must be identity. But by Theorem 5.1.6 it is given by

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$$

for $n \geq 2$, and

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

for $n = 1$. It is impossible for the identity to factor like this. □

Theorem 5.1.10 (Brouwer fixed point)**.** *Every continuous map $f \colon D^n \to D^n$ which is the identity on ∂D^n must have a fixed point.*

Proof. This is a proof by contradiction. If f had no fixed point, we could define a continuous retraction $r \colon D^n \to \partial D^{n-1}$ by drawing a ray from $f(x)$ to x and sending x to the points where this ray meets the boundary, cf. Figure 5.1. If $x \in \partial D^n$ this evidently gives back x, and I'll leave it to you to check that it is continuous. The existence of r contradicts the previous proposition. □

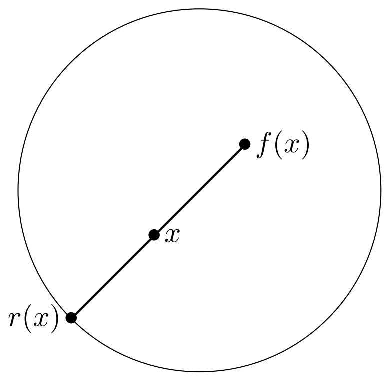


Figure 5.1 The map r in the proof of the Brouwer fixed point theorem.

5.2 The Eilenberg–Steenrod axioms

5.2.1 The axioms

We abstract the properties that we have proven for singular homology to the notino of a homology theory:

Definition 5.2.1. A *homology theory* is given by

- a functor $H_*(-) \colon \mathsf{HoTop}_2 \to \mathsf{GrAb}$,
- a natural transformation $\partial \colon H_*(X, A) \to H_{*-1}(A, \varnothing)$,

satisfying the following properties:

- *long exact sequence of a pair*: for each pair $(X, A) \in \mathsf{Top}_2$ there is a long exact sequence

$$\cdots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, \varnothing) \longrightarrow H_n(X, \varnothing) \longrightarrow H_n(X, A) \xrightarrow{\partial} \cdots$$

- *excision*: for each excisive triad $U \subset A \subset X$ the inclusion induces an isomorphism
$$H_*(X \setminus U, A \setminus A) \xrightarrow{\cong} H_*(X, A).$$
- *wedge axiom*: the inclusions $X_i \to \bigsqcup_{i \in I} X_i$ induces an isomorphism
$$\bigoplus_i H_*(X_i) \xrightarrow{\cong} H_*\left(\bigsqcup_i X_i\right),$$
- *dimension*: $H_*(\mathrm{pt})$ vanishes except in degree 0.

Remark 5.2.2. Of course we also know that $H_0(\mathrm{pt}) = \mathbb{Z}$. This and the previous three axioms determine $H_*(-)$ uniquely on reasonable spaces (those with the homotopy type of CW-complexes). The dimension axiom allows for other abelian groups so as to eventually accomodate Chapter 9.

5.2.2 The locality principle

We will next discuss the proof of excision as a consequence of the *locality principle*, whose underlying idea is that you should be able to compute homology using only simplices that are small with respect to a cover in the following sense:

Definition 5.2.3. A collection $\mathcal{A}$ of subsets of X is a *cover* if their interiors cover X.

Given $\mathcal{A}$, we say that $\sigma\colon \Delta^n \to X$ is $\mathcal{A}$-*small* if its image lies in an element of $\mathcal{A}$. It is evident that each $d_i(\sigma) = \sigma \circ \delta_i$ is again $\mathcal{A}$-small. Thus the subgroups $S_n^{\mathcal{A}}(X)$ spanned by $\mathcal{A}$-small simplices form a subcomplex
$$S_*^{\mathcal{A}}(X) \subset S_*(X).$$

Theorem 5.2.4 (Locality principle)**.** *The inclusion $S_*^{\mathcal{A}}(X) \to S_*(X)$ induces an isomorphism on homology.*

Let us deduce excision from this, which said that if $U \subset A \subset X$ was an excisive triple, then $H_*(X, A) \cong H_*(X \setminus U, A \setminus U)$.

Proof of Theorem 5.1.2. Write $B = X \setminus U$ and take $\mathcal{A} = (A, B)$, this is a cover because $U \subset A \subset X$ is an excisive triple. Observe that $S_*(A), S_*(B) \subset S_*^{\mathcal{A}}(X)$, and furthermore $S_*^{\mathcal{A}}(X)/S_*(A) = S_*(B)/S_*(A \cap B)$. The map of short exact sequences of chain complexes
$$\begin{tikzcd}
0 \arrow[r] & S_*(A) \arrow[r] \arrow[d, equal] & S_*^{\mathcal{A}}(X) \arrow[r] \arrow[d, "\cong_{H_*}"] & S_*^{\mathcal{A}}(X)/S_*(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & S_*(X)/S_*(A) \arrow[r] & 0,
\end{tikzcd}$$
combined with the five-lemma proves that the map
$$S_*(B)/S_*(A \cap B) \xrightarrow{\cong} S_*^{\mathcal{A}}(X)/S_*(A) \longrightarrow S_*(X)/S_*(A)$$
induces an isomorphism on homology. Taking homology and recalling $A \cap B = A \setminus U$ gives excision. $\square$

5.3 Problems

Problem 5.3.1 (A result in point-set topology)**.** Prove that if $X \to X'$ is a quotient of topological spaces (that is, it is surjective and X' is given the quotient topology) and A is compact Hausdorff, then $X \times A \to X' \times A$ is also a quotient.

Problem 5.3.2 (Wedges of spheres)**.**

(i) Compute $H_*(S^n \vee S^m)$ for $n, m \geq 0$.

(ii) State the generalization to $\bigvee_{i=1}^r S^{n_i}$.

Configuration spaces

Definition 5.3.3. The *configuration space of k points in X* is given by

$$\mathrm{Conf}_k(X) \coloneqq \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Problem 5.3.4. Let us investigate the configuration spaces of the Euclidean space $\mathbb{R}^n$ for $n \geq 2$.

(i) Prove that $\mathrm{Conf}_2(\mathbb{R}^n)$ is homotopy equivalent to S^{n-1}.

(ii) Show that $H_*(\mathrm{Conf}_k(\mathbb{R}^n))$ is a summand of $H_*(\mathrm{Conf}_{k+1}(\mathbb{R}^n))$.

(iii) Compute $H_*(\mathrm{Conf}_3(\mathbb{R}^n))$.

The Frobenius–Perron theorem

The following is a version of the Frobenius–Perron theorem:

Theorem. *Let A be an $(n \times n)$-matrix with non-negative real entries such that there is an $m \geq 1$ so that the entries of A^m are positive. Then A has a positive eigenvalue λ and we can pick an eigenvector with eigenvalue λ which has positive entries.*

Problem 5.3.5. Recall $\Delta^{n-1} \subset \mathbb{R}^n$ is given by $\{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } x_1 + \cdots + x_n = 1\}$. In other words, if we set $|x|_1 = |x_1| + \cdots + |x_n|$ then $\Delta^{n-1} = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } |x|_1 = 1\}$.

(i) Prove that

$$\begin{aligned} f \colon \Delta^{n-1} &\longrightarrow \Delta^{n-1} \\ x &\longmapsto \frac{Ax}{|Ax|_1} \end{aligned}$$

is well-defined and continuous.

(ii) Deduce the above theorem by applying the Brouwer fixed point theorem to f.

Chapter 6

The locality principle and the Mayer–Vietoris theorem

In this chapter we outline the proof of the locality principle, and deduce from it the Mayer–Vietoris theorem, (P3): $H_*(U \cup V)$ "is" $H_*(U) + H_*(V) - H_*(U \cap V)$.

6.1 The locality principle

Of all the Eilenberg–Steenrod axioms, it remains to prove excision. We wanted to deduce this from the *locality principle*. Let us repeat its statement.

Recall that a collection $\mathcal{A}$ of subsets of X is a *cover* if their interiors cover X. Given a cover $\mathcal{A}$, we said $\sigma\colon \Delta^n \to X$ was $\mathcal{A}$*-small* if its image lies in the interior of an element of $\mathcal{A}$. Then the subgroups $S_n^{\mathcal{A}}(X) \subset S_n(X)$ spanned by $\mathcal{A}$-small simplices form a subcomplex

$$S_*^{\mathcal{A}}(X) \subset S_*(X),$$

and the locality principle says this can also be used to compute the homology of X: the inclusion $S_*^{\mathcal{A}}(X) \to S_*(X)$ induces an isomorphism on homology.

We also used this to prove Mayer–Vietoris, an very helpful computational tool expressing the homology of $X = U \cap V$ in terms of the homology of U, V and $U \cap V$, through a long exact sequence.

6.1.1 An outline of the proof of the locality principle

We will not give the details of the proof of the locality principle, sticking to the main ideas instead. The reason for this is that, like the proof homotopy invariance, in practice the result is more important that the arguments used to prove it.

Proof of Theorem 5.2.4. The underlying idea is that by subdivision, we can replace an n-cycle a representing a homology class with a different representative which has "smaller" simplices. If we do this enough times, the representative will be $\mathcal{A}$-small.

Let us describe the first two cases of subdivision; $\mathbb{S}(\Delta^0)$ is just Δ^0 and to obtain $\mathbb{S}(\Delta^1)$ we add a 0-simplex at its center $(1/2, 1/2)$ (resulting in two copies of Δ^1 glued along endpoints). Another way to describe this is as follows: we take the *barycenter* in the interior, and then take the convex hull of this center and each of the simplices in the boundary. This is leads to an inductive construction of the subdivision: knowing how to subdivide k-simplices Δ^k for $k < n$, we will get a subdivision $\$(\partial\Delta^n)$ of $\partial\Delta^n$. To do so, we take the barycenter $(1/(n+1), \dots, 1/(n+1)) \in \Delta^n$, and take the convex hull of this and the k-simplices in $\$(\partial\Delta^n)$ to get the $(k+1)$-simplices of $\$(\Delta^n)$.

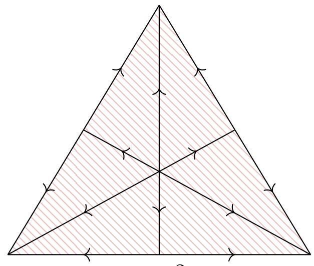


Figure 6.1 The subdivision of Δ^2.

Recall that ι_n is our notation for the identity map $\Delta^n \to \Delta^n$ considered as an element of $S_n(\Delta^n)$. We let $\$(\iota_n)$ denote the sum of the n-simplices in $\$(\Delta^n)$. Then, we define the subdivision of an n-simplex $\sigma\colon \Delta^n \to X$ by naturality:

$$\$(\sigma) \coloneqq \sigma_*(\$(\iota_n)).$$

We extend this linearly to a homomorphism $\$\colon S_n(X) \to S_n(X)$. This satisfies $\$\circ d = d\circ\$$, because the boundary of a subdivided n-simplex is obtained by subdividing each of the faces. Thus we have given chain map

$$\$\colon S_*(X) \longrightarrow S_*(X).$$

There is a chain homotopy T from $\$$ to id, so $\$$ induces the identity on homology. This is constructed first for $S_*(\Delta^n)$ and then extended to all topological spaces X by naturality. By iterating this chain homotopy, we also see that the iterated subdivision $\$^k$ is chain homotopic to the identity.

We claim that for any simplex $\sigma\colon \Delta^n \to X$, there exists an integer $k \geq 0$ such that $\$^k(\sigma)$ is $\mathcal{A}$-small. To see this, observe that the pullback of the cover $\mathcal{A}$ along σ gives a cover $\mathcal{B}$ of Δ^n. By the Lebesgue number lemma (Problem 6.5.3 below), there exists an $\epsilon > 0$ such that every subset of Δ^n of radius $< \epsilon$ is contained in an element of $\mathcal{B}$. But the simplices in the iterated subdivision $\$^k(\Delta^n)$ become arbitrarily small, in particular have radius $< \epsilon$ for k large enough.

We can now start the proof that the inclusion $S_*^{\mathcal{A}}(X) \to S_*(X)$ induces an isomorphism on homology. We first prove surjectivity. Suppose we are given an n-cycle, a finite sum $a = \sum_i n_i\sigma_i$. Pick an integer $k \geq 0$ large enough such that each simplex in $\$^k(\sigma_i)$ is $\mathcal{A}$-small. Then $[\$^k(a)] \in H_*(X)$ lies in the image of the homomorphism $H_*^{\mathcal{A}}(X) \to H_*(X)$, and since $\$^k$ is chain homotopic to the identity, $[\$^k(a)] = [a]$.

As is usual in algebraic topology, the proof of injectivity is a "relative form" of the proof of surjectivity. Suppose that an n-cycle $a \in S_*^{\mathcal{A}}(X)$ becomes a boundary in $S_*(X)$, that is, $a = db$ with $b = \sum_i n_i \tau_i$. As before, there exists an integer $k \geq 0$ such that $\$^k(b) \in S_*^{\mathcal{A}}(X)$. But then $0 = [d\,\$^k(b)] = [\$^k(a)] = [a]$. This computation requires that $\$ \colon S_*(X) \to S_*(X)$ restricts to a chain map $\$ \colon S_*^{\mathcal{A}}(X) \to S_*^{\mathcal{A}}(X)$, as does the chain homotopy T between $\$$ and id. This follows from the naturality of the constructions of $\$$ and T: $\$(\sigma) = \sigma_*(\$(\iota_n))$, so all n-simplices in it are in the image of σ and thus $\mathcal{A}$-small, and similarly for T. $\square$

6.2 The Mayer–Vietoris theorem

The Mayer–Vietoris theorem is closely related to the locality principle and excision. Here we deduce it from the locality principle, but it may also be deduced from excision.

Theorem 6.2.1 (Mayer–Vietoris)**.** *Suppose a topological space X is covered by two open subsets U, V. Then there is a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & H_n(U) \oplus H_n(V) \arrow[r] & H_n(X) \arrow[dll, rounded corners] \\ H_{n-1}(U \cap V) \arrow[r] & H_{n-1}(U) \oplus H_{n-1}(V) \arrow[r] & \cdots . \end{tikzcd}$$

Proof. We apply the locality principle with $\mathcal{A} = \{U, V\}$. Then there is a short exact sequence of chain complexes:

$$0 \longrightarrow H_*(U \cap V) \longrightarrow H_*(U) \oplus H_*(V) \longrightarrow H_*^{\mathcal{A}}(X) \longrightarrow 0.$$

Here the first map is given by $a \mapsto (a, -a)$ and the second map is given by $(a, b) \mapsto a + b$. When passing to homology, this gives the desired long exact sequence. $\square$

The proof also tells us what the morphisms in the long exact sequence are: the left one is given by $((j_1)_*, (j_2)_*)$ for the inclusions $j_1 \colon U \cap V \hookrightarrow U$ and $j_2 \colon U \cap V \hookrightarrow V$, while the right map is given by $(i_1)_* + (i_2)_*$ for the inclusions $i_1 \colon U \hookrightarrow X$ and $I_2 \colon V \hookrightarrow X$.

Example 6.2.2. Let us consider the topological space $S^2 \vee S^1$. This has an open cover by $U = S^2 \cup \{\text{little bit of } S^1\}$ and $V = \{\text{little bit of } S^2\} \cup S^1$. This has the property that $U \simeq S^2$, $V \simeq S^1$ and $U \cap V \simeq *$. Thus the relevant part of the Mayer–Vietoris long exact sequence looks like:

$$\begin{tikzcd} H_2(U \cap V) = 0 \arrow[r] & H_2(U) \oplus H_2(V) = \mathbb{Z} \arrow[r] & H_2(X) \arrow[dll, rounded corners] \\ H_1(U \cap V) = 0 \arrow[r] & H_1(U) \oplus H_1(V) = \mathbb{Z} \arrow[r] & H_1(X) \arrow[dll, rounded corners] \\ H_0(U \cap V) = \mathbb{Z} \arrow[r, "{(\mathrm{id}, -\mathrm{id})}"] & H_0(U) \oplus H_0(V) = \mathbb{Z}^2 \arrow[r, "+"] & H_0(S^2 \vee S^1) = \mathbb{Z}. \end{tikzcd}$$

So we conclude that

$$H_*(S^2 \vee S^1) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

The previous example is an instance of a general result. We say that pointed spaces X is *well-pointed* if the basepoint $x_0 \in X$ has a small neighborhood which deformation retracts onto it.

Proposition 6.2.3. *If X and Y are well-pointed, then the natural map*

$$\widetilde{H}_*(X) \oplus \widetilde{H}_*(Y) \longrightarrow \widetilde{H}_*(X \vee Y)$$

is an isomorphism.

6.3 The suspension isomorphism

Our first application of Mayer–Vietoris is a generalization of the following computation for spheres

$$\tilde{H}_{*+1}(S^{n+1}) \cong \tilde{H}_*(S^n).$$

We can obtain S^{n+1} from S^n by taking $S^n \times [-1,1]$ and collapsing both of the subsets $S^n \times \{1\}$ and $S^n \times \{-1\}$ to a point. Replacing S^n with X we get the unreduced suspension:

Definition 6.3.1. Let X be a topological space. Its *unreduced suspension SX* is given by

$$SX \coloneqq (X \times [-1,1])/{\sim},$$

with $\sim$ the equivalence relation generated by $(x,1) \sim (x',1)$ and $(y,-1) \sim (y,-1)$.

Example 6.3.2. This can also be obtained by gluing together two cones:

$$CX = (X \times [0,1])/{\sim},$$

with $\sim$ the equivalence relation generated by $(x,1) \sim (x',1)$. This contains a copy of X as the image of $X \times \{0\}$ under the quotient map $X \times [0,1] \to CX$. Then SX is homeomorphic to $CX \cup_X CX$.

Proposition 6.3.3 (Suspension isomorphism)**.** *There is an isomorphism*

$$\tilde{H}_{*+1}(SX) \cong \tilde{H}_*(X).$$

Proof. Let $U \subset SX$ be the image of $X \times [-1,1/2)$ in SX, and V be the image of $X \times (-1/2,1]$. These are both contractible, deformation retracting onto one of the two cone points. Their intersection $U \cap V$ is $(-1/2,1/2) \times X$, which is homotopy equivalent to X.

The Mayer–Vietoris sequence for reduced homology looks like:

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_{m+1}(U) \oplus \tilde{H}_{m+1}(V) \arrow[r] & \tilde{H}_{m+1}(U \cap V) \arrow[dll, rounded corners] \\
\tilde{H}_m(U \cap V) \arrow[r] & \tilde{H}_m(U) \oplus \tilde{H}_m(V) \arrow[r] & \tilde{H}_m(U \cap V) \arrow[dll, rounded corners] \\
\tilde{H}_{m-1}(U \cap V) \arrow[r] & \tilde{H}_{m-1}(U) \oplus \tilde{H}_{m-1}(V) \arrow[r] & \cdots
\end{tikzcd}$$

and substituting our values for $U \cap V$, U, V, and $U \cap V$, we get

$$\cdots \longrightarrow 0 \longrightarrow \tilde{H}_{m+1}(SX) \longrightarrow \tilde{H}_m(X) \longrightarrow 0 \longrightarrow \tilde{H}_m(SX) \longrightarrow \tilde{H}_{m-1}(X) \longrightarrow 0 \longrightarrow \cdots$$

We conclude that the connecting homomorphisms give isomorphisms $\tilde{H}_{m+1}(SX) \xrightarrow{\cong} \tilde{H}_m(X)$. □

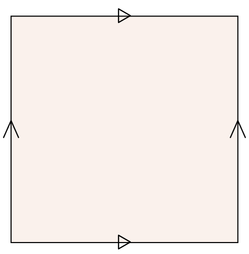
6.4 Many examples

Let us compute some examples, so as to get used to our tools.

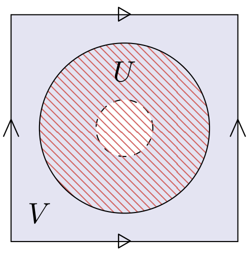
6.4.1 The torus, the real projective plane, and the Klein bottle

What the three topological spaces in the title of this subsection have in common is that there are obtained from a square $[0,1]^2$ by make an identification on its boundary. I will not give formula's for them, but just draw pictures.

Example 6.4.1 (Torus). In this case we have



We can cover the torus $\mathbb{T}^2$ by two open subsets: a little disk around the center U, and the complement of the center V:



Then it is clear that $U \simeq *$, $U \cap V \simeq S^1$, and V deformation retracts onto the boundary. This is a wedge of two circles, $S^1 \vee S^1$: the two horizontal edges get identified to a circle, as do the two vertical edges, and they get glued together at the corner. We know how to compute the homology of each of these.

The Mayer–Vietoris sequence looks like

$$\begin{tikzcd}
H_2(U \cap V) = 0 \arrow[r] & H_2(U) \oplus H_2(V) = 0 \arrow[r] & H_2(U \cup V) = ? \arrow[dll, rounded corners] \\
H_1(U \cap V) = \mathbb{Z} \arrow[r] & H_1(U) \oplus H_1(V) = \mathbb{Z}^2 \arrow[r] & H_1(U \cup V) = ? \arrow[dll, rounded corners] \\
H_0(U \cap V) = \mathbb{Z} \arrow[r] & H_0(U) \oplus H_0(V) = \mathbb{Z} \arrow[r] & H_0(U \cup V) = \mathbb{Z}
\end{tikzcd}$$

We did not draw higher degrees, as those will consists only of zeroes: $H_m(\mathbb{T}^2) = 0$ for $m > 2$. We also know the homomorphisms on the bottom row; the left one is $n \mapsto (n, -n)$, the right one is $(n, m) \mapsto n + m$. In particular, the left one is injective. Thus it only remains to understand the homomorphism

$$H_1(U \cap V) = \mathbb{Z} \longrightarrow H_1(U) \oplus H_1(V) = H_1(V) = \mathbb{Z}^2.$$

We can do this by recalling that the isomorphism $H_1(V) \cong \mathbb{Z}^2$ can be given by $H_1(S^1 \vee S^1) \to H_1(S^1) \oplus H_1(S^1)$ induced by the two maps $p_0, p_1 \colon S^1 \vee S^1 \to S^1$ collapsing one of the two wedge summands to a point. But the composition

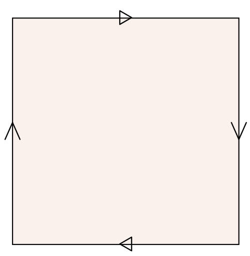
$$S^1 \xrightarrow{\simeq} U \cap V \longrightarrow V \xrightarrow{\simeq} S^1 \vee S^1 \xrightarrow{p_i} S^1$$

is given by the domain S^1 going along target S^1 twice; once in one direction and once in the other. This is null-homotopic. Hence the map $H_1(U \cap V) \to H_1(V)$ is 0.

Feeding this into the Mayer–Vietoris long exact sequence we get

$$H_*(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}^2 & \text{if } * = 1, \\ \mathbb{Z} & \text{if } * = 2, \\ 0 & \text{if } * > 2. \end{cases}$$

Example 6.4.2 (The real projective plane). In this case we have



We can cover the real projective plane $\mathbb{R}P^2$ by the same two open subsets: a little disk around the center U, and the complement of the center V. We still have $U \simeq *$, $U \cap V \simeq S^1$, but now also $V \simeq S^1$.

The Mayer–Vietoris sequence now looks like

$$\begin{tikzcd}
H_2(U\cap V)=0 \arrow[r] & H_2(U)\oplus H_2(V)=0 \arrow[r] & H_2(U\cup V)=\,? \arrow[dll, rounded corners] \\
H_1(U\cap V)=\mathbb{Z} \arrow[r] & H_1(U)\oplus H_1(V)=\mathbb{Z} \arrow[r] & H_1(U\cup V)=\,? \arrow[dll, rounded corners] \\
H_0(U\cap V)=\mathbb{Z} \arrow[r] & H_0(U)\oplus H_0(V)=\mathbb{Z} \arrow[r] & H_0(U\cup V)=\mathbb{Z}
\end{tikzcd}$$

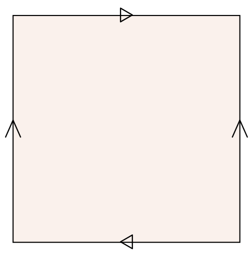
As before, it remains to understand the map

$$S^1 \xrightarrow{\simeq} U\cap V \longrightarrow V \xrightarrow{\simeq} S^1.$$

It is the map that sends the domain S^1 around the target S^1 twice. This implies that the map $H_1(U\cap V)\to H_1(V)$ is $2\colon \mathbb{Z}\to\mathbb{Z}$. This is injective with cokernel $\mathbb{Z}/2$, and feeding this into the Mayer–Vietoris long exact sequence we get

$$H_*(\mathbb{R}P^2)=\begin{cases}\mathbb{Z} & \text{if } *=0,\\ \mathbb{Z}/2 & \text{if } *=1,\\ 0 & \text{if } *>1.\end{cases}$$

Example 6.4.3 (The Klein plane). In this case we have



We again cover the Klein plane $\mathbb{K}$ by a little disk around the center U, and the complement of the center V. We still have $U\simeq *$, $U\cap V\simeq S^1$, but now $V\simeq S^1\vee S^1$ again.

The Mayer–Vietoris sequence thus looks like that for $\mathbb{T}^2$, and we need to understand the two maps

$$S^1 \xrightarrow{\simeq} U\cap V \longrightarrow V \xrightarrow{\simeq} S^1\vee S^1 \xrightarrow{p_i} S^1.$$

One of these is null-homotopic, but the other is homotopic to the map that sends the domain S^1 around the target S^1 twice. By the discussion of degree, this implies that the map $H_1(U\cap V)\to H_1(V)$ is given by $(0,2)\colon \mathbb{Z}\to\mathbb{Z}^2$. This is injective with cokernel $\mathbb{Z}\oplus\mathbb{Z}/2$, and feeding this into the Mayer–Vietoris long exact sequence we get

$$H_*(\mathbb{K})=\begin{cases}\mathbb{Z} & \text{if } *=0,\\ \mathbb{Z}\oplus\mathbb{Z}/2 & \text{if } *=1,\\ 0 & \text{if } *>1.\end{cases}$$

6.4.2 Knot complements

A knot K is a smoothly embedded circle in $\mathbb{R}^3$. One might wonder whether the homology of its complement $\mathbb{R}^3 \setminus K$ is an interesting invariant.

Example 6.4.4. For the trivial knot K_0, $\mathbb{R}^3 \setminus K_0$ is homotopy equivalent to $S^2 \vee S^1$, and its homology is $\mathbb{Z}$ in degrees $0, 1, 2$ and vanishes in other degrees.

The following says it is not:

Proposition 6.4.5. *We have*

$$H_*(\mathbb{R}^3 \setminus K) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 1, 2, \\ 0 & \text{if } * > 2. \end{cases}$$

Proof. We need a result in differential topology which says that there exists an open neighborhood V of K which is homeomorphic to $K \times \operatorname{int}(D^2)$. This is in particular homotopy equivalent to S^1.

Let us cover $\mathbb{R}^3$ by $U = \mathbb{R}^3 \setminus K$ and V as above. Then $U \cap V \simeq S^1 \times S^1 = \mathbb{T}^2$. The interesting part of the Mayer–Vietoris long exact sequence looks like

$$\begin{tikzcd}
\cdots \arrow[rr] & & H_3(U \cup V) = 0 \arrow[dll] \\
H_2(U \cap V) = \mathbb{Z} \arrow[r] & H_2(U) \oplus H_2(V) = ? \oplus 0 \arrow[r] & H_2(U \cup V) = 0 \arrow[dll] \\
H_1(U \cap V) = \mathbb{Z}^2 \arrow[r] & H_1(U) \oplus H_1(V) = \mathbb{Z} \oplus ? \arrow[r] & H_1(U \cup V) = 0 \arrow[dll] \\
H_0(U \cap V) = \mathbb{Z} \arrow[r] & H_0(U) \oplus H_0(V) = \mathbb{Z} \arrow[r] & H_0(U \cup V) = \mathbb{Z}.
\end{tikzcd}$$

The proposition now easily follows. □

Question 6.4.6. What is the generalization to an embedded S^k in $\mathbb{R}^n$?

6.5 Problems

Problem 6.5.1 (Surfaces)**.** Recall that Σ_g denotes an orientable surfaces of genus g.

(i) Compute $H_*(\Sigma_g \setminus \operatorname{int}(D^2))$ for $g \geq 1$.

(ii) Prove that the inclusion $\partial(\Sigma_g \setminus \operatorname{int}(D^2)) \to \Sigma_g \setminus \operatorname{int}(D^2)$ induces the zero map on H_1.

(iii) Compute $H_*(\Sigma_g)$ for $g \geq 1$. (Hint: (1.3) gives the answer.)

Problem 6.5.2 (Products with circles)**.**

(i) Prove that $H_*(X \times S^1) \cong H_*(X) \oplus H_{*-1}(X)$.

(ii) Let $\mathbb{T}^n = (S^1)^n$ be the n-torus. Give without proof a formula for $H_*(\mathbb{T}^n)$.

The following is used in the proof of the locality principle.

Problem 6.5.3 (Lebesgue number lemma)**.** Let $\{U_i\}_{i\in I}$ be an open cover of a compact metric space (X, d). Prove that there exists a real number $\epsilon > 0$ such that each subset of X of radius $< \epsilon$ is contained in some U_i. (Hint: explain why you can assume I is finite and consider $x \mapsto \max_{i\in I} d(x, X - U_i)$.)

The Kakutani fixed point theorem

Let $\mathcal{P}(X)$ denote the set of subsets of X. We will prove a version of the Kakutani fixed point theorem, which is used in game theory and economics:

Theorem (The Kakutani fixed point theorem)**.** *Recall that $\Delta^{n-1} \subset \mathbb{R}^n$ is the convex hull of the standard basis vectors $e_1, \ldots, e_n$ and suppose we are given*

$$f\colon \Delta^{n-1} \longrightarrow \mathcal{P}(\Delta^{n-1})$$

satisfying

(i) for all $x \in \Delta^{n-1}$, $f(x) \subset \Delta^{n-1}$ is convex,

(ii) the set $\{(x, y) \mid y \in f(x)\} \subset (\Delta^{n-1})^2$ is closed.

Then there exists an $x \in \Delta^{n-1}$ such that $x \in f(x)$.

Observe that Δ^{n-1} is the convex hull $\mathrm{conv}(e_1, \ldots, e_n)$ of the standard basis vectors. This means any point $x \in \Delta^{n-1}$ can be written uniquely as $\sum_{i=1}^n t_i e_i$ for $t_i \in [0, 1]$. Pick $y_i \in f(e_i)$ for $1 \le i \le n$, and define $f^0\colon \Delta^{n-1} \to \Delta^{n-1}$ by sending $\sum t_i e_i \mapsto \sum t_i y_i$.

Problem 6.5.4.

(i) More generally, for $k \ge 1$ take the kth barycentric subdivision $\mathbb{S}^k(\Delta^{n-1})$ and for each of its vertices v_i pick $y_i \in f(v_i)$. Modify the definition of f^0 and construct a continuous map $f^k\colon \Delta^{n-1} \to \Delta^{n-1}$ such that $f^k(v_i) = y_i$.

(ii) Prove that f^k has a fixed point x^k.

(iii) Pick an $(n-1)$-simplex $\mathrm{conv}(v_1^k, \ldots, v_n^k)$ of $\mathbb{S}^k(\Delta^{n-1})$ containing x^k, with corresponding $y_i^k \in f(v_i^k)$. Write $x^k = \sum_{i=1}^n t_i^k v_i^k$. Prove that the sequence

$$(x^k, (t_i^k), (v_i^k), (y_i^k)) \in \Delta^{n-1} \times [0,1]^n \times (\Delta^{n-1})^n \times (\Delta^{n-1})^n$$

has a convergent subsequence.

(iv) Prove that its first component converges to an x satisfying $x \in f(x)$.

Remark 6.5.5. Using suitable extension theorems for functions, you can deduce from the previous result a stronger one, replacing Δ^{n-1} with a compact convex subset $P \subset \mathbb{R}^n$.

Good open covers

Definition 6.5.6. An open cover $U_1, \dots, U_r$ of X is *good* if for all subsets $I \subset \{1, \dots, r\}$ either $\bigcap_{i \in I} U_i$ is empty or has trivial reduced homology.

Problem 6.5.7 (Good open covers)**.**

(i) Prove that if X has a good open cover with r elements, then $\tilde{H}_i(X) = 0$ for $i \geq r-1$.

(ii) Give an example showing this is optimal, i.e. find an X with a good open cover with r elements and $\tilde{H}_{r-2}(X) \neq 0$.

(iii) Suppose X has a good open cover with r elements. Give a non-trivial bound on the rank on $\tilde{H}_1(X)$ in terms of r (it does not need to be optimal).

Mapping tori

Definition 6.5.8. Given $f, g\colon X \to Y$, let Z be the topological space obtained as the quotient

$$Z \coloneqq (Y \sqcup X \times [0,1])/{\sim}$$

with $\sim$ the equivalence relation generated by $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$.

If $g = \mathrm{id}$, this is called the *mapping torus of f*.

Problem 6.5.9 (The homology of mapping tori)**.** We shall construct a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "f_*-g_*"] & H_n(Y) \arrow[r, "i_*"] & H_n(Z) \arrow[dll] \\ H_{n-1}(X) \arrow[r, "f_*-g_*"] & H_{n-1}(Y) \arrow[r, "i_*"] & \cdots \end{tikzcd} \tag{6.1}$$

with $i\colon Y \to Z$ the composition $Y \hookrightarrow Y \sqcup X \times [0,1] \to Z$ of the inclusion and the quotient map.

(i) The composition $X \times [0,1] \to Y \sqcup X \times [0,1] \to Z$ maps $X \times \{0,1\}$ to Y; thus there is a map of pairs $(X \times [0,1], X \times \{0,1\}) \to (Z, Y)$. Prove that this induces an isomorphism

$$H_*(X \times [0,1], X \times \{0,1\}) \xrightarrow{\cong} H_*(Z, Y).$$

This fits into a map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(X \times [0,1], X \times \{0,1\}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_n(X \times \{0,1\}) \arrow[r] \arrow[d] & H_n(X \times [0,1]) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_{n+1}(Z, Y) \arrow[r] & H_n(Y) \arrow[r, "i_*"] & H_n(Z) \arrow[r] & \cdots \end{tikzcd}$$

(ii) Prove that the boundary homomorphism $\partial\colon H_{n+1}(X \times [0,1], X \times \{0,1\}) \to H_n(X \times \{0,1\})$ is an isomorphism onto the subgroup of elements of the form $(a, -a)$.

(iii) Prove the existence of long exact sequence (6.1).

Problem 6.5.10 (Examples of mapping tori)**.**

(i) The torus $\mathbb{T}^2$ is the mapping torus of the identity map $\mathrm{id}\colon S^1 \to S^1$. Use this to compute $H_*(\mathbb{T}^2)$.

(ii) The Klein bottle $\mathbb{K}$ is a mapping torus of a map $f\colon S^1 \to S^1$. Which map? Use this to compute $H_*(\mathbb{K})$.

Chapter 7

CW-complexes

Having established the axioms which homology satisfies, we will now develop powerful computational tools. The most important of these is CW-homology, which gives a very small chain complex computing the homology of a topological space obtained from gluing together D^n's. In this chapter we define that class of topological spaces.

7.1 Attaching cells

7.1.1 Gluings and pushouts

Given a pair (B, A) and a map $f\colon A \to X$ we can form a new topological space

$$X \cup_f B$$

by taking $(X \sqcup B)/\sim$ where $\sim$ is the equivalence relation generated by $a \sim f(a)$ for all $a \in A$. We give this the quotient topology. This fits in a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & X \cup_f B \end{tikzcd}$$

Example 7.1.1. If $A = \varnothing$ then $X \cup_f B = X \sqcup B$.

Example 7.1.2. If $X = A$ and $f = \mathrm{id}$, then $X \cup_f B = B$.

Example 7.1.3. If $X = *$ then $X \cup_f B = B/A$.

Example 7.1.4. If $(B, A) = (D^n, S^{n-1})$, $X = D^n$ and f is the inclusion $S^{n-1} \to D^n$, then $X \cup_f B \cong S^n$.

This is an instance of a pushout, a concept in category theory. Given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ B & \end{tikzcd}$$

in C, a *pushout* is an object P with a pair of morphisms $f\colon X \to P$, $g\colon B \to P$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r, "g"] & P \end{tikzcd}$$

commutes, and for every other commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "\bar{f}"] \\ B \arrow[r, "\bar{g}"] & Q \end{tikzcd}$$

there exists a unique morphism $u\colon P \to Q$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[ddr, bend left, "\bar{f}"] & \\ B \arrow[r, "g"'] \arrow[drr, bend right, "\bar{g}"'] & P \arrow[dr, dashed, "u"] & \\ & & Q \end{tikzcd}$$

commutes. We refer to the latter as the *universal property* of the pushout P.

Remark 7.1.5. It is not the case that "being a pushout" is a property of the object P; instead, it is a property of P and the morphisms f, g. This is not included in the notation.

Lemma 7.1.6. *P is unique up to isomorphism.*

Proof. If P' has the same universal property, then we get morphisms u and u' in

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[ddr, bend left, "f'"] & \\ B \arrow[r, "g"'] \arrow[drr, bend right, "g'"'] & P \arrow[dr, dashed, "u"] & \\ & & P' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f'"] \arrow[ddr, bend left, "f"] & \\ B \arrow[r, "g'"'] \arrow[drr, bend right, "g"'] & P' \arrow[dr, dashed, "u'"] & \\ & & P. \end{tikzcd}$$

Thus we get a morphisms $u' \circ u\colon P \to P$. By the universal property of P this is the unique morphism which fits into the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[ddr, bend left, "f"] & \\ B \arrow[r, "g"'] \arrow[drr, bend right, "g"'] & P \arrow[dr, dashed, "u"] & \\ & & P \end{tikzcd}$$

But so does $\mathrm{id}_P\colon P \to P$, hence $u' \circ u = \mathrm{id}_P$. A similar argument gives $u \circ u' = \mathrm{id}_{P'}$, so u and u' are mutually inverse isomorphisms between P and P'. $\square$

The usual gluing construction is indeed a pushout:

Lemma 7.1.7. *The object $X \cup_f B$ (with its maps $j\colon X \to X \cup_f B$ and $g\colon B \to X \cup_f B$) is a pushout of*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \\ B & \end{tikzcd}$$

Proof. We must prove that given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d, "\bar{j}"] \\ B \arrow[r, "\bar{g}"] & Q \end{tikzcd}$$

there is a unique map $u\colon X \cup_f B \to Q$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "j"] \arrow[ddr, bend left, "\bar{j}"] & \\ B \arrow[r, "g"] \arrow[drr, bend right, "\bar{g}"] & X \cup_f B \arrow[dr, dashed, "u"] & \\ & & Q \end{tikzcd}$$

commutes. Indeed, to make the right triangle commute we must set $u([x]) = \bar{j}(x)$ and to make the left triangle commute we must set $u([b]) = \bar{g}(x)$. This is unique since we have given the value on all points of $X \cup_f B$.

