Multijets and approximations of polynomials

Part I – Multijets and Thom transversality theorem

Let X and Y be smooth manifolds.

- A k-jet is an equivalence class of a triple (x, U, f) where $x \in U \overset{\text{open}}{\subset} X$ and $f : U \to Y$ a smooth map, and with equivalence relation $(x, U, f) \simeq (x', U', f')$ if $x = x'$ and for some chart containing x and $f(x)$ the two maps have the same derivatives up to degree k.
- Denote by $J^k(X, Y)$ the set of all k-jets. Since $J^k(\mathbb{R}^m, \mathbb{R}^n)$ is a vector space, the charts of X and Y give charts of $J^k(X, Y)$ which induce a topology so that $J^k(X, Y)$ is a smooth manifold.
- There are the following maps

$$\begin{aligned} s : J^k(X, Y) &\longrightarrow X & \qquad [x, U, f] &\longmapsto x \\ t : J^k(X, Y) &\longrightarrow Y & \qquad [x, U, f] &\longmapsto f(x) \end{aligned}$$

  and any smooth map $f : X \to Y$ induces a section $j^k f : X \longrightarrow J^k(X, Y)$ of s, called the k-prolongation

There weak/strong C^∞ topology can be defined as union over all k of the subspace topologies of

$$C^\infty(X,Y) \longrightarrow \mathrm{Map}(X, J^k(X,Y)), \qquad f \longmapsto j^k f$$

with respect to the compact-open/fine topology on the space of continuous maps.

Then

- For compact manifolds X both topologies coincide;
- The space of smooth maps $C^\infty(X,Y)$ is a Baire space with respect to both topologies.

Definition

X is a Baire space if for every countable collection of open, dense set $\{U_i\}_{i\in\mathbb{N}}$, the intersection $\bigcap_i U_i$ is dense in X.

Theorem (Thom Transversality I)

Let X and Y be smooth manifolds and $W \subset J^k(X, Y)$ a smooth submanifold. Then $\{f \in C^\infty(X, Y) \,|\, j^k f \pitchfork W\}$ is dense in $C^\infty(X, Y)$ (with respect to the weak and strong topology). If W is closed, then the set is open in the strong topology.

The reduction to the local case is based on the Baire property:

- Choose countable cover of W by open sets W_i such that
  - The closure of W_i in $J^k(X, Y)$ is contained in W
  - $\overline{W}_i$ is compact
  - $s \times t(W_i)$ is contained in $U_i \times V_i \subset X \times Y$ for charts (U_i, φ_i) and (V_i, ψ_i).
  - $\overline{U}_i$ is compact.
- $\{f \in C^\infty(X, Y) \,|\, j^k f \pitchfork W\} = \bigcap_i \{f \in C^\infty(X, Y) \,|\, j^k f \pitchfork \overline{W}_i\}$
- Prove that $\{f \in C^\infty(X, Y) \,|\, j^k f \pitchfork \overline{W}_i\}$ is open and dense

Definition

The space of multijets is

$$ {}_rJ^k(X,Y) := \{(z_1,\ldots,z_r)\in J^k(X,Y)\,|\,sz_i \neq sz_j \text{ for } i\neq j\} $$

There are source and target maps

$$ s\times t : {}_rJ^k(X,Y) \longrightarrow X^{(r)}\times Y^r $$

and for any smooth map $f : X\to Y$ there is a prolongation which is a section of $s : {}_rJ^k(X,Y)\to X^{(r)}$

$$ {}_rj(f) : X^{(r)} \longrightarrow {}_rJ^k(X,Y) $$

Multijet transversality

Definition

Let $C \subset X$ be a closed subset and $f_0 \in C^\infty(X, Y)$, consider

$$C^\infty(X, Y; f_0, C) := \{f \in C^\infty(X, Y) \,|\, f|_C = f_0|_C\}$$

as the subspace with respect to strong topology on $C^\infty(X, Y)$.

Theorem (Thom Transversality II)

For X, Y, C, f_0 as above and $W \subset {}_rJ^k(X, Y)$ a submanifold for some $r \geq 1$ and finite $k \geq 0$. Then

$$\{f \in C^\infty(X, Y; f_0, C) \,|\, {}_rj(f|_{X \setminus C}) \pitchfork W\}$$

is dense in $C^\infty(X, Y; f_0, C)$.

