DIFFEOMORPHISMS OF DISKS

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Abstract. The simplest manifold is arguably the disk, but the homotopy type of their diffeomorphism groups remains mysterious. I will discuss what is known about them, as well as work-in-progress joint with Oscar Randal-Williams, which has the goal of computing the rational homotopy groups of the group of diffeomorphisms of the $2n$-dimensional disk fixing the boundary pointwise for $2n \geq 6$.

Contents

1. Diffeomorphisms of disks 1
2. Low dimensions 3
3. Exotic spheres 4
4. Algebraic K-theory 6
5. Moduli spaces of manifolds and embedding calculus 7
References 9

1. Diffeomorphisms of disks

If you are reading this, you probably are familiar with the d-dimensional disk D^d:

$$D^d := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_i x_i^2 \leq 1 \right\}.$$

Disks are arguably the simplest smooth manifolds, and serve as the basic building blocks of all other smooth manifolds. This talk is about its group of diffeomorphisms:

Definition 1.1. $\mathrm{Diff}_\partial(D^d)$ is the topological group of C^∞-diffeomorphisms which fix a neighborhood of the boundary ∂D^d pointwise, in the C^∞-topology.

Example 1.2. The group $\pi_0(\mathrm{Diff}_\partial(D^d))$ of path components is the group of diffeomorphisms up to the equivalence relation of isotopy: ϕ_0 and ϕ_1 are *isotopic* if there is a smooth family ϕ_t of diffeomorphisms interpolating between them.

Like any topological group, this has a classifying space $B\mathrm{Diff}_\partial(D^d)$, unique up to homotopy equivalence. It is called a classifying space because it classifies smooth disk bundles: there

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are bijections, natural in X,

(1)

$$[X, B\mathrm{Diff}_\partial(D^d)] \overset{\cong}{\longleftrightarrow} \frac{\{\text{principal } \mathrm{Diff}_\partial(D^d)\text{-bundles}\}}{\text{isomorphism}} \overset{\cong}{\longleftrightarrow} \frac{\left\{\begin{array}{c}\text{smooth } D^d\text{-bundles with} \\ \text{trivialized boundary}\end{array}\right\}}{\text{isomorphism}}.$$

Explicitly, the first of these bijections is given by

$$\begin{aligned} [X, B\mathrm{Diff}_\partial(D^d)] &\longrightarrow \frac{\{\text{principal } \mathrm{Diff}_\partial(D^d)\text{-bundles}\}}{\text{isomorphism}} \\ [f] &\longmapsto [f^*\xi_{\mathrm{univ}}], \end{aligned}$$

pulling back a principal $\mathrm{Diff}_\partial(D^d)$-bundle $\xi_{\mathrm{univ}} \colon E\mathrm{Diff}_\partial(D^d) \to B\mathrm{Diff}_\partial(D^d)$, called the *universal bundle*. The second of these bijections is given by

$$\begin{aligned} \frac{\{\text{principal } \mathrm{Diff}_\partial(D^d)\text{-bundles}\}}{\text{isomorphism}} &\longrightarrow \frac{\left\{\begin{array}{c}\text{smooth } D^d\text{-bundles with} \\ \text{trivialized boundary}\end{array}\right\}}{\text{isomorphism}} \\ [\xi] &\longmapsto [\xi \times_{\mathrm{Diff}_\partial(D^d)} D^d], \end{aligned}$$

the associated bundle construction.

Smooth disk bundles are interesting geometric objects, and one way to interpret these bijections is that if you understand the homotopy type of $B\mathrm{Diff}_\partial(D^d)$ you can classify disk bundles by obstruction theory. Thus we are led to ponder the following question:

Question 1.3. What is the homotopy type of $B\mathrm{Diff}_\partial(D^d)$?