To see this is well-defined, we must verify that u takes the same value on a and $f(a)$; it does because $\bar{j} \circ f = \bar{g} \circ i$. This is continuous by the definition of the quotient topology; it is continuous when the composition $X \sqcup B \to X \cup_f B \to Q$ is, but this is $\bar{j} \sqcup \bar{g}$ $\square$

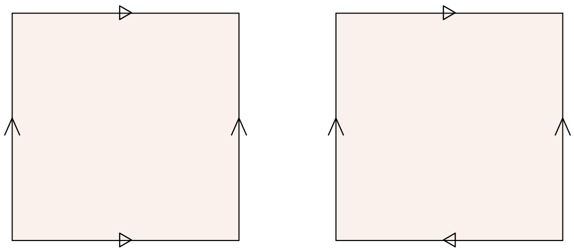
7.1.2 Cell attachments

Let us take $(B, A) = (D^n, S^{n-1})$, and call the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] & X \cup_f D^n \end{tikzcd}$$

the result of *attaching an n-cell to X*. We call $f\colon S^{n-1} \to X$ the *attaching map* and $D^n \to X \cup_f D^n$ the *characteristic map.*

Example 7.1.8. We have seen a number of these already: starting with a single 0-cell D^0, S^1 is obtained by attaching a single 1-cell, and $S^1 \vee S^1$ by attaching two 1-cells. From this we can build the torus $\mathbb{T}^2$ and the Klein bottle $\mathbb{K}$ by attaching a 2-cell: the following figures identify D^2 with $[0,1]^2$ (by a homeomorphism) and tell us what the attaching maps are



In the these figures we only indicated the attaching maps but did not give formula's. One justification for this is the following lemma:

Lemma 7.1.9. *If $f_0, f_1 \colon S^{n-1} \to X$ are homotopic, then $X \cup_{f_0} D^n$ and $X \cup_{f_1} D^n$ are homotopy equivalent.*

Proof. Let $H \colon S^{n-1} \times [0,1] \to X$ be a homotopy from f_1 to f_0. Parametrizing D^n by radial coordinates $(r,\theta) \in [0,1] \times S^{n-1}$, I will write a map $u \colon X \cup_{f_0} D^n \to X \cup_{f_1} D^n$, and leave it to you to produce the homotopy inverse and the required homotopies.

The idea will be to insert the homotopy on the collar of the boundary. We first give a map

$$\begin{aligned} \bar{g} \colon D^n &\longrightarrow X \cup_{f_1} D^n \\ (r,\theta) &\longmapsto \begin{cases} H(\theta, 2r-1) & \text{if } x = (r,\theta) \text{ with } r \geq 1/2, \\ (2r,\theta) & \text{if } x = (r,\theta) \text{ with } r \leq 1/2. \end{cases} \end{aligned}$$

This is well-defined when $r = 1/2$ because in $X \cup_{f_1} D^n$ the points $H(\theta,0) = f_1(\theta) \in X$ and $(1,\theta) \in D^n$ were identified.

We claim that this fits into a commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "f_0"] \arrow[d, "i"] & X \arrow[d, "j"] \arrow[rdd, bend left, "\bar{j}"] & \\ D^{n-1} \arrow[r, "g"] \arrow[rrd, bend right, "\bar{g}"'] & X \cup_{f_0} D^n \arrow[rd, dotted, "u"] & \\ & & X \cup_{f_1} D^n, \end{tikzcd}$$

with $\bar{j} \colon X \to X \cup_{f_1} D^n$ the inclusion. This would produce uniquely the continuous map u. This claim amounts to the statement $\bar{g}(1,\theta) = H(\theta,1) = f_0(\theta)$, using that H is a homotopy from f_1 to f_0. $\square$

We can also attach multiple n-cells at the same time: set $(B,A) = (\bigsqcup_i D_i^n, \bigsqcup_i S_i^{n-1})$ and take the pushout

$$\begin{tikzcd} \bigsqcup_i S_i^{n-1} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \bigsqcup_i D_i^n \arrow[r] & X'. \end{tikzcd}$$

7.1.3 Increasing unions and sequential colimits

Given a sequence of inclusions

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

we can topologize the union $X \coloneqq \bigcup_n X_n$ by declaring a set $C \subset X$ to be a closed if and only if all of its intersections $C \cap X_n$ are. This is a particular example of a sequential colimit, again a concept in category theory. Given a diagram

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

in C, a *sequential colimit* in an object P with morphisms $f_n \colon X_n \to P$ such that

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[dl, "f_1"] & X_2 \arrow[r] \arrow[dll, "f_2"] & \cdots \\ P \end{tikzcd}$$

commutes, and for every other commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[dl, "\bar{f}_1"] & X_2 \arrow[r] \arrow[dll, "\bar{f}_2"] & \cdots \\ Q \end{tikzcd}$$

there exists a unique morphism $u \colon P \to Q$ such that

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dd, bend left] & X_1 \arrow[r] \arrow[dl] \arrow[ddl, bend left] & X_2 \arrow[r] \arrow[dll] \arrow[ddll, bend left] & \cdots \\ Q \arrow[d, dashed, "u"] \\ P \end{tikzcd}$$

commutes. This is the universal property of a sequential colimit, and we usually denote P as $\mathrm{colim}_{n\to\infty} X_n$. The caveat of Remark 7.1.5 holds; being a sequential colimit is not just a property of P but also the maps f_n. As in Lemma 7.1.6 it is unique up to isomorphism.

As the topology on the union $X = \bigcup_n X_n$ has the property that a map $g \colon X \to Y$ is continuous if and only if all of its restrictions $g|_{X_n} \colon X_n \to Y$ are, it follows that this is the sequential colimit of the diagram

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

with horizontal maps given by the inclusions.

7.2 CW-complexes

7.2.1 Definitions

A CW-complex is a topological space obtained by cell attachments, performed in order of dimension.

Definition 7.2.1. A *CW-complex* is a topological space with a sequence of subspaces

$$\varnothing = \mathrm{sk}_{-1}(X) \subset \mathrm{sk}_0(X) \subset \mathrm{sk}_1(X) \subset \cdots \subset X$$

such that

- for each n there is a pushout diagram

$$\begin{tikzcd} \bigsqcup_{i\in I_n} S_i^{n-1} \arrow[r, "f"] \arrow[d] & \mathrm{sk}_{n-1}(X) \arrow[d] \\ \bigsqcup_{i\in I_n} D_i^n \arrow[r] & \mathrm{sk}_n(X). \end{tikzcd}$$

- X is the sequential colimit $\mathrm{colim}_{n\to\infty} \mathrm{sk}_n(X)$.

Observe that a CW-complex is really a topological space equipped with some additional structure. A topological space X can be made into a CW-complex in many ways. As a set, X is the union of the interior of its cells.

Example 7.2.2. An n-sphere S^n can be given a CW-structure with a single 0-cell and a single n-cell, but also a CW-structure with two k-cells for $0 \leq k \leq n$. The latter is built inductively: S^n is obtained from S^{n-1} by attaching two hemispheres.

The advantage of the "more wasteful" CW-structure is that it is equivariant with respect to reflection. Reflection swaps the two k-cells, and so we get a CW-structure on the quotient $S^n/\{\pm 1\}$. This is a model for the real projective space $\mathbb{R}P^n$ of lines through the origin in $\mathbb{R}^{n+1}$. That is, $\mathbb{R}P^n$ can be made into a CW-complex with a single k-cell for $0 \leq k \leq n$.

Example 7.2.3. It is a consequence of Morse theory that every smooth manifold admits the structure of a CW-complex. For example, Poincaré homology sphere S^3/I^* has a CW-structure with five 0-cells, ten 1-cells, six 2-cells, and a single 3-cell.

The subspace $\mathrm{sk}_n(X)$ is called the *n-skeleton*. By the universal property of sequential colimits, $f\colon X \to Y$ is continuous if and only if all restrictions $f|_{\mathrm{sk}_n(X)}\colon \mathrm{sk}_n(X) \to Y$ are. Iteratively using the universal property of pushouts, this can be rephrased in terms of the characteristic maps $g_i\colon D_i^n \to X$: $f\colon X \to Y$ is continuous if and only if each composition $f \circ g_i\colon D_i^n \to X$ is. This is equivalent to saying that

$$\bigsqcup_n \bigsqcup_{i\in I_n} g_i \colon \bigsqcup_n \bigsqcup_{i\in I_n} D_i^n \longrightarrow X$$

is a quotient map.

7.2.2 Point-set topological properties

Let us end with some point-set topological properties. We first introduce some terminology:

Definition 7.2.4. A CW-complex X is

- *finite-dimensional* if $\mathrm{sk}_n(X) = X$ for some n,
- *of finite type* if it has finitely many cells in each dimension,
- *finite* if it has finitely many cells.

Proposition 7.2.5. *CW-complexes are Hausdorff, and compact if and only if they are finite.*

A subspace A of a CW-complex is a *subcomplex* if it is obtained by taking a subset of the cells. More precisely, for $A \subset X$ to be a subcomplex there should be subsets $J_k \subset I_k$ of the k-cells of X such that for each $n \geq 0$, the sequence $\mathrm{sk}_n(A) \coloneqq A \cap \mathrm{sk}_n(X)$ provides Y with a CW-structure with characteristic maps for the k-cells given by g_i with $i \in J_k$. More generally, a *cellular map* is a continuous map $f \colon X \to Y$ which satisfies $f(\mathrm{sk}_n(X)) \subset \mathrm{sk}_n(Y)$, so the inclusion $A \hookrightarrow X$ of a subcomplex is a cellular map.

Let us prove a generalization of the second part of the above proposition:

Proposition 7.2.6. *Let X be a CW-complex. Then every compact subset $K \subset X$ is contained in a finite subcomplex.*

Proof. We first prove that K is contained in finitely many cells, i.e. in the image of $g_i(D^n_i)$ of finitely many characteristic maps. If not, we can find an infinite sequence $x_0, x_1, \ldots$ of point in K such that all x_i lie in distinct $g_i(D^n_i)$. We claim that $S \coloneqq \{x_0, x_1, \ldots\}$ is closed. This is because its inverse image in the domain $\bigsqcup_n \bigsqcup_{i \in I_n} D^n_i$ consists of at most one point in ecah D^n_i. The same argument works for any subset of S, so S is discrete. Discrete and compact implies finite, leading to a contradiction.

Now we need to prove that every finite collection of cells lies in a finite subcomplex. We prove this by induction over the maximal dimension n. To go from n to $n+1$, we first observe that since a finite union of finite subcomplexes is finite, it suffices to prove that each $(n+1)$-cell lies in a finite subcomplex. The boundary of an $(n+1)$-cell is compact and by the previous part hence is contained in finitely many cells, necessarily of lower dimension. By the induction step these are contained in a finite subcomplex. Adding the $(n+1)$-cell to this gives a finite subcomplex which contains it. $\square$

Another useful property concerns the inclusion $\mathrm{sk}_{n-1}(X) \hookrightarrow \mathrm{sk}_n(X)$; this satisfies the hypothesis for the isomorphism

$$H_*(\mathrm{sk}_n(X), \mathrm{sk}_{n-1}(X)) \xrightarrow{\cong} \tilde{H}_*(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X)).$$

This is useful, as there is a homeomorphism

$$\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X) \xrightarrow{\cong} \bigvee_{i \in I_n} S^n_i.$$

Lemma 7.2.7. *There is an open neighborhood B of $\mathrm{sk}_{n-1}(X)$ in $\mathrm{sk}_n(X)$ which deformation retracts onto $\mathrm{sk}_{n-1}(X)$.*

Proof. Let $A^n := \{(r,\theta) \mid 1/2 \leq r \leq 1\} \subset D^n$, then B is given by the pushout

$$\begin{array}{ccc} \bigsqcup_{i \in I_n} S_i^{n-1} & \xrightarrow{f} & \mathrm{sk}_{n-1}(X) \\ \downarrow & & \downarrow \\ \bigsqcup_{i \in I_n} A_i^n & \longrightarrow & \mathrm{sk}_n(X). \end{array}$$

That is, we add collars on the boundaries of the n-cells D_i^n. The deformation retraction is given by shrinking the size of these collars; I'll leave it to you to give a formula. ☐

7.3 Problems

Problem 7.3.1. Let *complex projective n-space* $\mathbb{C}P^n$ be the topological space of complex lines in $\mathbb{C}^{n+1}$, i.e. the quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$. Give a CW-structure on $\mathbb{C}P^n$ with a single $2k$-cell for $0 \leq k \leq n$.

Problem 7.3.2 (Homology of CW-complexes)**.** Prove the following properties by induction over the skeleton:

(i) If X is a finite-dimensional CW-complex, then there exists an N such that $H_*(X) = 0$ for $* > N$.

(ii) If X is a finite CW-complex, then each homology group $H_n(X)$ is finitely generated.

Problem 7.3.3 (Uniqueness of homology theories on CW-complexes)**.** Suppose E_* and F_* are homology theories, and $\eta_* \colon E_* \to F_*$ is a natural transformation such that (i) $\partial \circ \eta_* = \eta_{*-1} \circ \partial$ and (ii) $\eta_* \colon E_*(\mathrm{pt}) \to F_*(\mathrm{pt})$ is an isomorphism.

(i) Prove that $\eta_* \colon E_*(S^n) \to F_*(S^n)$ is an isomorphism for all $n \geq 0$.

(ii) Prove that $\eta_* \colon E_*(X) \to F_*(X)$ is an isomorphism for all finite CW-complexes X.

(iii) Outline a proof that $\eta_* \colon E_*(X) \to F_*(X)$ is an isomorphism for all CW-complexes X.

Homology of sequential colimits

Definition 7.3.4. Given a sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots$$

of abelian groups and homomorphisms, the (sequential) *colimit* $\mathrm{colim}_{i\to\infty} A_i$ is the quotient

$$\left(\bigsqcup A_i\right)/\sim$$

with $\sim$ the equivalence relation generated by $(i,a) \sim (i+1, f_i(a))$.

This is an abelian group with homomorphisms $A_i \to \mathrm{colim}_{i\to\infty} A_i$ fitting into a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[rr] \arrow[d] & & A_2 \arrow[r] \arrow[dll] & A_3 \arrow[r] \arrow[dlll] & \cdots \\
\mathrm{colim}_{i\to\infty} A_i. & & & &
\end{tikzcd}$$

Problem 7.3.5 (Sequential colimits of abelian groups)**.** Prove that, as the notation suggests, $\mathrm{colim}_{i\to\infty} A_i$ is the sequential colimit in the category of abelian groups.

Problem 7.3.6 (Homology of sequential colimits of chain complexes)**.**

(i) Let $A^0_* \to A^1_* \to A^2_* \to \cdots$ by a sequence of chain complexes and chain maps. Define the sequential colimit chain complex $\mathrm{colim}_{n\to\infty} A^n_*$.

(ii) Use the universal property of colimits to construct a map

$$\mathrm{colim}_{n\to\infty} H_*(A^n_*) \longrightarrow H_*(\mathrm{colim}_{n\to\infty} A^n_*).$$

(iii) Prove that this is an isomorphism. We say that "homology commutes with sequential colimits."

Problem 7.3.7 (Homology of sequential colimits of spaces)**.**

(i) Suppose that a topological space Y has a filtration $Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y$ such that every continuous map $f\colon K \to Y$ with K compact, factors over some Y_i. Prove that
$$H_n(Y) \cong \mathrm{colim}_{i\to\infty} H_n(Y_i).$$

(ii) Prove that if X is a CW-complex, then $\mathrm{sk}_0(X) \subset \mathrm{sk}_1(X) \subset \mathrm{sk}_2(X) \subset \cdots \subset X$ has the above property. Conclude that $H_n(X) \cong \mathrm{colim}_{i\to\infty} H_n(\mathrm{sk}_i(X))$.

(iii) Use this to prove the following generalization of Problem 7.3.2 (ii): if X is of finite type, then each homology group $H_n(X)$ is finitely generated.

Chapter 8

CW-homology

In this chapter, we describe a small chain complex which computes the homology of a CW-complex. It is based on the fundamental observation that $H_*(D^n, S^{n-1})$ is concentrated in the single degree n, given by $\mathbb{Z}$.

8.1 The degree of a map between spheres

In Theorem 5.1.6 we computed that $\widetilde{H}_*(S^n)$ vanishes unless $* = n$, in which case it is given by $\mathbb{Z}$. By identifying this group with $\widetilde{H}_n(\partial\Delta^{n+1})$, Example 5.1.7 gives a canonical generator which we denote $[S^n]$. We can use this to define an invariant of maps $S^n \to S^n$.

Definition 8.1.1. The *degree* of a map $f\colon S^n \to S^n$ is the integer $\deg(f) \in \mathbb{Z}$ such that $f_*([S^n]) = \deg(f)[S^n]$.

Since $\widetilde{H}_*$ is a homotopy-invariant functor, this has the following properties:

Lemma 8.1.2.

- $\deg(f)$ *only depends on the homotopy class of* f*,*
- $\deg(\mathrm{id}_{S^n}) = 1$*,*
- $\deg(g \circ f) = \deg(g)\deg(f)$*.*

Example 8.1.3. If $f\colon S^n \to S^n$ is a homotopy equivalence, by definition there exists a $g\colon S^n \to S^n$ such that $g \circ f \simeq \mathrm{id}_{S^n}$. Thus $\deg(g)\deg(f) = \deg(g \circ f) = \deg(\mathrm{id}_{S^n}) = 1$, so $\deg(f) = \pm 1$. It is in fact the case that f is a homotopy equivalence if and only if $\deg(f) = \pm 1$, and proving this amounts to verifying that $\deg\colon [S^n, S^n] \to \mathbb{Z}$ is an isomorphism when $n \geq 1$.

8.1.1 Applications

The degree of orthogonal maps

An orthogonal matrix $A \in O(n+1)$ induces a homeomorphism $f_A\colon S^n \to S^n$, so its degree must be ± 1.

Lemma 8.1.4. *The degree of* f_A *is* 1 *if and only if* $A \in SO(n)$*.*

Proof. Since $O(n)$ has two path components, it suffices to prove that the degree of a reflection is -1. To see this, use that under the identifications $\widetilde{H}_n(S^n) \cong \widetilde{H}_n(\partial\Delta^{n+1}) \cong H_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1})$ the generator $[D^{n+1}, S^n]$ is represented by $\mathrm{id}_{\Delta^{n+1}}$. Under these identification, we can obtain the reflection on S^n as the restriction of the boundary of a map of Δ^{n+1} obtained by exchanging two vertices. This sends $[\mathrm{id}_{\Delta^{n+1}}]$ to $-[\mathrm{id}_{\Delta^{n+1}}]$. □

Thus the degree of $-\mathrm{id}\colon S^n \to S^n$ is $(-1)^{n+1}$. This has the following application:

Proposition 8.1.5. *If $f\colon S^n \to S^n$ has no fixed point then* $\deg(f) = (-1)^{n+1}$.

Proof. Such a map f is homotopic to $-\mathrm{id}$ via the homotopy

$$f_t(x) = \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}.$$

The hypothesis implies that the denominator never vanishes. □

Local computation of degree

The degree of a map can be computed from local data. Suppose that $n \geq 1$, $U \subset S^n$ is open, $x \in S^n$, and $f\colon U \to S^n$ such that $f^{-1}(x)$ is compact, then we have a homomorphism

$$\begin{aligned} \widetilde{H}_n(S^n) &\longrightarrow H_n(S^n, S^n \setminus f^{-1}(x)) \xleftarrow{\cong} H_n(U, U \setminus f^{-1}(x)) \\ &\xrightarrow{f_*} H_n(S^n, S^n \setminus \{x\}) \xrightarrow{\cong} \widetilde{H}_n(S^n) \end{aligned}$$

with associated degree $\deg_x(f)$. Comapactness is used to justify excision in the first isomorphism.

We enlarge $f^{-1}(x)$ to a compact K and shrink U to an open V, as long as $K \subset V$, and get the same degree. This follows from the commutative diagram

$$\begin{tikzcd} & H_n(S^n, S^n \setminus f^{-1}(x)) & H_n(U, U \setminus f^{-1}(x)) \arrow[l] \arrow[dr] & \\ \widetilde{H}_n(S^n) \arrow[ur] \arrow[dr] & & & H_n(S^n, S^n \setminus \{x\}) \\ & H_n(S^n, S^n \setminus K) \arrow[u] & H_n(V, V \setminus K) \arrow[l] \arrow[u] \arrow[ur] & \end{tikzcd}$$

In particular, if $U = S^n$ we may take $K = S^n$ and deduce that $\deg_x(f) = \deg(f)$.

If V is a disjoint union $\bigcup_{j=1}^r V_j$ and all $f^{-1}(V_j)$ are disjoint, this gives a formula for the degree of f as a sum of local degrees

$$\deg_x(f) = \sum_{j=1}^{r} \deg_x(f|_{V_j}).$$

Example 8.1.6. This can be used to construct maps $S^n \to S^n$ of arbitrary degrees $k \in \mathbb{Z}$. Thus the homomorphism $\deg\colon [S^n, S^n] \to \mathbb{Z}$ is surjective.

We will want to apply this when $f\colon S^n \to S^n$ is smooth near $f^{-1}(x)$. Let me use some terminology from differential topology. We then want to assume that x is a regular value of f. This means that the derivative $d_y f$ of f is surjective at all points y in $f^{-1}(x)$. By the inverse function theorem there are orientation-preserving charts around x and disjoint orientation-preserving around each y such that f looks like a (necessarily invertible) linear map. Letting V_y denote the image of the chart around y, an argument as in Lemma 8.1.4 shows the local degree $\deg_x(V_y) = \pm 1$ and equal to 1 if and only if $d_y f$ is preserves the orientation. In that case, this above formula says that

$$\begin{aligned}\deg_x(f) = \#\{y \in f^{-1}(x) \text{ with } d_y f \text{ orientation-preserving}\}\\ - \#\{y \in f^{-1}(x) \text{ with } d_y f \text{ orientation-reserving}\}.\end{aligned}$$

The fundamental theorem of algebra

The unit complex numbers endow S^1 with a multiplication

$$\mu\colon S^1 \times S^1 \longrightarrow S^1.$$

Thus, given two maps $f, g\colon S^1 \to S^1$ we can form another map

$$\mu(f,g)\colon S^1 \xrightarrow{\Delta} S^1 \times S^1 \xrightarrow{f\times g} S^1 \times S^1 \xrightarrow{\mu} S^1,$$

with $\Delta\colon S^1 \to S^1 \times S^1$ the diagonal map $x \mapsto (x,x)$.

Lemma 8.1.7. $\deg(\mu(f,g)) = \deg(f) + \deg(g)$.

Proof. By a generalization of Problem 6.5.2, we know that the induced map

$$(\pi_1)_* \oplus (\pi_2)_*\colon H_n(S^n \times S^n) \longrightarrow H_n(S^n) \oplus H_n(S^n) = \mathbb{Z} \oplus \mathbb{Z}$$

is an isomorphism. Generators are given by $(i_1)_*[S^1]$ and $(i_2)_*[S^1]$ with $i_j\colon S^1 \to S^1 \times S^1$ the inclusion whih takes the other component to be 1. From this, we deduce that $\Delta_*[S_1] = (i_1)_*[S^1] + (i_2)_*[S^1]$ and $\mu_*(a(i_1)_*[S^1] + b(i_2)_*[S^1]) = (a+b)[S^1]$. The result then follows from the computation

$$\begin{aligned}(f \times g)_*((i_1)_*[S^1] + (i_2)_*[S^1]) &= (i_1 \circ f)_*[S^1] + (i_2 \circ g)_*[S^1]\\ &= \deg(f) i_*[S^1] + \deg(g) i_*[S^1]. \square\end{aligned}$$

Note that the map $z \mapsto z^2$ is $\mu(\mathrm{id}, \mathrm{id})$ so has degree 2. By induction $\deg(z \mapsto z^k) = k$.

Corollary 8.1.8 (Fundamental theorem of algebra)**.** *Any non-constant complex polynomial has a zero.*

Proof. For contradiction, let p be a non-constant polynomial without a zero and write $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. By replacing $p(z)$ by $p(Rz)$ for small $R > 0$ if necessary, we may assume that $|a_n| + \cdots + |a_1| < |a_0|$. First consider the map

$$\begin{aligned}\widehat{p}\colon S^1 &\longrightarrow S^1\\ z &\longmapsto \frac{p(z)}{||p(z)||}.\end{aligned}$$

Taking $\frac{p(tz)}{||p(tz)||}$, the map $\widehat{p}$ is homotopic to a constant map so has degree 0.

Next consider the polynomials

$$p_t(z) := a_n z^n + (1-t)(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$$

for $t \in [0,1]$, which have no zeroes on the unit circle since $|a_n| + \cdots + |a_1| < |a_0|$. Hence taking the maps $S^1 \to S^1$ given by $\frac{p_t(z)}{||p_t(z)||}$, we see that $\widehat{p}$ is homotopic to $z \mapsto z^n$ so has degree n. Since $n > 0$ because p is not constant, this gives a contradiction. $\square$

8.2 CW-homology

Let X be a CW-complex. Recall that $\mathrm{sk}_{n-1}(X) \subset \mathrm{sk}_n(X)$ has an open neighborhood which deformation retracts onto $\mathrm{sk}_{n-1}(X)$ and that $\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X)$ is homeomorphic to a wedge $\bigvee_{i \in I_n} S_i^n$ of n-spheres indexed by the n-cells of X. Thus $H_*(\mathrm{sk}_n(X), \mathrm{sk}_{n-1}(X)) = 0$ vanishes unless $* = n$, in which case there is a natural isomorphism

$$H_n(\mathrm{sk}_n(X), \mathrm{sk}_{n-1}(X)) \xrightarrow{\cong} \mathbb{Z}^{\oplus I_n}.$$

We shall denote these groups by $C_n^{\mathrm{CW}}(X)$ and will show that they assemble to a chain complex

$$\cdots \longrightarrow C_n^{\mathrm{CW}}(X) \xrightarrow{d} C_{n-1}^{\mathrm{CW}}(X) \longrightarrow \cdots$$

whose homology is isomorphic to $H_*(X)$. Though the notation does not reflect this, $C_*(X)$ of course depends on the choice of CW-structure on X; a different CW-structure gives a different chain complex.

Let us first define the differential

$$d\colon C_n^{\mathrm{CW}}(X) \longrightarrow C_{n-1}^{\mathrm{CW}}(X)$$

and verify that $d^2 = 0$. For the sake of readibility, we shorten notation to

$$X^n := \mathrm{sk}_n(X).$$

It is obtained by pasting together pieces from two long exact sequences of pairs

$$\begin{tikzcd} H_n(X^n, X^{n-1}) \arrow[r, "\partial"'] \arrow[rr, bend left, "d"] & H_{n-1}(X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}, X^{n-2}). \end{tikzcd}$$

To see it squares to zero, consider the commuting diagram with $C_*(X)$ for $* = n+1, n, n-1$

appearing on the diagonal:

$$
\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
H_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial"'] \arrow[dr, "d"] & & H_{n-1}(X^{n-2}) \arrow[d] \\
H_n(X^{n-1}) \arrow[r] \arrow[d] & H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[dr, "d"'] & H_{n-1}(X^{n-1}) \arrow[d] \\
H_n(X^{n+1}) \arrow[d] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}
$$

Now observe that the composition of the middle two horizontal maps is zero, as they are part of the same long exact sequence.

Theorem 8.2.1. *There are isomorphisms $H_*(C_*(X)) \cong H_*(X)$.*

Proof. By induction over the long exact sequence of the triples $X^k \subset X^{k+1} \subset X^n$ we conclude that $H_*(X^n, X^k) = 0$ if $* \leq k$ or $* \geq n+1$. Taking $k \leq n-2$, this give us the zero entries in the commutative diagram (similar to the one used above, but constructed from long exact sequences of triples instead of pairs)

$$
\begin{tikzcd}
& \vdots \arrow[d] & & \\
& H_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial_{n+1}"'] \arrow[dr, "d_{n+1}"] & & 0 \arrow[d] \\
0 \arrow[r] & H_n(X^n, X^k) \arrow[r, "i_*"'] \arrow[d] & H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & H_{n-1}(X^{n-1}, X^k) \arrow[d, "j_*"] \\
& H_n(X^{n+1}, X^k) \arrow[d] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d] \\
& 0 & & \vdots
\end{tikzcd}
$$

This gives that

$$
\begin{aligned}
H_n(X^{n+1}, X^k) &= H_n(X^n, X^k)/\mathrm{im}(\partial^1) & & \\
&= \mathrm{im}(i_*)/\mathrm{im}(i_*\partial_{n+1}) & & i_* \text{ is injective} \\
&= \ker(\partial_n)/\mathrm{im}(d_{n+1}) & & \text{row is exact} \\
&= \ker(j_*\partial_n)/\mathrm{im}(d_{n+1}) & & j_* \text{ is injective} \\
&= \ker(d_n)/\mathrm{im}(d_{n+1}). & &
\end{aligned}
$$

It remains to identify $H_n(X^{n+1}, X^k)$ with $H_n(X, X^k)$. To do so, one takes a colimit as $n \to \infty$ to see that $H_*(X, X^k) = 0$ if $* \leq k$ and uses the long exact sequence

$$\cdots \longrightarrow H_*(X^{n+1}, X^k) \longrightarrow H_*(X, X^k) \longrightarrow H_*(X, X^{n+1}) \longrightarrow \cdots$$

we conclude $H_*(X^{n+1}, X^k) \to H_*(X, X^k)$ is an isomorphism when $* \leq n$. □

If $f\colon X \to Y$ is a cellular map between CW-complexes, it induces maps

$$f_*\colon H_*(X^n, X^{n-1}) \longrightarrow H_*(Y^n, Y^{n-1}),$$

compatible with the long exact sequence used to construct the differential. Thus it induces a chain map $f_*\colon C_*(X) \to C_*(Y)$. The previous argument is natural in cellular maps, and proves that the following diagram commutes

$$\begin{tikzcd} H_*(C_*(X)) \arrow[r, "f_*"] \arrow[d, "\cong"'] & H_*(C_*(Y)) \arrow[d, "\cong"] \\ H_*(X) \arrow[r, "f_*"] & H_*(Y). \end{tikzcd}$$

Let us end with the explicit identification of the differential. It is a homomorphism

$$d\colon \mathbb{Z}^{\oplus I_n} = H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2}) = \mathbb{Z}^{\oplus I_{n-1}},$$

so is uniquely determined by the components

$$\begin{tikzcd} \mathbb{Z} \arrow[r, dashed] \arrow[d, "\text{inclusion of } i \in I_n"'] & \mathbb{Z} \\ \mathbb{Z}^{\oplus I_n} \arrow[r, "d"] & \mathbb{Z}^{\oplus I_{n-1}}. \arrow[u, "\text{projection to } i' \in I_{n-1}"'] \end{tikzcd}$$

As a homomorphism, the dashed map is multiplication by some integer. From the commutative diagram

$$\begin{tikzcd} H_n(D_i^n, S_i^{n-1}) \arrow[r, "\partial", "\cong"'] \arrow[d] & H_{n-1}(S_i^{n-1}) \arrow[d] \arrow[drr] & & \\ H_n(X^n, X^{n-1}) \arrow[r, "\partial"] & H_{n-1}(X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & H_{n-1}(D_{i'}^{n-1}, S_{i'}^{n-2}) \end{tikzcd}$$

we see it is the degree of the map

$$S_i^{n-1} \longrightarrow D_{i'}^{n-1}/S_{i'}^{n-2}$$

given by the composition of the attaching map of the ith n-cell with the map $X^{n-1} \to D_{i'}^{n-1}/S_{i'}^{n-2}$ collapsing all except the interior of the i'th $(n-1)$-cell to a point.

In practice, the techniques of Section 8.1.1 are used to compute this: after a homotopy we may assume the attaching map is smooth near the pre-image of a point in the interior of the i'th $(n-1)$-cell and has this point as a regular value, and then we count the pre-images with sign. In practice, this means counting how many times the attaching map "goes across the $(n-1)$-cell, counted with orientation." For example, this is really the technique we were using in Section 6.4.1.

8.3 Examples

8.3.1 Real projective spaces

In Example 7.2.2 we gave a CW-structure on S^n with two k-cells for all $0 \leq k \leq n$. Some reflection upon the attaching maps shows that the cellular chain complex $C_*(S^n)$ is given by

$$\mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}} \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \mathbb{Z}^2 \longleftarrow \cdots \longleftarrow \mathbb{Z}^2$$

with differential given alternatively by the indicated matrices. One can recover the homology of S^n from this, but that's not goal. Instead, we want to compute the homology of $\mathbb{R}P^n = S^n/\{\pm 1\}$. The indicated CW-structure has a single k-cell for all $0 \leq k \leq n$ and the differential can be determined by observing that the quotient map $S^n \to \mathbb{R}P^n$ is a cellular map; it identifies the generators of the copies of $\mathbb{Z}$ (up to a sign in even degrees, because reflection then has odd degree). Thus we get that the cellular chain complex $C_*(\mathbb{R}P^n)$ is given by

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \longleftarrow \cdots \longleftarrow \mathbb{Z}$$

with differential given alternatively by 0 and 2. We conclude that

$$H_*(\mathbb{R}P^{2n}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}/2 & \text{if } * = 1, 3, \cdots, 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$H_*(\mathbb{R}P^{2n+1}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2n+1, \\ \mathbb{Z}/2 & \text{if } * = 1, 3, \cdots, 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

8.3.2 Euler characteristics

If A_* is a graded abelian group which has finitely-many non-zero entries, each of which is finitely-generated, we have an invariant

$$\chi(A_*) = \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{rk}(A_n)$$

called the *Euler characteristic*. Let us say A_* is *finitely-generated* if it satisfies the above assumptions.

If X is a finite CW-complex, then its homology $H_*(X)$ satisfies the above assumptions and we can define

$$\chi(X) \coloneqq \chi(H_*(X)).$$

Proposition 8.3.1. *For a finite CW-complex X, we have*

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \#\{n\text{-}cells\}.$$

This follows from the application of the following lemma to $C_*(X)$.

Lemma 8.3.2. *If A_* is a finitely-generated chain complex, then $\chi(A_*) = \chi(H_*(A_*))$.*

Proof. This uses the simple fact that if $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely-generated abelian groups, $\chi(A) - \chi(B) + \chi(C) = 0$. (This follows from example from the classification of finitely-generated abelian groups.)

We write B_n, Z_n, and H_n, for the boundaries, cycles, and homology of A_* in degree n. Then $0 \to B_n \to Z_n \to H_n \to 0$ and $0 \to Z_n \to A_n \to B_{n-1}$ are exact. Thus we can write

$$\begin{aligned}
\chi(H_*(A_*)) &= \sum_{n=0}^{\infty} (-1)^n \mathrm{rk}(H_n) \\
&= \sum_{n=0}^{\infty} (-1)^n (\mathrm{rk}(Z_n) - \mathrm{rk}(B_n)) \\
&= \sum_{n=0}^{\infty} (-1)^n (\mathrm{rk}(A_n) - \mathrm{rk}(B_n) - \mathrm{rk}(B_{n-1})) \\
&= \sum_{n=0}^{\infty} (-1)^n \mathrm{rk}(A_n) \\
&= \chi(A_*),
\end{aligned}$$

where the fourth equality uses that the terms $\mathrm{rk}(B_n)$ cancel out. □

Example 8.3.3. Since Σ_g has a CW-structure with one 0-cell, $2g$ 1-cells, and one 2-cell, we have that $\chi(\Sigma_g) = 2 - 2g$. This is consistent with (1.3).

To compute Euler characteristics, one often uses the following consequence of the fact stated at the beginning of the proof of the previous lemma. One might apply it to the long exact sequence of a pair, or Mayer–Vietoris.

Lemma 8.3.4. *Suppose $0 \to A_* \to B_* \to C_* \to 0$ is a short exact sequence of chain complexes. Then if any two of them are finitely-generated so is the third, and*

$$\chi(A_*) - \chi(B_*) + \chi(C_*) = 0.$$

8.4 Problems

Problem 8.4.1 (The degree of a map without antipodal points)**.** Suppose that $f\colon S^n \to S^n$ has no antipodal points, that is, $f(x) \neq -x$ for all $x \in S^n$. Prove that $\deg(f) = 1$.

Problem 8.4.2 (Top homology of CW-complexes)**.** Use CW-homology to prove that if X is an n-dimensional CW-complex, then $H_n(X)$ is a free abelian group.

Problem 8.4.3 (Homology of complex projective planes)**.** Use Problem 7.3.1 to compute $H_*(\mathbb{C}P^n)$.

Problem 8.4.4 (The homology of geometric realizations)**.** Recall $||X_\bullet||$ from Definition 2.4.8.

(i) Show that $||X_\bullet||$ admits a CW structure with a single k-cell for each k-simplex in X_k.

(ii) Using CW-homology to prove that $H_*(||\mathrm{Sin}_\bullet(X)||) \cong H_*(X)$.

Problem 8.4.5 (Euler characteristic is multiplicative)**.** Prove that if X and Y are finite CW-complexes, then

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

Fundamental theorem of algebra for quaternions

The quaternions $\mathbb{H}$ are the $\mathbb{R}$-algebra with generators i, j, k and relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$ and $ki = j$. Every quaternion can be written uniquely as $a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$. There is a conjugation operation $x \mapsto \bar{x}$ sending $a + bi + cj + dk$ to $a - bi - cj - dk$. Then $x\bar{x}$ is always real (that is, $b = c = d = 0$) and we can define a norm $||x|| = \sqrt{x\bar{x}}$. The quaternions $S(\mathbb{H}) \cong S^3$ of unit norm thus have a multiplication on them. Warning: though this multiplication is associative, it is *not* commutative.

Problem 8.4.6 (Fundamental theorem of algebra for quaternions)**.**

(i) Prove that the map $z \mapsto z^k$ on $S(\mathbb{H})$ has degree k.