If ∂X, ∂Y and C are empty, then this is due to Mather (who said its due to Morlet).

Parametrized multijet transversality

Consider a smooth map $f : X \times M \to Y$ as a family of smooth maps $f_p : X \to Y$ for $p \in M$. Consider the associated multijet

$$_r\tilde{j}(f) : X^{(r)} \times M \longrightarrow {}_rJ(X, Y), \quad {}_r\tilde{j}(f)(x_1, \ldots, x_r, p) = {}_rj(f_p)(x_1, \ldots, x_r),$$

i.e. the jet with respect to the coordinates in X.

Theorem (Thom transversality III)

Let X, Y, C, W be as above and $r \geq 1$ and finite $k \geq 0$. For any $f_0 \in C^\infty(X \times M, Y)$, the set

$$\{f \in C^\infty(X \times M, y; f_0, C \times M) \mid {}_r\tilde{j}(f|_{(X-C)\times M}) \pitchfork W\}$$

is dense in $C^\infty(X \times M, y; f_0, C \times M)$.

"Idea of the proof": Reduce to the second version of the transversality theorem.

$$\begin{tikzcd}
X^{(r)} \times M \arrow[d] \arrow[dddd, bend right=60, "{{}_r\tilde{j}(f)}"'] & \\
(X \times M)^{(r)} \arrow[d, "{{}_r j(f)}"] & \\
{}_r J^k(X \times M, Y) \arrow[d, "\psi"] & \\
J^k(X, Y)^r \times M^r \arrow[r] \arrow[d] & M^r \\
J^k(X, Y)^r &
\end{tikzcd}$$

- The composition is the same as ${}_r\tilde{j}(f)$.
- ψ is a bundle projection and the image is an open subset.
- Hence, $W' := \psi^{-1}(W \times \Delta)$ is a submanifold of ${}_r J^k(X \times M, Y)$.
-

$$\begin{aligned} {}_r j(f) \pitchfork W' &\Leftrightarrow \psi \circ {}_r j(f) \pitchfork W \times \Delta \\ &\Leftrightarrow {}_r\tilde{j}(f) \pitchfork W \end{aligned}$$

With the obvious modifications for the subspace condition, we can then apply the multijet version of the Thom transversality.

Corollary

Let X, Y, C, M, f_0 *be as above and* W_α *a countable collection of submanifolds of* ${}_{r_\alpha}J^{m_\alpha}(X, Y)$*. Then for every neighbourhood* N *of* $f_0 \in C^\infty(X \times M, Y; f_0, C \times M)$ *there exists a smooth homotopy* f^u *such that*

- $f^0 = f_0$
- $f^u \in N$ *for all* u
- ${}_{r_\alpha}j^{m_\alpha}(f^1|_{(X-C)\times M}) \pitchfork W_\alpha$ *for all* α*.*

Proof. Any neighbourhood of f_0 contains such an f^1 by density. Any neighbourhood contains a path connected neighbourhood.

Part II - Approximation by polynomials

Some motivation first

Preview: The stratification is constructed from subsets of the multijet spaces

- Construct subsets $S(Z,P,N) \subset {}_rJ^\infty(\mathbb{R}\times P,N) = \lim_{\infty\leftarrow m} {}_rJ^m(\mathbb{R}\times P,N)$ which are supposed to detect points in a fibrered concordance whose image lie above each other.
- These are determined locally by subsets $Z \subset {}_rJ^\infty(\mathbb{R}\times\mathbb{R}^p,\mathbb{R}^n)$ so that $Z = (p_m^\infty)^{-1}(Z')$ for $Z' \subset {}_rJ^m(\mathbb{R}\times\mathbb{R}^p,\mathbb{R}^n)$ and Z' is a closed algebraic subset defined by real polynomials.
  Then $S(Z,P,N)$ is determined as follows: Cover ${}_rJ^\infty(\mathbb{R}\times P,N)$ by sets ${}_rJ^\infty(\mathbb{R}\times U,V)$ for open subsets $U\subset P$ and $V\subset N$ with embeddings $\varphi : U \hookrightarrow \mathbb{R}^p$ and $\psi : V \to \mathbb{R}^n$. Let $z \in {}_rJ^\infty(\mathbb{R}\times U,V)$, then