(ii) Prove that for non-zero quaternions $a_0, \ldots, a_k$, the map

$$z \mapsto \frac{a_0 z a_1 z \cdots a_{k-1} z a_k}{||a_0 z a_1 z \cdots a_{k-1} z a_k||}$$

on $S(\mathbb{H})$ has degree k.

(iii) A polynomial on the quaternions is a map $\mathbb{H} \to \mathbb{H}$ of the form

$$p(z) = a_0 z a_1 z \cdots a_{k-1} z a_k + \phi(z)$$

where ϕ is a finite sum of terms $b_0 z b_1 z \cdots a_{k'-1} z b_{k'}$ for $k' < k$. Prove that any non-constant such polynomial has a zero in $\mathbb{H}$.

The Poincaré homology sphere

Definition 8.4.7. Let $I \subset SO(3)$ be the finite group of rotations which are symmetries of the icosahedron (placed at the origin). Then the *Poincaré homology sphere* P is the quotient $SO(3)/I$.

For the following problem, you might as well ignore the previous definition.

Problem 8.4.8 (The homology of the Poincaré homology sphere)**.** Another way to construct P is to take the solid dodecahedron[1] in $\mathbb{R}^3$, and identify opposite faces (which are pentagons) by a 36° clockwise rotation.

[1]The icosahedron and dodecahedron are dual platonic solids, so the appearance of the dodecahedron here should not greatly surprise you.

Figure 8.1 Find the Dodecahedron. This is Escher's Reptiles, https://en.wikipedia.org/wiki/Reptiles_(M._C._Escher).

(i) Explain how this gives rise to a CW structure on P with five 0-cells, ten 1-cells, six 2-cells, and a single 3-cell.

(ii) Compute $\chi(P)$.

(iii) Compute $H_*(P)$.

Remark 8.4.9. P is a 3-dimensional closed orientable manifold whose fundamental group is the binary icosahedral group I^*, an extension of I by $\mathbb{Z}/2$. In particular, P is *not* homotopy equivalent to S^3.

The Vietoris–Rips complex

The following is an important concept in *topological data analysis.* You might want to use Problem 8.4.4.

Definition 8.4.10. Given a finite metric space (M, d) and real number $\epsilon > 0$, the *Vietoris–Rips complex* $V(M, \epsilon)$ is the topological space defined by the following procedure.

First we pick an arbitrary order $\prec$ on the finitely many points in M. Next, we define a semisimplicial set $V_\bullet(M, \epsilon)$ by letting $V_k(M, \epsilon)$ by the set of $(k + 1)$-tuples $x_0 \prec \cdots \prec x_k$ of distinct elements such that $\max(d(x_i, x_j) \mid 0 \leq i, j \leq k) \leq \epsilon$, and taking $d_i(x_0 \prec \cdots \prec x_k) \coloneqq x_0 \prec \cdots \prec \hat{x}_i \prec \cdots \prec x_k$. Finally, we take the geometric realization to get a topological space:

$$V(M, \epsilon) \coloneqq ||V_\bullet(M, \epsilon)||.$$

Problem 8.4.11 (Homology of the Vietoris–Rips complex)**.**

(i) Explain why $V(M, \epsilon)$ is independent, up to homeomorphism, of the order $\prec$. You do not need to give proofs.

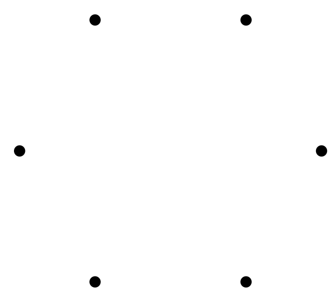
(ii) Suppose M is non-empty. Prove that there exists an ϵ_1 such that $V(M,\epsilon)$ is contractible for every $\epsilon > \epsilon_1$. Similarly, prove that there exist an ϵ_0 such that $V(M,\epsilon) \simeq M$ (the underlying set, i.e. considered as a discrete topological space) for every $\epsilon < \epsilon_0$.

(iii) Prove that each homology group $H_i(V(M,\epsilon))$ is finitely-generated for each finite metric space (M,d) and ϵ.

(iv) Give a bound N depending on the number $\#M$ of points in M, such that $H_i(V(M,\epsilon)) = 0$ for $i > N$. Prove that it is sharp.

The idea is to extract information about the "shape" of (M,d) from the homology of $V(M,\epsilon)$ as ϵ varies.

(v) Let M be the subset of $\mathbb{R}^2$ given below, with its induced metric:



Explain how $H_*(V(M,\epsilon))$ varies as ϵ does, and explain what this tells you about the "shape" of (M,d). You do not need to give proofs.

Chapter 9

Homology with coefficients

Today we modify the definition of homology to take values in R-modules instead of abelian groups; this may simplify our computations or focus our attention on features of particular interest.

9.1 Homology with coefficients in a ring

By construction, the homology groups $H_n(X)$ of a topological space X are abelian groups. Abelian groups can be rather complicated, though we do have a classification of finitely generated ones. The classification of vector spaces over a field $\mathbb{F}$ is much easier: for each cardinal κ, there is up to isomorphism a single vector space of dimension κ.

It is easy to see how to replace $\mathbb{Z}$ by $\mathbb{F}$, or in fact any commutative ring R, by recalling the construction of homology as a composition of functors

$$\mathsf{HoTop} \xrightarrow{\mathrm{Sin}_\bullet} \mathsf{ssSet} \xrightarrow{\mathbb{Z}[-]} \mathsf{Ch}_{\mathbb{Z}} \longrightarrow \mathsf{GrAb}.$$

The first functor takes the singular semisimplicial set $X_\bullet$ with $X_n = \{\Delta^n \to X\}$, and the second constructs from this a chain complex $S_*(X) = \mathbb{Z}[\mathrm{Sin}_\bullet(X)]$ of abelian groups:

$$S_n(X) = \mathbb{Z}[\{\Delta^n \to X\}], \qquad d = \sum_{i=0}^{n} (-1)^i d_i.$$

We can replace the free abelian group $\mathbb{Z}[\{\Delta^n \to X\}]$ by the free R-module $R[\{\Delta^n \to X\}]$ and use the same formula for the differential d, which now is a homomorphism of R-modules; the result is a chain complex $S_*(X;R)$ of R-modules. Taking homology we now get *homology groups with coefficients in* R:

$$H_n(X;R) := \frac{\ker[d\colon S_n(X;R) \to S_{n-1}(X;R)]}{\mathrm{im}[d\colon S_{n+1}(X;R) \to S_n(X;R)]}.$$

That is, we take the composition of functors

$$\mathsf{HoTop} \xrightarrow{\mathrm{Sin}_\bullet} \mathsf{ssSet} \xrightarrow{R[-]} \mathsf{Ch}_R \longrightarrow \mathsf{GrMod}_R,$$

where Ch_R is the category of chain complexes in R and GrMod_R is the category of graded R-modules.

Remark 9.1.1. Since we have taken R to be a commutative, we can afford to be a bit careless with the distinction between left and right R-modules. We will continue doing so throughout this lecture.

Homology with coefficients in R satisfies the Eilenberg–Steenrod axioms. The dimension axiom says the homology of a point vanishes unless $* = 0$; for $H_*(-)$ this is $\mathbb{Z}$, but for $H_*(-; R)$ it is rather R. This has a number of predictable consequences: for example,

$$H_*(S^n; R) = \begin{cases} R & \text{if } * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the computational techniques we developed for homology of CW-complexes go through as long as we replace $\mathbb{Z}$'s for every cell by R's.

9.1.1 Real projective spaces revisited

The common choices of commutative rings R are: the integers $\mathbb{Z}$, fields such as $\mathbb{Q}$ and $\mathbb{F}_p$, and localizations like $\mathbb{Z}[1/p]$. You use the field $\mathbb{F}_p$ when are you only interested in p-torsion, and $\mathbb{Z}[1/p]$ if you are interested in everything except p-torsion.

We illustrate by the example $\mathbb{R}P^n$. We know that 2-torsion appears, so let us compute $H_*(\mathbb{R}P^n; \mathbb{F}_2)$. ($\mathbb{F}_2$ is particular nice, since there are no sign issues.) If we use the standard CW structure, there is a single k-cell of each dimension $0 \leq k \leq n$. Thus, the cellular chain complex $C_*(\mathbb{R}P^n; \mathbb{F}_2)$ is given by $C_k(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2$ for $0 \leq k \leq n$. The degree of each attaching map is either 0 or multiplication by 2, both of which reduce to 0 modulo 2. Thus the differential vanishes,

$$\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \cdots \xleftarrow{0} \mathbb{F}_2 \longleftarrow 0 \longleftarrow \cdots,$$

and we conclude that

$$H_*(\mathbb{R}P^n; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } 0 \leq * \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Not only do we see the 2-torsion appear, but it in fact doubled. This phenomena will be made precise when we prove the universal coefficients theorem, which tells us how to compute $H_*(X; R)$ from $H_*(X)$.

On the other hand, let us compute $H_*(\mathbb{R}P^n; \mathbb{Z}[1/2])$. As before, the cellular homology chain complex $C_*(\mathbb{R}P^n; \mathbb{Z}[1/2])$ is given by $C_k(\mathbb{R}P^n; \mathbb{Z}[1/2]) = \mathbb{Z}[1/2]$ for $0 \leq k \leq n$. The degree of each attaching map is either 0 or multiplication by 2, and now 2 is an isomorphism.

$$\mathbb{Z}[1/2] \xleftarrow{0} \mathbb{Z}[1/2] \xleftarrow{\cong} \mathbb{Z}[1/2] \xleftarrow{0} \mathbb{Z}[1/2] \cdots \longleftarrow \mathbb{Z}[1/2] \longleftarrow 0 \longleftarrow \cdots,$$

and now there are two cases, depending on whether n is even or odd:

$$H_*(\mathbb{R}P^{2n}; \mathbb{Z}[1/2]) = \begin{cases} \mathbb{Z}[1/2] & \text{if } * = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_*(\mathbb{R}P^{2n+1}; \mathbb{Z}[1/2]) = \begin{cases} \mathbb{Z}[1/2] & \text{if } * = 0, 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

9.2 Homology with coefficients in a module

To understand the relation between homology groups with different coefficients, we need to study the tensor product construction. In this section we recall some facts about tensor products, and use these to generalize $H_*(X;R)$ to $H_*(X;M)$ for an R-module M.

9.2.1 Tensor products

Let us recall some facts from commutative algebra.

Definition 9.2.1. Let R be a commutative ring and M, N be R-modules. Then *tensor product* $M \otimes_R N$ is the R-module defined by taking the quotient of the free R-module

$$R[\{(m,n) \mid m \in M, n \in N\}]$$

by the equivalence relation generated by

- $(rm,n) \sim r(m,n) \sim (m,rn)$,
- $(m+m',n) \sim (m,n)+(m',n)$,
- $(m,n+n') \sim (m,n)+(m,n')$.

We denote the equivalence class of (m,n) by $m \otimes n$.

Warning 9.2.2. A general element of $M \otimes_R N$ is a finite sum $\sum_i r_i m_i \otimes n_i$, and can't be reduced to a single term $m \otimes n$ by applying the relations. A single term $m \otimes n$ is also referred to as an *indecomposable tensor*.

This definition may seem to come out of nowhere, but is determined by a universal property. That is, we will give a characterization of homomorphisms of R-modules out of $M \otimes_R N$, which determines it uniquely up to isomorphism.

Definition 9.2.3. Let P be an R-module. A function $\beta\colon M \times N \to P$ is *bilinear* if it satisfies

- $\beta(rm,n) = r\beta(m,n) = \beta(m,rn)$,
- $\beta(m+m',n) = \beta(m,n)+\beta(m',n)$,
- $\beta(m,n+n') = \beta(m,n)+\beta(m,n')$.

These are exactly the relations that the symbols $m \otimes n$ satisfy. Writing β_0 for the function $M \times N \to M \otimes_R N$ given by $(m,n) \mapsto m \otimes n$, this says that β_0 is bilinear. It is in fact the initial bilinear map:

Proposition 9.2.4. *There is bijection between homomorphisms $b\colon M \otimes_R N \to P$ and bilinear maps $\beta\colon M \times N \to P$, in one direction given by $b \mapsto b \circ \beta_0$.*

Sketch of proof. That $b \circ \beta_0$ is bilinear follows easily from the fact that b is a homorphism and β_0 is bilinear.

To prove that $b \mapsto b \circ \beta_0$ is a bijection, we need to show that it is injective and surjective. Injectivity follows from the fact that the map $\beta\colon M \times N \to M \otimes_R N$ has all generators $m \otimes n$ in its image. For surjectivity, we observe that given a bilinear map $\beta \colon M \times N \to P$ the formula

$$m_i \otimes n_i \longmapsto \beta(m_i, n_i)$$

is compatible with the equivalence relation and hence determines a unique homomorphism $b \colon M \otimes_R N \to P$ such that $\beta = b \circ \beta_0$. $\square$

Like pushouts, tensor products are determined uniquely up to isomorphism by this universal property. You are not likely to use the universal property to compute tensor products. It is instead used to prove properties of the tensor products.

Lemma 9.2.5. *Given R-module homomorphisms $f \colon M \to M'$ and $g \colon N \to N'$, there is a unique R-module homomorphism $f \otimes g \colon M \otimes_R N \to M' \otimes_R N'$ such that the following diagram commutes*

$$\begin{tikzcd}
M \times N \arrow[r, "\beta_0"] \arrow[d, "f \times g"] & M \otimes_R \arrow[d, dotted, "f \otimes g"] \\
M' \times N' \arrow[r, "\beta_0'"] & M' \otimes_R N'.
\end{tikzcd}$$

This is compatible with composition: if we are also given R-module homomorphisms $f' \colon M' \to M''$ and $g' \colon N' \to N''$

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g).$$

Proof. By the universal property, it suffices to observe that $\beta_0' \circ (f \times g)$ is a bilinear map $M \times N \to M' \otimes_R N'$. To see it is compatible with composition, observe that both can be inserted in the dotted arrow to make the diagram commute; by uniqueness they are equal:

$$\begin{tikzcd}
M \times N \arrow[r, "\beta_0"] \arrow[d, "f \times g"] & M \otimes_R N \arrow[dd, dotted] \\
M' \times N' \arrow[d, "f' \times g'"] & \\
M'' \times N'' \arrow[r, "\beta_0'"] & M'' \otimes_R N''.
\end{tikzcd}$$

$\square$

This says that tensor product gives a functor

$$\otimes_R \colon \mathsf{Mod}_R \times \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_R.$$

Using the similar arguments using the universal properties, one proves that the tensor product indeed behaves like a product ought to behave:

Proposition 9.2.6. *The tensor product has the following properties:*

- *associativity: $M \otimes_R (N \otimes_R P) \cong (M \otimes_R N) \otimes_R P$,*
- *unitality: $M \otimes_R R \cong M \cong R \otimes_R M$,*

- *commutativity* $M \otimes_R N \cong N \otimes_R M$.

Proof. I will only prove the commutativity, and leave the rest as an exercise for the diligent reader. A homomorphism $f\colon M \otimes_R N \to N \otimes_R M$ is uniquely determined by a bilinear map $\phi\colon M \times N \to N \times N$, as the homomorphism fitting in a commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\beta_0"] \arrow[d, "\phi"] & M \otimes_R N \arrow[d, dotted, "f"] \\ N \times M \arrow[r, "\beta_0'"] & N \otimes_R M. \end{tikzcd}$$

Of course, we will take $\phi(m,n) = (n,m)$. To find the inverse we take $\chi(n,m) = (m,n)$ as a function $N \times M \to M \times N$, from which the universal property produces a homomorphism $g\colon N \otimes_R M \to M \otimes_R N$.

We now prove these are mutually inverse. The composition $g \circ f$ fits into a commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\beta_0"] \arrow[d, "\phi"] & M \otimes_R N \arrow[dd, dotted, "g \circ f"] \\ N \times M \arrow[d, "\chi"] & \\ M \times N \arrow[r, "\beta_0"] & M \otimes_R N, \end{tikzcd}$$

but so does $\mathrm{id}_{M \otimes_R N}$, so $g \circ f = \mathrm{id}$. Similarly, $f \circ g = \mathrm{id}$, so f and g are indeed mutually inverse isomorphisms. □

Remark 9.2.7. Given R-modules $M_1, \ldots, M_n$, we can tensor these together in different order and possible include some R's. By the previous proposition, the resulting module is isomorphic to $M_1 \otimes_R \cdot \otimes_R M_n$. This isomorphism is in fact unique. This is an example of a *coherence result*.

Similar arguments using the universal property tell you that the tensor product is distributive in appropriate senses. Firstly, $- \otimes N$ distributes over direct sums.

Lemma 9.2.8. $(\bigoplus_{i \in I} M_i) \otimes N \cong \bigoplus_{i \in I} (M_i \otimes N)$.

Secondly, $- \otimes g$ distributes over sums of homomorphisms.

Lemma 9.2.9. $(f + f') \otimes g = f \otimes g + f' \otimes g$.

9.2.2 Some examples

We will work out some examples for $R = \mathbb{Z}$. The unitality of tensor product tells us that $A \otimes_{\mathbb{Z}} \mathbb{Z} \cong A$, an isomorphism which is induced by the bilinear map

$$\begin{aligned} A \times \mathbb{Z} &\longrightarrow A \\ (a, m) &\longmapsto ma. \end{aligned}$$

Similarly, the bilinear map

$$\begin{aligned} A \times \mathbb{Z}/n &\longrightarrow A/nA \\ (a, m) &\longmapsto ma \end{aligned}$$

induces an isomorphism

$$A \otimes_{\mathbb{Z}} \mathbb{Z}/n \xrightarrow{\cong} A/nA.$$

To see this, we produce an inverse. From their construction, we see that tensor products preserve surjections. Thus any abelian group A, there is a surjection

$$A \cong A \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Z}/n.$$

Since $n(a \otimes m) = (a \otimes nm) = 0$ in the right hand side, its kernel contains nA and this factors over A/nA. This produces an inverse $A/nA \to A \otimes_{\mathbb{Z}} \mathbb{Z}/n$

Example 9.2.10. We can take $A = \mathbb{Z}/m$. The subgroup of $\mathbb{Z}$ generated by m and n is that generated by $\gcd(m, n)$, so we conclude that $\mathbb{Z}/m \otimes \mathbb{Z}/n \cong \mathbb{Z}/\gcd(m, n)$.

9.2.3 Homology with coefficients in a module

We can use tensor products to define $H_*(X; M)$ for M an R-module. The chain complex $S_*(X; R)$ given by $R[\mathrm{Sin}_\bullet(X)]$ is one of R-modules, so we can take

$$S_n(X; M) \coloneqq S_n(X; R) \otimes_R M.$$

When we define a differential on this by $d \otimes_R M$, the compatibility of tensor products with composition tells us we again have a chain complex of R-modules. Using distributivity, we see that $d = \sum_i (-1)^i d_i \otimes_R M$. *Homology with coefficients in* M is defined as the homology of this chain complex

$$H_n(X; M) \coloneqq \frac{\ker[d\colon H_n(X; M) \to H_{n-1}(X; M)]}{\mathrm{im}[d\colon H_{n+1}(X; M) \to H_n(X; M)]} \in \mathsf{Mod}_R.$$

As before, this satisfies the Eilenberg–Steenrod axioms.

9.2.4 The Bockstein long exact sequence

The construction of the chain complex $S_*(X; M)$ is natural in M: for any homomorphism of R-modules $f\colon M \to M'$ we get a chain map $S_*(X; M) \to S_*(X; M')$ and hence an induced map on homology.

As a first attempt to relate these homology groups, we consider a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0. \tag{9.1}$$

This induces short exact sequences

$$0 \longrightarrow S_n(X) \otimes M \longrightarrow S_n(X) \otimes M' \longrightarrow S_n(X) \otimes M'' \longrightarrow 0$$

since each $S_n(X)$ is a free abelian group and thus this is a just a direct sum of many copies of (9.1). Thus we get a short exact sequence of chain complex

$$0 \longrightarrow S_*(X;M) \longrightarrow S_*(X;M') \longrightarrow S_*(X;M'') \longrightarrow 0$$

and hence a long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X;M') \arrow[r] & H_{n+1}(X;M'') \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_n(X;M) \arrow[r] & H_n(X;M') \arrow[r] & H_n(X;M'') \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_{n-1}(X;M) \arrow[r] & H_{n-1}(X;M') \arrow[r] & \cdots
\end{tikzcd}$$

This is the *Bockstein long exact sequence.*

Of particular interest is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0,$$

giving a long exact sequence relating $H_*(X)$ with $H_*(X;\mathbb{Z}/p)$. In this case the long exact sequence looks like

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X) \arrow[r] & H_{n+1}(X;\mathbb{Z}/p) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_n(X) \arrow[r, "p"] & H_n(X) \arrow[r] & H_n(X;\mathbb{Z}/p) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_{n-1}(X) \arrow[r, "p"] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

with $p\colon H_n(X) \to H_n(X)$ the multiplication-by-p homomorphism of abelian groups. Let me justify this: every $[a] \in H_n(X)$ is represented by some $a = \sum n_i\sigma_i \otimes 1 \in C_n(X) \subset S_n(X) = S_n(X) \otimes \mathbb{Z}$. By construction the left map $H_n(X) \to H_n(X)$ is given by sending this to $\sum n_i\sigma_i \otimes p = p\left(\sum n_i\sigma_i \otimes 1\right)$, which represents $p[a] \in H_n(X)$.

Example 9.2.11. For $X = \mathbb{R}P^n$ with $n > 2$ and $p = 2$, the beginning of this long exact sequence looks like

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\mathbb{R}P^n) = 0 \arrow[r] & H_2(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2 \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_1(\mathbb{R}P^n) = \mathbb{Z}/2 \arrow[r, "2=0"] & H_1(\mathbb{R}P^n) = \mathbb{Z}/2 \arrow[r, "\cong"] & H_1(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2 \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_0(\mathbb{R}P^n) = \mathbb{Z} \arrow[r, "2"] & H_0(\mathbb{R}P^n) = \mathbb{Z} \arrow[r] & H_0(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2.
\end{tikzcd}$$

Another interesting case is

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

Then the long exact sequence has an interesting connecting homomorphism

$$\beta\colon H_n(X;\mathbb{Z}/p) \longrightarrow H_{n-1}(X;\mathbb{Z}/p),$$

called the *Bockstein homomorphism*. Knowledge of β and $H_*(X;\mathbb{Z}/p)$ amounts to knowledge of $H_*(X;\mathbb{Z}/p^2)$.

9.3 Problems

Problem 9.3.1 (A pinch map)**.** Prove that the quotient map $\mathbb{R}P^2 \to \mathbb{R}P^2/\mathbb{R}P^1$ induces the trivial map on $\widetilde{H}_*(-)$, but not on $\widetilde{H}_*(-;\mathbb{Z}/2)$.

Problem 9.3.2 (The square of the Bockstein)**.** Prove that $\beta^2 = 0$.

Chapter 10

Towards the universal coefficients theorem

In this chapter we start with the proof of the universal coefficients theorem, which explains how to compute $H_*(X; M)$ from $H_*(X)$ when R is a PID.

10.1 The failure of right-exactness

10.1.1 Tor_1^R as a measure of the failure of right-exactness

It is not true that tensoring a chain complex of R-modules with M commutes with taking homology: $H_*(C_* \otimes_R M) \neq H_*(C_*; R) \otimes_R M$ in general.

Example 10.1.1. Let us take the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

which you may expand with 0's to the left and right to give a chain complex with trivial homology. If we tensor it with $\mathbb{Z}/2$ and recall that $A \otimes_{\mathbb{Z}} \mathbb{Z}/2 = A/2A$, we get

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

which is not exact. In particular, its homology is not just 0's (which is what we would get by tensoring 0's with $\mathbb{Z}/2$).

That is, $- \otimes_R M$ is not *exact*, i.e. does not preserve exact sequence. However, not all is lost as it is still *right-exact*, which means it is preserves cokernels. (There is also a notion of *left-exact* functors, which preserve kernels).

Lemma 10.1.2. *$N \mapsto N \otimes_R M$ preserves cokernels, so is right-exact.*

Proof. Given an exact sequence

$$N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

N'' is the cokernel of $N' \to N$ if and only if satisfies the following universal property: every homomorphism $N \to P$ such that $N' \to N \to P$ is zero factors uniquely over N'':

$$\begin{tikzcd} N' \arrow[r] \arrow[dr, "0"'] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[dl, dotted, "\exists!"] & 0 \\ & P. & & \end{tikzcd}$$

Thus we ought to prove the same in

$$\begin{tikzcd} N' \otimes_R M \arrow[r] \arrow[dr, "0"'] & N \otimes_R M \arrow[r] \arrow[d] & N'' \otimes_R M \arrow[r] & 0 \\ & P. & & \end{tikzcd}$$

Homomorphisms $N'' \otimes_R M \to P$ are in bijection with bilinear maps $N'' \times M \to P$, so it suffices to show that there is a unique bilinear map which makes the following diagram commute:

$$\begin{tikzcd} N' \times M \arrow[r, "g \times \mathrm{id}"] \arrow[dr, "0"'] & N \times M \arrow[r] \arrow[d, "\beta"] & N'' \times M \arrow[r] \arrow[dl, dotted, "\beta''"] & 0 \\ & P. & & \end{tikzcd}$$

Indeed, from a bilinear map $\beta\colon N \times M \to P$ sending each entry $(g(n'), m)$ to 0 we can construct a function $\beta''\colon N'' \times M \to P$ by lifting $(n'', m) \in N'' \times M$ to $N \times M$ and applying β; all choices get sent to the same element. It is easy to verify that this is bilinear because β is. $\square$

Whenever a desired property fails in mathematics, we should quantify its failure. To do so, we will use that if M is free then $- \otimes_R M$ *is* exact: if $M = R[S]$ is free on a set S, then $N \otimes_R M = \bigoplus_S N$. Thus tensoring an exact sequence

$$0 \longrightarrow N'' \longrightarrow N \longrightarrow N \longrightarrow 0$$

with $M = R[S]$ amounts to taking an S-indexed direct sum of this exact sequence, which is easily seen to still be exact.

To use this, we resolve M by free modules. By picking generators of M we can construct a surjection $F_0 \to M$ with F_0 free. If $F_1 = \ker(F_0 \to M)$ happened to be free as well (this is the case for PID's such as $\mathbb{Z}$), we could define a measure of the failure of exactness by

$$\operatorname{Tor}_1^R(M, N) \coloneqq \ker\left[F_1 \otimes_R N \to F_0 \otimes_R N\right].$$

Indeed, when $0 \to N \to N' \to N'' \to 0$ is exact, then by considering

$$0 \longrightarrow F_* \otimes_R N \longrightarrow F_* \otimes_R N' \longrightarrow F_* \otimes_R N'' \longrightarrow 0$$

as a short exact sequence of chain complexes, we get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Tor}_1^R(M,N) \arrow[r] & \operatorname{Tor}_1^R(M,N') \arrow[r] & \operatorname{Tor}_1^R(M,N'') \arrow[dll, rounded corners] & \\ & M \otimes_R N \arrow[r] & M \otimes_R N' \arrow[r] & M \otimes_R N'' \arrow[r] & 0. \end{tikzcd}$$

Writing $\operatorname{Tor}_0^R(M, N) \coloneqq M \otimes_R N$, we see that Tor_0^R and Tor_1^R are the homology of the rather small chain complex

$$\cdots \longrightarrow 0 \longrightarrow F_1 \otimes_R N \longrightarrow F_0 \otimes_R N \longrightarrow 0 \longrightarrow \cdots$$

obtained by tensoring with N the *truncated free resolution*

$$\cdots \longrightarrow 0 \longrightarrow F_1 \longrightarrow F_0$$

obtained from the *free resolution*

$$\cdots \longrightarrow 0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M.$$

To see that the cokernel of $F_1 \otimes_R N \to F_0 \otimes_R N$ is indeed $M \otimes_R N$, one uses that $- \otimes_R N$ is right-exact.

We will soon prove that the definition of these Tor-groups is independent of the choice of truncated free resolutions, and we will generalize it from PID's to general commutative rings R.

10.1.2 Examples

Let us first do some examples.

Example 10.1.3. If M is a free module F, then we can take $F_0 = F$ and $F_1 = 0$, from which it follows that $\mathrm{Tor}_1^R(F, N) = 0$. Thus is equivalent to the fact that if F is free, then $F \otimes_R -$ is exact.

Example 10.1.4. Suppose that R is a field $\mathbb{F}$. Then R-modules are $\mathbb{F}$-vector spaces, so always free. Thus we can take $F_0 = M$ and $F_1 = 0$, and compute $\mathrm{Tor}_1^{\mathbb{F}}(M, N) = 0$. Equivalently, over a field $M \otimes_{\mathbb{F}} -$ *is* exact.

Example 10.1.5. Let us take $R = \mathbb{Z}$ and $M = \mathbb{Z}/n$. This has a free resolution

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n.$$

Thus $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n, N)$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, N)$ are the cokernel or kernel of the multiplication-by-n homomorphism $n\colon N \to N$. We conclude that

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n, N) = N/nN,$$

which is indeed $\mathbb{Z}/n \otimes_{\mathbb{Z}} N$, and that

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, N) = \ker[n\colon N \to N].$$

In particular, we have that

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(n, m), \qquad \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(n, m).$$

10.2 Tor-groups

10.2.1 Tor-groups for general R

Before proving that our Tor-groups are well-defined, let us define them for a general commutative ring R.

Give an R-module M, we start by building a free resolution of M: picking generators we find a surjection $F_0 \to M$ with F_0 free, which has a kernel K_0. Picking generators again, we find a surjection $F_1 \to K_0$ with F_1 free, etc. The result is a free resolution of M,

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M.$$

By construction, it is an exact chain complex: the image of $F_{i+1} \to F_i$ is K_i, the kernel of $F_i \to F_{i-1}$ (interpreting F_{-1} as M). If we remove M to obtain the truncated free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0$$

we get a chain complex with a chain map to

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow N$$

which induces an isomorphism on homology, i.e. is a quasi-isomorphism.

Definition 10.2.1. Let $F_* \to M$ be a free resolution of an R-module M and N another R-module, then

$$\mathrm{Tor}_n^R(M, N) = H_n(F_* \otimes_R N).$$

Remark 10.2.2. A free resolution of a module contains a lot of information about M: generators, relations between these, relations between the relations ("syzygies"), etc. From the point of view of homotopy theory or homological algebra, all of these are of a similar nature.

10.2.2 The fundamental theorem of homological algebra

To show Tor-groups are well-defined, we shall prove:

Theorem 10.2.3 (Fundamental theorem of homological algebra)**.** *Let M, N be R-modules,*

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$$

be a free resolution, and

$$\cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow N$$

be exact at each E_i. Then each homomorphism $f\colon M \to N$ can be lifted to a chain map $f_*\colon E_* \to F_*$, unique up to chain homotopy.

Proof. Write $F_{-1} = M$ and $E_{-1} = N$, then we will construct components $f_k\colon F_k \to E_k$ of the chain map f_* by induction over k. The initial case $k = -1$ is provided to us.

To complete the induction step from $k-1$ to k, consider the commutative diagram

$$\begin{tikzcd}
F_k \arrow[r, "d"] \arrow[d, dotted] & F_{k-1} \arrow[r, "d"] \arrow[d, "f_{k-1}"] & F_{k-2} \arrow[d, "f_{k-2}"] \\
E_k \arrow[r, "d"] & E_{k-1} \arrow[r, "d"] & E_{k-2}.
\end{tikzcd}$$

Since the right square commutes, $K_{k-1} = \ker[d\colon F_{k-1} \to F_{k-2}]$ is mapped into $L_{k-1} = \ker[d\colon E_{k-1} \to E_{k-2}]$ by f_{k-1}. Since F_* is a free resolution, F_k surjects to K_{k-1}, and similarly since E_* is exact, E_k surjects onto L_{k-1}. For each generator x_i of F_k pick a lift y_i of $f_{k-1}(d(x_i))$ to E_k. Because F_k is free, there is a homomorphism $f_k\colon F_k \to E_k$ uniquely defined by sending x_i to y_i. Since

$$d(f_k(x_i)) = d(y_i) = f_{k-1}(d(x_i)),$$

this homomorphism makes the left square commute.

For the uniqueness up to chain homotopy, we implement the philosophy that uniqueness is just relative existence. We will thus reduce the uniqueness up to chain homotopy to small elaboration of the previous argument.

Just like tensor products of R-modules, there is a tensor product $C_* \otimes_R D_*$ of chain complexes: in degree n it is given by $\bigoplus_{p+q=n} C_p \otimes_R D_q$ and differential on $C_p \otimes_R D_q$ given by $d \otimes \mathrm{id} + (-1)^p \mathrm{id} \otimes d$. Let us define a chain complex Λ_* as

$$\cdots \longrightarrow 0 \longrightarrow \Lambda_1 = R\{t\} \xrightarrow{d} \Lambda_0 = R\{t_0, t_1\}$$

with $d(t) = t_0 - t_1$. A chain homotopy from f_*^0 to f_*^1 is then the same as a chain map $H_*\colon \Lambda_* \otimes F_* \to E_*$ such that $H_*(t_0 \otimes -) = f_*^0$, $H_*(t_1 \otimes -) = f_*^1$. The maps $h_n\colon C_n \to D_{n+1}$ can be extracted as $H_{n+1}(t \otimes -)$.

Now we are in the following situation: instead of F_* we have a free resolution $F_*'' = \Lambda_* \otimes F_*$ with a subcomplex $F_*' = R\{t_0, t_1\} \otimes F_* \subset F_*''$ so that F_k' admits a free complement C_k in F_k'', and instead of looking for f_*, we are looking for an extension $H_*\colon F_*'' \to E_*$ of a given chain map $F_*' \to E_*$. (In the previous case, $F_*'' = F_*$ and $F_*' = 0$.)

Our initial case is now $* = -2$, where we take $H_{-2} = 0$. There is something to check, as this only gives a chain map if $dH_{-1} = f_{-1}^0 - f_{-1}^1 = 0$; this is the case because both chain maps $f_*^0, f_*^1\colon F_* \to E_*$ lift $f\colon M \to N$. For the induction step from $k-1$ to k, consider the commutative diagram

$$\begin{tikzcd}
F_k' \arrow[r] \arrow[d, hook] \arrow[dd, bend right] & F_{k-1}' \arrow[r] \arrow[d, hook] & F_{k-2}' \arrow[d, hook] \\
F_k'' \arrow[r, "d"] \arrow[d, dotted] & F_{k-1}'' \arrow[r, "d"] \arrow[d, "H_{k-1}"] & F_{k-1}'' \arrow[d, "H_{k-2}"] \\
E_k \arrow[r, "d"] & E_{k-1} \arrow[r, "d"] & E_{k-2}.
\end{tikzcd}$$

We still need to construct H_k on the complement C_k to F_k' (in this case given by $R\{t\} \otimes F_{k-1}$). As before, we can pick generators x_i of C_k, and lift $H_{k-1}(d(x_i))$ to E_k. $\square$

Remark 10.2.4. In this argument we didn't need that the entries F_k were free, only that they are *projective*: an R-module P is projective if in each diagram

$$\begin{tikzcd}
 & M \arrow[d, two heads] \\
P \arrow[r] \arrow[ur, dotted] & N
\end{tikzcd}$$

with right map surjective, a dotted map exists making the diagram commute. Free R-modules are always projective, and an R-module is projective if and only if it is a summand of a free module. Over a PID (e.g. $\mathbb{Z}$ or a field) every projective module is free.

10.2.3 The well-defined of Tor-groups

Let us now prove that $\mathrm{Tor}_i^R(M,N)$ is well-defined, i.e. independent of free resolution F_* of M. We will simultaneously prove that they are functorial in both entries.

Suppose that we are given a homomorphism $f\colon M \to M'$, and we pick arbitrary free (or even projective) resolutions $F_* \to M$ and $F'_* \to M'$. Then Theorem 10.2.3 provides a chain map $f_*\colon F_* \to F'_*$ lifting f, unique up to chain homotopy. Thus we get a chain map

$$f_* \otimes \mathrm{id}\colon F_* \otimes_R N \longrightarrow F'_* \otimes_R N$$

unique up to chain homotopy. This induces a unique map $H_*(f_* \otimes \mathrm{id})\colon H_*(F_* \otimes_R N) \to H_*(F'_* \otimes_R N)$, i.e. well-defined homomorphisms

$$f_*\colon \mathrm{Tor}_*^R(M,N) \longrightarrow \mathrm{Tor}_*^R(M',N).$$

In particular, if we had taken $f = \mathrm{id}\colon M \to M$ but different free resolutions of M, we get $f_*\colon F_* \to F'_*$ and also $g_*\colon F'_* \to F_*$. Then $(f_* \otimes \mathrm{id}) \circ (g_* \otimes \mathrm{id})$ lifts the identity on M to a self-map of F_*; but so does the identity of F_* and hence this is chain-homotopic to the identity. Using this and the same argument for $(g_* \otimes \mathrm{id}) \circ (f_* \otimes \mathrm{id})$, we conclude that $f_* \otimes \mathrm{id}$ and $g_* \otimes \mathrm{id}$ induce mutually inverse isomorphisms on $\mathrm{Tor}_*^R(M,N)$. This proves that Tor-groups are well-defined.

Collecting all we have proven, we get the following theorem:

Theorem 10.2.5. *There is a functor* $\mathrm{Tor}_*^R(-,N)\colon \mathsf{Mod}_R \to \mathsf{GrMod}_R$ *extending* $- \otimes_R N$ *in degree* 0*.*

The tensor product is symmetric since R is commutative. The same is true for Tor-groups: $\mathrm{Tor}_*^R(M,N) \cong \mathrm{Tor}_*^R(N,M)$. This can be proven using Theorem 10.2.3.

10.3 The universal coefficients theorem

We will use this technology to finally clarify the relationship between $H_*(X;M)$ and $H_*(X;R)$. You should imagine taking $R = \mathbb{Z}$, and $M = \mathbb{Z}/n$, $\mathbb{Q}$ or $\mathbb{Z}[1/n]$.

Theorem 10.3.1 (Universal coefficients theorem)**.** *Let R be a PID, then there is a natural short exact sequence*

$$0 \longrightarrow H_n(X;R) \otimes_R M \longrightarrow H_n(X;M) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(X);M) \longrightarrow 0,$$

of R-modules, which is split but not naturally so.