$$
{}_rj^\infty(1_\mathbb{R}\times\varphi)(sz) \in {}_rJ^\infty(\mathbb{R}\times U,\mathbb{R}\times\mathbb{R}^p)
$$
$$
{}_rj^\infty(\psi)(tz) \in {}_rJ^\infty(V,\mathbb{R}^n)
$$

and z is in $S(Z,P,N)$ if there is a $\tilde{z}\in Z$ so that

$$
{}_rj^\infty(\psi)(tz)\circ z = \tilde{z}\circ {}_rj^\infty(1_\mathbb{R}\times\varphi)(sz) \in {}_rJ^\infty(\mathbb{R}\times U,\mathbb{R}^n)
$$

Some remarks:

- For some reasons (better properties of varieties over $\mathbb{C}$) this is actually done with complex multijets ${}_rJ^\infty(\mathbb{C}\times\mathbb{C}^p,\mathbb{C}^n)$ so that there is some content in Z' being defined by real polynomials
- Obviously, the sets $Z\subset {}_rJ^\infty(\mathbb{R}\times\mathbb{R}^p,\mathbb{R}^n)$ need to satisfy invariance properties with respect to the above choices (next talk).

Example:

$$Z^0=\{(z_1,z_2)\in {}_2J^\infty(\mathbb{C}\times\mathbb{C}^p,\mathbb{C}^n)\,|\,t(z_1)=t(z_2)\}$$

- Next: Introduce the operations A,B,C and D that can be applied to sets Z above that measure, for example, if there are several points whose image lie above each other.
- Applying these operations to Z^0 gives a countable sequence of varieties $Z_\alpha\subset {}_{r_\alpha}J^\infty(\mathbb{C}^{p+1},\mathbb{C}^n)$ for which we can form $S(Z_\alpha,P,N)$
- Make a fibred concordance transverse to the submanifolds of ${}_rJ^\infty(\mathbb{R}\times P,N)$ that come from the non-singular parts of these varieties.

Operation D

Motivation: Let $Z \subset {}_rJ^m(\mathbb{C}\times\mathbb{C}^p,\mathbb{C}^n)$ and set $K = \mathrm{Poly}_{r\binom{p+1+m}{m}-1}(\mathbb{C}^{p+1})$ and

$$X = \left\{(x,f) \in (\mathbb{C}^{p+1})^{(r)} \times K^n \,|\, {}_rj^\infty(f)(x) \in Z\right\}$$

Then the closure $\overline{X} \subset (\mathbb{C}^{p+1})^r \times K^n$ potentially contains points on the fat diagonal. Collisions can be encoded by surjective maps $\phi : \{1,\dots,r\}\to\{1,\dots,r'\}$

$$Y_\phi := \{(y,f) \in \mathbb{C}^{p+1})^{(r')} \times K^n \,|\, (\phi^* y, f) \in \overline{X}\}.$$

Then set

$$D_\phi(Z) := \Big\{(z_1,\dots,z_{r'}) \in {}_{r'}J^\infty(\mathbb{C}^{p+1},\mathbb{C}^n)\,|\,\exists\,(y,f) \in Y_\phi \text{ s.t.}$$
$$p^\infty_{(m+1)|\phi^{-1}(i')|-1}(z_i') = j^{(m+1)|\phi^{-1}(i')|-1}(f)(y_{i'})\Big\}$$

Claim: For a convergent sequence $\{f^\nu\}_{\nu\geq 1}$ in $C^\infty(\mathbb{R}^{p+1},\mathbb{R}^n)$ with limit f, and a sequence $\{x^\nu\}_{\nu\geq1} \in (\mathbb{R}^{p+1})^{(r)}$ converging to $x = \phi^* y$ for $y \in (\mathbb{R}^{p+1})^{(r')}$. Then

$${}_rj^\infty(f^\nu)(x^\nu) \in Z \quad \forall\nu \Rightarrow {}_{r'}j^\infty(f)(y) \in D_\phi(Z)$$

Lemma 37. Let $m \geq 0$, $r \geq r' \geq 1$, and $k_i \geq 0$ for $1 \leq i \leq r$ be integers and denote $K = \sum_i k_i$. Let $\phi : \{1, \ldots, r\} \to \{1, \ldots, r'\}$ be surjective, $U \subset \mathbb{C}^m$ open, $\{f^\nu\}_{\nu \geq 1}$ a convergent sequence in the space $H(U)$ of holomorphic functions on U. Let $\{x^\nu\}_{\nu \geq 1} \in U^{(r)}$ converging to $x = \phi^* y$ for $y \in U^{(r')}$.