Example 10.3.2. Let us use this to compute $H_*(\mathbb{R}P^2;\mathbb{Z}/2)$ from $H_*(\mathbb{R}P^2)$ for $* = 1,2$ (the only interesting cases). For $* = 1$, we get

$$0 \longrightarrow H_1(\mathbb{R}P^2) = \mathbb{Z}/2 \longrightarrow H_1(\mathbb{R}P^2;\mathbb{Z}/2) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z};\mathbb{Z}/2) = 0 \longrightarrow 0$$

the right term vanishing because $\mathbb{Z}$ is free, so $H_1(\mathbb{R}P^2;\mathbb{Z}/2)=\mathbb{Z}/2$. For $*=2$, we get

$$0 \longrightarrow H_2(\mathbb{R}P^2)=0 \longrightarrow H_2(\mathbb{R}P^2;\mathbb{Z}/2) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2;\mathbb{Z}/2)=\mathbb{Z}/2 \longrightarrow 0$$

the right term given by $\mathbb{Z}/\gcd(2,2)$, so $H_2(\mathbb{R}P^2;\mathbb{Z}/2)=\mathbb{Z}/2$ as well.

10.4 Problems

Problem 10.4.1 (Do fields detect integral homology?)**.** Is it true that $\tilde{H}_*(X;\mathbb{Q})$ vanishes and $\tilde{H}_*(X;\mathbb{F}_p)$ vanishes for all primes p, if and only if $\tilde{H}_*(X)$ vanishes? Give a proof or find a counterexample.

10.4.1 Euler characteristic revisited

Problem 10.4.2. (i) Suppose that X is a topological space such that $H_i(X)$ is finitely generated for each i. Prove that $H_i(X;\mathbb{F})$ is finite-dimensional for each field $\mathbb{F}$.

(ii) Suppose that X is a topological space such that $H_i(X)=0$ when $i\geq N$. Prove that $H_i(X;\mathbb{F})=0$ when $i\geq N+1$ for each field $\mathbb{F}$.

Thus under the assumptions of parts (i) and (ii), the following Euler characteristics are well-defined:

$$\chi(X;\mathbb{Z}) := \sum_{i=0}^{\infty}(-1)^i \operatorname{rk} H_i(X) \quad \text{and}$$
$$\chi(X;\mathbb{F}) = \sum_{i=0}^{\infty}(-1)^i \dim H_i(X;\mathbb{F}).$$

(iii) Under the assumptions of parts (i) and (ii), prove that

$$\chi(X;\mathbb{Z})=\chi(X;\mathbb{F}).$$

Conclude that the Euler characteristic is independent of the choice of coefficients.

(iv) Verify by computation that $\chi(\mathbb{R}P^n;\mathbb{Q})=\chi(\mathbb{R}P^n;\mathbb{Z})=\chi(\mathbb{R}P^n;\mathbb{F}_2)$.

We will thus denote the common values of all these Euler characteristics by $\chi(X)$.

Problem 10.4.3. (i) Suppose that X and Y are topological spaces such that $H_i(X)$ is finitely generated for each i. Prove that $H_i(X\times Y)$ is also finitely generated for each i.

(ii) Suppose that X and Y are topological spaces such that $H_i(X)=0$ for $i\geq N$ and $H_i(Y)=0$ for $i\geq M$. Prove that $H_i(X\times Y)=0$ for $i\geq N+M+1$ as well. Give an example showing that you can't improve this to $i\geq N+M$.

Thus under the assumptions of parts (i) and (ii), we have well-defined Euler characteristics $\chi(X)$, $\chi(Y)$ and $\chi(X\times Y)$.

(ii) Under the assumptions of parts (i) and (ii), prove that

$$\chi(X\times Y)=\chi(X)\chi(Y).$$

(iii) Show that $\chi(\Sigma_g\times X)$ is divisible by 2.

Chapter 11

The universal coefficients and Künneth theorems

In this chapter we give the proof of the universal coefficients theorem, which tells us how to relate $H_*(X; R) \otimes_R M$ with $H_*(X; M)$. With the tools developed to do so, we also prove the Künneth theorem, which tells us how to relate $H_*(X; R) \otimes_R H_*(Y; R)$ to $H_*(X \times Y; R)$.

11.1 The universal coefficients theorem

Let C_* be a chain complex of R-modules, eventually $R = \mathbb{Z}$ and $C_* = S_*(X)$. The universal coefficients theorem concerns the failure of the natural map

$$\begin{aligned} \alpha \colon H_n(C_*) \otimes_R M &\longrightarrow H_n(C_* \otimes_R M) \\ [a] \otimes m &\longmapsto [a \otimes m] \end{aligned}$$

to be an isomorphism, under some conditions on R and C_*. These are satisfied in the examples we are interesting in.

Theorem 11.1.1 (Universal coefficients theorem)**.** *Let R be a PID and C_* a chain complex of free R-modules. Then there is a natural exact sequence of R-modules*

$$0 \longrightarrow H_n(C_*) \otimes_R M \longrightarrow H_n(C_* \otimes_R M) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \longrightarrow 0.$$

Furthermore, these are split (but not naturally so).

What is the point of the addendum? Usually you know the outer terms in the short exact sequence, and there is a number of possibilities for the middle, e.g.

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow (\mathbb{Z}/2)^2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The addendum says that the middle term is always the direct sum of the outer ones (that is, the latter of the two examples is what happens), though not canonically so.

Proof. We resolve M by free R-modules. We can get away with a two-step resolution since R is a PID: $0 \to F_1 \to F_0 \to M \to 0$. We will use that for F free

$$\alpha\colon H_n(C_*) \otimes_R F \longrightarrow H_n(C_* \otimes_R F)$$

is an isomorphism.

Tensoring the free resolution with C_* we get a short exact sequence of chain complexes

$$0 \longrightarrow C_* \otimes_R F_1 \longrightarrow C_* \otimes_R F_0 \longrightarrow C_* \otimes_R M \longrightarrow 0.$$

This uses the assumption that each C_n is free (otherwise it would not be exact at the left). We obtain from this a long exact sequence on homology

$$\begin{tikzcd} \cdots \arrow[r] & H_n(C_* \otimes_R F_0) \arrow[r] & H_n(C_* \otimes_R M) \arrow[dll, rounded corners] \\ H_{n-1}(C_* \otimes_R F_1) \arrow[r] & H_{n-1}(C_* \otimes_R F_0) \arrow[r] & \cdots \end{tikzcd}$$

Exactness at the right terms gives us a short exact sequence

$$\begin{tikzcd} 0 \arrow[d] \\ \operatorname{coker}[H_n(C_* \otimes_R F_1) \to H_n(C_* \otimes_R F_0)] \arrow[d] \\ H_*(C_* \otimes_R M) \arrow[d] \\ \ker[H_{n-1}(C_* \otimes_R F_1) \to H_{n-1}(C_* \otimes_R F_0)] \arrow[d] \\ 0 \end{tikzcd} .$$

Using that α is an isomorphism for free R-modules, we see this is naturally isomorphic to

$$\begin{tikzcd} 0 \arrow[d] \\ \operatorname{coker}[H_n(C_*) \otimes_R F_1 \to H_n(C_*) \otimes_R F_0] \arrow[d] \\ H_*(C_* \otimes_R M) \arrow[d] \\ \ker[H_{n-1}(C_*) \otimes_R F_1 \to H_{n-1}(C_*) \otimes_R F_0] \arrow[d] \\ 0 \end{tikzcd} .$$

But the first term is just $\mathrm{Tor}_0^R(H_n(C_*), M) = H_n(C_*) \otimes_R M$ and the last term is just $\mathrm{Tor}_1^R(H_{n-1}(C_*), M)$.

For the addendum, we use that since R is a PID and each entry C_n is free, then $d(C_n) = B_{n-1}(C_*) \subset C_{n-1}$ is free. By lifting generators, we can find a map β' splitting

$$\begin{tikzcd} 0 \arrow[r] & Z_n(C_*) \arrow[r] & C_n \arrow[l, bend right, "\beta"'] \arrow[r, "d"'] & B_{n-1}(C_*) \arrow[l, bend right, "\beta'"'] \arrow[r] & 0 \end{tikzcd},$$

from which we get β as $x \mapsto x - \beta'(d(x))$.

This induces a splitting map

$$\begin{aligned} H_n(C_* \otimes_R M) &\longrightarrow H_n(C_*) \otimes_R M \\ [a \otimes m] &\longmapsto [\beta(a)] \otimes m. \end{aligned}$$

That is, we lift a class in $H_n(C_* \otimes_R M)$ to $Z_n(C_* \otimes R) \subset C_n \otimes_R M$, map it to $Z_n(C_*) \otimes_R M$ using $\beta \otimes_R \mathrm{id}$, and then take its equivalence class in $H_n(C_*) \otimes_R M$ by taking the quotient by $B_n(C_*) \otimes_R M$. Here we use that tensor products preserve cokernels. This is independent of the choice of lift: by definition $d(\sum_i r_i \otimes m_i) = \sum_i d(a_i) \otimes m_i$, and this evidently gets mapped to 0 when we take the quotient by $B_n(C_*) \otimes_R M$. $\square$

Example 11.1.2. Recall that Klein bottle $\mathbb{K}$ has homology given by

$$H_*(\mathbb{K}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get that

$$H_1(\mathbb{K}; \mathbb{Z}/2) \cong H_1(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_0(\mathbb{K}), \mathbb{Z}/2) = (\mathbb{Z}/2)^2$$

coming from the first term, and

$$H_2(\mathbb{K}; \mathbb{Z}/2) \cong H_2(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \oplus \mathrm{Tor}_2^{\mathbb{Z}}(H_1(\mathbb{K}), \mathbb{Z}/2) = \mathbb{Z}/2$$

coming from the second term, as $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$. Thus $\mathbb{K}$ and $\mathbb{T}^2$ have the same homology with $\mathbb{Z}/2$-coefficients.

11.2 The Künneth theorem

Recall that for homotopy invariance we constructed, rather inexplicitly, a bilinear cross product

$$\times \colon S_*(X) \times S_*(Y) \longrightarrow S_*(X \times Y).$$

The same construction goes through with coefficients.

At that point, we took $X = [0, 1]$ but also wondered what this map tells us about the homology of a product. There are two steps involved in answering this question:

- Understand how far the homology of $S_*(X) \otimes S_*(Y)$ is from $H_*(X) \otimes H_*(Y)$.

- Understand how far the homology of $S_*(X) \otimes S_*(Y)$ is that of $S_*(X \times Y)$.

The first can be answered by techniques used to prove the universal coefficients theorem, so we will not give the proof. We will drop the subscript R from the tensor products for the sake of readability.

Theorem 11.2.1 (Künneth for chain complexes)**.** *Let R be a PID, C_* a chain complex of free R-modules and D_* any chain complex of R-modules. There are short exact sequences*

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \to H_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_*), H_q(D_*)) \to 0$$

which are split (but not naturally so).

Corollary 11.2.2. *Let C_*, C'_*, D_* be chain complexes of R-modules. If $f_* \colon C_* \to C'_*$ is induces an isomorphism on homology, so does $C_* \otimes D_* \to C'_* \otimes D_*$.*

Proof. Applying Theorem 11.2.1, we get that f_* induces isomorphisms on the outer terms. By the five lemma, it also induces an isomorphism on the middle term. □

This reduces our program to proving the following theorem:

Theorem 11.2.3. *The cross product $\times \colon S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ induces an isomorphism on homology.*

This is proven using the *method of acyclic models*, which we prove in the next section. Combining this with Theorem 11.2.1, Künneth for chain complexes, we get:

Corollary 11.2.4 (Künneth theorem)**.** *Let R be a PID, then there are short exact sequences*

$$0 \to \bigoplus_{p+q=n} H_p(X;R) \otimes H_q(Y;R) \to H_n(X \times Y;R) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(X;R), H_q(Y;R)) \to 0$$

which are split (but not naturally so).

Example 11.2.5. We compute $H_*(\mathbb{R}P^3 \times \mathbb{R}P^3; \mathbb{Z}/2)$. To do so, recall that $\operatorname{Tor}_1^{\mathbb{F}}(M,N)$ always vanishes when $\mathbb{F}$ is a field, because every $\mathbb{F}$-module is free. Thus the Künneth short exact sequence reduces to an isomorphism of graded $\mathbb{F}$-vector spaces

$$H_*(X;\mathbb{F}) \otimes H_*(Y;\mathbb{F}) \to H_*(X \times Y;\mathbb{F}).$$

Using that $\mathbb{Z}/2$ is a field, and that

$$H_*(\mathbb{R}P^3;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \le * \le 3, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain that

$$H_*(\mathbb{R}P^3 \times \mathbb{R}P^3; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } * = 0, \\ (\mathbb{Z}/2)^2 & \text{if } * = 1, \\ (\mathbb{Z}/2)^3 & \text{if } * = 2, \\ (\mathbb{Z}/2)^4 & \text{if } * = 3, \\ (\mathbb{Z}/2)^3 & \text{if } * = 4, \\ (\mathbb{Z}/2)^2 & \text{if } * = 5, \\ \mathbb{Z}/2 & \text{if } * = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Example 11.2.6. Next, we compute $H_*(\mathbb{R}P^2 \times \mathbb{R}P^2)$. Recall that

$$H_*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$, we compute that

$$H_*(\mathbb{R}P^2 \times \mathbb{R}P^2) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ (\mathbb{Z}/2)^2 & \text{if } * = 1, \\ \mathbb{Z}/2 & \text{if } * = 2, \\ \mathbb{Z}/2 & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{cases}$$

11.3 Acyclic models

Recall the following lemma, which we used to prove that Tor-groups are independent of the choice of free resolution. It is a special case of Theorem 10.2.3:

Lemma 11.3.1. *If both $F_\bullet \to M$ and $G_\bullet \to N$ are free resolutions, then any homomorphism $f\colon M \to M'$ can be lifted to a chain map $f_\bullet\colon F_\bullet \to G_\bullet$, unique up to chain homotopy.*

Proof hint. To lift $f\colon M \to N$ to $f_0\colon F_0 \to G_0$, we shall use that $G_0 \to N$ is surjective. Picking a generating set $\{x_i\}$ of the free R-module F_0, we can lift to G_0 the image under $F_0 \to M \to N$ of each x_i. Thus so far we have obtained

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[d, "f"] \\ \cdots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & N. \end{tikzcd}$$

To next produce $f_1\colon F_1 \to G_1$ we do something similar: each element of a generating set of F_1 is mapped under $F_1 \to F_0 \to G_0$ to a cycle. By exactness at G_0 it is a boundary and hence in the image of $G_1 \to G_0$; we pick a lift.

This process can be continued until we have found all $f_k \colon F_k \to G_k$. There are clearly choices involved, but we can use a similar argment to find the components $h_k \colon F_k \to G_{k+1}$ of the chain homotopy between two different choices. $\square$

Remark 11.3.2. To obtain Theorem 10.2.3, note you can weaken the hypotheses: $F_\bullet$ needs to consist of projective modules instead of free ones, and $G_\bullet \to N$ only needs to be exact.

The acyclic models theorem is a generalization of this argument from chain complexes to functors with values in chain complexes. We fix a category C. We can think of a resolution $\cdots \to P_1 \to P_0 \to M$ of an R-module M as non-negatively graded chain complex $P_\bullet$ such that the chain complex

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P \longrightarrow 0$$

is exact.

Its generalisation to a functor $F \colon \mathsf{C} \to \mathsf{Mod}_R$ instead of an R-module M is as functor $F_\bullet \colon \mathsf{C} \to \mathsf{Ch}_R^{\geq 0}$ such that for each $X \in \mathrm{ob}(\mathsf{C})$ the chain complex

$$\cdots \longrightarrow F_2(X) \longrightarrow F_1(X) \longrightarrow F_0(X) \longrightarrow F(X) \longrightarrow 0$$

is exact.

The acyclic models theorem gives a condition under which a natural transformation

$$f \colon F \longrightarrow G$$

of functors $\mathsf{C} \to \mathsf{Mod}_R$ can be lifted to a natural transformation $f_\bullet \colon F_\bullet \to F'_\bullet$ of functors $\mathsf{C} \to \mathsf{Ch}_R^{\geq 0}$, unique up to natural chain homotopy. The hypothesis we give is one which reduces the problem from one about functors and natural transformations, to one about modules and homomorphisms.

We fix a collection $\mathcal{M}$ of objects C which we call *models*, and use these to specify a particular class of functors:

Definition 11.3.3. A *representable functor* $F \colon \mathsf{C} \to \mathsf{Mod}_R$ is one of the form $R[\mathrm{Hom}_{\mathsf{C}}(X, -)]$.

We then demand that $F_\bullet$ and $G_\bullet$ are degreewise $\mathcal{M}$-free, i.e. a direct sum of representable functors $\mathbb{Z}[\mathrm{Hom}_{\mathsf{C}}(M, -)]$ with $M \in \mathcal{M}$. Then we futher demand that the extended complexes

$$\cdots \longrightarrow F_2(M) \longrightarrow F_1(M) \longrightarrow F_0(M) \longrightarrow F(M) \longrightarrow 0,$$

$$\cdots \longrightarrow G_2(M) \longrightarrow G_1(M) \longrightarrow G_0(M) \longrightarrow G(M) \longrightarrow 0,$$

are exact for all $M \in \mathcal{M}$, i.e. $\mathcal{M}$-*exact*: we say that $F_\bullet \to F$ and $G_\bullet \to G$ are $\mathcal{M}$-*free resolutions.*

Theorem 11.3.4 (Acylic models)**.** *If $F_\bullet \to F$ and $G_\bullet \to G$ are $\mathcal{M}$-free resolutions, then every natural transformation $f \colon F \to G$ of functors $\mathsf{C} \to \mathsf{Mod}_R$ can be lifted to a natural transformation $f_\bullet \colon F_\bullet \to G_\bullet$ of functors $\mathsf{C} \to \mathsf{Mod}_R$, unique up to natural chain homotopy.*

Remark 11.3.5. You can weaken the conditions on $F_\bullet \to F$ and $G_\bullet \to G$ as above: $F_\bullet$ needs to be $\mathcal{M}$-projective and $G_\bullet$ to be $\mathcal{M}$-exact.

Example 11.3.6. If we take C to be the category with a unique object $*$ and only an identity morphisms, and we take $\mathcal{M} = \{*\}$, the acyclic models theorem is equivalent to Lemma 11.3.1.

To prove Theorem 11.3.4, we recall that the important property of the free modules was lifting against surjective maps. Let us prove the corresponding statement for $\mathcal{M}$-free functors:

Definition 11.3.7. A natural transformation $\eta\colon F \to G$ of functors $\mathsf{C} \to \mathsf{Mod}_R$ is an $\mathcal{M}$*-epimorphism* if $F(M) \to G(M)$ is surjective for all $M \in \mathcal{M}$.

Lemma 11.3.8. *If we have a diagram*

$$\begin{tikzcd} & G \arrow[d, two heads, "g"] \\ F \arrow[r] \arrow[ur, dotted] & H \end{tikzcd}$$

of functors $\mathsf{C} \to \mathsf{Mod}_R$ *where* F *is* $\mathcal{M}$*-free and* g *is an* $\mathcal{M}$*-epimorphism, then there exists a dotted lift.*

Proof. First we observe it suffices to treat the case

$$F = R[\mathrm{Hom}(M, -)],$$

as the set of natural transformations out of a direct sum of functors is the product the set of natural transformations out of the individual summands.

Then let us first evaluate at M, to get

$$\begin{tikzcd} & G(M) \arrow[d, two heads, "g"] \\ R[\mathrm{Hom}(M, M)] \arrow[r] \arrow[ur, dotted] & H(M). \end{tikzcd}$$

Since the right map is surjective, we can find a lift x of the generators id_M. We claim that this choice determines an entire natural transformation $F \to G$ by naturality. Naturality demands that a generator $f \in R[\mathrm{Hom}(M, X)]$ is sent to $G(f)(x)$, as we can write f as $F(f)(\mathrm{id}_M)$. □

What this argument uses is the Yoneda lemma, which says that if $F = \bigoplus_i R[\mathrm{Hom}(M_i, -)]$ is a direct sum of representable, then

$$\mathrm{Nat}(F, G) = \prod_i G(M_i),$$

naturally in G. To prove Theorem 11.3.4 we repeat the proof of Lemma 11.3.1 with some modifications:

Proof of Theorem 11.3.4. To lift the natural transformation $f\colon F \to G$ to $f_0\colon F_0 \to G_0$, we write $F_0 = \bigoplus_i R[\mathrm{Hom}(M_i, -)]$. By Yoneda, a natural transformation $F_0 \to G$ is uniquely determined by a collection of elements $x_i \in G(M_i)$. Since $G_0 \to G$ is an $\mathcal{M}$-epimorphism, the maps $G_0(M_i) \to G(M_i)$ are surjective. Thus we can lift the x_i to $G_0(M_i)$, and using Yoneda these lifts determine a unique natural transformation $F_0 \to G_0$. Thus so far we have obtained

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] \arrow[d, "f_0"] & F \arrow[d, "f"] \\ \cdots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & G. \end{tikzcd}$$

To next produce $f_1\colon F_1 \to G_1$ we do something similar: we shall write $F_1 = \bigoplus_{i'} R[\mathrm{Hom}(M'_{i'}, -)]$ and use Yoenda identify the natural transformation $F_1 \to F_0 \to G_0$ as a collection of elements $x'_{i'} \in G_0(M'_{i'})$. By construction these lie in the kernel of $d\colon G_0(M'_{i'}) \to G(M'_{i'})$. Since the bottom row is $\mathcal{M}$-exact, each $x'_{i'}$ is a boundary and hence in the image of $G_1(M'_{i'}) \to G_0(M'_{i'})$. We pick a lifts and use Yoneda to produce $f_1\colon F_1 \to G_1$.

This process can be continued until we have found all $f_k\colon F_k \to G_k$. There are clearly choices involved, but we can use a similar argment to find the components $h_k\colon F_k \to G_{k+1}$ of the chain homotopy between two different choices. □

11.3.1 Proving the Künneth theorem

To prove the Künneth theorem we apply Theorem 11.3.4 to

$$\begin{aligned} \mathsf{C} &= \mathsf{Top}^2, \\ \mathcal{M} &= \{(\Delta^p, \Delta^q) \mid p, q \geq 0\}, \\ F &= ((X,Y) \mapsto H_0(X) \otimes H_0(Y)) \\ F_\bullet &= ((X,Y) \mapsto S_*(X) \otimes S_*(Y)) \\ G &= (X,Y) \mapsto H_0(X \times Y) \\ G_\bullet &= ((X,Y) \mapsto S_*(X \times Y)) \\ f &= \text{natural iso}\colon H_0(X) \otimes H_0(Y) \xrightarrow{\cong} H_0(X \times Y) \end{aligned}$$

and its inverse, to get a chain homotopy equivalence

$$S_*(X) \otimes S_*(Y) \longrightarrow S_*(X \times Y)$$

unique up to chain homotopy.

We need to verify hypotheses that both

$$(X,Y) \longmapsto S_n(X \times Y; R) \qquad \text{and} \qquad (X,Y) \longmapsto \bigoplus_{p+q=n} S_p(X;R) \otimes S_q(Y;R)$$

are $\mathcal{M}$-free resolutions. That they are $\mathcal{M}$-exact follows because the reduced homology of the models vanish. To see the first is $\mathcal{M}$-free, we write it as a direct sum of the representables

$$\begin{aligned} S_p(X;R) \otimes S_q(Y;R) &= R[\{(\Delta^p \to X, \Delta^q \to Y)\}] \\ &= R[\mathrm{Hom}_{\mathsf{Top}^2}((\Delta^p, \Delta^q), (X, Y))]. \end{aligned}$$

To second the second is $\mathcal{M}$-free, we recognize it as the representable

$$\begin{aligned} S_n(X \times Y; R) &= R[\{\Delta^n \to X \times Y\}] \\ &= R[\{\Delta^n \to X, \Delta^n \to Y\}] \\ &= R[\mathrm{Hom}_{\mathsf{Top}^2}((\Delta^n, \Delta^n), (X, Y))]. \end{aligned}$$

Theorem 11.3.9. *There are natural chain maps*

$$S_*(X;R) \otimes S_*(X;R) \rightleftarrows S_*(X \times Y; R)$$

which are unique up to chain homotopy, and induce mutually inverse isomorphisms on homology.

11.4 Problems

Problem 11.4.1 (Moore spaces)**.** The *Moore space* $M(\mathbb{Z}/m, n)$ is obtained by attaching an $(n+1)$-cell to S^n along a map of degree m. Here we use that homotopy classes of continuous maps $S^n \to S^n$ are classified by their degree, so this describes a topological space which is well-defined up to homotopy equivalence.

(i) Compute $H_*(M(\mathbb{Z}/m, n))$ and $H^*(M(\mathbb{Z}/m, n))$.

(ii) Prove that the quotient map $M(\mathbb{Z}/m, n) \to M(\mathbb{Z}/m, n)/S^n$ induces the zero map on reduced homology, but not on reduced cohomology.

(iii) Deduce that the splitting in the universal coefficient theorem for cohomology can not be natural.

Problem 11.4.2 (Alexander–Whitney map)**.** In Problem 3.5.1 we gave an explicit construction of a cross product $\mathrm{EZ}\colon S_*(X) \times S_*(Y) \to S_*(X \times Y)$, the *Eilenberg–Zilber* map. This generalizes to homology with coefficients. There is an explicit inverse $\mathrm{AW}\colon S_*(X \times Y; R) \to S_*(X;R) \otimes S_*(Y;R)$ up to homotopy, called the *Alexander–Whitney* map:

$$S_*(X \times Y; R) \ni \sigma \longmapsto \sum_{p+q}^{n} \pi_1 \circ \sigma|_{\Delta^p} \otimes \pi_2 \circ \sigma|_{\Delta^q} \in S_*(X;R) \otimes S_*(Y;R),$$

where $\Delta^p \subset \Delta^n$ is the first face and $\Delta^q \subset \Delta^n$ the last one.

(i) Verify AW is a chain map.

(ii) Check that $\mathrm{AW} \circ \mathrm{EZ} = \mathrm{id}$ when working with normalized chains as in Problem 4.4.7.

In fact, it is also possible to give an explicit chain homotopy between $\mathrm{EZ} \circ \mathrm{AW}$ and the identity, see [GDR99].

Chapter 12

Recap and an extended example

We will soon move on to cohomology; a dual to homology which has values in graded algebras instead of graded abelian groups. Before doing so, we recap what we are proven so far and work out an interesting example.

12.1 Recap of homology

It is fair to divide our work between axioms which we have established and computational tools which we have developed. Let's start with the former, stated with coefficients in an R-module M.

Long exact sequence of a pair For every $A \subset X$ we get a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(X;M) \arrow[r] & H_n(X,A;M) \arrow[dll, rounded corners] \\ H_{n-1}(A;M) \arrow[r] & H_{n-1}(X;M) \arrow[r] & \cdots \end{tikzcd}$$

This is natural in the pair $A \subset X$.

Excision When $U \subset A \subset X$ is an excisive triple (i.e. $\mathrm{cl}(U) \subset \mathrm{int}(A)$), the inclusion induces an isomorphism

$$H_*(X \setminus U, A \setminus U; M) \xrightarrow{\cong} H_*(X, A; M).$$

This is natural in the excisive triple $U \subset A \subset X$.

Wedge For disjoint unions the inclusions induce an isomorphism

$$\bigoplus_{i \in I} H_*(X_i; M) \xrightarrow{\cong} H_*\left(\bigsqcup_{i \in I} X_i; M\right).$$

These are called the *Eilenberg–Steenrod axioms*, and determine $H_*(-;M)$ uniquely up to natural isomorphism in terms of value at a point, at least on spaces homotopy equivalent to CW-complexes. From this, we developed several techniques to determine homology groups:

Mayer–Vietoris If two open subsets U, V cover X, then there is a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(U;M) \oplus H_n(V;M) \arrow[r] & H_n(X;M) \arrow[dll, rounded corners] \\ H_{n-1}(U \cap V;M) \arrow[r] & H_{n-1}(U;M) \oplus H_{n-1}(V;M) \arrow[r] & \cdots \end{tikzcd}$$

This is natural in the data $U, V \subset X$.

Cellular homology If X is given a CW-structure, then one can compute $H_*(X;M)$ using the chain complex $C_*(X;M)$, given in degree n by $R[\{n\text{-cells}\}] \otimes_R M$:

$$\cdots \longrightarrow R[\{n\text{-cells}\}] \otimes_R M \longrightarrow R[\{(n-1)\text{-cells}\}] \otimes_R M \longrightarrow \cdots .$$

For an n-cell e, the differential of $e \otimes m$ is given the sum over $(n-1)$-cells e' of $d(e,e')e' \otimes m$ with $d(e,e')$ the degree of the map

$$S^{n-1} \cong \partial e \to \mathrm{sk}_{n-1}(X)/\mathrm{sk}_{n-2}(X) \to e'/\partial e' \cong S^{n-1}.$$

This is natural with respect to cellular maps.

Universal coefficients theorem If R is a PID, then there are short exact sequences

$$0 \to H_n(X;R) \otimes_R M \to H_n(X;M) \to \mathrm{Tor}_1^R(H_{n-1}(X;R), M) \to 0$$

which are split but not naturally so. These are natural in X.

Künneth theorem If R is a PID, then there are short exact sequences

$$0 \to \bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R) \to H_n(X \times Y;R) \to \bigoplus_{p+q=n_1} \mathrm{Tor}_1^R(H_p(X;R), H_q(Y;R)) \to 0$$

which are split but not naturally so. These are natural in X and Y.

It is also worth cataloguing the examples we have computed so far:

- the spheres S^n,
- the real and complex projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$,
- the genus g surface Σ_g,
- the Poincaré homology sphere P.

We can create more examples by taking suspensions, disjoint unions, wedges and products of these. Today we will add the special orthogonal groups to this list.

12.2 Special orthogonal groups

Recall that an $(n \times n)$-matrix A with real entries is orthogonal if $AA^t = \mathrm{id}$; it follows that $A^tA = \mathrm{id}$ and that $\det(A) = \pm 1$. The *orthogonal group* $O(n)$ is the group of orthogonal $(n \times n)$-matrices, topologized as a subspace of $\mathbb{R}^{n^2}$. This has two path-components, distinguished by the determinant, and the *special orthogonal group* $SO(n)$

is the subgroup of orthogonal $(n \times n)$-matrices with determinant 1. We will compute as much as the homology of these spaces as we can. We could do better with more tools, as we do not yet exploit the fact that the $SO(n)$ are topological groups which map to each other by group homomorphisms.

Remark 12.2.1. The study of the homology and homotopy of $SO(n)$ was of great importance in algebraic topology. It led to characteristic classes, Bott periodicity, topological K-theory, the image of J, etc. Their importance is due to the fact that we can compute all they are spaces of geometric importance whose homology and homotopy are both computable; a combination that is quite rare.

12.2.1 $SO(2)$

It is a result of Euler that $SO(n)$ is generated by rotations around some axis. Thus $SO(2)$ is the group of the rotations in $\mathbb{R}^2$, and by specifying an angle are homeomorphic to S^1. For example using the suspension isomorphism, we see that

$$H_*(SO(2)) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

12.2.2 $SO(3)$

Similarly, $SO(3)$ is the group in rotations in $\mathbb{R}^3$. It is in fact homeomorphic to $\mathbb{R}P^3$ as we will now explain.

Recall that the quaternions $\mathbb{H}$ are the $\mathbb{R}$-algebra with generators i, j, k and relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$ and $ki = j$. Every quaternion can be written uniquely as $a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$. Those with $b = c = d = 0$ are called *real*, those with $a = 0$ *imaginary*. There is a conjugation operation $x \mapsto \bar{x}$ sending $a + bi + cj + dk$ to $a - bi - cj - dk$. Then $x\bar{x}$ is always real, and we can define a norm $||x|| = \sqrt{x\bar{x}}$. The quaternions $S(\mathbb{H}) \cong S^3$ of unit norm form a group under composition: $x^{-1} = \bar{x}$. The unit quaterions act on the imaginary quaternions: $x \in S(H)$ sends $y \in \mathrm{Im}(\mathbb{H})$ to xyx^{-1}. This preserves the norm, so gives a homomorphism

$$S^3 \cong S(H) \longrightarrow \{\text{orthogonal linear maps on } \mathrm{Im}(\mathbb{H})\} \cong SO(3).$$

This is surjective with kernel ± 1, and induces a homeomorphism $\mathbb{R}P^3 = S^3/\pm 1 \cong SO(3)$.

There is a CW structure on $\mathbb{R}P^n$ with a single k-cell for $0 \leq k \leq n$, and attaching maps alternatively of degree 2 and 0. In particular, for $n = 3$ we have a cellular chain complex

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}.$$

This gives us that

$$H_*(SO(3)) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}/2 & \text{if } * = 1, \\ 0 & \text{if } * = 2, \\ \mathbb{Z} & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{cases}$$

12.2.3 $SO(4)$

There is a similar exceptional isomorphism involving quaternions for $SO(4)$: letting $x, x' \in S(\mathbb{H}) \times S(\mathbb{H})$ act on $\mathbb{H}$ by $y \mapsto xy(x')^{-1}$, one sees that as a topological group $SO(4)$ is isomorphic to

$$(S^3 \times S^3)/\{\pm\mathrm{id}\}.$$

If we forget the group sturcture, this is turn homeomorphic to $S^3 \times \mathbb{R}P^3$.

Thus we can compute its homology using the Künneth theorem, Theorem 11.2.4. Because the homology of S^3 is free in each degree, there are no non-zero Tor-terms and we get an isomorphism

$$H_*(SO(4)) \cong H_*(S^3) \otimes H_*(\mathbb{R}P^3).$$

12.2.4 $SO(n)$

At this point our exceptional isomorphisms run out, which the exception of $SO(8) \cong SO(7) \times S^7$, and we have to confront the special orthogonal groups head on. We will do so by describing a CW structure on them, following Section 3.D of Hatcher.

For $v \in S^{n-1}$, let $r(v)$ denote the reflection of $\mathbb{R}^n$ across the plane orthogonal to v. This is orthogonal but has determinant -1, which can be fixed by using instead $r(v)r(e_1)$. Thus we obtain a continuous map

$$\begin{aligned} \rho\colon \mathbb{R}P^{n-1} &\longrightarrow SO(n) \\ v &\longmapsto r(v)r(e_1). \end{aligned}$$

This is injective because the map $v \mapsto r(v)$ is, and on $O(n)$ multiplication by $r(e_1)$is a homeomorphism.

More generally, for any sequence $I = (i_1, \ldots, i_r)$ with $0 < i_j < n$, we have a continuous map

$$\begin{aligned} \rho_I\colon \mathbb{R}P^{i_1} \times \cdots \times \mathbb{R}P^{i_r} &\longrightarrow SO(n) \\ (v_1, \ldots, v_r) &\longmapsto \rho(v_1) \cdots \rho(v_r). \end{aligned}$$

Let $\phi\colon D^i \to \mathbb{R}P^i$ be the characteristic map of the top cell in the standard CW structure on $\mathbb{R}P^i$. These can be combined to a map

$$\phi_I\colon D^{i_1} \times \cdots \times D^{i_r} \longrightarrow \mathbb{R}P^{i_1} \times \cdots \times \mathbb{R}P^{i_r}$$

and we can compose this with ρ_I as well as the usual homeomorphism of the domain with $D^{i_1+\cdots+i_r}$, we obtain a map

$$\chi_I\colon D^{i_1+\cdots+i_r} \longrightarrow SO(n).$$

Some of these maps have overlapping images, and so we focus on I which are *admissible*, which means that $n > i_1 > \cdots > i_r > 0$. Note that I may be empty. This is [Hat02, Proposition 3D.1].

Proposition 12.2.2. *The maps $\chi_I\colon D^{i_1+\cdots+i_r} \to SO(n)$, ranging over admissible I, are the characteristic maps of a CW structure on $SO(n)$ such that the map*

$$\rho\colon \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-2} \times \cdots \times \mathbb{R}P^1 \longrightarrow SO(n)$$

is cellular if we use the standard CW structure on the domain.

We shall denote the cells with characteristic maps χ_I by e^I, or if we have $I = n > i_1 > \cdots > i_r > 0$ by $e^{i_1} \times \cdots \times e^{i_r}$. Observe that each cell of $SO(n)$ is by definition obtained by taking cells $\chi_i\colon D^i \to \mathbb{R}P^i \to SO(n)$ and multiplying these using the group structure of $SO(n)$. In particular, the map ρ is surjective.

Example 12.2.3. For $n = 2$, there are only two admissible sequences: (1) and $()$, corresponding to cells of dimension 1 and 0. These are the cells for the ordinary CW-decomposition of $SO(2) = S^1$, and the map $\rho\colon \mathbb{R}P^1 \to SO(2)$ is given by composing sending a line v to $\rho(v)\rho(e_1)$, which is the same as rotation through twice the angle between v and e_1.

Example 12.2.4. For $n = 3$, there are only four admissible sequences: $(2, 1)$, (2), (1) and $()$, corresponding to cells of dimension 3, 2, 1 and 0. These are the cells for the ordinary CW-decomposition of $\mathbb{R}P^3$.

Proposition 12.2.5. *The homology group $H_i(SO(n); \mathbb{Z}/2)$ is the free $\mathbb{Z}/2$-vector space on the set of admissible sequences I with $i_1 + \cdots + i_r = i$.*

Proof. Since the map $\rho\colon \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-2} \times \cdots \times \mathbb{R}P^1 \longrightarrow SO(n)$ is a cellular map, there is an induced map of cellular chain complexes with $\mathbb{Z}/2$-coefficients

$$C_*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-2} \times \cdots \times \mathbb{R}P^1; \mathbb{Z}/2) \longrightarrow C_*(SO(n); \mathbb{Z}/2),$$

which is surjective since ρ is.

Recall that the cells of a product are the products of cells in each of the terms so each entry in $C_*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-2} \times \cdots \times \mathbb{R}P^1; \mathbb{Z}/2)$ is generated by $e^{i_1} \times \ldots \times e^{i_{n-1}}$ with e^{i_j} a i_j-cell of $\mathbb{R}P^{n-j}$, and the differential is given by[1]

$$d(e^{i_1} \times \ldots \times e^{i_{n-1}}) = \sum_j (-1)^{i_1+\cdots+i_{j-1}} e^{i_1} \times \cdots \times d(e^{i_j}) \times \cdots \times e_{i_{n-1}}.$$

Since $d(e^{i_j})$ is always 0 modulo 2, we conclude that the differential vanishes in the domain. Since the map of chain complexes is surjective, the differential must also vanish in the target. In other words, there is an isomorphism of graded vector spaces

$$C_*(SO(n); \mathbb{Z}/2) \cong H_*(SO(n); \mathbb{Z}/2).$$

The proposition then easily follows. $\square$

Example 12.2.6. Let us write

$$f(x) = \sum_{i=0}^{\infty} \dim H_i(SO(n); \mathbb{Z}/2) \cdot x^i \in \mathbb{Z}[[x]]$$

[1] For a detailed proof, see Proposition 3B.1 of Hatcher.

for the generating series of the dimension. Since in an admissible sequence we can decide to either include the integer i or not, we see that

$$f(x) = \prod_{i=1}^{n-1} (1 + x^i).$$

By expanding this product we can easily read off the dimensions of $H_*(SO(n); \mathbb{Z}/2)$. For example, when $n = 6$ we get

$$x^{15}+x^{14} + x^{13} + 2x^{12} + 2x^{11} + 3x^{10} + 3x^9 + 3x^8 + \\ 3x^7 + 3x^6 + 3x^5 + 2x^4 + 2x^3 + x^2 + x + 1.$$

This is compatible with $SO(6)$ being 15-dimensional.

In fact, these dimensions are the same as those of the graded $\mathbb{Z}/2$-algebra

$$\mathbb{Z}/2[\bar{w}_1, \ldots, \bar{w}_{n-1}]/(\bar{w}_1^2, \ldots, \bar{w}_{n-1}^2)$$

where the degree of $\bar{w}_i$ is i. If we had worked out the homomorphisms

$$H_*(SO(n); \mathbb{Z}/2) \otimes H_*(SO(n); \mathbb{Z}/2) \cong H_*(SO(n) \times SO(n); \mathbb{Z}/2) \longrightarrow H_*(SO(n); \mathbb{Z}/2),$$

which make $H_*(SO(n); \mathbb{Z}/2)$ into a $\mathbb{Z}/2$-algebra, we would have found this description.

It is harder to compute the homology with $\mathbb{Z}$-coefficients; we shall not give a complete answer but outline a procedure to compute these groups:

Theorem 12.2.7. *There are "small" chain complexes C_*^{2i} and C_*^{2i+1} such that*

$$C_*(SO(2k+1)) \cong C_*^2 \otimes C_*^4 \otimes \cdots \otimes C_*^{2k}$$
$$C_*(SO(2k+2)) \cong C_*^2 \otimes C_*^4 \otimes \cdots \otimes C_*^{2k} \otimes C_*^{2k+1}.$$

Proof. We will only do the case $n = 2k+1$ in detail, as the case $n = 2k+2$ is similar. Recall that $e^i \subset SO(2k+1)$ denotes an i-cell, and $e^i e^j$ for $i > j$ is a product of two such cells.

Let us consider the four cells $e^0, e^{2i-1}, e^{2i}, e^{2i}e^{2i-1}$. These are in the image of the cellular map $\mathbb{R}P^{2i} \times \mathbb{R}P^{2i-1} \to SO(2k+1)$, of the cells $e^0 \times e^0$, $e^0 \times e^{2i-1}$, $e^{2i} \times e^0$, $e^{2i} \times e^{2i-1}$. The boundary of e^i with $i > 0$ is $2e^{i-1}$ when i is even and 0 when i is odd. Thus $d(e^0 \times e^{2i-1}) = 0$, $d(e^{2i} \times e^0) = 2e^{2i-1} \times e^0$, and

$$d(e^{2i} \times e^{2i-1}) = 2e^{2i-1} \times e^{2i-1}.$$

When we map $e^{2i-1} \times e^{2i-1} \subset \mathbb{R}P^{2i} \times \mathbb{R}P^{2i-1}$ to $SO(2k+1)$, we land in the much lower-dimensional cell e^{2i-1} as we are multiplying rotations which happen to lie in the same $2i$-dimensional subspace. Thus $d(e^{2i}e^{2i-1}) = 0$.

In other words, the four cells $e^0, e^{2i-1}, e^{2i}, e^{2i}e^{2i-1}$ span a subcomplex $C_*^{2i} \subset C_*(SO(2k+1))$. We now claim that

$$C_*(SO(2k+1)) \cong C_*^2 \otimes C_*^4 \otimes \cdots \otimes C_*^{2k}.$$

The proof is by induction over k; for $k = 1$ this $SO(2k+1) = \mathbb{R}P^3$ and $C_*(SO(2k+1)) = C^2_*$. For the induction step, observe that $C_*(SO(2k-1)) \subset C_*(SO(2k+1))$ is the subcomplex spanned by all cells e^I with I not containing $2k-1$ or $2k$. Inspecting the differentials, we see that

$$C_*(SO(2k+1)) = C_*(SO(2k-1)) \otimes C^{2k}_*.$$

Applying the inductive hypothesis we are done.

For the case $n = 2k+2$, one adds C^{2n+1}_* spanned by the cells e^0, e^{2n+1} and with trivial differential. □

Let us draw some corollaries:

Corollary 12.2.8. *The rational homology groups are (additively) given by*

$$H_*(SO(2k+1); \mathbb{Q}) \cong \mathbb{Q}[\bar{p}_3, \cdots \bar{p}_{4k-1}]$$
$$H_*(SO(2k+2); \mathbb{Q}) \cong \mathbb{Q}[\bar{p}_3, \cdots \bar{p}_{4k-1}, \bar{e}_{2k+1}]$$

Proof. Again we do the case $n = 2k+1$ only: $H_*(C^2_* \otimes C^4_* \otimes \cdots \otimes C^{2k}_* \otimes \mathbb{Q})$ can be computed by algebraic Künneth theorem as the tensor product of the homologies of $C^{2i}_* \otimes \mathbb{Q}$ and $C^{2i+1}_* \otimes \mathbb{Q}$:

$$H_*(C^{2i}_*) = \begin{cases} \mathbb{Q} & \text{if } * = 0, 4i-1, \\ 0 & \text{otherwise.} \end{cases}$$
$$H_*(C^{2i+1}_*) = \begin{cases} \mathbb{Q} & \text{if } * = 0, 2i+1, \\ 0 & \text{otherwise.} \end{cases}$$

□

In fact, this computation works with coefficients in $\mathbb{Z}[1/2]$. This is consistent with the following result:

Corollary 12.2.9. *The homology groups $H_*(SO(n))$ are direct sums of $\mathbb{Z}$'s and $\mathbb{Z}/2$'s.*

Proof. As before we do the case $n = 2k+1$ only: $H_*(C^2_* \otimes C^4_* \otimes \cdots \otimes C^{2k}_*)$ can be computed by algebraic Künneth theorem from the homology of C^{2i}_*:

$$H_*(C^{2i}_*) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 4i-1, \\ \mathbb{Z}/2 & \text{if } * = 2i-1 \\ 0 & \text{otherwise.} \end{cases}$$

The contribution to $H_*(C^2_* \otimes C^4_* \otimes \cdots \otimes C^{2k}_*)$ are then tensor products of such groups, which will only be $\mathbb{Z}$'s or $\mathbb{Z}/2$'s, as well as $\mathrm{Tor}^{\mathbb{Z}}_1$-terms, which will only be $\mathbb{Z}/2$'s. □

Example 12.2.10 ($SO(5)$). By the argument in the previous corollary, it suffices to compute $H_*(C_*^2 \otimes C_*^4)$ using the algebraic Künneth theorem. We first compute

$$H_*(C_*^2) \otimes H_*(C_*^4) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 3, 7, 10, \\ \mathbb{Z}/2 & \text{if } * = 1, 3, 4, 6, 8, \\ 0 & \text{otherwise.} \end{cases}$$

Next we add Tor-terms, which only occur between $H_1(C_*^2)$ and $H_3(C_*^4)$ and contribute a $\mathbb{Z}/2$ to degree 5. We conclude that

$$H_*(SO(5)) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 3, 7, 10, \\ \mathbb{Z}/2 & \text{if } * = 1, 3, 4, 5, 6, 8, \\ 0 & \text{otherwise.} \end{cases}$$

12.3 Problems

Problem 12.3.1 (Stiefel manifolds)**.** Let $V_{n,k}$ be the topological space of orthonormal k-tuples of vectors in $\mathbb{R}^n$. These are called *Stiefel manifolds.*

(i) Use the action of $SO(n)$ on the k-tuple $(e_{n-k+1}, \dots, e_n)$ to give a homeomorphism

$$SO(n)/SO(n-k) \xrightarrow{\cong} V_{n,k}.$$

(ii) Use this to give a CW-structure on $V_{n,k}$.

(iii) Prove that C_*^{2k} is the cellular chain complex of $V_{2k+1,2}$ and use this to compute $H_*(V_{2k+1,2})$.

Chapter 13

Cohomology

In this chapter we define cohomology, the dual to homology, and explain how to relate the two.

13.1 Cohomology as the dual of homology

Cohomology is an invariant of topological spaces analogous to homology, which is *contravariant* rather than covariant: a continuous map $f\colon X \to Y$ induces a homomorphism $f^*\colon H^*(Y) \to H^*(X)$. That is, as a functor it will have domain $\mathsf{HoTop}^{\mathrm{op}}$ instead of HoTop. There are several reasons to want this:

(1) Many geometric objects pull back along continuous map $f\colon X \to Y$, such as differential forms, covering spaces, or vector bundles. Invariants of such objects that are compatible with pull back, must take values in an abelian group which is contravariantly assigned to X.

(2) One can build invariants of topological spaces from geometric objects, such as differential forms (deRham cohomology), or vector bundles (topological K-theory). Such invariants will be contravariant, and to compare homology we know we need contravariant variations of it.

(3) For every topological these is a diagonal map $X \to X \times X$. Cohomology will turn this into a map $\Delta^*\colon H^*(X \times X) \to H^*(X)$, from which we can extract a product on $H^*(X)$. This additional algebraic structure is a helpful invariant.

Let us fix an R-module M to use as coefficients. We define the *singular n-cochains on X with values in M* to be the module of R-module homomorphisms $S_n(X;R) \to M$:

$$S^n(X;M) \coloneqq \mathrm{Hom}_R(S_n(X;R), M).$$

These assemble to a *cochain complex*, which is like a chain complex but its differential increases degree rather than decreasing it:

$$\begin{aligned} d\colon S^n(X;M) &\longrightarrow S^{n+1}(X;M) \\ \alpha &\longmapsto (-1)^{n+1}\alpha \circ d. \end{aligned}$$

The purpose of this sign is not be clear, but it will be used to guarantee that the Kronecker pairing is well-defined. Anyway, this differential satisfies $d^2 = 0$, and we take cohomology by taking the quotient of the *cocycles* by the *coboundaries*

$$Z^n(X;M) \coloneqq \ker[d\colon S^n(X;M) \to S^{n+1}(X;M)],$$
$$B^n(X;M) \coloneqq \operatorname{im}[d\colon S^{n-1}(X;M) \to S^n(X;M).$$

Definition 13.1.1. The *cohomology groups of X with coefficients in M* are given by

$$H^n(X;M) \coloneqq \frac{Z^n(X;M)}{B^n(X;M)}.$$

This amounts to inserting the functor $\operatorname{Hom}_R(-,M)\colon \mathsf{Ch}_R^{\mathrm{op}} \to \mathsf{CoCh}_R$ in the construction of homology: cohomology is given as the composition

$$\mathsf{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Sin}_\bullet} \mathsf{ssSet}^{\mathrm{op}} \xrightarrow{R[-]} \mathsf{Ch}_R^{\mathrm{op}} \xrightarrow{\operatorname{Hom}_R(-,M)} \mathsf{CoCh}_R \xrightarrow{H^*} \mathsf{GrAb}.$$

Cochain homotopies are defined similarly to chain homotopy, but now decrease degree instead of increasing it. Dualizing a chain homotopy gives a cochain homotopy. Thus cohomology is homotopy-invariant and factors over $\mathsf{HoTop}^{\mathrm{op}}$.

Remark 13.1.2. It should be clear at this point that we should have defined cohomology to be negatively graded:

$$S^\vee_{-n}(X;M) \coloneqq S^n(X;M).$$

With this grading convention the differential decreases degree, so we get a chain complex. As we will see later, using this convention makes arguments go through more smoothly.

Example 13.1.3. $H^0(X;M)$ consists of those functions $\alpha\colon \mathrm{Sin}_0(X) \to M$ such that $d(\alpha) = 0$. The equation $d(\alpha) = 0$ means that $\alpha(d(\sigma))$ for every 1-simplex $\sigma \in X$: thus it is expressing that α is constant on path components. We conclude that

$$H^0(X;M) = \{\text{functions } \pi_0(X) \to M\}.$$

13.1.1 Eilenberg–Steenrod axioms for cohomology

Cohomology is a dual version of homology. To make this precise we need to first define a relative version. To make sense of $H^*(X,A;M)$ for a pair $A \subset X$ we replace $S_*(X;R)$ by $S^*(X,A;R) \coloneqq S_*(X;R)/S_*(A;R)$. In other words, $S^n(X,A;M)$ consists of those R-linear functionals $S_n(X;R) \to M$ which vanish on $S_n(A;R) \subset S_n(X;R)$. By definition, there is then a short exact sequence of cochain complexes

$$0 \longrightarrow S^*(X,A;M) \longrightarrow S^*(X;M) \longrightarrow S^*(A;M) \longrightarrow 0$$

and from our usual homological algebra machinery, we get a long exact sequence where now all maps go the opposite way from the one in homology:

$$\begin{tikzcd}
\cdots & H^n(X;M) \arrow[l] & H^n(X,A;M) \arrow[l] \arrow[dll, rounded corners, to path={}] \\
H^{n-1}(X,A;M) \arrow[urr] & H^{n-1}(X;M) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

This gives one of the three Eilenberg–Steenrod axioms for cohomology: the *long exact sequence for a pair*. The other two are versions of the excision and wedge axioms. Excision for homology was proven using the subdivision operation $\mathbb{S}$ and a chain homotopy between the identity and $\mathbb{S}$, and we dualize these to give excision for cohomology. The precise statement of *excision* is that for an excisive triple $U \subset A \subset X$, the homomorphism

$$H^*(X, A; M) \longrightarrow H^*(X \setminus U, A \setminus U; M)$$

is an isomorphism.

The *wedge axiom* will look slightly different, as $\mathrm{Hom}_R(-, M)$ takes direct sums to direct products: the natural maps $H^*(\bigsqcup_{i \in I} X_i) \to H^*(X_i)$ induced by the inclusions assemble to an isomorphism

$$H^*\left(\bigsqcup_{i \in I} X_i\right) \longrightarrow \prod_{i \in I} H^*(X_i).$$

13.1.2 First computational tools

With Eilenberg–Steenrod axioms for cohomology come the same computational tools as for homology: Mayer–Vietoris takes the shape

$$\begin{array}{l} \cdots \longleftarrow H^n(U; M) \oplus H^n(V; M) \longleftarrow H^n(X; M) \longleftarrow \\ \longleftarrow H^{n-1}(U \cap V; M) \longleftarrow H^{n-1}(U; M) \oplus H^{n-1}(V; M) \longleftarrow \cdots \end{array}$$

and from this we can deduce a suspension isomorphism

$$\widetilde{H}^n(SX; M) \cong \widetilde{H}^{n-1}(X; M).$$

This gives us the cohomology of spheres from $H^*(S^0; M)$: for $n \geq 1$ we have

$$H^*(S^n; M) = \begin{cases} M & \text{if } * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

From this we can construct the machinery of cellular cohomology: for CW-complexes there is a cellular cochain complex computing $H^*(X; M)$ obtained by applying $\mathrm{Hom}_R(-, M)$ to the cellular chain complex $C_*(X; R)$.

Example 13.1.4. Let us compute $H^*(\mathbb{R}P^3)$. Dualizing the cellular chain complex $C_*(\mathbb{R}P^3)$ for the standard CW structure, we get the cellular cochain complex:

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Thus we see that

$$H^*(\mathbb{R}P^3) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ 0 & \text{if } * = 1, \\ \mathbb{Z}/2 & \text{if } * = 2, \\ \mathbb{Z} & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{cases}$$

That is, in comparison with homology the free summands stay in the same degrees, but the torsion shifts up one degree: this is a general phenomenon we will shall explain shortly.

13.2 The Kronecker pairing

For any R-module N there is an evaluation pairing

$$\begin{aligned} \mathrm{Hom}_R(N, M) \times N &\longrightarrow M \\ (\alpha, n) &\longmapsto \alpha(n). \end{aligned}$$

This is also known as the *Kronecker pairing*. It is bilinear, so induces a map out of the tensor product

$$\mathrm{Hom}_R(N, M) \otimes_R N \longrightarrow M.$$

We get an evaluation pairing $S^n(X; M) \otimes_R S_n(X; R) \to M$ between singular chain and cochains. To make this compatible with gradings, with M in degree 0, we use the regraded version $S^\vee_{-n}(X; M) = S^n(X; M)$:

$$S^\vee_{-n}(X; M) \otimes_R S_n(X; R) \longrightarrow M.$$

Proposition 13.2.1. *These maps assemble into map of chain complexes*

$$S^\vee_*(X; M) \otimes_R S_*(X; R) \longrightarrow M$$

with M considered as a chain complex concentrated in degree 0.

Proof. We must verify that the evaluation pairings are compatible with the differential. There is only something to check on the term $S^\vee_{-n}(X; M) \otimes_R S_n(X; R)$ in degree 0. The differential on the tensor product is given on generators by

$$d(\alpha \otimes \sigma) = d(\alpha) \otimes \sigma + (-1)^n \alpha \otimes d(\sigma) = (-1)^{n+1}(\alpha \circ d) \otimes \sigma + (-1)^n \alpha \otimes d(\sigma).$$

Getting the sign on the right hand side was the reason for modifying the sign on the differential in the singular cochain complex. Under the evaluation pairing this goes to

$$(-1)^{n+1}\alpha(d(\sigma)) + (-1)^n \alpha(d(\sigma)) = 0. \qquad \square$$

We can now take homology and get maps

$$H_*(S^\vee_*(X; M)) \otimes_R H_*(X; R) \longrightarrow H_*(S^\vee_*(X; M) \otimes_R S_*(X; R)) \longrightarrow M.$$

Recalling that $H_{-n}(S^\vee_*(X; M)) = H^n(X; M)$, this gives homomorphisms

$$\langle -, - \rangle \colon H^n(X; M) \otimes_R H_n(X; R) \longrightarrow M.$$

13.3 Ext and universal coefficients for cohomology

The Kronecker pairing provides a map

$$\begin{aligned} \beta\colon H^n(X;M) &\longrightarrow \mathrm{Hom}_R(H_n(M),R) \\ \alpha &\longmapsto \langle \alpha, - \rangle. \end{aligned}$$

We will investigate to what extent it is an isomorphism. This is possible using a version of the universal coefficients theorem, obtained by studying the functor $\mathrm{Hom}_R(-,M)$ instead of $-\otimes_R M$. Let us give the algebraic version:

Theorem 13.3.1 (Universal coefficients for cochain complexes)**.** *Let R be a PID, M an R-module, and C_* be a chain complex of free R-modules. There are short exact sequences*

$$0 \longrightarrow \mathrm{Ext}_1^R(H_{n-1}(C_*),M) \longrightarrow H^n(\mathrm{Hom}_R(C_*,M)) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(C_*),M) \longrightarrow 0$$

which are split but not naturally so.

Taking $C_* = S_*(X;R)$, we get a universal coefficients theorem for singular cohomology

$$0 \longrightarrow \mathrm{Ext}_1^R(H_{n-1}(X;R),M) \longrightarrow H^n(X;M) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(X;R),M) \longrightarrow 0.$$

The statement should look familiar: it is similar to that of the universal coefficients theorem in homology. However, instead of Tor_1^R, which is the derived functor of $-\otimes_R M$, we see a new functor Ext_1^R, which is the derived functor of $\mathrm{Hom}_R(-,M)$. The issue is similar to before: $\mathrm{Hom}_R(-,M)$ is not exact, but not all hope is lost:

Lemma 13.3.2. $\mathrm{Hom}_R(-,M)$ *takes cokernels to kernels.*

Proof. If N'' is the cokernel of $g\colon N \to N'$, we need to prove that every homomorphism $f\colon P \to \mathrm{Hom}_R(N',M)$ such that

$$P \xrightarrow{f} \mathrm{Hom}_R(N',M) \xrightarrow{-\circ g} \mathrm{Hom}_R(N,M)$$

is zero, factors uniquely over $\mathrm{Hom}_R(N'',M)$. Indeed, if $f(p)$ vanishes on the image of g, then it factors uniquely as a map $\widetilde{f(p)}\colon N'/g(N) = N'' \to M$. □

We can now call on the machinery of homological algebra to measure the failure of exactness of $\mathrm{Hom}_R(-,M)$. We take a free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0,$$

and after truncating this and applying $\mathrm{Hom}_R(-,M)$ get a chain complex

$$0 \longrightarrow \mathrm{Hom}_R(F_0,M) \longrightarrow \mathrm{Hom}_R(F_1,M) \longrightarrow \cdots .$$

Definition 13.3.3. We define the Ext-*groups of N against M over R* by

$$\mathrm{Ext}_n^R(N,M) \coloneqq H_n(\mathrm{Hom}_R(F_*,M)).$$

Since $\mathrm{Hom}_R(-, M)$ sends cokernels to kernels, we see that

$$\mathrm{Ext}_0^R(N, M) = \mathrm{Hom}_R(N, M).$$

Example 13.3.4. If $R = \mathbb{Z}$ and $M = \mathbb{Z}$, then of course $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = 0$. However, $\mathrm{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$ does not vanish. To see this, we resolve $\mathbb{Z}/n$ by

$$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n$$

and see that $\mathrm{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$ is the cokernel of the homomorphism

$$\mathbb{Z} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{n} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z},$$

so $\mathrm{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$. Through this phenomenon, Theorem 13.3.1 explains the following, as long as all $H_*(X)$ are finitely-generated abelian groups:

$$\text{free part of } H^n(X) \cong \text{free part of } H_n(X),$$

$$\text{torsion part of } H^n(X) \cong \text{torsion part of } H_{n-1}(X).$$

The same techniques as for Tor imply:

Proposition 13.3.5. $\mathrm{Ext}_n^R(-, M)\colon \mathsf{Mod}_R^{\mathrm{op}} \to \mathsf{Mod}_R$ *is independent of choice of resolution and functorial.*

Remark 13.3.6. As should be clear from the definitions, $\mathrm{Ext}_n^R(N, M)$ is also covariantly functorial in M.

Proof of Theorem 13.3.1. Let us shorten the homology, cycles and boundaries in C_* to H_n, Z_n and B_n. Then $H_n = Z_n/B_n$ is a submodule of C_n/B_n, and its quotient is isomorphic to $C_n/Z_n = B_{n-1}$ via the differential. We thus have a short exact sequence

$$0 \longrightarrow H_n \longrightarrow C_n/B_n \xrightarrow{d} B_{n-1} \longrightarrow 0.$$

Since $B_{n-1} \subset C_{n-1}$ is free, this remains short exact after applying $\mathrm{Hom}_R(-, M)$:

$$0 \longrightarrow \mathrm{Hom}_R(B_{n-1}, M) \longrightarrow \mathrm{Hom}_R(C_n/B_n, M) \longrightarrow \mathrm{Hom}_R(H_n, M) \longrightarrow 0.$$

We can map $H^n(\mathrm{Hom}_R(C_*, M))$ into the right term using β. Recall that $H^n(\mathrm{Hom}_R(C_*, M))$ is a quotient of the cycles $Z^n(\mathrm{Hom}_R(C_*, M))$ in $\mathrm{Hom}_R(C_*, M)$. These are functionals of C_n which vanish on B_n, so are equal to the middle term. We thus get a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & B^n(\mathrm{Hom}_R(C_*, M)) \arrow[r] \arrow[d, "\beta'"] & Z^n(\mathrm{Hom}_R(C_*, M)) \arrow[r] \arrow[d, "\cong"] & H^n(\mathrm{Hom}_R(C_*, M)) \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(B_{n-1}, M) \arrow[r] & \mathrm{Hom}_R(C_n/B_n, M) \arrow[r] & \mathrm{Hom}_R(H_n, M) \arrow[r] & 0.
\end{tikzcd}$$

Extending vertically by 0's to a short exact sequence of chain complexes, we get a long exact sequence on homology. Concentrating on the entry of this long exact sequence coming from $H^n(\mathrm{Hom}_R(C_*, M))$, we see that β is surjective and its kernel is $\mathrm{coker}(\beta')$. In other words, it fits into a short exact sequence

$$0 \longrightarrow \mathrm{coker}(\beta') \longrightarrow H^n(\mathrm{Hom}_R(C_*, M)) \xrightarrow{\beta} \mathrm{Hom}_R(H_n, M) \longrightarrow 0.$$

So it remains to identify $\mathrm{coker}(\beta')$ with an Ext-group. To do so, observe that $B^n(\mathrm{Hom}_R(C_*, M))$ consist of those functionals $C_n \to M$ that factor as $C_n \xrightarrow{d} C_{n-1} \to M$. Since $C_n \to C_{n-1}$ factors over Z_{n-1}, which is a direct summand since C_n is free, this is the same as functionals $C_n \to M$ that factor as $C_n \xrightarrow{d} Z_{n-1} \to M$. Thus the cokernel of β' is equal to the cokernel of $\mathrm{Hom}_R(Z_{n-1}, M) \to \mathrm{Hom}_R(B_{n-1}, M)$ given by restriction to $B_{n-1} \subset Z_{n-1}$:

$$\mathrm{coker}(\beta') = \mathrm{coker}[\mathrm{Hom}_R(Z_{n-1}, M) \to \mathrm{Hom}_R(B_{n-1}, M)].$$

Since both Z_{n-1} and B_{n-1} are free, as submodules of the free R-module C_{n-1}, we see that

$$0 \longrightarrow B_{n-1} \longrightarrow Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0$$

is a free resolution of H_{n-1} and thus the above cokernel computes $\mathrm{Ext}_1^R(H_{n-1}, M)$.

Finally, for the addendum about the splitting we use that the short exact sequence

$$0 \longrightarrow H_n \longrightarrow C_n/B_n \longrightarrow B_{n-1} \longrightarrow 0$$

admits a splitting because B_{n-1} is free, and proceed as in the previous universal coefficients theorem. $\square$

Example 13.3.7. Let us compute $H^*(\mathbb{R}P^3)$ again, using that

$$H_*(\mathbb{R}P^3) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}/2 & \text{if } * = 1, \\ 0 & \text{if } * = 2, \\ \mathbb{Z} & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ gives a $\mathbb{Z}$ in degrees 0 and 3, but no $\mathbb{Z}/2$ as that can't map non-trivially into $\mathbb{Z}$. However $\mathrm{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2$, so we get a $\mathbb{Z}/2$ in degree 2. This agrees with the answer obtained above using cellular cohomology.

13.4 Problems

Problem 13.4.1 (H^1)**.** Prove that $H^1(X; M) \cong \mathrm{Hom}_R(H_1(X; R), M)$. Conclude that $H^1(X)$ is always torsion-free.

Problem 13.4.2 (The degree in cohomology)**.** A map $f\colon S^n \to S^n$ induces a self-map of both $H_n(S^n)$ and $H^n(S^n)$, both of which are given by multiplication with an integer. Prove that these integers are equal.

Chapter 14

The cup product

One motivation for cohomology is that its contravariance should allow us to produce a product from the diagonal map. We will do so in this chapter.

14.1 The cross product

As before, we drop the subscript R from tensor products for brevity. Recall from Theorem 11.3.9 that there is a natural chain homotopy equivalence

$$AW_{X,Y} \colon S_*(X \times Y; R) \longrightarrow S_*(X; R) \otimes S_*(Y; R),$$

unique up to chain homotopy.

Remark 14.1.1. In Problem 11.4.2 we gave an explicit formula, called the *Alexander–Whitney* map:

$$S_*(X \times Y; R) \ni \sigma \longmapsto \sum_{p+q}^{n} \pi_1 \circ \sigma|_{\Delta^p} \otimes \pi_2 \circ \sigma|_{\Delta^q} \in S_*(X; R) \otimes S_*(Y; R),$$

where $\Delta^p \subset \Delta^n$ is the first face and $\Delta^q \subset \Delta^n$ the last one. Though our "agnostic" approach through acylic models is better for proofs, I recommend you think in terms of this definition instead.

Dualizing this gives a map of cochain complexes

$$AW^*_{X,Y} \colon \mathrm{Hom}_R(S_*(X) \otimes S_*(Y), R) \longrightarrow \mathrm{Hom}_R(S_*(X \times Y; R), R) = S^*(X \times Y; R). \quad (14.1)$$

Since we can dualize a chain homotopy to a cochain homotopy, this is a cochain homotopy equivalence.

To map into the domain $S_*(X; R) \otimes S_*(Y; R)$, we use the chain map

$$a_{C_*,D_*} \colon \mathrm{Hom}_R(C_*, R) \otimes \mathrm{Hom}_R(D_*, R) \longrightarrow \mathrm{Hom}_R(C_* \otimes D_*; R)$$

given by

$$\alpha \otimes \beta \longmapsto \begin{cases} (x \otimes y \mapsto (-1)^{pq} \alpha(x)\beta(y) & \text{if } \deg(\alpha) = p = \deg(x),\ \deg(\beta) = q = \deg(y). \\ 0 & \text{otherwise.} \end{cases}$$

Taking $C_* = S_*(X;R)$ and $D_* = S_*(Y;R)$, this gives a cochain map

$$S^*(X;R) \otimes S^*(Y;R) \longrightarrow \mathrm{Hom}_R(S_*(X) \otimes S_*(Y), R). \tag{14.2}$$

Composing with (14.1), we get a cochain map $S^*(X;R) \otimes S^*(Y;R) \to S^*(X \times Y;R)$, and passing to homology a map

$$H^*(S^*(X;R) \otimes S^*(Y;R)) \longrightarrow H^*(X \times Y;R).$$

As in Künneth, there is a natural map $H^*(X;R) \otimes H^*(Y;R) \to H^*(S^*(X;R) \otimes S^*(Y;R))$, and precomposing with this we finally obtain the *cohomology cross product*

$$\times\colon H^*(X;R) \otimes H^*(Y;R) \to H^*(X \times Y;R).$$

14.2 The cup product

The diagonal is the continuous map $\Delta\colon X \to X \times X$ given by $x \mapsto (x,x)$. It induces a map $\Delta^*\colon H^*(X \times X;R) \to H^*(X;R)$ on cohomology, and we can precompose it with the cohomology cross product:

Definition 14.2.1. The *cup product*

$$\cup\colon H^*(X;R) \otimes H^*(X;R) \longrightarrow H^*(X;R).$$

is the composition of the cohomology cross product with Δ^*.

Remark 14.2.2. The formula for the Alexander–Whitney maps from Problem 11.4.2 gives an explicit formula for the cup product as the map induced on cohomology by

$$\begin{aligned} S^p(X;R) \otimes S^q(X;R) &\longrightarrow S^{p+q}(X \times Y;R) \\ \alpha \otimes \beta &\longmapsto (\sigma \mapsto (-1)^{pq}\alpha(\pi_X \circ \sigma|_{\Delta^p}))\beta(\pi_Y \circ \sigma|_{\Delta^q}))\,. \end{aligned}$$

By construction the cup product is natural in X, that is, $f^*\colon H^*(Y;R) \to H^*(X;R)$ satisfies $f^*(x \cup y) = f^*(x) \cup f^*(y)$. It has the following further properties:

Proposition 14.2.3. *The cup product is associative, unital, and graded-commutative. The unit is given by element of* $H^0(X;R) = \mathrm{Map}(\pi_0(X), R)$ *that assigns* 1 *to every connected component.*

Here graded-commutativity means that $\alpha \cup \beta = (-1)^{pq}\beta \cup \alpha$ when $\deg(\alpha) = p$ and $\deg(\beta) = q$. Note that the naturality of f^* can be rephrased as saying that it is a homomorphism $H^*(Y;R) \to H^*(X;R)$ of graded-commutative R-algebras.

The properties of the cup product can all be proven using acyclic models. Let us prove the associativity this way. At this point, I will also drop R from the coefficients and the subscript on Hom, to make all diagrams fit on the page. Our cup product is a composition of the dual (14.1) of AW, the chain map α (14.2) and the map induced by the diagonal:

$$\begin{tikzcd}
H^*(X)\otimes H^*(X) \arrow[d, "a_{X,X}"] \\
H^*(\mathrm{Hom}(S_*(X)\otimes S_*(X),R)) \arrow[d, "AW^*_{X,X}"] \\
H^*(X\times X) \arrow[d, "\Delta^*"] \\
H^*(X)
\end{tikzcd}$$

When we start with three copies of $H^*(X)$, we can either take $\cup\circ(\cup\otimes\mathrm{id})$ or $\cup\circ(\mathrm{id}\otimes\cup)$ to end up at $H^*(X)$: associativity asserts these are equal.

Since AW^* and a are natural, the following diagrams commute

$$\begin{tikzcd}
H^*(X\times X)\otimes H^*(X) \arrow[r, "AW^*_{X\times X,X}\circ a_{X\times X,X}"] \arrow[d, "\Delta^*\otimes\mathrm{id}"] & H^*(X\times X\times X) \arrow[d, "(\Delta\times\mathrm{id})^*"] \\
H^*(X)\otimes H^*(X) \arrow[r, "AW^*_{X,X}\circ a_{X,X}"] & H^*(X\times X)
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(X)\otimes H^*(X\times X) \arrow[r, "AW^*_{X,X\times X}\circ a_{X,X\times X}"] \arrow[d, "\mathrm{id}\otimes\Delta^*"] & H^*(X\times X\times X) \arrow[d, "(\mathrm{id}\times\Delta)^*"] \\
H^*(X)\otimes H^*(X) \arrow[r, "AW^*_{X,X}\circ a_{X,X}"] & H^*(X\times X).
\end{tikzcd}$$

These can pasted together to top-right and bottom-left squares of

$$\begin{tikzcd}
 & H^*(X)\otimes H^*(X\times X) \arrow[r, "\mathrm{id}\otimes\Delta^*"] \arrow[d, "AW^*_{X,X\times X}\circ a_{X,X\times X}"'] & H^*(X)\otimes H^*(X) \arrow[d, "AW^*_{X,X}\circ a_{X,X}"'] \\
H^*(X\times X)\otimes H^*(X) \arrow[r, "AW^*X\times X{,}X\circ a_{X\times X,X}"] \arrow[d, "\Delta^*\otimes\mathrm{id}"] & H^*(X\times X\times X) \arrow[r, "(\mathrm{id}\times\Delta)^*"] \arrow[d, "(\Delta\times\mathrm{id})^*"] & H^*(X\times X) \arrow[d, "\Delta^*"] \\
H^*(X)\otimes H^*(X) \arrow[r, "AW^*_{X,X}\circ a_{X,X}"] & H^*(X\times X) \arrow[r, "\Delta^*"] & H^*(X)
\end{tikzcd}$$

where the bottom-right corner commutes since the following diagram of spaces does

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"] & X\times X \arrow[d, "\mathrm{id}\times\Delta"] \\
X\times X \arrow[r, "\Delta\times\mathrm{id}"] & X\times X\times X.
\end{tikzcd}$$

It thus suffices to prove that the following completion of the top-left corner commutes

$$\begin{tikzcd} H^*(X)\otimes H^*(Y)\otimes H^*(Z) \arrow[r, "\mathrm{id}\otimes(AW^*_{Y,Z}\circ a_{Y,Z})"] \arrow[d, "(AW^*_{X,Y}\circ a_{X,Y})\otimes \mathrm{id}"] & H^*(X)\otimes H^*(Y\times Z) \arrow[d, "AW^*_{X,Y\times Z}\circ \alpha_{X,Y\times Z}"] \\ H^*(X\times Y)\otimes H^*(Z) \arrow[r, "AW^*_{X\times Y,Z}\circ a_{X\times Y,Z}"] & H^*(X\times Y\times Z), \end{tikzcd}$$

and specialize X, Y, Z all to X.

Since α is natural in chain maps and AW^* is given by precomposition with a chain map, we see that they commute. It thus suffices to prove that the following two diagrams commute up to chain homotopy

$$\begin{tikzcd} S^*(X)\otimes S^*(Y)\otimes S^*(Z) \arrow[r, "a_{X,Y}\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes a_{Y,Z}"] & \mathrm{Hom}(S_*(X)\otimes S_*(Y), R)\otimes S^*(Z) \arrow[d, "a_{X\times Y,Z}"] \\ S^*(X)\otimes \mathrm{Hom}(S_*(Y)\otimes S_*(Z), R) \arrow[r, "a_{X,Y\times Z}"] & \mathrm{Hom}(S_*(X)\otimes S_*(Y)\otimes S_*(Z), R). \end{tikzcd}$$

$$\begin{tikzcd} S_*(X\times Y\times Z) \arrow[r, "AW_{X\times Y,Z}"] \arrow[d, "AW_{X,Y\times Z}"] & S^*(X\times Y)\otimes S^*(Z) \arrow[d, "AW_{X,Y}\otimes \mathrm{id}"] \\ S_*(X)\otimes S_*(Y\times Z) \arrow[r, "\mathrm{id}\otimes AW_{X,Y\times Z}"] & S^*(X)\otimes S^*(Y)\otimes S^*(Z). \end{tikzcd}$$

The first is commutative by a simple inspecting on the formula for a. However, the second is not obviously commutative (how can it be given our construction?) so a chain homotopy is needed. We find it using acyclic models:

Lemma 14.2.4. *The two chain homotopy equivalences* $(\mathrm{id}\otimes AW_{Y,Z})\circ AW_{X,Y\times Z}$, $(AW_{X,Y}\otimes \mathrm{id})\circ AW_{X\times Y,Z}$*, both chain maps*

$$S_*(X\times Y\times Z; R)\longrightarrow S_*(X;R)\otimes S_*(Y;R)\otimes S_*(Y;R),$$

are naturally chain homotopic.