Then there exists a convergent sequence $g^\nu \in \mathrm{Poly}_{K-1}(\mathbb{C}^m)$ with limit g so that

- $g^\nu - f^\nu$ vanishes to order k_i at x_i^ν for all i, ν
- $g - f$ vanishes to order $\sum_{\phi(i)=j} k_i$ at y_j for all j.

Remark: The same holds for smooth functions on $U \subset \mathbb{R}^m$ instead of $H(U)$ and real valued polynomials $\mathrm{Poly}_{K-1}(\mathbb{R}^m)$ instead of complex ones.

Special case $m = 1$

Lemma. Let $U \subset \mathbb{C}$ and $(a_1, \dots, a_k) \in U^k$ and $f \in H(U)$, there is a unique polynomial $A_{a,f} \in \mathrm{Poly}_{k-1}(\mathbb{C})$ such that $f - A_{a,f}$ vanishes at $a_1, \dots, a_k$ with multiplities.

Proof.

- $k = 1$ then $A_{a,f} = f(a)$
- $k \geq 1$ then $A_{a_1,\dots,a_k,f} = f(a_k) + (t - a_k)A_{a_1,\dots,a_{k-1},\tilde{f}}$ where

$$\tilde{f}(t) = \begin{cases} \frac{f(t)-f(a_k)}{t-a_k} & t \neq a_k \\ f'(a_k) & t = a_k \end{cases}$$

For $m > 1$ there is no unique construction, but Lemma 37 says that such a polynomial always exists that there is a continuity property in choosing it.

Lemma. Let $m \geq 0$, $r \geq 1$, and $k_i \geq 0$ for $1 \leq i \leq r$ be integers and denote $K = \sum_i k_i$. Let $(x_1, \dots, x_r) \in (\mathbb{C}^m)^{(r)}$. Then the subspace of $\mathrm{Poly}_{K-1}(\mathbb{C}^m)$ of polynomials that vanish to order $\geq k_i$ at x_i has codimension $\sum_i \binom{m+k_i-1}{k_i-1}$.

Proof. It is the preimage of the map

$$\mathrm{Poly}_{K-1}(\mathbb{C}^m) \longrightarrow \prod_{i=1}^{r} \mathrm{Poly}_{k_i-1}(\mathbb{C}^m), \quad f \longmapsto (T_{x_i}^{\leq k_i-1}(f))_i$$

which explains the codimension. The statement is equivalent to proving that one can always construct a polynomial with prescribed Taylor polynomial at x_i, and we prove it by induction over r where the case $r = 1$ is trivial.

- By induction assumption, we can construct polynomials with prescribed Taylor approximation at $x_1, \dots, x_{r-1}$ so that we can assume without loss of generality that the Taylor approximation is non-trivial only at x_r.
- Choose linear functions L_i with $L_i(x_j) \neq 0$ for $i \neq j$ and $L_j(x_j) = 0$ and set $g = h \prod_{i=1}^{r-1} L_i^{k_i}$. Since $\prod_{i=1}^{r-1} L_i^{k_i}(x_r) \neq 0$ we can find h so that g has the prescribed Taylor approximation at x_r.

Proof of Lemma 37

Lemma 37. Let $m \geq 0$, $r \geq r' \geq 1$, and $k_i \geq 0$ for $1 \leq i \leq r$ be integers and denote $K = \sum_i k_i$. Let $\phi : \{1, \dots, r\} \to \{1, \dots, r'\}$ be surjective, $U \subset \mathbb{C}^m$ open, $\{f^\nu\}_{\nu \geq 1}$ a convergent sequence in the space $H(U)$ of holomorphic functions on U. Let $\{x^\nu\}_{\nu \geq 1} \in U^{(r)}$ converging to $x = \phi^* y$ for $y \in U^{(r')}$.

Then there exists a convergent sequence $g^\nu \in \text{Poly}_{K-1}(\mathbb{C}^m)$ with limit g so that

- $g^\nu - f^\nu$ vanishes to order k_i at x_i^ν for all i, ν
- $g - f$ vanishes to order $\sum_{\phi(i)=j} k_i$ at y_j for all j.