Proof. We apply Theorem 11.3.4 to

$$\begin{aligned} \mathsf{C} &= \mathsf{Top}^3, \\ \mathcal{M} &= \{(\Delta^p, \Delta^q, \Delta^r) \mid p,q,r\geq 0\}, \\ F &= (X,Y,Z)\mapsto H_0(X\times Y\times Z) \\ F_\bullet &= ((X,Y,Z)\mapsto S_*(X\times Y\times Z)) \\ G &= ((X,Y,Z)\mapsto H_0(X)\otimes H_0(Y)\otimes H_0(Z)) \\ G_\bullet &= ((X,Y,Z)\mapsto S_*(X)\otimes S_*(Y)\otimes S_*(Z)) \\ f &= \text{natural iso}\colon H_0(X\times Y\times Z)\xrightarrow{\cong} H_0(X)\otimes H_0(Y)\otimes H_0(Z). \end{aligned}$$

We do not only obtain a natural chain map $f_*\colon F_*\to G_*$ extending f, but this is unique up to chain homotopy. But both $(\mathrm{id}\otimes AW)\circ AW, (AW\otimes \mathrm{id})\circ AW$ extend f, so they must be chain-homotopic. ∎

This completes the proof of associativity. You are probably tired of commutative squares at this point, so we will be satisfied with saying that similar proofs give the unitality and graded-associativity of the cup product. The latter reduces to proving that the following diagram for AW commutes up to natural chain homotopy

$$\begin{tikzcd} S_*(X \times Y) \arrow[r, "T_*"] \arrow[d, "AW_{X,Y}"] & S_*(Y \times X) \arrow[d, "AW_{Y,X}"] \\ S_*(X) \otimes S_*(Y) \arrow[r, "\tau"] & S_*(Y) \otimes S_*(X), \end{tikzcd} \tag{14.3}$$

where $T(x,y) = (y,x)$ and $\tau(a \otimes b) = (-1)^{\deg(a)\deg(b)} b \otimes a$. Without this sign, the bottom map is not a chain map.

Remark 14.2.5. It is in fact impossible to make the cup-product on $S^*(X)$ graded-commutative with any cochain complex model for cohomology. The failure of this is measured by the so-called *Steenrod squares* [MT68].

14.3 Examples

Let us compute some first examples of cohomology rings.

14.3.1 Spheres

If a topological space has rather little cohomology, it is easy to determine the cup product. For example, let $n \geq 1$ and let us write 1 for the generator of $H^0(S^n)$ and x_n for the generator of $H^n(S^n)$ in

$$H^*(S^n) = \begin{cases} \mathbb{Z} & \text{if } * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then 1 is the unit, and $x \cup x$ vanishes because it lies in degree $2n$. Letting $\Lambda_\mathbb{Z}(x_n)$ denote the *exterior algebra* $\mathbb{Z}[x_n]/(x_n^2)$, we see that

$$H^*(S^n) \cong \Lambda_\mathbb{Z}(x_n).$$

14.3.2 Products

We know that over a field $\mathbb{F}$, $H^*(X \times Y; \mathbb{F}) \cong H^*(X; \mathbb{F}) \otimes_\mathbb{F} H^*(Y; \mathbb{F})$ as graded vector spaces. The same is true as $\mathbb{F}$-algebras. To make sense for this, we first need to define the tensor product of graded-commutative algebras:

Definition 14.3.1. Let A^* and B^* be graded-commutative R-algebras, then we make $A^* \otimes_R B^*$ into a graded-commutative R-algebra by taking its product to be

$$(a \otimes b)(a' \otimes b') = (-1)^{pq} aa' \otimes bb',$$

where $\deg(b) = p$ and $\deg(a') = q$.

Proposition 14.3.2. *The cross product*

$$\times\colon H^*(X \times Y; R) \longrightarrow H^*(X; R) \otimes H^*(Y; R)$$

is an R-algebra homomorphism. It is an isomorphism when $H^*(X; R)$ consists of free R-modules (e.g. when R is a field $\mathbb{F}$).

Proof. We need to prove that

$$(a \times b) \cup (a' \times b') = (-1)^{\deg(b)\deg(a')}(a \cup a') \times (b \cup b').$$

There is a commutative diagram

$$\begin{tikzcd}
X \times Y \arrow[rr, "\Delta_{X\times Y}"] \arrow[dr, "\Delta_X \times \Delta_Y"'] & & X \times Y \times X \times Y \arrow[dl, "\mathrm{id}\times T\times \mathrm{id}"] \\
& X \times X \times Y \times Y, &
\end{tikzcd}$$

with $T(x, y, x', y') = (x, x', y, y')$.

By definition, we have

$$\begin{aligned}
(a \times b) \cup (a' \times b') &= \Delta^*_{X\times Y}(a \times b \times a' \times b') \\
&= (\Delta_X \times \Delta_Y)^*(\mathrm{id} \times T \times \mathrm{id})^*(a \times b \times a' \times b') \\
&= (\Delta_X \times \Delta_Y)^*(a \times T^*(b \times a') \times b') \\
&= (-1)^{\deg(a')\deg(b)}(\Delta_X \times \Delta_Y)^*(a \times a' \times b \times b').
\end{aligned}$$

Here the sign appears due to commutative diagram (14.3). The naturality of the cross product says that the following diagram commutes

$$\begin{tikzcd}
H^*(X \times X \times Y \times Y) \arrow[r, "(\Delta_X\times\Delta_Y)^*"] \arrow[d, "\times"] & H^*(X \times X) \otimes H^*(Y \times Y) \arrow[d, "\Delta_X^*\otimes\Delta_Y^*"] \\
H^*(X \times Y) \arrow[r, "\times"] & H^*(X) \otimes H^*(Y),
\end{tikzcd}$$

so that

$$(-1)^{\deg(a')\deg(b)}(\Delta_X \times \Delta_Y)^*(a \times a' \times b \times b') = (-1)^{\deg(a')\deg(b)}(a \cup a') \times (b \cup b'),$$

as desired. □

Example 14.3.3. We see that for $n, m \geq 1$,

$$H^*(S^n \times S^m) \cong \Lambda_{\mathbb{Z}}(x_n) \otimes \Lambda_{\mathbb{Z}}(y_m) =: \Lambda_{\mathbb{Z}}(x_n, y_m)$$

with $\deg(x_n) = n$, $\deg(y_m) = m$. Observe that the latter exterior algebra is the free graded-commutative algebra on two generators, satisfying $x_n y_m = (-1)^{nm} y_m x_n$.

Corollary 14.3.4. *For $n, m \geq 1$, every map $f\colon S^{n+m} \to S^n \times S^m$ is trivial on cohomology in positive degrees.*

Proof. It can only be non-zero in degree $n + m$. In this degree $H^{n+m}(S^n \times S^m)$ is generated by $x_n \cup y_m$. Now we use that $f^*(x_n \cup y_m) = f^*(x_n) \cup f^*(y_m) = 0 \cup 0 = 0$. □

Thus the cup product distinguishes $S^n \times S^m$ from $S^n \vee S^m \vee S^{n+m}$, which have the same homology. In particular, these topological spaces are not homotopy equivalent.

14.4 Problems

Problem 14.4.1 (The degree of a self-map of $\mathbb{C}P^n$)**.** Recall that $H^{2n}(\mathbb{C}P^n) = \mathbb{Z}$. Any map $f\colon \mathbb{C}P^n \to \mathbb{C}P^n$ induces a map $f^*\colon H^{2n}(\mathbb{C}P^n) = \mathbb{Z} \to H^{2n}(\mathbb{C}P^n) = \mathbb{Z}$. This is given by multiplication with an integer which we denote $\deg(f)$.

Prove that $\deg(f) \geq 0$ when n is even.

14.4.1 The James construction on spheres

Definition 14.4.2. The *James construction* $J(X)$ of a based topological space X is the quotient

$$\left(\coprod_{k\geq 0} X^k\right)/\!\sim$$

with $(x_1, \dots, x_k) \simeq (x_1, \dots, \hat{x}_i, \dots, x_k)$ if x_i is the basepoint of X.

We let $J_m(X) \subset J(X)$ be the subspace given by the image of $\coprod_{0\leq k\leq m} X^k$

You may use the following facts about $J(X)$: given a CW-structure on X with the basepoint a 0-cell, we get a product CW-structure on X^m such that the surjective map $X^m \to J_m(X)$ is cellular. This map is given by identifying the m subcomplexes of X^m with the product CW structure, where one of the entries is the basepoint. The inclusion $J_m(X) \to J_{m+1}(X)$ is the inclusion of a subcomplexes, as is the inclusion $J_m(X) \to J(X)$.

Problem 14.4.3. Let $n \geq 2$.

(i) Describe the k-cells of $J(S^n)$, if we give S^n the CW-structure with a single 0- and n-cell.

(ii) Compute $H_*(J(S^n))$ and $H^*(J(S^n))$ as graded abelian groups.

Problem 14.4.4. We will now compute the cup product when n is even.

(i) For $m \geq 1$, let

$$q_m\colon (S^n)^m \to J_m(S^n)$$

be the quotient map. Let $a_i \in H^n((S^n)^m)$ denote the generator of the ith sphere and x_1 the generator of $H^n(J_m(S^n))$. Prove that $q_m^*(x_1) = a_1 + \cdots + a_m$.

(ii) Let x_m denote the generator $H^{nm}(J_m(S^n))$. Prove that $q^*(x_m) = a_1 \cdots a_m$.

(iii) Prove that $x_1^m = m!x_m$.

Problem 14.4.5. Finally, we identify the algebra, still supposing throughout this problem that n is even. The *divided power algebra* $\Gamma_{\mathbb{Z}}[x]$ on a generator x is the subring of the polynomial ring $\mathbb{Q}[x]$ generated by the elements $x^i/i!$ for $i \geq 1$.

(i) Prove that $H^*(J(S^n))$ is isomorphic to $\Gamma_{\mathbb{Z}}[n]$ with $\deg(x) = n$.

(ii) Prove that $H^*(J(S^n); \mathbb{Q}) \cong \mathbb{Q}[x]$ with $\deg(x) = n$.

Remark 14.4.6. $J(X)$ is homotopy equivalent to the "free topological group on X with basepoint as the identity." See [Hat02, Section 4.J] for more information.

Chapter 15

Cup products in two examples

In this chapter we work out the cup products on $\mathbb{R}P^n$ and surfaces. Both have interesting applications: the first will allow us to prove a result about division algebras and the second will points us towards Poincaré duality.

15.1 The cup product on the cohomology of $\mathbb{R}P^n$

15.1.1 The relative cross product

In Problem 15.3.2 you will construct a relative cup product

$$\cup\colon H^*(X,A;R)\otimes H^*(X,B;R)\longrightarrow H^*(X,A\cup B;R)$$

when $A,B\subset X$ are open. This fits into a commutative diagram

$$\begin{array}{ccc} H^*(X,A;R)\otimes H^*(X,B;R) & \xrightarrow{\cup} & H^*(X,A\cup B;R) \\ \downarrow & & \downarrow \\ H^*(X;R)\otimes H^*(X;R) & \xrightarrow{\quad\cup\quad} & H^*(X;R). \end{array}$$

This is constructed from a relative cross product, which I will outline below.

Recall that the relative cochains $S^*(X,A;R)$ are given by those linear functionals on $S_*(X;R)$ which vanish on $S_*(A;R)$. By naturality, the cross product of an element of $S^*(X,A;R)\otimes S^*(Y,B;R)$ will be a functional on $S_*(X\times Y;R)$ which vanishes on $S_*(X\times B;R)+S_*(A\times Y;R)$. We know from the locality principle that when A and B are open, the chain map

$$S_*(X\times B;R)+S_*(A\times Y;R)\longrightarrow S_*(X\times B\cup A\times Y;R)$$

induces an isomorphism on homology. This implies that

$$\begin{array}{c} S^*(X\times Y,\times B\cup A\times Y;R) \\ \downarrow \\ \ker[S^*(X\times Y;R)\to \operatorname{Hom}(S_*(X\times B;R)+S_*(A\times Y;R),R)] \end{array}$$

induces an isomorphism on homology. Thus we get a *relative cross product*

$$\times\colon H^*(X,A;R)\otimes H^*(Y,B;R)\longrightarrow H^*(X\times Y, X\times B\cup A\times Y;R).$$

We will use one property of this relative cross product, which we will not prove for the sake of time: recall that $H^1(\mathbb{R},\mathbb{R}\setminus\{0\};R)\cong H^1([0,1],\partial[0,1];R)\cong R$, and all otherwise cohomology groups vanish. We already know that the left and right hand sides of the following lemma are isomorphic, the content is the statement that the relative cross product induces this isomorphism.

Lemma 15.1.1. *If $A\subset X$ is open, the homomorphism*

$$\times\colon H^1(\mathbb{R},\mathbb{R}\setminus\{0\};R)\otimes H^n(X,A;R)\longrightarrow H^{n+1}(X\times\mathbb{R}, X\times(\mathbb{R}\setminus\{0\})\cup A\times\mathbb{R};R)$$

is an isomorphism.

Example 15.1.2. Observe that when we have $(X,A)=(\mathbb{R},\mathbb{R}\setminus\{0\})$ and $(Y,B)=(\mathbb{R}^{n-1},\mathbb{R}^{n-1},\setminus\{0\})$, we get $(X\times Y, X\times B\cup A\times Y)=(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})$. Thus the homomorphism

$$\times\colon H^1(\mathbb{R},\mathbb{R}\setminus\{0\};R)\otimes H^{n-1}(\mathbb{R}^{n-1},\mathbb{R}^{n-1}\setminus\{0\};R)\longrightarrow H^n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\};R).$$

is an isomorphism.

15.1.2 The cohomology of $\mathbb{R}P^n$

We will use this lemma, together with some geometric arguments, to compute the cup product on $\mathbb{R}P^n$.

Theorem 15.1.3. *The cohomology ring of $\mathbb{R}P^n$ with $\mathbb{Z}/2$-coefficients is given by*

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2)\cong\mathbb{Z}/2[x]/(x^{n+1}).$$

Proof. Let's drop the $\mathbb{Z}/2$'s for the sake of brevity. By induction over n and naturality with respect to the inclusion $\mathbb{R}P^{n-1}\to\mathbb{R}P^n$, it suffices to prove that the cup product

$$H^1(\mathbb{R}P^n)\otimes H^{n-1}(\mathbb{R}P^n)\longrightarrow H^n(\mathbb{R}P^n)$$

is an isomorphism.

Recall that $\mathbb{R}P^n$ is the topological space of lines in $\mathbb{R}^{n+1}$. The subspaces $\mathbb{R}^{n-1}\times\{0\}$ and $\{0\}\times\mathbb{R}^2$, which intersect in $\{0\}\times\mathbb{R}\times\{0\}$, induce inclusions

$$\begin{aligned}\mathbb{R}P^{n-1}&=\{[x_0:\cdots:x_n]\mid x_n=0\}\subset\mathbb{R}P^n,\text{ and}\\ \mathbb{R}P^1&=\{[x_0:\cdots:x_n]\mid x_0=\cdots=x_{n-2}=0\}\subset\mathbb{R}P^n\end{aligned}$$

intersecting the point $p=[0:\cdots:0:1:0]$. We will consider the open subsets $\mathbb{R}P^n\setminus\mathbb{R}P^{n-1}\subset\mathbb{R}P^n$ and $\mathbb{R}P^n\setminus\mathbb{R}P^1\subset\mathbb{R}P^n$, whose union is $\mathbb{R}P^n\setminus\{p\}$.

We also observe that $\mathbb{R}P^n\setminus\mathbb{R}P^{n-1}$ consists of points $[x_0:\cdots:x_n]$ with $x_n\neq 0$ so deformation retracts onto $\mathbb{R}P^0=[0:\cdots:0:1]$. Similarly, $\mathbb{R}P^n\setminus\mathbb{R}P^1$ consists of those points $[x_0 : \cdots : x_n]$ with not all of $x_0, \ldots, x_{n-2}$ equal to 0, so deformation retracts onto $\mathbb{R}P^{n-2} = \{[x_0 : \cdots : x_n] \mid x_{n-1} = 0 = x_n\}$. Finally, $\mathbb{R}P^n \setminus \{p\} \simeq \mathbb{R}P^{n-1}$.

Using cellular cohomology, this implies that in the commutative diagram

$$\begin{tikzcd}
H^1(\mathbb{R}P^n, \mathbb{R}P^n \setminus \mathbb{R}P^{n-1}) \otimes H^{n-1}(\mathbb{R}P^n, \mathbb{R}P^n \setminus \mathbb{R}P^1) \arrow[r, "\cup"] \arrow[d, "\cong"] & H^n(\mathbb{R}P^n, \mathbb{R}P^n \setminus \{p\}) \arrow[d, "\cong"] \\
H^1(\mathbb{R}P^n) \otimes H^{n-1}(\mathbb{R}P^n) \arrow[r, "\cup"] & H^n(\mathbb{R}P^n)
\end{tikzcd}$$

the vertical maps are isomorphisms. It hence suffices to prove that the top relative cup product is an isomorphism.

To do so, we restrict to the chart $\mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto [x_1 : x_2 : \cdots : x_{n-1} : 1 : x_n] \in \mathbb{R}P^n$:

$$\begin{tikzcd}
H^1(\mathbb{R}P^n, \mathbb{R}P^n \setminus \mathbb{R}P^{n-1}) \otimes H^{n-1}(\mathbb{R}P^n, \mathbb{R}P^n \setminus \mathbb{R}P^1) \arrow[r, "\cup"] \arrow[d, "\cong"] & H^n(\mathbb{R}P^n, \mathbb{R}P^n \setminus \{p\}) \arrow[d, "\cong"] \\
H^1(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^{n-1}) \otimes H^{n-1}(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}) \arrow[r, "\cup"] & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})
\end{tikzcd}$$

The vertical maps are isomorphisms by the commutative diagram (for $i = n-1$, 1 and 0 respectively):

$$\begin{tikzcd}
H^{n-i}(\mathbb{R}P^n) \arrow[d, "\cong"] & H^{n-i}(\mathbb{R}P^n, \mathbb{R}P^n \setminus \mathbb{R}P^i) \arrow[l, "\cong"'] \arrow[r] \arrow[d] & H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \arrow[d, "\cong"] \\
H^{n-i}(\mathbb{R}P^{n-i}) & H^{n-i}(\mathbb{R}P^{n-i}, \mathbb{R}P^{n-i} \setminus \{p\}) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H^{n-i}(\mathbb{R}^{n-i}, \mathbb{R}^{n-i} \setminus \{0\}).
\end{tikzcd}$$

The left horizontal maps are isomorphisms by the deformation retractions given above, the left vertical map by a cellular cohomology computation, the bottom right horizontal map by excision, and the right vertical map since it is induced by the homotopy equivalence.

Recalling that the cup product is induced by restricting the cross product along the diagonal, we look at the commutative diagram

$$\begin{tikzcd}
H^1(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^{n-1}) \otimes H^{n-1}(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}) \arrow[d, "\times"] & H^1(\mathbb{R}, \mathbb{R} \setminus \{0\}) \otimes H^{n-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\}) \arrow[l, "\cong"'] \arrow[d, "\times", "\cong"'] \\
H^n(\mathbb{R}^{2n}, \mathbb{R}^n \times (\mathbb{R}^n \setminus \mathbb{R}) \cup (\mathbb{R}^n \setminus \mathbb{R}^{n-1}) \times \mathbb{R}^n) \arrow[d, "\Delta^*"] & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[l] \arrow[dl, dotted] \\
H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) &
\end{tikzcd}$$

with the right vertical map an isomorphism by Example 15.1.2 and the top horizontal map induced by the projection $\mathbb{R}^n \to \mathbb{R}$ onto the last coordinate and $\mathbb{R}^n \to \mathbb{R}^{n-1}$ onto the first $(n-1)$-coordinates. The middle horizontal map is induced by $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_n, y_1, \ldots, y_{n-1})$. This means that the composition of it with Δ^*, the dotted map, is induced by permutation of coordinates and hence an isomorphism. Thus the left vertical composition, which is the cup product, is an isomorphism. □

Remark 15.1.4. Replacing $\mathbb{R}$ with $\mathbb{C}$ or $\mathbb{H}$, this argument proves that

$$H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1}) \quad \text{and} \quad H^*(\mathbb{H}P^n) = \mathbb{Z}[y]/(y^{n+1})$$

with $|x| = 2$ and $|y| = 4$.

15.1.3 Division algebras

Definition 15.1.5. A *division algebra structure* on $\mathbb{R}^n$ is a bilinear map $m\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that right and left multiplication with a non-zero element are invertible.

Example 15.1.6. Note we do not require commutativity or even associativity. The ordinary multiplications of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ hence give division algebra structures on $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^4$, and $\mathbb{R}^8$.

Theorem 15.1.7. *There can only be a division algebra structure on $\mathbb{R}^n$ if $n = 2^k$.*

Proof. Since multiplication by $a \neq 0$ is an invertible linear map, it sends $\mathbb{R}^n \setminus \{0\}$ to itself. Thus $(a, b) \mapsto \frac{\mu(a,b)}{||\mu(a,b)||}$ gives a continuous map

$$S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}.$$

We can quotient by ± 1 to get a continuous map

$$\overline{\mu}\colon \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^{n-1}$$

which has the property that when restricted to $\mathbb{R}P^{n-1} \times \{b\}$ or $\{a\} \times \mathbb{R}P^{n-1}$ it is a homeomorphism. This implies that the induced map on cohomology with $\mathbb{Z}/2$-coefficients

$$\overline{\mu}^*\colon H^*(\mathbb{R}P^{n-1}; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^n) \longrightarrow H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2]/(x_1^n, x_2^n)$$

pulls back the generator $x \in H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2)$ to the element $x_1 + x_2 \in H^1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{Z}/2)$.

Since pullback is a homomorphism, we must have $(x_1 + x_2)^n = 0$. Observing that squaring is linear modulo 2, and writing $n = \sum a_i 2^i$ with $a_i \in \{0, 1\}$, we see that $\prod (x_1^{2^i} + x_2^{2^i})^{a_i}$ must vanish. However, no terms can cancel when expanding this product, so we must have that each term $(x_1^{2^i} + x_2^{2^i})^{a_i}$ vanishes. This only happens when $n = 2^k$. $\square$

Remark 15.1.8. In fact, it is a famous result in algebraic topology that there can only be a division algebra structure on $\mathbb{R}^n$ if $n = 1, 2, 4, 8$. This is done by rewriting the problem in terms of the homotopy groups of spheres, where it becomes the Hopf invariant one problem [Ada60].

15.2 The cup product on the cohomology of surfaces

15.2.1 Towards Poincaré duality

Recall that an n-dimensional topological manifold is a second countable Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$. The following is a special case of the next big theorem which we will prove:

Theorem 15.2.1 (Poincaré duality with $\mathbb{Z}/2$-coefficients)**.** *Let M be a compact n-dimensional topological manifold. Then there is a unique class $[M] \in H_n(M;\mathbb{Z}/2)$ such that composition*

$$H^i(M;\mathbb{Z}/2) \otimes H^{n-i}(M;\mathbb{Z}/2) \xrightarrow{\cup} H^n(M;\mathbb{Z}/2) \xrightarrow{\langle -,[M]\rangle} \mathbb{Z}/2$$

is a perfect pairing, i.e. the induced map $H^i(M;\mathbb{Z}/2) \to \mathrm{Hom}(H^{n-i}(M;\mathbb{Z}/2),\mathbb{Z}/2)$ is an isomorphism.

The class $[M]$ is called the *fundamental class*.

Example 15.2.2 (Real projective spaces)*.* We show this is true for real projective spaces: the fundamental class $[\mathbb{R}P^n]$ is the generator of $H_n(\mathbb{R}P^n;\mathbb{Z}/2)$, and it follows from our previous computation that

$$H^i(\mathbb{R}P^n;\mathbb{Z}/2) \otimes H^{n-i}(\mathbb{R}P^n;\mathbb{Z}/2) \xrightarrow{\cup} H^n(\mathbb{R}P^n;\mathbb{Z}/2) \xrightarrow{\langle -,[\mathbb{R}P^n]\rangle} \mathbb{Z}/2$$

is given by

$$x^i \otimes x^{n-i} \longmapsto x^n \longmapsto 1.$$

Example 15.2.3 (Torus)*.* Recall that the torus $\mathbb{T}^2$ is just homeomorphic to $S^1 \times S^1$, so its cohomology ring with $\mathbb{Z}/2$-coefficients is given by

$$H^*(\mathbb{T}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha,\beta]/(\alpha^2,\beta^2).$$

Thus in this case the fundamental class is the generator of $H_2(\mathbb{T}^2;\mathbb{Z}/2)$ and the isomorphism $H^1(M;\mathbb{Z}/2) \to \mathrm{Hom}(H^1(M;\mathbb{Z}/2),\mathbb{Z}/2)$ is given by sending the basis α,β to the dual basis elements e_β, e_α respectively.

15.2.2 The intersection product

We can combine the Poincaré duality map with universal coefficient theorem for cohomology, to get an isomorphism

$$H^i(M;\mathbb{Z}/2) \xrightarrow{\cong} \mathrm{Hom}(H^{n-i}(M;\mathbb{Z}/2);\mathbb{Z}/2) \xleftarrow{\cong} H_{n-i}(M;\mathbb{Z}/2).$$

Using these isomorphisms, we can transform the cup product on cohomology into an *intersection product* on homology

$$\begin{tikzcd}
H_p(M;\mathbb{Z}/2) \otimes H_q(M;\mathbb{Z}/2) \arrow[r, "\pitchfork"] \arrow[d, "\cong"] & H_{p+q-n}(M;\mathbb{Z}/2) \arrow[d, "\cong"] \\
H^{n-p}(M;\mathbb{Z}/2) \cong H^{n-q}(M;\mathbb{Z}/2) \arrow[r, "\cup"] & H^{2n-p-q}(M;\mathbb{Z}/2).
\end{tikzcd}$$

This is called the intersection product because if $a \in H_p(M;\mathbb{Z}/2)$ and $b \in H_q(M;\mathbb{Z}/2)$ can be represented as the image of fundamental class of some submanifolds A and B of dimension p and q respectively, $a \pitchfork b$ can be computed as follows: make A and B transverse, take their intersection to get a $(p+q-n)$-dimensional dimensional submanifold $A \cap B$, and take the image of its fundamental class.

Example 15.2.4. For the torus, the generators a, b of $H_1(\mathbb{T}^2; \mathbb{Z}/2)$ are represented by circles which intersect transversally in a single point. Hence their intersection product is $1 \in H_0(\mathbb{T}^2; \mathbb{Z}/2)$. This reflects the fact that $\alpha \cup \beta$ is the generator of $H^2(\mathbb{T}^2; \mathbb{Z}/2)$.

Example 15.2.5. For a genus g surface Σ_g, there are $2g$ circles $a_1, b_1, \cdots, a_g, b_g$ which generate $H_1(\Sigma_g; \mathbb{Z}/2)$. Each pair a_i, b_i intersects transversally in a single point, while elements of different pairs do not intersect. Thus $a_i \pitchfork b_i = 1$ and all other intersection products vanish. We conclude that $H^*(\Sigma_g; \mathbb{Z}/2)$ is generated by elements a_i, b_i of degree 1, satisfying $a_i a_j = 0 = b_i b_j$ for all $1 \leq i, j \leq g$, as well as $a_i b_j = 0$ and $a_i b_i = a_j b_j$ for all $1 \leq i \neq j \leq g$.

15.2.3 The classification of surfaces

To every compact connected surface we can associate a perfect symmetric bilinear form

$$\langle - \cup -, [\Sigma] \rangle \colon H^1(\Sigma; \mathbb{Z}/2) \otimes H^1(\Sigma; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2,$$

and homeomorphic surfaces get assigned isomorphic forms. Thus we get a map of sets

$$\begin{array}{c} \mathsf{Surf} := \dfrac{\{\text{connected compact surfaces}\}}{\text{homeomorphism}} \\ \downarrow \\ \mathsf{Bil} := \dfrac{\{\text{perfect symmetric bilinear forms over } \mathbb{Z}/2\}}{\text{isomorphism}} \end{array}$$

Both sides have a commutative monoid structure: the right hand side by orthogonal sum $\oplus$, and the left hand side by connected sum $\#$. The connected sum $\Sigma \# \Sigma'$ of Σ and Σ' is given by removing a disk from both sides and gluing the boundaries together. These are compatible:

Lemma 15.2.6. *There is an isomorphism* $H^1(\Sigma \# \Sigma'; \mathbb{Z}/2) \cong H^1(\Sigma; \mathbb{Z}/2) \oplus H^1(\Sigma'; \mathbb{Z}/2)$ *of perfect symmetric bilinear forms over* $\mathbb{Z}/2$.

A bilinear form over $\mathbb{Z}/2$ can be encoded up to isomorphism by the matrix over $\mathbb{Z}/2$, whose entries are given by the values of the form on pairs of basis elements. For example, the form on $H^1(\mathbb{T}^2; \mathbb{Z}/2)$ is given by hyperbolic form $H = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and the one on $H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ is given by the diagonal form $D = [\,1\,]$.

Lemma 15.2.7. *Any perfect symmetric bilinear form over* $\mathbb{Z}/2$ *is a direct sum of copies of* H *and* D.

Proof. Any perfect symmetric bilinear form is finite-dimensional, so the proof is by induction over dimension of V. If we can find a vector v such that $v \cdot v = 1$, we have found a copy of D. If we can't find such a vector we will have to settle for a non-zero $v \in V$ such that $v \cdot v = 0$. If so, pick a w such that $v \cdot w = 1$; necessarily $w \cdot w = 0$ as well, and we have found a copy of H. Now we use that if the form restricted to $V' \subset V$ is perfect, then $V \cong V' \oplus (V')^{\perp}$. $\square$

There is a relation between H and D in Bil: $D+H=3D$. To see that this is the only relation, one observes that the dimension and the number of vectors satisfying $v\cdot v=0$ are invariants. Thus we see that

$$\mathsf{Bil} = \frac{\mathbb{Z}_{\geq 0}\{H,D\}}{D+H=3D}.$$

Since we can hit H by $\mathbb{T}^2$ and D by $\mathbb{R}P^2$, the map $\mathsf{Surf} \to \mathsf{Bil}$ is surjective. The classification of surfaces says it is also injective:

Theorem 15.2.8 (Classification of surfaces)**.** *The map* $\mathsf{Surf} \to \mathsf{Bil}$ *is an isomorphism of commutative monoids.*

Example 15.2.9. The relation $D+H=3D$ translates to $\mathbb{R}P^2 \# T^2 \cong \mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$. This can be seen as follows: $\mathbb{T}^2$ is obtained by gluing the ends of a cylinder as usual, while $\mathbb{R}P^2\#\mathbb{R}P^2 = \mathbb{K}$ is obtained by gluing the ends of a cylinder after a reflection. This means that we can think of $\mathbb{R}P^2\#T^2$ as removing two nearby disks and gluing their boundary to a cylinder as usual, while for $\mathbb{R}P^2\#\mathbb{K}$ we introduce a reflection for one. However, when we move one of the disks along the generator of $\pi_1(\mathbb{R}P^2)$ its returns with a reflection so we get the same surface whether we glue the cylinder along the two boundaries as usual or with one reflection.

15.3 Problems

Problem 15.3.1 (Maps between projective spaces)**.** In this problem you will answer three questions about continuous maps between projective spaces.

(i) The standard inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ induces a continuous map $i\colon \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ between real projective spaces. Does this admit a retraction, i.e. is there a continuous map $r\colon \mathbb{R}P^n \to \mathbb{R}P^{n-1}$ such that $r\circ i = \mathrm{id}_{\mathbb{R}P^{n-1}}$?

(ii) The Segre embedding is the continuous map

$$\begin{aligned} S\colon \mathbb{C}P^1 \times \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^3 \\ ([x_0:x_1],[y_0,y_1]) &\longmapsto ([x_0y_0 : x_1y_0 : x_0y_1 : x_1y_1]). \end{aligned}$$

What is the induced map $S^*\colon H^*(\mathbb{C}P^3) \to H^*(\mathbb{C}P^1\times\mathbb{C}P^1)$ on cohomology?

(iii) Complexification $\mathbb{R}^n \to \mathbb{C}^n$ induces a continuous map $j\colon \mathbb{R}P^n \to \mathbb{C}P^n$. What is the induced map $j^*\colon H^*(\mathbb{C}P^n) \to H^*(\mathbb{R}P^n)$ on cohomology?

Problem 15.3.2 (Cup products on suspensions)**.**

(i) Let $A, B \subset X$ be open subspaces. Construct a relative cohomology cup product

$$H^*(X,A;R)\otimes_R H^*(X,B;R) \longrightarrow H^*(X, A\cup B;R),$$

which fits into a commutative diagram

$$\begin{array}{ccc} H^*(X,A;R)\otimes_R H^*(X,B;R) & \longrightarrow & H^*(X,A\cup B;R) \\ \downarrow & & \downarrow \\ H^*(X;R)\otimes_R H^*(X;R) & \longrightarrow & H^*(X;R). \end{array}$$

(Hint: it might be more convenient to use the explicit formula rather than acyclic models, and you will need the locality principle.)

(ii) Use this to prove that in the cohomology of SY, all cup products of elements of positive degree vanish.

(iii) Deduce that $\mathbb{C}P^n$ for $n \geq 2$ can not be homotopy equivalent to a suspension.

In the following problem you may not use the classification of surfaces.

Problem 15.3.3 (The Poincaré duality pairing and connected sums)**.** Recall Poincaré duality says that on a connected compact surface Σ there is a unique class $[\Sigma] \in H_2(\Sigma; \mathbb{Z}/2)$ such that the bilinear form

$$- \cdot -\colon H^1(\Sigma; \mathbb{Z}/2) \otimes H^1(\Sigma; \mathbb{Z}/2) \xrightarrow{\cup} H^2(\Sigma; \mathbb{Z}/2) \xrightarrow{\langle -, [\Sigma] \rangle} \mathbb{Z}/2$$

is non-degenerate (equivalently, it is a perfect pairing).

Also recall that the connected sum $\Sigma \# \Sigma'$ of two connected compact surfaces is given by removing an open disk from both Σ and Σ' and gluing their boundaries together. You may assume this is a well-defined operation, up to homeomorphism.

(i) Use Mayer–Vietoris to prove that $H^1(\Sigma \# \Sigma'; \mathbb{Z}/2) \cong H^1(\Sigma; \mathbb{Z}/2) \oplus H^1(\Sigma'; \mathbb{Z}/2)$ as $\mathbb{Z}/2$-vector spaces.

(ii) Prove that this isomorphism is one of $\mathbb{Z}/2$-vector spaces with bilinear forms, if we interpret the right hand side as the orthogonal direct sum.

The cup product for a wedge

You may use the following proposition:

Proposition 15.3.4. *Let X and Y be pointed path-connected spaces. Then as a graded-commutative R-algebra, $H^*(X \vee Y; R)$ is given by*

$$(H^*(X; R) \oplus H^*(Y; R)) /\!\sim$$

where $\sim$ identifies $r \cdot 1_X \in H^0(X; R)$ with $r \cdot 1_Y \in H^0(Y; R)$.

Problem 15.3.5. Use cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

Maps between surfaces

You may use the following computation:

Theorem. *The cohomology ring of a genus g surface Σ_g with coefficients in a commutative ring R is the graded-commutative R-algebra with generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of degree 1, and relations*

(a) $\alpha_i \alpha_j = 0 = \beta_i \beta_j$ *for* $1 \leq i, j \leq g$,

(b) $\alpha_i \beta_j = 0$ *for* $1 \leq i \neq j \leq g$, *and*

(c) $\alpha_i \beta_i = \alpha_j \beta_j$ *for* $1 \leq i, j \leq g$.

Problem 15.3.6. Use cup products to prove that if $g < h$ there exists no continuous map $\Sigma_g \to \Sigma_h$ which induces an injection on H_2.

The Hopf invariant

Definition 15.3.7. Let $n \geq 2$. Given a continuous map $f\colon S^{2n-1} \to S^n$, we define $C(f)$ to be $S^n \cup_f D^{2n}$. This has a canonical CW-structure with 3 cells, which give generators $1, e_n, e_{2n}$ for $H^*(C(f))$ with $* = 0, n, 2n$ respectively. Thus we have

$$e_n \cup e_n = H(f) \cdot e_{2n}$$

for some integer $H(f) \in \mathbb{Z}$. This integer is the *Hopf invariant* of f.

Problem 15.3.8.

(i) Prove that $H(f)$ only depends on the homotopy class of f.

The Hopf invariant in fact gives a homomorphism from the homotopy group $\pi_{2n-1}(S^n)$, the set of based homotopy classes of based continuous maps from S^{2n-1} to S^n, to $\mathbb{Z}$. Understanding its image thus amounts to producing maps with small Hopf invariant.

(ii) Use $\mathbb{C}P^2$ to give an example of a continuous map $S^3 \to S^2$ with Hopf invariant one.

The same ideas applied to $\mathbb{H}P^2$ and $\mathbb{O}P^2$ tells us that continuous maps of Hopf invariant one exist for $n = 2, 4, 8$. Thus in these dimensions $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ is surjective. A theorem of Adams says these are the only cases [Ada60].

Thus the next best possible result is the following: for each even $n \geq 2$, there exists a continuous map of Hopf invariant two. You will construct it now:

(iii) Inside $S^n \times S^n$ there are two copies of S^n; $S^n \times \{x_0\}$ and $\{x_0\} \times S^n$. Let X be obtained by identifying these in the obvious manner. Show that X has the structure of a CW-complex with a single 0-, n- and $2n$-cell, and hence is of the form $C(f)$ for some $f\colon S^{2n-1} \to S^n$.

(iv) Prove that $H(f) = 2$ when n is even. (Hint: use naturality of cup products with respect to the quotient map $S^n \times S^n \to X$.)

Chapter 16

Covering spaces and orientations

In this chapter we start our proof of Poincaré duality, and we first need to confront the notion of an orientation of a manifold. We approach this through covering spaces.

16.1 Covering spaces

The exponential map

$$\begin{aligned} \mathbb{R} &\longrightarrow S^1 \\ \theta &\longmapsto e^{2\pi i\theta} \end{aligned}$$

has the property that the inverse image of a small segment of the single is homeomorphic to a disjoint union of many copies of this segment. The definition a covering space formalizes this example:

Definition 16.1.1. A continuous map $p\colon E \to B$ is a *covering map* if it has the following properties:

- for all $b \in B$ the subspace $p^{-1}(b) \subset E$ is discrete,
- each $b \in B$ has an open neighborhood $U \subset B$ such that there is a homeomorphism $\phi\colon p^{-1}(U) \to U \times p^{-1}(b)$ which fits into a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "\phi", "\cong"'] \arrow[dr, "p"'] & & U \times p^{-1}(b) \arrow[dl, "\pi_1"] \\ & U. & \end{tikzcd}$$

We call E the *total space*, B the *base*, the subspaces $p^{-1}(b)$ the *fibers*, p the *projection map*, and the ϕ's *local trivializations*.

Example 16.1.2. For any set X we can form the product

$$\pi_1\colon B \times X \to B.$$

This is a covering map, which we call the *trivial* covering map with fiber X.

Covering maps arise as the quotient maps for sufficiently nice group actions. A discrete group G acts *freely* on E when $g(e) = e$ implies $g = \mathrm{id}$. A stronger version is that G acts *properly discontinuously*; each $e \in E$ should have an open neighborhood V such that $g(V) \cap V \neq \varnothing$ if and only if $g = \mathrm{id}$.When G acts properly discontinuously, the quotient map

$$q\colon E \to E/G$$

is a covering map (the U's will be images of the V's). Its fibers can be identified with G. The following is a fact from point-set topology: when G is finite and E is Hausdorff, free implies properly discontinuous.

Example 16.1.3. We can apply this to the quotient map

$$S^n \longrightarrow S^n/\pm 1 = \mathbb{R}P^n,$$

which exhibits S^n as a double cover of $\mathbb{R}P^n$. Since S^n is simply-connected for $n \geq 2$, it is then also the universal cover.

Example 16.1.4. Similar, we can apply this to the quotient map

$$S^3 \longrightarrow S^3/I^*,$$

which exhibits the Poincaré homology sphere as the base of a covering map with total space S^3. The inverse image of a point in S^3/I^* has cardinality $\#I^* = 120$, so we say this is *120-sheeted cover.*

16.2 The fundamental group and the classification of covering spaces

Suppose $p\colon E \to B$ is a covering map and we are given a commutative diagram

$$\begin{tikzcd} \{0\} \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"] & B. \end{tikzcd}$$

That is, we are given a path $\gamma\colon [0,1] \to B$ with a lift to E of its starting point. Using the local trivializations we can lift this path to E:

Proposition 16.2.1. *Given a commutative diagram as above, there exists a unique map* $\tilde{\gamma}\colon [0,1] \to E$ *such that* $\tilde{\gamma}(0) = e_0$ *and* $p \circ \tilde{\gamma} = \gamma$.

In other words, it's a lift in

$$\begin{tikzcd} \{0\} \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"] \arrow[ur, dotted, "\tilde{\gamma}"] & B. \end{tikzcd}$$

Proof. We can cover the image of γ by the U's of local trivializations. Since $\gamma([0,1])$ is compact, it's covered by finitely many of these, $\{U_i\}_{i=1}^n$, and we can give $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_i, t_{i+1}]) \subset U_{i+1}$. By induction over n, we may thus assume that the image of γ is contained in U of a local trivialization. But then it is clear that there is a unique lift: under the homeomorphism $\phi\colon p^{-1}(U) \cong U \times \pi^{-1}(b)$, e_0 lies in a unique sheet $U \times \{e\}$ and the unique lift is given by taking $\tilde{\gamma}$ to be the composition of

$$[0,1] \xrightarrow{\gamma} U \xleftarrow{\cong} U \times \{e\} \hookrightarrow U \times \pi^{-1}(b) \xrightarrow{\phi^{-1}} p^{-1}(U) \hookrightarrow E. \qquad \square$$

Remark 16.2.2. Using little squares or cubes instead of little intervals, one proves there are unique lifts in

$$\begin{tikzcd} {[0,1]^k \times \{0\}} \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ {[0,1]^k \times [0,1]} \arrow[r, "\gamma"] \arrow[ur, dotted, "\tilde{\gamma}"] & B, \end{tikzcd}$$

and by induction over cells we may even replace $[0,1]^k \cong D^K$ with a finite CW-complex X. A map p with the property that (not necessarily unique) lifts exists in such diagrams is called a *Serre fibration.* Thus every covering map is a Serre fibration.

This construction relates the fibers over points b, b' that can be connected by a path γ: we map $e \in p^{-1}(b)$ to endpoint of the lift $\tilde{\gamma}$ starting at e. Taking $b = b'$, this gives rise to an action of the fundamental group $\pi_1(B, b)$ on $p^{-1}(b)$. Before giving hte detail, let us recall the definition of the fundamental group:

Definition 16.2.3. Given a based space (B, b), the *fundamental group* $\pi_1(B, b)$ is given by the set of "loops" $\gamma\colon [0,1] \to B$ such that $\gamma(0) = b = \gamma(1)$, up to homotopy fixing the endpoints. Concatenation makes this into a group.

This is natural in based spaces, in the sense that a continuous map induces a homomorphism. More precisely, the fundamental group gives a functor

$$\pi_1\colon \mathsf{Top}_* \longrightarrow \mathsf{Grp}.$$

The action of $[\gamma] \in \pi_1(B, b)$ on $p^{-1}(b)$ is then given as above: given $e \in p^{-1}(b)$, we lift a representative $\gamma\colon [0,1] \to B$ to a path $\tilde{\gamma}\colon [0,1] \to E$ with starting point e, and define the (right) action as

$$e \cdot [\gamma] = \tilde{\gamma}(0).$$

The uniqueness of lifts guarantees $- \cdot [\gamma]$ is a permutation of $p^{-1}(b)$. It is independent of the choice of representative because we can uniquely lift a homotopy between two representatives, so their endpoints need to be the same point of $p^{-1}(b)$. This construction is natural in the covering map p:

Definition 16.2.4. For a topological space B, the category $\mathsf{Cov}_{/B}$ of *covering spaces over* B has objects covering maps $p\colon E \to B$ and morphisms commutative diagrams

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B. & \end{tikzcd}$$

Definition 16.2.5. For a discrete group G, the category Set_G of G*-sets* has objects sets X with G-actions and morphisms given by G-equivariant functions.

If B has base point b, then sending p to $p^{-1}(b)$ gives a functor

$$\mathsf{Cov}_{/B} \longrightarrow \mathsf{Set}_{\pi_1(B,b)}.$$

For this to be an equivalence of categories, i.e. have an inverse up to natural isomorphism, at least B needs to be path-connected. This is almost enough, but we also need B to be locally nice: it needs to be *semi-locally simply-connected*, i.e. for every $b \in B$, each open neighborhood U of b contains another open neighborhood V of p such that $\pi_1(V, b) \to \pi_1(U, b)$ is trivial.

Theorem 16.2.6. *If B is path-connected and semi-locally simply-connected, the functor*

$$\mathsf{Cov}_{/B} \longrightarrow \mathsf{Set}_{\pi_1(B,b)}$$

is an equivalence of categories.

The inverse is given in terms of a "universal cover" of B. This is a covering map

$$p\colon \tilde{B} \longrightarrow B,$$

such that $\tilde{B}$ comes with a fiberwise free and transitive left $\pi_1(B, b)$-action over B. The inverse functor then sends $\pi_1(B, b)$-set X to $X \times_{\pi_1(B,b)} \tilde{B}$. This universal cover is characterized by the property that $\tilde{B}$ is simply-connected, and in fact isomorphism classes of coverings $p\colon E \to B$ with E path-connected are in bijection with conjugacy classes of subgroups $H \subset \pi_1(B, b)$ (the relation is that $p^{-1}(b)$ is $\pi_1(B, b)/H$ as a $\pi_1(B, b)$-set).

16.3 Orientations

Suppose that M is an n-dimensional manifold. I claim that

$$H_*(M, M \setminus \{m\}; R) \cong \begin{cases} R & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

To see this, pick a neighborhood U of M which is homeomorphic to $\mathbb{R}^n$ (a *chart*). Then the map $H_*(U, U \setminus \{p\}; R) \to H_*(M, M \setminus \{m\}; R)$ is an isomorphism by excision, and $H_*(U, U \setminus \{p\}; R) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$. These sets for varying m assemble to a cover of M:

Definition 16.3.1. The *R-orientation cover* $\mathcal{O}_{M,R}$ of M is the topological space with underlying set

$$\bigsqcup_{m \in M} H_*(M, M \setminus \{m\}; R)$$

and topology given as follows: a basis is indexed by triples (U, a) with $U \subset M$ open, $\bar{a} \in H_n(M, M \setminus \overline{U}; R)$ and given by the elements $(m, a) \in \bigsqcup_{m \in M} H_*(M, M \setminus \{m\}; R)$ such that $\bar{a}$ maps to a under the map $H_n(M, M \setminus \overline{U}; R) \to H_*(M, M \setminus \{m\}; R)$.

Sending $(m, a) \in \mathcal{O}_{M,R}$ to m gives a continuous map which is a covering map. The local trivializations are produced from the charts of M.

The addition in the fibers makes this an "abelian group over M." Recall that given maps $p\colon E \to B$, $p'\colon E' \to B$, the pullback $E \times_B E'$ is given by the subspace $(e, e') \in E \times E'$ such that $p(e) = p'(e')$. That is, it consists of pairs of points in the fibers of p. That $\mathcal{O}_{M,R}$ is abelian group over M can be made precise by saying that there are continuous maps

$$+\colon \mathcal{O}_{M,R} \times_M \mathcal{O}_{M,R} \longrightarrow \mathcal{O}_{M,R}$$

satisfying the usual properties for addition. Similarly, we have map $R \times \mathcal{O}_{M,R} \to \mathcal{O}_{M,R}$, which combines with the addition to make $\mathcal{O}_{M,R}$ in an R-module over M.

Let $\Gamma(M, \mathcal{O}_{M,R})$ denote the set of sections of $p\colon \mathcal{O}_{M,R} \to M$. These are the continuous maps $s\colon M \to \mathcal{O}_{M,R}$ satisfying $p \circ s = \mathrm{id}_M$. The addition and scalar multiplication in the fibers make this into an R-module.

Now we observe that there is a canonical map

$$H_n(M; R) \longrightarrow \Gamma(M, \mathcal{O}_{M,R})$$

given by sending $b \in H_n(M; R)$ to the map sending m to the image of b under the map $H_n(M; R) \to H_n(M, M \setminus \{m\}; R)$. This is a map of R-modules. The first step in proving Poincaré duality is given by establishing:

Theorem 16.3.2 (Orientation theorem)**.** *If M is compact then the homomorphism*

$$H_n(M; R) \longrightarrow \Gamma(M, \mathcal{O}_{M;R})$$

is an isomorphism.

Let us now compute the right hand side. The classification of covering spaces can be applied not just to sets over M but also R-modules over M to give (we use that all manifolds are semi-locally simply-connected):

Theorem 16.3.3. *If M is path-connected with basepoint m, the functor*

$$\begin{aligned} \mathsf{Mod}_{R,/B} &\longrightarrow \mathsf{Mod}_{R[\pi_1(M,m)]} \\ (p\colon E \to B) &\longmapsto p^{-1}(m) \end{aligned}$$

is an equivalence of categories.

Under this equivalence, passing to section of $p\colon E \to B$ amounts to taking the R-module $(p^{-1}(m))^{\pi_1(M,m)}$ of $\pi_1(M, m)$-invariants: the section picks out an element of $p^{-1}(m)$, as well as canonical lifts of paths.

Example 16.3.4. Let us specialize to $R = \mathbb{Z}$. Then the action of $\pi_1(M, m)$ on $p^{-1}(m) \cong \mathbb{Z}$ amounts to a homomorphism

$$w_1\colon \pi_1(M, m) \longrightarrow \mathrm{Aut}(p^{-1}(m)) \cong \{\pm 1\}.$$

This is essentially the first Stiefel–Whitney class. Either it is trivial or surjective. In the first case we say M is *orientable* and we have $\Gamma(M, \mathcal{O}_{M;\mathbb{Z}}) \cong \mathbb{Z}$. In the second case we say M is *non-orientable* and we have $\Gamma(M, \mathcal{O}_{M;\mathbb{Z}}) \cong \mathbb{Z}^{\pm 1} \cong 0$.

Inside $\mathcal{O}_{M;\mathbb{Z}}$ there is a subcovering $\mathcal{O}^{\times}_{M;\mathbb{Z}}$ of M consisting of those elements of the local homology groups $H_n(M, M \setminus \{m\})$ which are generators. M is orientable if and only this admits a section; such a section is called an *orientation* and picks out consistent generators of $H_n(M, M \setminus \{m\})$ for all $m \in M$. Since each fiber has two elements, $\mathcal{O}^{\times}_{M;\mathbb{Z}}$ admits a section if and only if it is trivial.

Let us denote by $\mathcal{O}_{M;\mathbb{Z}} \otimes R$ the fiberwise tensor product with R. The universal coefficient theorem tells us that $\mathcal{O}_{M;\mathbb{Z}} \otimes R \cong \mathcal{O}_{M;R}$. Thus the action

$$\pi_1(M, m) \longrightarrow \mathrm{Aut}(p^{-1}(m)) \cong \mathrm{Aut}(R)$$

factors over $\{\pm 1\}$. Then we have

$$\Gamma(M, \mathcal{O}_{M;R}) \cong \begin{cases} R & \text{if } M \text{ is orientable,} \\ \text{2-torsion in } R & \text{if } M \text{ is non-orientable.} \end{cases}$$

As before, there is a subcover $\mathcal{O}^{\times}_{M,R} \subset \mathcal{O}_{M,R}$ consisting of the generators in each $H_n(M, M \setminus \{m\}; R)$.

Definition 16.3.5. An R-orientation of M is a section of $\mathcal{O}^{\times}_{M,R}$.

Example 16.3.6. If $R = \mathbb{Z}/2$, we get

$$\Gamma(M, \mathcal{O}_{M;R}) \cong \mathbb{Z}/2$$

whether M is orientable or not. There is a unique $\mathbb{Z}/2$-orientation given by picking the unique non-zero element in $H_n(M, M \setminus \{m\}; \mathbb{Z}/2)$ for each $m \in M$.

16.4 Problems

Problem 16.4.1 (Transfer maps)**.** Let $p\colon E \to B$ be a covering map with N sheets, and R be a commutative ring.

(i) Prove that each simplex $\sigma\colon \Delta^p \to B$ has exactly N distinct lifts to E. Denoting these lifts by $\sigma_1, \ldots, \sigma_N$, prove that

$$\begin{aligned} S_*(B; R) &\longrightarrow S_*(E; R) \\ \sigma &\longmapsto \sum_{i=1}^{N} \sigma_i \end{aligned}$$

is a chain map. Conclude it induces a map $\tau\colon H_*(B; R) \to H_*(E; R)$ on homology.

(ii) Prove that $p_* \circ \tau$ is given by multiplication with N.

(iii) Prove that if G is a finite group acting freely on S^n, then $H_*(S^n/G; \mathbb{F}_p) = 0$ for $0 < * < n$ when p does not divide $|G|$.

(iv) Prove that if G is a finite group acting freely on a manifold M, then $H_*(M/G;\mathbb{Q})$ injects into the G-invariants $H_*(M;\mathbb{Q})^G$.

(v) Suppose $A \subset B$ is a subspace. Set $E_A \coloneqq p^{-1}(A)$, and let $p_A \colon E_A \to B$ be the restriction of p to E_A. Construct a transfer map $\tau \colon H_*(B, A; R) \to H_*(E, E_A; R)$ which is compatible with the exact sequence of pairs, in the sense that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A;R) \arrow[r] \arrow[d, "\tau"] & H_n(B;R) \arrow[r] \arrow[d, "\tau"] & H_n(B,A;R) \arrow[r] \arrow[d, "\tau"] & \cdots \\
\cdots \arrow[r] & H_n(E_A;R) \arrow[r] & H_n(E;R) \arrow[r] & H_n(E,E_A;R) \arrow[r] & \cdots .
\end{tikzcd}$$

Chapter 17

The orientation theorem

We continue our proof of Poincaré duality. Our goal in this chapter is to prove the orientation theorem, which extracts a fundamental class from an orientation. We then discuss a cap product map, which will appear in the general formulation of Poincaré duality.

17.1 The orientation theorem

Recall that for each n-dimensional topological manifold M, there is a canonical covering map

$$p\colon \mathcal{O}_{M,R} \longrightarrow M.$$

with has fiber over $m \in M$ given by $H_n(M, M \setminus \{m\}; R)$. An *orientation* of M is a section s of $\mathcal{O}_{M,R}$ such that each $s(m) \in H_n(M, M \setminus \{m\}; R)$ is an R-module generator.

Our goal is to extract a fundamental class $[M] \in H_n(M; R)$ from an orientation. We will do so through the map

$$\begin{aligned} j\colon H_n(M; R) &\longrightarrow \Gamma(M, \mathcal{O}_{M,R}) \\ a &\longmapsto [m \mapsto (j_m)_*(a)], \end{aligned}$$

with j_m the map of pairs $(M, \varnothing) \to (M, M \setminus \{m\})$. Last chapter we saw that $\mathcal{O}_{M,R}$ is an R-module over M. This makes $\Gamma(M, \mathcal{O}_{M,R})$ into an R-module, just like the left hand side $H_n(M; R)$.

Theorem 17.1.1 (Orientation theorem)**.** *If M is compact, the map $j\colon H_n(M; R) \to \Gamma(M, \mathcal{O}_{M,R})$ is an isomorphism of R-modules.*

Corollary 17.1.2. *If M is compact, an orientation of M gives rise to a* fundamental class *$[M] \in H_n(M; R)$ which restricts to a generator of $H_n(M, M \setminus \{m\}; R)$ for each $m \in M$.*

We will prove a more general theorem, and introduce for $A \subset M$ closed the notation

$$H_n(M|A) := H_n(M, M \setminus A; R).$$

As before, there is a map

$$\begin{aligned} j_A\colon H_n(M|A) &\longrightarrow \Gamma(A, \mathcal{O}_{M,R}) \\ a &\longmapsto [m \mapsto (j_m)_*(a)]. \end{aligned}$$

Theorem 17.1.3. *For $q > n$, $H_q(M|A) = 0$, and for $q = n$, the map $j_A\colon H_n(M|A) \to \Gamma(A, \mathcal{O}_{M,R})$ is an isomorphism of R-modules.*

The case $A = \varnothing$ is the orientation theorem, while the case $A = \{m\}$ is obvious. Our proof will be induction over increasingly general A. We start with two lemma's which make this induction possible:

Lemma 17.1.4. *Let $A, B \subset M$ be closed. If Theorem 17.1.3 holds for A, B and $A \cap B$, it holds for $A \cup B$.*

Proof. This follows from Mayer–Vietoris for relative homology: the pairs $(M, M \setminus A)$ and $(M, M \setminus B)$ cover $(M, M \setminus (A \cap B))$ with intersection $(M, M \setminus (A \cup B))$, and give rise to a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(M|A) \oplus H_n(M|B) \arrow[r] & H_n(M|A \cap B) \arrow[dll, rounded corners] \\ H_{n-1}(M|A \cup B) \arrow[r] & H_{n-1}(M|A) \oplus H_{n-1}(M|B) \arrow[r] & \cdots \end{tikzcd}$$

Our hypotheses tell us that $H_q(M|A \cup B) = 0$ for $q > n$, and for $q = n$ there is a short exact sequence

$$0 \longrightarrow H_n(M|A \cup B) \longrightarrow H_n(M|A) \oplus H_n(M|B) \longrightarrow H_n(M|A \cap B)$$

This fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_n(M|A \cup B) \arrow[r] \arrow[d] & H_n(M|A) \oplus H_n(M|B) \arrow[r] \arrow[d, "\cong"] & H_n(M|A \cap B) \arrow[d, "\cong"] \\ 0 \arrow[r] & \Gamma(A \cup B; \mathcal{O}_{M,R}) \arrow[r] & \Gamma(A; \mathcal{O}_{M,R}) \oplus \Gamma(B; \mathcal{O}_{M;R}) \arrow[r] & \Gamma(A \cap B; \mathcal{O}_{M,R}). \end{tikzcd}$$

The bottom row is exact: a section over $A \cup B$ is determined by its restriction to A and B, and sections over A and B that agree over the closed set $A \cap B$ can be glued to a unique section over $A \cup B$. By the five lemma, the map $H_n(M|A \cup B) \to \Gamma(A \cup B; \mathcal{O}_{M,R})$ is an isomorphism. □

Lemma 17.1.5. *l If $A_1 \supset A_2 \supset \cdots$ is a sequence of compact subsets with $A = \bigcap_i A_i$. Theorem 17.1.3 holds for each A_i, it holds for A.*

Proof. Recall the construction of a colimit of abelian groups B_i, $i \in \mathbb{N}$: $\operatorname{colim}_i B_i$ has as elements equivalence classes of $b \in B_i$ under the equivalence relation generated by saying $b \in B_i$ is equivalence to its image in B_{i+1}.

As all $H_n(X|A_i)$ have compatible restrictions map to $H_n(X|A)$, and similarly for the $\Gamma(A_i, \mathcal{O}_{M,R})$ there are induced maps

$$\operatorname{colim}_i H_n(X|A_i) \longrightarrow H_n(X|A), \qquad \operatorname{colim}_i \Gamma(A_i, \mathcal{O}_{M,R}) \longrightarrow \Gamma(A, \mathcal{O}_{M,R})$$

which fit in a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_i H_n(X;A_i) \arrow[r] \arrow[d, "\cong"] & H_n(X|A) \arrow[d] \\
\operatorname{colim}_i \Gamma(A_i, \mathcal{O}_{M,R}) \arrow[r] & \Gamma(A, \mathcal{O}_{M,R}).
\end{tikzcd}$$

The vertical map is an isomorphism by the hypotheses, as a colimit of isomorphisms is an isomorphism.

It thus suffices to prove that the horizontal maps are isomorphisms. Let us start with the top one. Since colimits commute with homology by Problem 7.3.6, it suffices to prove that

$$\operatorname{colim}_i S_*(X;A_i) \longrightarrow S_*(X|A),$$

is an isomorphism. Recall that $S_n(X|A)$ is $S_n(X;R)/S_n(X \setminus A;R)$, so it suffices to prove that if $\sigma\colon \Delta^p \to X$ has image in $X \setminus A$, it has image in $X \setminus A_i$ for some i. This follows by compactness: the $X \setminus A_i$ form an open cover of $X \setminus A$ and $\sigma(\Delta^p)$ is compact so lies in one of these.

To see that the bottom map is an isomorphism, we first prove it is surjective. Since A is compact, it is covered by finitely many open subsets U of local trivializations. By uniqueness of path-lifting, given $s\colon A \to \mathcal{O}_{M,R}$ it extends to their union (or at least all path components of U which intersect A). This contains A_i (as the intersection of the compact subsets $A_i \setminus U$ is empty, hence one of the $A_i \setminus U$ must already be empty). Hence s is the in image of $\Gamma(A_i, \mathcal{O}_{M,R})$. For injectivity, we observe that if the restriction of $s_i \in \Gamma(A_i, \mathcal{O}_{M,R})$ vanishes over A, by the previous arguments it vanishes over some open neighborhood of A and hence on some A_j for $j \geq i$. Hence the equivalence class of s_i is 0 in the colimit. □

We can prove Theorem 17.1.3:

Proof of Theorem 17.1.3. We say $A \subset M$ is *Euclidean* if it lies in a chart. Our proof will be induction over the following cases:

(1) $M = \mathbb{R}^n$, A is compact and convex,

(2) $M = \mathbb{R}^n$, A is a finite union of compact and convex subsets,

(3) $M = \mathbb{R}^n$, A is compact,

(4) M is arbitrary, A is a compact Euclidean subset,

(5) M is arbitrary, A is compact.

For (1), we observe that by translation and scaling we may assume that $0 \in A$. Since A is contractibe, it follows that

$$H_n(\mathbb{R}^n|0) \longrightarrow H_n(\mathbb{R}^n|A)$$

is an isomorphism. This fits in a commutative diagram

$$\begin{tikzcd}
H_n(\mathbb{R}^n|0) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_n(\mathbb{R}^n|A) \arrow[d] \\
\Gamma(0, \mathcal{O}_{\mathbb{R}^n,R}) \arrow[r, "\cong"] & \Gamma(A, \mathcal{O}_{\mathbb{R}^n,R}).
\end{tikzcd}$$

The left vertical map and bottom horizontal map are isomorphisms, because $\mathcal{O}_{\mathbb{R}^n,R}$ is trivial and hence any section is determined by its value on a single point.

For (2), do an induction over the number of subsets using Lemma 17.1.4; the initial case is (1). For (3), use Lemma 17.1.5 and the observation that any compact subset is the intersection A_i which are unions of finitely many closed disks.

For (4), use excision to reduce it to (3). For (5), we cover A by disks D_i in finitely many charts and take $A_i = A \cap D_i$. Then we do an induction over the number of subsets using Lemma 17.1.4; the initial case is (4). □

Example 17.1.6. Every simply-connected manifold is orientable, so if M is path-connected we have $H_n(M;R) \cong R$. Thus for such manifold, you always know $H_0(M;R)$ and $H_n(M;R)$.

17.2 The cap product and the formulation of Poincaré duality

The cap product will be obtained by combining the cup product and the Kronecker pairing. All coefficients are in R, but we suppress this for the sake of brevity. The *cap product* will be a map

$$\begin{aligned} H^p(X) \otimes H_n(X) &\longrightarrow H_{n-p}(X) \\ \alpha \otimes b &\longmapsto \alpha \cap b. \end{aligned}$$

It is given by taking homology of the chain map

$$\begin{tikzcd}
S^p(X) \otimes S_n(X) \arrow[d, "\mathrm{id}\otimes AW_X \circ \Delta_*"] \\
S^p(X) \otimes \bigoplus_{k+l=n} S_k(X) \otimes S_l(X) \arrow[d, "\text{projection on term } k=p"] \\
S^p(X) \otimes S_p(X) \otimes S_{n-p}(X) \arrow[d, "\langle -,- \rangle \otimes \mathrm{id}"] \\
S_{n-p}(X).
\end{tikzcd}$$

Using the explicit formula for the Alexander–Whitney map, it is given by

$$\alpha \otimes b \longmapsto \alpha(b|_{\Delta^p}) \cdot b|_{\Delta^{n-p}}.$$

This has the following properties, which can be verified using this formula:

Lemma 17.2.1. *The cap product satisfies:*

(i) It makes $H_*(X)$ into a graded $H^*(X)$-module, i.e. $(\alpha \cup \beta) \cap x = \alpha \cap (\beta \cap x)$ and $1 \cap x = x$.

(ii) Given a map $f\colon X \to Y$, $\alpha \in H^p(Y)$ and $x \in H_n(X)$, $f_*(f^*(\alpha) \cap x) = \alpha \cap f_*(x)$.

(iii) For the augmentation $\epsilon\colon H_*(X) \to R$, $\epsilon(\alpha \cap x) = \langle \alpha, x \rangle$.

(iv) We have $\langle \alpha \cup \beta, x \rangle = \langle \alpha, \beta \cap x \rangle$.

Proof. We will only prove (ii). We write $\beta \in C^p(Y)$ and $\sigma \in C_n(X)$ for representatives. Then we have that

$$\begin{aligned} f_*(f^*(\beta) \cap \sigma) &= f_*\left(f^*(\beta)(\sigma|_{\Delta^p})\sigma|_{\Delta^{n-p}}\right) \\ &= f_*\left(\beta(f \circ \sigma|_{\Delta^p})\sigma|_{\Delta^{n-p}}\right) \\ &= \beta(f \circ \sigma|_{\Delta^p}) f_*(\sigma|_{\Delta^{n-p}}) \\ &= \beta(f \circ \sigma|_{\Delta^p})(f \circ \sigma)|_{\Delta^{n-p}}) \\ &= \beta \cap f_*(\sigma). \end{aligned}$$

□

If M is compact, the orientation theorem takes an R-orientation in $\Gamma(M, \mathcal{O}_{M,R})$ to an element $[M] \in H_n(M; R)$. Poincaré duality says that the cap product with this class is an isomorphism:

Theorem 17.2.2 (Poincaré duality)**.** *Suppose M is compact and we are given an R-orientation of M. Then there is a unique class $[M] \in H_n(M; R)$ which restricts to the orientation in each $H_n(M, M \setminus \{m\}; R)$, and the cap product*

$$- \cap [M]\colon H^p(M; R) \longrightarrow H_{n-p}(M; R)$$

is an isomorphism for all p.

Example 17.2.3. Take R to be a field $\mathbb{F}$. Then this says that $H^p(M; \mathbb{F})$ is isomorphic to $H_{n-p}(M; \mathbb{F})$, and in particular has the same dimension. Since $\dim H^p(M; \mathbb{F}) \cong \dim H_p(M; \mathbb{F})$ by the universal coefficients theorem, we conclude the equality of Betti numbers

$$b_p(M) = b_{n-p}(M)$$

that we have been using as motivation for Poincaré duality.

17.3 Problems

Problem 17.3.1 (A verification)**.** Prove Lemma 17.2.1 (iv).

Chapter 18

Poincaré duality

In this chapter we prove Poincaré duality. To do so, we need to establish that the Čech cohomology groups $\check{H}^*(K)$ for $K \subset M$ satisfy versions of the Eilenberg–Steenrod axioms.

18.1 Poincaré duality and Čech cohomology

Recall Poincaré duality says:

Theorem 18.1.1 (Poincaré duality)**.** *Suppose M is compact and we are given an R-orientation of M. Then there is a unique class $[M] \in H_n(M;R)$ which restricts to the orientation in each $H_n(M, M \setminus \{m\}; R)$, and the cap product*

$$- \cap [M]\colon H^p(M;R) \longrightarrow H_{n-p}(M;R)$$

is an isomorphism for all p.

We prove this by formulating a relative version, which we approach through an induction as in the proof of the orientation theorem. As before, we drop R from the notation for the sake of brevity.

We start with the construction of a *relative cap product*

$$\cap\colon H^p(X) \otimes H_n(X,A) \longrightarrow H_{n-p}(X,A).$$

To define this, we need that property (ii)

given a map $f\colon X \to Y$, $\alpha \in H^p(Y)$ and $x \in H_n(X)$, $f_*(f^*(\alpha)\cap x) = \alpha \cap f_*(x)$,

of the cap product is true before passing to homology. In particular, when applied to f given by $i\colon A \to X$, it says that the following diagram commutes

$$\begin{tikzcd} S^p(X)\otimes S_n(A) \arrow[r, "i^*\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes i_*"] & S^p(A)\otimes S_n(A) \arrow[r, "\cap"] & S_{n-p}(A) \arrow[d, "i_*"] \\ S^p(X)\otimes S_n(X) \arrow[rr, "\cap"] & & S_{n-p}(X). \end{tikzcd}$$

Thus we get an induced map on cokernels of the vertical maps

$$S^p(X) \otimes S_n(X, A) \longrightarrow S_{n-p}(X, A).$$

(Here we use that $S^p(X) \otimes -$ preserves the exact sequence $0 \to S_n(A) \to S_n(X) \to S_n(X, A) \to 0$ since it is split.) This is the *relative cap product* and it has the same properties as the absolute one. In particular, $H_*(X, A)$ is a graded $H^*(X)$-module.

Observe that excision tells us that when $K \subset U \subset X$ is excisive triple, the map

$$H_*(U, U \setminus K) \longrightarrow H_*(X, X \setminus K)$$

is an isomorphism. Thus $H_*(X, X \setminus K)$ is not only an $H^*(X)$-module but also an $H^*(U)$-module. These module structures are compatible: if $K \subset U \subset X$ and $K \subset V \subset X$ are excisive triples with $V \subset U$, then the following diagram commutes

$$\begin{tikzcd}
H^p(U) \otimes H_n(X, X \setminus K) \arrow[d] \arrow[dr] & \\
& H_{n-p}(X, X \setminus K) \\
H^p(V) \otimes H_n(X, X \setminus K). \arrow[ur] &
\end{tikzcd}$$

In particular, we can take the colimit over open subsets U of X containing K. A *poset* is a "partially ordered set," i.e. a set with a partial order $\preceq$, which can be considered as a category with a unique morphisms from x to y if $x \preceq y$. Let $\mathcal{U}_K$ denote the poset of such open subsets, ordered by reverse inclusion.

Definition 18.1.2. The *Čech cohomology* of K is given by

$$\check{H}^p(K) \coloneqq \operatorname*{colim}_{U \in \mathcal{U}_K} H^p(U).$$

Example 18.1.3. That is, an element of $\check{H}^p(K)$ is an equivalence class of pairs (U, α) with $U \in \mathcal{U}_K$ and $\alpha \in H^*(U)$. The equivalence relation is that for $V \subset U$, (U, α) is equivalent to $(V, i^*_{V \subset U}\alpha)$.

The restriction maps $H^*(U) \to H^*(V)$ are maps of graded-commutative R-algebras, so $\check{H}^*(K)$ is also a graded-commutative R-algebra. The above observations tell us that the relative homology group $H_*(X, X \setminus K)$ form a graded $\check{H}^*(X)$-module.

Remark 18.1.4. In general $\check{H}^p(K)$ is not equal to $H^p(K)$. However, this is the case when each open neighborhood U of K contains another open neighborhood V such that $K \to V$ is a homotopy equivalence.

The relative version of Poincaré duality is then stated in terms of Čech cohomology as follows. By the orientation theorem, if M is an n-dimensional topological manifold and $A \subset M$ is compact, then

$$\begin{aligned} H_n(M|A) &\longrightarrow \Gamma(A, \mathcal{O}_{M;R}) \\ a &\longmapsto [m \mapsto (j_m)_*(a)] \end{aligned}$$

is an isomorphism. An R-orientation of M along A is a section in $\Gamma(A, \mathcal{O}_{M;R})$ which restricts to a generator of each fiber $H_n(M, M \setminus \{m\}; R)$ for $m \in A$. This isomorphism converts such an R-orientation into a fundamental class $[M|A] \in H_n(M|A)$ of M along A. The fully relative version of Poincaré duality says that cap product with this induces an isomorphism:

Theorem 18.1.5 (Relative Poincaré duality)**.** *Let M be as above and A be a compact subset. Given an R-orientation of M along A we get a fundamental class $[M|A]$, and the cap product*

$$- \cap [M|A] \colon \check{H}^p(A; R) \longrightarrow H_{n-p}(M, M \setminus A; R)$$

is an isomorphism.

If we assume M is compact and $A = M$, we recover absolute Poincaré duality.

18.2 Properties of Čech cohomology

Čech cohomology is not a cohomology theory. In fact, saying this wouldn't even make sense as $\check{H}^*(K)$ depends on the topological space X containing K. To see this, recall it was defined in terms of the poset $\mathcal{U}_K$ of open subsets U of X containing K.

However, it still satisfies properties reminiscent of the Eilenberg–Steenrod axioms as long as restrict our attention to closed subsets of a fixed topological space X. In this section we establish some of these, as we will use them in the proof of Poincaré duality.

18.2.1 Functoriality

The first observation is that if $L \subset K$ is an inclusion, then each open neighborhood of K is an open neighborhood of L. This gives rise to a map of posets $\mathcal{U}_K \to \mathcal{U}_L$, which we claim induces a restriction map on colimits

$$\check{H}^*(K) \longrightarrow \check{H}^*(L),$$

making $\check{H}^*(-)$ into a functor $\mathsf{Cl}_X \longrightarrow \mathsf{GrAb}$, where Cl_X is the poset of closed subsets of X ordered by reverse inclusion.

This is true more generally. Suppose that I and J are posets and $\phi \colon \mathsf{I} \to \mathsf{J}$ is a functor (or equivalently, a weakly-order preserving map). Then from any functor $A \colon \mathsf{J} \to \mathsf{Ab}$, we obtain a functor $A \circ \phi \colon \mathsf{I} \to \mathsf{Ab}$. There is then a natural map

$$\operatorname*{colim}_{i \in \mathsf{I}} A \circ \phi \longrightarrow \operatorname*{colim}_{j \in \mathsf{J}} A,$$

sending an element represented by $a \in A(\phi(i))$ for $i \in \mathsf{I}$ to the element represented by $a \in A(\phi(i))$ with $\phi(i) \in \mathsf{I}$.

Taking $A = H^p(-)$, $\mathsf{I} = \mathcal{U}_K$ and $\mathsf{J} = \mathcal{U}_L$, we get a map

$$\check{H}^p(K) = \operatorname*{colim}_{U \in \mathcal{U}_K} H^p \longrightarrow \operatorname*{colim}_{U \in \mathcal{U}_L} H^p = \check{H}^p(L).$$

18.2.2 The long exact sequence of a pair

To state version sof the long exact sequence of a pair and excision, we need to define *relative Čech cohomology groups.* This is straightforward: for $L \subset K$ closed, let $\mathcal{U}_{K,L}$ be the poset of pairs of open subsets $V \subset U$ of X such that $L \subset V$ and $K \subset U$, ordered by reverse inclusion. Then we define

$$\check{H}^*(K,L) := \operatorname*{colim}_{(U,V)\in\mathcal{U}_{K,L}} H^*(U,V).$$

Lemma 18.2.1 (Long exact sequence of a pair)**.** *Suppose that $L \subset K$ are closed subsets of X, then there is a natural long exact sequence*

$$\cdots \longrightarrow \check{H}^p(K,L) \longrightarrow \check{H}^p(K) \longrightarrow \check{H}^p(L) \longrightarrow \check{H}^{p+1}(K,L) \longrightarrow \cdots.$$

The difficulty with deducing this lemma from the usual long exact sequence of a pair in cohomology lies with the fact that the colimits that appear are taken over different posets; $\mathcal{U}_{K,L}$, $\mathcal{U}_K$ and $\mathcal{U}_L$ respectively. We thus need to know what happens when we vary the indexing poset. That is, we will want to answer the following:

Question 18.2.2. Given a map of posets $\phi\colon \mathsf{I} \to \mathsf{J}$, when is

$$\operatorname*{colim}_{i\in\mathsf{I}} A\circ\phi \longrightarrow \operatorname*{colim}_{j\in\mathsf{J}} A$$

an isomorphism?

The condition will be that I is "dense" in J:

Definition 18.2.3. A map of posets $\phi\colon \mathsf{I} \to \mathsf{J}$ is *cofinal* if for all $j \in \mathsf{J}$ there exists a $i \in \mathsf{I}$ such that $j \leq \phi(i)$.

Example 18.2.4. Surjective maps of posets are cofinal.

Proposition 18.2.5. *If $\phi\colon \mathsf{I} \to \mathsf{J}$ is cofinal then*

$$\operatorname*{colim}_{i\in\mathsf{I}} A\circ\phi \longrightarrow \operatorname*{colim}_{j\in\mathsf{J}} A$$

is an isomorphism.

Proof. We first prove surjectivity. Each element of $\operatorname{colim}_{j\in\mathsf{J}} A$ is represented by some $a \in A_j$. By cofinality it is also represented by an $a' \in A_{\phi(i)}$ with $j \leq \phi(i)$; this is clearly in the image of $\operatorname{colim}_{i\in\mathsf{I}} A\circ\phi$.