Proof. Double induction on m and K.

- $m = 1$ and K arbitrary follows from explicit formulas in the special for tuples

$$a^\nu = (\underbrace{x_1^\nu, \dots, x_1^\nu}_{k_1}, \dots, x_r^\nu, \dots, x_r^\nu) \qquad \Rightarrow g^\nu = A_{a^\nu, f^\nu}$$

- Now let $m \geq 2$ and $K \geq 1$ and assume its true for $(m-1, K)$ and for $(m, K-1)$.

Step 1. Assume

$$x_r^\nu = 0 \quad \forall \nu, \qquad\qquad x_i = 0 \quad \forall i$$

- Find a projection $p : \mathbb{C}^m = \mathbb{C}^{m-1} \times \mathbb{C} \to \mathbb{C}^{m-1}$ so that $p_1(x_i^\nu) \neq p_1(x_j^\nu)$ for $i \neq j$. Write the coordinates with respect to p_1 as (x, t).
- Find neighbourhoods $U_1 \subset \mathbb{C}^{m-1}$ and $U_2 \subset \mathbb{C}$ of zero ($U_1 \times U_2$ contains all but finitely many x_i^ν by the second condition) where one can write

$$f^\nu(x,t) = f_1^\nu(x) + tf_2^\nu(x,t)$$
$$f(x,t) = f_1(x) + tf_2(x,t)$$

- Apply induction hypothesis to f_1^ν and f_1 with $(m-1, K)$, and to f_2^ν and f_2 with $(m, K-1)$.
- Then

$$g^\nu(x,t) = g_1^\nu(x) + tg_2^\nu(x,t)$$
$$g(x,t) = g_1(x) + tg_2(x,t)$$

satisfies the conditions of the theorem.

Step 2. Assume

$$x_i^\nu \in D_\epsilon(0) \subset U \quad \forall \nu, \qquad x_i = 0 \quad \forall i$$

Then we can reduce to step 1 by replacing the sequence with $\tilde{x}_i^\nu = x_i^\nu - x_r^\nu$ and $\tilde{f}^\nu(\xi) := f^\nu(\xi + x_r^\nu)$ and $\tilde{f} = f$ on $D_\epsilon(0)$.

Step 3. Assumptions as above but x_i is some complex number $\forall i$
—> Translate to zero

Step 4. The general case

- The basic idea is to use the previous steps for the sequences x_i^ν for all i with $\phi(i) = j$ for some fixed j to obtain functions g_j^ν and g_j so that
$$f^\nu - g_j^\nu \text{ vanish at } x_i \text{ to order } k_i \text{ for all } i \text{ with } \phi(i) = j.$$
- If we can also arrange that these g_i^ν satisfy
$$g_j^\nu \text{ vanish at } x_i \text{ to order } k_i \text{ for all } i \text{ with } \phi(i) \neq j$$
then $\sum_{j=1}^{r'} g_j^\nu$ is a sequence of functions we are looking for.
- Choose a linear polynomial $L \in \mathrm{Poly}_1(\mathbb{C}^m)$ so that $L(x_i^\nu) \neq L(x_{i'}^\nu)$ for $i \neq i'$ and $L(y_j) \neq L(y_{j'})$ for $j \neq j'$ and define
$$q_j^\nu := \prod_{i \in \phi^{-1}(j)} (L - L(x_i))^{k_i}$$
$$q_j := \prod_{i \in \phi^{-1}(j)} (L - L(y_i))^{k_i}$$
which satisfy
$$q_j^\nu(x_i) \neq 0 \text{ if } \phi(i) = j, \text{ and } q_j^\nu(x_i) \text{ vanish to order } k_i \text{ at } x_i \text{ for } \phi(i) \neq j.$$

- Apply previous step to the sequence f^ν/q_j^ν and configurations (x_i^ν) for all $i \in \phi^{-1}(j)$ to obtain an approximation h_j^ν of polynomials satisfying

  $$h_j^\nu - f^\nu/q_j^\nu \text{ vanishes to degree } k_i \text{ at } x_i \text{ for } i \in \phi^{-1}(j)$$

- Then set $g_j^\nu := q_j h_j^\nu$ which vanishes by construction at x_i to order k_i for i with $\phi(i) \neq j$.