For injectivity, suppose that $a \in \operatorname{colim}_{i\in\mathsf{I}} A\circ\phi$ is represented by $a \in A(\phi(i))$ and becomes identified with 0 in $\operatorname{colim}_{j\in\mathsf{J}} A$. This means its image $a' \in A(j)$ is 0 for some $\phi(i) \leq j$. By cofinality we can pick a further i' such that $j \leq \phi(i')$ so that its further image $a'' \in A(\phi(i'))$ is 0. But a'' also represents $a \in \operatorname{colim}_{i\in\mathsf{I}} A\circ\phi$ so this was already 0. $\square$

Proof of Lemma 18.2.1. By the previous lemma we may replace $\mathcal{U}_K$ and $\mathcal{U}_L$ in the definitions of $\check{H}^*(K)$ and $\check{H}^*(L)$ by $\mathcal{U}_{K,L}$, as both functors

$$\mathcal{U}_K \longleftarrow \mathcal{U}_{K,L} \longrightarrow \mathcal{U}_L$$

are surjective. Then for each $(U,V) \in \mathcal{U}_{K,L}$, we get a long exact sequence

$$\cdots \longrightarrow H^p(U,V) \longrightarrow H^p(U) \longrightarrow H^p(V) \longrightarrow H^{p+1}(U,V) \longrightarrow \cdots,$$

and since colimits preserve long exact sequences, we can take the colimit over $\mathcal{U}_{K,L}$ to get the desired long exact sequence on Čech cohomology. □

18.2.3 Excision and Mayer–Vietoris

Excision is proven in a similar manner, and Mayer–Vietoris is a formal consequence of excision and the long exact sequence of a pair.

Lemma 18.2.6 (Excision)**.** *Suppose that X is normal and $A, B \subset X$ are closed. Then the map*

$$\check{H}^p(A \cup B, A) \longrightarrow \check{H}^p(B, A \cap B)$$

is an isomorphism.

Proof. The assumption on X implies that both maps of posets

$$\mathcal{U}_{A\cup B,A} \longleftarrow \mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{B}} \longrightarrow \mathcal{U}_{B,A\cap B},$$

left one given by $(U,U') \mapsto (U \cup U', U)$ and right one by $(U,U') \mapsto (U', U \cap U')$, are cofinal. Thus we may replace $\mathcal{U}_{A\cup B,A}$ and $\mathcal{U}_{B,A\cap B}$ by $\mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{B}}$ on both sides of the statement of excision, and obtain the result from excision for ordinary cohomology by taking a colimit over $\mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{B}}$. □

A version of Mayer–Vietoris follows formally from excision and the long exact sequence of a pair:

Lemma 18.2.7 (Mayer–Vietoris)**.** *Suppose that X is normal and $A, B \subset X$ are closed. Then there is a natural long exact sequence*

$$\cdots \longrightarrow \check{H}^p(A \cup B) \longrightarrow \check{H}^p(A) \oplus \check{H}^p(B) \longrightarrow \check{H}^p(A \cap B) \longrightarrow \check{H}^{p+1}(A \cup B) \longrightarrow \cdots.$$

18.3 Fully relative Poincaré duality

18.3.1 A fully relative cap product

We already constructed a relative cap product

$$\cap\colon \check{H}^p(K) \otimes H_n(X, X \setminus K) \longrightarrow H_{n-p}(X, X \setminus K).$$

This was in fact the motivation for defining the Čech cohomology group $\check{H}^p(K)$. For the sake of induction arguments, we need a more relative version. This should involve relative Čcech cohomology: if $L \subset K$ are closed subsets of X, then it is given by

$$\cap\colon \check{H}^p(K, L) \otimes H_n(X, X \setminus K) \longrightarrow H_{n-p}(X \setminus L, X \setminus K).$$

This should fit into a commutative diagram

$$\begin{tikzcd}
\check{H}^p(K, L) \otimes H_n(X, X \setminus K) \arrow[r, "\cap"] \arrow[d] & H_{n-p}(X \setminus L, X \setminus K) \arrow[d] \\
\check{H}^p(K) \otimes H_n(X, X \setminus K) \arrow[r, "\cap"] & H_{n-p}(X, X \setminus K).
\end{tikzcd}$$

The Čech cohomology cap product came from compatible cap products $\cap\colon H^p(U) \otimes H_n(U, U \setminus K) \to H_{n-p}(U, U \setminus K)$. These came from a chain map

$$S^p(U) \otimes S_n(U)/S_n(U \setminus K) \longrightarrow S_{n-p}(U)/S_{n-p}(U \setminus K),$$

which fits into a commutative diagram

$$\begin{tikzcd}
S^p(U) \otimes S_n(U)/S_n(U \setminus K) \arrow[r] & S_{n-p}(U)/S_{n-p}(U \setminus K) \\
S^p(U, V) \otimes S_n(U)/S_n(U \setminus K) \arrow[r] \arrow[u] \arrow[dr, dotted] & S_{n-p}(U)/S_{n-p}(U \setminus K) \arrow[u] \\
S^p(U, V) \otimes \left(S_n(U \setminus L) + S_n(V)\right)/S_n(U \setminus K) \arrow[r] \arrow[u, "\simeq"] & S_{n-p}(U \setminus L)/S_{n-p}(U \setminus K) \arrow[u],
\end{tikzcd}$$

where we use in the bottom row that cap product with elements in $S^p(U, V)$ kills $S_n(V)$. Since $U = (U \setminus L) \cup V$ is an open cover, the locality principle makes the left bottom vertical map a chain homotopy equivalence.

Thus upon taking homology, there will be a dotted map which induces the fully relative cap product after composition with the Künneth map $H^p(U, V) \otimes H_n(U, U \setminus K) \to H^*(S^*(U, V) \otimes S_*(U)/S_*(U \setminus K))$ and taking colimits.

It is compatible with the long exact sequence of pairs and Mayer–Vietoris in the following sense:

Lemma 18.3.1. *$x_K \in H_n(X, X \setminus K)$, the following commutes*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K, L) \arrow[r] \arrow[d, "- \cap x_K"] & \check{H}^p(K) \arrow[r] \arrow[d, "- \cap x_K"] & \check{H}^p(L) \arrow[r] \arrow[d, "- \cap x_L"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(X \setminus L, X \setminus K) \arrow[r] & H_{n-p}(X, X \setminus K) \arrow[r] & H_{n-p}(X, X \setminus L) \arrow[r] & \cdots,
\end{tikzcd}$$

with x_L the image in $H_n(X, X \setminus L)$ of x_K.

Recall the shorthand $H_n(X|A)$ for $H_n(X, X \setminus A)$.

Lemma 18.3.2. *For $x_{A\cup B} \in H_n(X|A \cup B)$, the following commutes*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(A \cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \check{H}^p(A) \oplus \check{H}^p(B) \arrow[r] \arrow[d, "-\cap x_A \oplus -\cap x_B"] & \check{H}^p(A \cap B) \arrow[r] \arrow[d, "-\cap x_{A\cap B}"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(X|A \cup B) \arrow[r] & H_{n-p}(X|A) \oplus H_{n-p}(X|B) \arrow[r] & H_{n-p}(X|A \cap B) \arrow[r] & \cdots ,
\end{tikzcd}$$

with x_A, x_B and $x_{A\cap B}$ the image in $H_n(X|A)$, $H_n(X|B)$ and $H_n(X|A \cap B)$ of $x_{A\cup B}$.

We will use the latter during our proof of fully relative Poincaré duality.

18.3.2 The proof of full relative Poincaré duality

We now formulate fully relative Poincaré duality. Recall that the orientation theorem says that if M is an n-dimensional topological manifold and $A \subset M$ is compact, then

$$\begin{aligned} H_n(M|A) &\longrightarrow \Gamma(A, \mathcal{O}_{M;R}) \\ a &\longmapsto [m \mapsto (j_m)_*(a)] \end{aligned}$$

is an isomorphism. From this, an R-orientation of M along A is gives rise to a fundamental class $[M|A] \in H_n(M|A)$ of M along A. The fully relative version of Poincaré duality says that cap product with this induces an isomorphism.

Theorem 18.3.3 (Fully relative Poincaré duality)**.** *Let M be as above and $L \subset K$ a pair of compact subsets. Given an R-orientation of M along A we get a fundamental class $[M|A]$, and the cap product*

$$- \cap [M|K] \colon \check{H}^p(K, L; R) \longrightarrow H_{n-p}(M \setminus L, M \setminus K; R)$$

is an isomorphism.

Remark 18.3.4. You will generally want to assume M is compact, $L = \varnothing$ and $K = M$, to get that given an R-orientation of M with associated fundamental class $[M]$ the cap product

$$- \cap [M] \colon H^p(M; R) \longrightarrow H_{n-p}(M; R)$$

is an isomorphism. However, the fully relative version is easier to prove because it is more amenable to the type of inductive arguments that we also used to prove the orientation theorem.

We will first prove the relative version, where $L = \varnothing$. We write $K = A$. The argument will be analogous to that for the orientation theorem, an induction over increasingly general pairs of compact subsets $A \subset B$ in M. The orientation theorem used two lemma's: one allowed us to proceed from triples A, B, and $A \cap B$, to the union $A \cup B$, and the other from $A_1 \supset A_2 \supset \cdots$ to the intersection $\cap_i A_i$. Here are the corresponding statements for Poincaré duality, which use the shorthand

$$H_n(X|A) \coloneqq H_n(X, X \setminus A; R).$$

The first is the compability of the cap product with Mayer–Vietoris:

Lemma 18.3.5. *For $x_{A\cup B} \in H_n(X|A\cup B)$, the following commutes*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(A\cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \check{H}^p(A)\oplus \check{H}^p(B) \arrow[r] \arrow[d, "-\cap x_A \oplus -\cap x_B"] & \check{H}^p(A\cap B) \arrow[r] \arrow[d, "-\cap x_{A\cap B}"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(X|A\cup B) \arrow[r] & H_{n-p}(X|A)\oplus H_{n-p}(X|B) \arrow[r] & H_{n-p}(X|A\cap B) \arrow[r] & \cdots,
\end{tikzcd}$$

with x_A, x_B and $x_{A\cap B}$ the image in $H_n(X|A)$, $H_n(X|B)$ and $H_n(X|A\cap B)$ of $x_{A\cup B}$.

Proposition 18.3.6. *If relative Poincaré duality is true for compact A, B and $A\cap B$, then it is true for $A\cup B$.*

Proof. Apply the five-lemma to the long exact sequences in the previous Lemma. □

The following is a consequence of the fact that a colimit of colimits is itself a colimit.

Lemma 18.3.7. *If $A_1 \supset A_2 \supset \cdots$ is a sequence of compact subsets with intersection $A := \cap_i A_i$, then the map*

$$\operatorname*{colim}_{i\in\mathbb{N}} \check{H}^*(A_i) \longrightarrow \check{H}^*(A)$$

is an isomorphism.

Proposition 18.3.8. *If relative Poincaré duality is true for compact $A_1 \supset A_2 \supset \cdots$, then it is true for $A = \cap_i A_i$.*

Proof. Take colimits of both sides of the isomorphism

$$-\cap [M|A_i]\colon \check{H}^p(A_i) \longrightarrow H_{n-p}(M, M\setminus A_i),$$

and apply the previous lemma as well as the fact that homology commutes with colimits of chain complexes. □

These propositions complete the proof, after doing a straightforward initial case.

Proof of Theorem 18.3.3 for $L=\varnothing$, $K=A$. We will now prove by induction the following cases:

(1) $M=\mathbb{R}^n$, A is compact and convex,

(2) $M=\mathbb{R}^n$, A is a finite union of compact and convex subsets,

(3) $M=\mathbb{R}^n$, A is compact,

(4) M is arbitrary, A is a compact Euclidean subset,

(5) M is arbitrary, A is compact.

In case (1), A has the property that every open neighborhood $A\subset U$ contains another open neighborhood $A\subset V\subset U$ such that $A\to V$ is a homotopy equivalence. To see so, we observe that U contains an ϵ-neighborhood U_ϵ of A for ϵ small enough, and that assigning to $x\in U_\epsilon$ the unique closest point in A gives a homotopy inverse to the inclusion $A\hookrightarrow U_\epsilon$. This implies that

$$\check{H}^*(K) \longrightarrow H^*(K)$$

is an isomorphism.

Without loss of generality $0 \in K$. Then it suffices to prove that in the following commutative diagram the bottom horizontal map is an isomorphism

$$\begin{tikzcd} H^p(K) \arrow[r, "-\cap[\mathbb{R}^n|K]"] & H_{n-p}(\mathbb{R}^n, \mathbb{R}^n \setminus K) \\ H^p(*) \arrow[u, "\cong"] \arrow[r, "-\cap[\mathbb{R}^n|*]"] & H_{n-p}(\mathbb{R}^n, \mathbb{R}^n \setminus *). \arrow[u, "\cong"] \end{tikzcd}$$

For the bottom map, there is only something to prove when $p = 0$, and then result is $1 \cap [\mathbb{R}^n|*] = [\mathbb{R}^n|*]$, which is indeed the generator.

The remainder is as the orientation theorem:

$$(1) \overset{\text{Prop 18.3.6}}{\Longrightarrow} (2) \overset{\text{Prop 18.3.8}}{\Longrightarrow} (3) \overset{\text{excision}}{\Longrightarrow} (4) \overset{\text{Prop 18.3.8}}{\Longrightarrow} (5). \qquad \square$$

We now deduce the general case

Proof of Theorem 18.3.3. By Lemma 18.3.1, for $A \subset B$ compact in M, the following commutes

$$\begin{tikzcd} \cdots \arrow[r] & \check{H}^p(B, A) \arrow[r] \arrow[d, "-\cap[M|B]"] & \check{H}^p(B) \arrow[r] \arrow[d, "-\cap[M|B]"] & \check{H}^p(A) \arrow[r] \arrow[d, "-\cap[M|A]"] & \cdots \\ \cdots \arrow[r] & H_{n-p}(M \setminus A, M \setminus B) \arrow[r] & H_{n-p}(M, M \setminus B) \arrow[r] & H_{n-p}(M, M \setminus A) \arrow[r] & \cdots , \end{tikzcd}$$

with $[M|B]$ the image in $H_n(M, M \setminus A)$ of $[M|A]$. Thus by the five-lemma, the fully relative version follows from relative version. $\square$

18.4 Applications

18.4.1 Classical Poincaré duality

When we assume M is compact, $L = \varnothing$ and $K = M$, we get that given an R-orientation of M with associated fundamental class $[M]$, the cap product

$$- \cap [M] \colon H^p(M; R) \longrightarrow H_{n-p}(M; R)$$

is an isomorphism.

To get a more concrete statement, we combine this with the universal coefficient theorem. Let us take $R = \mathbb{F}$ a field. Then there is a pairing

$$H^p(M; \mathbb{F}) \otimes H^{n-p}(M; \mathbb{F}) \xrightarrow{\cup} H^n(M; \mathbb{F}) \xrightarrow{\langle -, [M] \rangle} \mathbb{F}.$$

Since $\langle a \cup b, [M] \rangle = \langle a, b \cap [M] \rangle$, this fits into a commutative diagram

$$\begin{tikzcd} H^p(M; \mathbb{F}) \otimes H^{n-p}(M; \mathbb{F}) \arrow[r] \arrow[d, "\cong"] & \mathbb{F} \\ H^p(M; \mathbb{F}) \otimes H_p(M; \mathbb{F}) \arrow[ur] & \end{tikzcd}$$

with vertical map given by $\mathrm{id} \otimes (- \cap [M])$, which is an isomorphism by Poincaré duality. By the universal coefficient theorem, the bottom map is a perfect pairing, and hence so is the top map.

This can generalized to the integers $\mathbb{Z}$, or more generally a PID R. In this case, we obtain from the universal coefficient theorem that

$$\frac{H^p(M;R)}{\text{tors}} \otimes \frac{H^{n-p}(M;R)}{\text{tors}} \xrightarrow{\cup} H^n(M;R) \xrightarrow{\langle -,[M]\rangle} R$$

is a perfect pairing.

Example 18.4.1. Every manifold has a unique $\mathbb{Z}/2$-orientation, as every fiber of $\mathcal{O}_{M,\mathbb{Z}/2}$ has a unique non-zero element. This is in particular true for $\mathbb{R}P^n$. Thus we see

$$\begin{aligned} H^p(\mathbb{R}P^n;\mathbb{Z}/2) \otimes H^{n-p}(\mathbb{R}P^n;\mathbb{Z}/2) &\longrightarrow \mathbb{Z}/2 \\ (a,b) &\longmapsto \langle a \cup b, [\mathbb{R}P^n]\rangle \end{aligned}$$

is a perfect pairing. If we denote the non-trivial element in $H^p(\mathbb{R}P^n;\mathbb{Z}/2)$ by x^p, then this implies that $x^p \cup x^{n-p} = x^n$. This confirms our computation of the cohomology ring of $\mathbb{R}P^n$ as

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{n+1}) \qquad \text{if } |x| = 1.$$

Example 18.4.2. $\mathbb{C}P^n$ has a $\mathbb{Z}$-orientation because it is simply-connected. It also has torsion-free integral cohomology. Then the same argument as for $\mathbb{R}P^n$ gives another proof that

$$H^*(\mathbb{C}P^n;\mathbb{Z}/2) = \mathbb{Z}[y]/(y^{n+1}) \qquad \text{with } |y| = 2.$$

.

18.4.2 Alexander duality

The relative version is interesting even when $M = \mathbb{R}^n$: for any compact subset $K \subset \mathbb{R}^n$

$$\check{H}^p(K;R) \xrightarrow{-\cap[\mathbb{R}^n|K]} H_{n-p}(\mathbb{R}^n, \mathbb{R}^n \setminus K; R)$$

is an isomorphism. The right-hand side is in turn isomorphic to $\tilde{H}_{n-p-1}(\mathbb{R}^n \setminus K; R)$ using the long exact sequence of a pair. In other words, we can use the relative version to compute homology of complements.

Suppose now that K is a nice compact subset, e.g. a smooth submanifold. Then $\check{H}^p(K;R) \cong H^p(K;R)$, and we conclude that

$$H^p(K;R) \xrightarrow{\cong} \tilde{H}_{n-p-1}(\mathbb{R}^n \setminus K; R).$$

Example 18.4.3. If K is an embedded S^1 in $\mathbb{R}^3$, i.e. a knot, then $\mathbb{R}^3 \setminus K$ has the homology of a circle. It need not be homotopy equivalent to a circle. More generally, if K is an embedded S^{n-2} in $\mathbb{R}^n$, then $\mathbb{R}^n \setminus K$ has the homology of a circle.

Example 18.4.4. If K is an embedded S^1 in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus K$ has the homology of S^0. This means it has two path components, which is the assertion of the Jordan curve theorem.

18.4.3 Poincaré–Lefschetz duality

Another application of the relative version starts with an oriented compact manifold M with boundary ∂M. This can be doubled to an oriented manifold

$$DM = M \cup_{\partial M} M$$

with compact subset K the second copy of M in DM. This is nicely embedded, so $\check{H}^*(K;R) \cong H^*(K,R)$. Then we get that

$$H^p(K;R) \xrightarrow{-\cap[DM|K]} H_{n-p}(DM, DM \setminus K; R)$$

is an isomorphism. Now using $K \cong M$ and applying excision to $(M, \partial M) \to (DM, DM \setminus K)$, we get Poincaré duality for manifolds with boundary, which is also known as Poincaré–Lefschetz duality:

$$H^p(M;R) \xrightarrow{\cong} H_{n-p}(M, \partial M; R).$$

Example 18.4.5. If M is contractible, then we get that

$$0 = H^p(M) \xrightarrow{\cong} H_{n-p}(M, \partial M) \longrightarrow \tilde{H}_{n-p-1}(\partial M)$$

unless $p = 0$, in which case the left hand side is $\mathbb{Z}$. Thus ∂M has the same homology as an $(n-1)$-sphere. The boundary of a contractible manifold need not be homeomorphic to S^{n-1}. In particular, Freedman showed that the Poincaré homology sphere bounds a contractible 4-manifold.

18.5 Problems

Problem 18.5.1 (The signature)**.** Let M be a path-connected compact oriented manifold of dimension $n = 4k$.

(i) Prove that the bilinear form

$$H^{2k}(M;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(M;\mathbb{R}) \xrightarrow{\cup} H^{4k}(M;\mathbb{R}) \xrightarrow{\langle -,[M]\rangle} \mathbb{R}$$

is symmetric and non-degenerate.

By the classification of bilinear forms over the real numbers, its associated quadratic form is equivalent to one of the form

$$(x_1, \ldots, x_{p+q}) \longmapsto \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2.$$

The number $p - q$ is called the *signature of* M, denoted $\sigma(M)$.

(ii) Give an example of a $4k$-dimensional manifold with signature 1.

Problem 18.5.2 (The $K3$-manifold)**.**

(i) Suppose that there exists a $(4k+1)$-dimensional compact oriented manifold W with boundary such that $\partial W = M$. Let $i\colon M \hookrightarrow W$ denote the inclusion. Prove that $i_*[M] = 0$.

With some more work, this leads to the result that $\sigma(M) = 0$ when M bounds a manifold W. You may use this fact in the next part:

(ii) The $K3$-manifold[1] is the 4-dimensional path-connected compact oriented submanifold M of $\mathbb{C}P^3$ cut out by the homogeneous equation $z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$. Its intersection form

$$\pitchfork\colon H_2(M) \otimes_{\mathbb{R}} H_2(M) \longrightarrow \mathbb{Z}$$

has Gram matrix (with entries $e_i \pitchfork e_j$ for some integral basis $e_1, \ldots, e_{22}$ of $H_2(M) \cong \mathbb{Z}^{22}$) given by $H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$ with

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad -E_8 = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \end{bmatrix}$$

Can it bound a compact oriented 5-dimensional manifold?

[1] It is in fact a 2-dimensional complex manifold, so also called a $K3$-surface. However, this is only one of many distinct complex structure on M.

Chapter 19

Past and future

In this chapter we will summarize what we have learned so far, and then I will explain some directions in which algebraic topology went next.

19.1 The past

In this course, we learned what the homology and cohomology groups of a topological space are, and how to compute them. Along the way we learned a number of important concepts and techniques in homological algebra.

19.1.1 Homology

The first object of interest was homology of a topological space X, defined by taking the homology of the chain complex $S_*(X)$ with entries

$$S_n(X) = \mathbb{Z}[\{\sigma\colon \Delta^n \to X\}],$$

and differential given by

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{\text{face opposite to } i\text{th vertex}}.$$

We saw three useful elaborations:

- *relative homology*: for $A \subset X$, define $H_*(X, A)$ as the homology of $S_*(X)/S_*(A)$,
- *coefficients*: for an R-module M, define $H_*(X; M)$ as the homology of $S_*(X) \otimes M$,
- *relative homology with coefficients*: combine the above two.

These satisfy the four Eilenberg–Steenrod axioms: long exact sequence of a pair, excision, wedge, and dimension. To prove these, we learned two important homological techniques. For the long exact sequence of a pair, we use that a short exact sequence of chain complexes induces a long exact sequence on homology. For excision, we constructred a chain homotopy from barycentric subdivision to the identity.

From the Eilenberg–Steenrod axioms we deduced the suspension isomorphism and the Mayer–Vietoris long exact sequence. These then led to cellular homology, which gives a rather small chain complex which computes the homology of a CW-complex.

Next we proved the universal coefficients theorem and Künneth theorem, which both involve Tor-terms: for coefficients in a PID R (which we drop from the notation for brevity), there are short exact sequences (split but not naturally so)

$$0 \to H_p(X) \otimes_R N \longrightarrow H_p(X; N) \longrightarrow \mathrm{Tor}_1^R(H_{p-1}(X), N) \to 0 \quad \text{for } N \text{ an } R\text{-module},$$

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(X)) \to 0.$$

The zeroth Tor-group is given by $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$ and the higher terms measure the failure of $M \otimes_R -$ to be exact. They can be computed by resolving M by free R-modules (though projective R-modules suffice). That is, one constructs chain complex $F_\bullet$ concentrated in non-negative degrees with a homomorphism $F_0 \to M$ such that the extended chain complex

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is exact, and takes

$$\mathrm{Tor}_i^R(M, N) \coloneqq H_i(F_\bullet \otimes_R N).$$

We call $F_\bullet \to M$ a *free resolution.* That the Tor-groups independent of the choice of free resolution $F_\bullet$ used the following result:

Lemma 19.1.1 (Fundamental theorem of homological algebra)**.** *If both $F_\bullet \to M$ and $G_\bullet \to N$ are free resolutions, then any homomorphism $f\colon M \to N$ can be lifted to a chain map $f_\bullet\colon F_\bullet \to G_\bullet$, unique up to chain homotopy.*

The acyclic models theorem generalizes this from chain complexes to functors with values in chain complexes and gives a condition under which a natural transformation $f\colon F \to G$ of functors $\mathsf{C} \to \mathsf{Mod}_R$ can be lifted to a natural transformation $f_\bullet\colon F_\bullet \to G_\bullet$ of functors $\mathsf{C} \to \mathsf{Mod}_R$, unique up to natural chain homotopy. To state it, we fix a collection $\mathcal{M}$ of objects C which we call *models*, and first we demand that $F_\bullet$ and $G_\bullet$ are degreewise $\mathcal{M}$-free, i.e. a direct sum of the functors $\mathbb{Z}[\mathrm{Hom}_{\mathsf{C}}(M, -)]$ with $M \in \mathcal{M}$. Then we futher demand that the extended complexes

$$\cdots \longrightarrow F_2(M) \longrightarrow F_1(M) \longrightarrow F_0(M) \longrightarrow F(M) \longrightarrow 0,$$

$$\cdots \longrightarrow G_2(M) \longrightarrow G_1(M) \longrightarrow G_0(M) \longrightarrow G(M) \longrightarrow 0,$$

are exact for all $M \in \mathcal{M}$, i.e. $\mathcal{M}$-exact: $F_\bullet \to F$ and $G_\bullet \to F'$ are $\mathcal{M}$-free resolutions.

Theorem 19.1.2 (Acylic models)**.** *If $F_\bullet \to F$ and $G_\bullet \to G$ are $\mathcal{M}$-free resolutions, then every natural transformation $f\colon F \to G$ of functors $\mathsf{C} \to \mathsf{Mod}_R$ can be lifted to a natural transformation $f_\bullet\colon F_\bullet \to G_\bullet$ of functors $\mathsf{C} \to \mathsf{Mod}_R$, unique up to natural chain homotopy.*

Example 19.1.3. To prove the Künneth theorem we applied acylic models to

$$\begin{aligned}
\mathsf{C} &= \mathsf{Top}^2, \\
\mathcal{M} &= \{(\Delta^p, \Delta^q) \mid p, q \geq 0\}, \\
F &= ((X,Y) \mapsto H_0(X) \otimes H_0(Y)) \\
F_\bullet &= ((X,Y) \mapsto S_*(X) \otimes S_*(Y)) \\
G &= (X,Y) \mapsto H_0(X \times Y) \\
G_\bullet &= ((X,Y) \mapsto S_*(X \times Y)) \\
f &= \text{natural iso}\colon H_0(X) \otimes H_0(Y) \xrightarrow{\cong} H_0(X \times Y)
\end{aligned}$$

and its inverse, to get a chain homotopy equivalence

$$S_*(X) \otimes S_*(Y) \longrightarrow S_*(X \times Y)$$

unique up to chain homotopy.

19.1.2 Cohomology

After proving these important results about homology, we discussed its dual, cohomology. It is the cohomology of the cochain complex $S^*(X)$ with entries

$$S^n(X) = \mathrm{Hom}(S_n(X), \mathbb{Z})$$

and differential given by

$$d(\alpha) = (-1)^{n+1} \alpha \circ d,$$

with d on the right hand side the differential for homology. There is a relative version with coefficients, which satisfy similar Eilenberg–Steenrod axioms. Thus for the computation of cohomology we have similar tools as for homology. Cohomology is related to homology through the Kronecker pairing

$$\begin{aligned}
\langle -, - \rangle \colon H^p(X;R) \otimes H_p(X;R) &\longrightarrow R \\
[\alpha] \otimes [\sigma] &\longmapsto \alpha(\sigma).
\end{aligned}$$

This gives a homomorphism $H^p(X;R) \to \mathrm{Hom}_R(H_p(X;R), R)$ which features in another universal coefficients theorem: for a PID R, there are short exact sequences

$$0 \longrightarrow \mathrm{Ext}_1^R(H_{p-1}(X;R), R) \longrightarrow H^p(X;R) \longrightarrow \mathrm{Hom}_R(H_p(X;R), R) \longrightarrow 0.$$

The Ext-groups are computed like the Tor-groups, but defined using Hom instead of $\otimes$.

The novel feature of cohomology is the existence of the cup product. There is an explicit formula for it, given by

$$\begin{aligned}
\cup\colon H^p(X;R) \otimes H^q(X;R) &\longrightarrow H^{p+q}(X;R) \\
[\alpha] \otimes [\beta] &\longmapsto \left[\sigma \mapsto \alpha(\sigma|_{\text{first } \Delta^p}) \beta(\sigma|_{\text{last } \Delta^q})\right].
\end{aligned}$$

This is clearly associative and unital, but not clearly graded commutative. This is proved by an application of the uniqueness clause in acyclic models to

$$\begin{aligned}
\mathsf{C} &= \mathsf{Top}^2,\\
\mathcal{M} &= \{(\Delta^p, \Delta^q) \mid p, q \geq 0\},\\
F &= (X, Y) \mapsto H_0(X \times Y)\\
F_\bullet &= ((X, Y) \mapsto S_*(X \times Y))\\
G &= ((X, Y) \mapsto H_0(X) \otimes H_0(Y))\\
G_\bullet &= ((X, Y) \mapsto S_*(X) \otimes S_*(Y))\\
f &= \text{natural iso}\colon H_0(X \times Y) \xrightarrow{\cong} H_0(X) \otimes H_0(Y),
\end{aligned}$$

and dualizing. Then we have two lifts:

$$\sigma \longmapsto \sum_{p+q=n} (\pi_1 \circ \sigma)|_{\text{first } \Delta^p} \otimes (\pi_2 \circ \sigma)|_{\text{last } \Delta^q},$$

$$\sigma \longmapsto \sum_{p+q=n} (-1)^{pq} (\pi_1 \circ \sigma)|_{\text{last } \Delta^p} \otimes (\pi_2 \circ \sigma)|_{\text{first } \Delta^q}.$$

The uniqueness clause tells you these are chain-homotopic. Some dualizations then give rise to the graded-commutativity of the cup product.

Finally, we observed that for manifolds there is another relation between homology and cohomology: *Poincaré duality*. An R-orientation on manifold M of dimension n is a section of the covering space $\mathcal{O}_{M,R}$ with fibers $H_n(M, M \setminus \{m\}; R)$, which has the property that its values are R-module generators. By the orientation theorem, this gives rise a unique fundamental class $[M] \in H_n(M; R)$. Poincaré duality then says that the cap product

$$\begin{aligned}
- \cap [M]\colon H^p(M; R) &\longrightarrow H_{n-p}(M; R)\\
[\alpha] &\longmapsto \alpha([M]|_{\text{first } \Delta^p})[M]|_{\text{last } \Delta^{n-p}}
\end{aligned}$$

is an isomorphism. (This involved a slight abuse of notation: $[M]$ is a linear combination of simplices in M, and restriction of this to a face means restricting each term to a face.)

19.2 The future

The results we have studied so far was mostly done in the 1950's and earlier. A lot has happened since, and below we will give an overview of the next couple of decades. If you are interested in reading more about this, [May99] is a good starting point.

19.2.1 Spectral sequences

We have done many computations of homology and cohomology groups throughout this course. However, we did not yet use the most powerful tool available: *spectral sequences* [McC01].

The goal of a spectral sequence is straightforward: suppose you have a filtered space X, i.e. a collection of subspaces

$$F_0(X) \subset F_1(X) \subset F_2(X) \subset \cdots$$

such that $\mathrm{colim}_{k\to\infty} F_k(X) \to X$ is a homeomorphism. Then you could attempt to compute $H_*(X)$ by computing $H_*(F_k(X))$ inductively using the long exact sequences

$$\cdots \longrightarrow H_i(F_{k-1}(X)) \longrightarrow H_i(F_k(X)) \longrightarrow H_i(F_k(X), F_{k-1}(X)) \longrightarrow \cdots$$

and taking $\mathrm{colim}_{k\to\infty} H_*(F_k(X))$. If the filtration is nice enough, this colimit is equal to $H_*(X)$ and you can identify $H_i(F_k(X), F_{k-1}(X))$ with $\widetilde{H}_i(F_k(X)/F_{k-1}(X))$.

A spectral sequence collects all these (infinitely many) computational steps into a single object. This allow one to state and use additional algebraic properties, which simplify the computations. The result is a machine

$$\bigoplus_{k\geq 0} \widetilde{H}_*(F_k(X)/F_{k-1}(X)) \xrightarrow{\text{spectral sequence}} H_*(X),$$

with the convention that $F_{-1}(X) = \varnothing$.

Example 19.2.1. In the case of the skeletal filtration

$$\mathrm{sk}_0(X) \subset \mathrm{sk}_1(X) \subset \mathrm{sk}_2(X) \subset \cdots$$

of a CW-complex X, the associated spectral sequence reduces to the statement that cellular chain complex computes homology.

Example 19.2.2. The workhorse of spectral sequences is the *Serre spectral sequence*. It applies to the following generalization of covering spaces: *bundles with fiber* F, for some topological space F. These are maps

$$p\colon E \longrightarrow B$$

such that there is an open cover of B by U's so that there is a homeomorphism $p^{-1}(U) \cong F \times U$ over U. We call B the base, E the total space, and F the fiber. If the base B is a CW-complex, we can filter the total space E by $p^{-1}(\mathrm{sk}_k(B))$. The associated spectral sequence is Serre spectral sequence and computes $H_*(E)$ from $H_*(B)$ and $H_*(F)$. If B is 1-connected, then the inut is more precisely given by the homology groups $H_*(B; H_*(F))$.

A famous example is Hopf fibration

$$S^3 \to S^2$$

with fiber S^1, given by the attaching map for the 4-cell of $\mathbb{C}P^2$.

19.2.2 Homotopy theory

More generally, the Sere spectral sequence applies to *Serre fibrations*. These are maps $p\colon E \to B$ such that in each commutative diagram

$$\begin{tikzcd} D^i \arrow[r] \arrow[d] & E \arrow[d] \\ D^i \times [0,1] \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

there exists a lift. We saw that covering spaces had this property (there the lift was even unique).

Serre fibrations are part of the subject of homotopy theory. The goal of this subject is classify topological spaces up to homotopy equivalence. The most important invariants are the homotopy groups

$$\pi_i(X, x_0) \coloneqq \frac{\{\text{based continuous maps } S^i \to X\}}{\text{based homotopy}}.$$

This is the most important invariant because a map $X \to Y$ between path-connected based spaces homotopy equivalent to CW complexes is a homotopy equivalence if and only if it induces isomorphisms on all homotopy groups; the *Whitehead theorem.*

A Serre fibration has an associated long exact sequence (ignoring basepoints and issues due to some homotopy groups being non-abelian)

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \pi_{i-1}(F) \longrightarrow \cdots.$$

There are also strong relationships between homotopy groups and (co)homology groups, so you can play homotopy theory and (co)homology theory off each other. For example, if $f\colon X \to Y$ induces an isomorphism on homology and both X and Y are simply-connected, then it induces an isomorphism on homotopy groups.

Over the years, homotopy theory has developed into the more general study of mathematical objects up to some notion of deformation, not just topological spaces up to homotopy equivalence. As such, it forms the foundation of higher category theory and derived algebraic geometry.

19.2.3 Characteristic classes

A motivating problem for homotopy theory was the classification of vector bundles. A vector bundle over B is a map $p\colon E \to B$ such that each fiber $p^{-1}(b)$ has the structure of a real (or complex) vector space, and there is an open cover U of B such that $p^{-1}(U) \cong \mathbb{R}^n \times U$ over U compatibly with these vector space structures.

Remark 19.2.3. You may noticed an interest in these questions from some of our examples: the projective spaces are related to line bundles, and the special orthogonal groups to oriented vector bundles.

To understand vector bundles, we need invariants that not only serve to distinguish them but also convey some geometric intuition (as, hopefully at this point, homology does for topological spaces for you). These are *characteric classes* [MS74]: they assign to a vector bundle $p\colon E \to B$ a cohomology class $x(E) \in H^r(B; R)$, naturally in the vector bundle.

Examples of characteristic classes are Stiefel–Whitney classe, which can tell you whether a vector bundle is orientable, and Euler classes, which tell you whether it has an everywhere non-zero section. Finding all characteristic classes and the properties they encode is related to understanding the cohomology of Lie groups such as $SO(n)$.

19.2.4 Topological K-theory

You can not only use topology to distinguish vector bundles, but you can also use vector bundles to distinguish topological spaces. This is an invariant called *topological K-theory* [Hata]: if X is compact Hausdorff, it is given by

$$K^0(X) := \frac{\mathbb{Z}[\text{iso. classes of vector bundles over } X]}{[E \oplus F] = [E] + [F]}.$$

As the notation suggests, these groups can be generalized to $K^i(X)$ for $i \in \mathbb{Z}$. There is also a relative version, and these form a cohomology theory which does *not* satisfy dimension axiom.

$$K^{-*}(\text{pt}) = \begin{cases} \mathbb{Z} & \text{if } * \equiv 0 \pmod 8 \\ \mathbb{Z}/2 & \text{if } * \equiv 1 \pmod 8 \\ \mathbb{Z}/2 & \text{if } * \equiv 2 \pmod 8 \\ 0 & \text{if } * \equiv 3 \pmod 8 \\ \mathbb{Z} & \text{if } * \equiv 4 \pmod 8 \\ 0 & \text{if } * \equiv 5 \pmod 8 \\ 0 & \text{if } * \equiv 6 \pmod 8 \\ 0 & \text{if } * \equiv 7 \pmod 8. \end{cases}$$

That is, it is a *generalized cohomology theory*. This means that even though all the computational tools for cohomology are present, the values will be quite different. For example, they are always 8-periodic (this is known as *Bott periodicity*). The suspension isomorphism tells you that $K^{-n}(\text{pt}) = K^0(S^n)$, so the above computation tells you that there are some interesting patterns in the classification of vector bundles on spheres.

More recently, topological K-theory was generalized to *algebraic K-theory*, which is built from algebraic vector bundles on schemes [Wei13]. Its values on a point are of great importance in number theory and geometric topology.

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