The homotopy type of $B\mathrm{Diff}_\partial(D^d)$ is closely related to that of $\mathrm{Diff}_\partial(D^d)$. The reason for this is that the universal bundle ξ_{univ} is in particular a fibration with contractible total space and fibers $\mathrm{Diff}_\partial(D^d)$:

$$\mathrm{Diff}_\partial(D^d) \longrightarrow E\mathrm{Diff}_\partial(D^d) \simeq * \longrightarrow B\mathrm{Diff}_\partial(D^d).$$

The long exact sequence of homotopy groups of this fibration then gives isomorphisms

$$0 = \pi_{i+1}(E\mathrm{Diff}_\partial(D^d)) \longrightarrow \pi_{i+1}(B\mathrm{Diff}_\partial(D^d)) \xrightarrow{\cong} \pi_i(\mathrm{Diff}_\partial(D^d)) \longrightarrow \pi_i(E\mathrm{Diff}_\partial(D^d)) = 0.$$

That is, the homotopy groups of $B\mathrm{Diff}_\partial(D^d)$ are those of $\mathrm{Diff}_\partial(D^d)$ shifted up one degree.

Example 1.4. Let's use (1) to classify disk bundles over the circle. These are in bijection with homotopy classes of maps $S^1 \to B\mathrm{Diff}_\partial(D^d)$. This can be computed as

$$[S^1, B\mathrm{Diff}_\partial(D^d)] = \frac{\pi_1(B\mathrm{Diff}_\partial(D^d))}{\text{conjugation}} = \frac{\pi_0(\mathrm{Diff}_\partial(D^d))}{\text{conjugation}}.$$

Here we have used that the general fact that unbased homotopy classes of maps of S^1 into a path-connected space X are based homotopy classes of maps of $S^1 \to X$ up to conjugation.

In terms of smooth disk bundles over the circle, the above identification is given by choosing an identification of a single fiber with D^d and computing the monodromy around the circle. The monodromy is well-defined up to isotopy, and changing the identification by a diffeomorphism ϕ changes the monodromy by conjugation with ϕ.

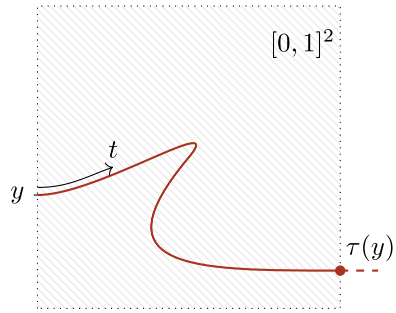

FIGURE 1. The red line is the flow-line along $\mathcal{Y}$ starting at $(0, y)$, and the $\tau(y)$ is the first and only time where this flowline hits the side $\{1\} \times [0,1]$ of $[0,1]^2$.

Of course, the isomorphisms $\pi_{i+1}(B\mathrm{Diff}_\partial(D^d)) \cong \pi_i(\mathrm{Diff}_\partial(D^d))$ are so far just isomorphisms between two unknown groups. In this talk I will try to explain some of what is known about these homotopy groups, as well as some recent progress.

2. Low dimensions

To get some feeling for the behaviour of diffeomorphisms of disks, one might start with the more familiar low dimensions $d \leq 3$.

2.1. **Dimension $d = 0$.** D^0 is a point and every diffeomorphism is the identity:

$$\mathrm{Diff}_\partial(D^0) = \{\mathrm{id}\}.$$

2.2. **Dimension $d = 1$.** D^1 is diffeomorphic to the unit interval $[0,1]$, and a smooth map $[0,1] \to [0,1]$ fixing a neighborhood of the boundary $\{0,1\}$ is a diffeomorphism if and only if its derivative is positive everywhere. This is a convex condition, so we can deformation retract $\mathrm{Diff}_\partial([0,1])$ onto $\{\mathrm{id}\}$ by linear interpolation ("tightening the string"):

$$\begin{aligned} H\colon [0,1] \times \mathrm{Diff}_\partial([0,1]) &\longrightarrow \mathrm{Diff}_\partial([0,1]) \\ (t, \phi) &\longmapsto (1-t) \cdot \phi + t \cdot \mathrm{id}. \end{aligned}$$

We conclude that:

Proposition 2.1. $\mathrm{Diff}_\partial(D^1)$ *is contractible.*

2.3. **Dimension $d = 2$.** In two dimensions the first difficulties arise: trying to linear interpolate a diffeomorphism to the identity can easily lead to the derivative being non-invertible. However, an argument of Smale uses a variation of "tightening" to again prove that [Sma59]:

Theorem 2.2 (Smale)**.** $\mathrm{Diff}_\partial(D^2)$ *is contractible.*

Sketch of proof. We may replace the disk D^2 with the square $[0,1]^2$, as there is a homotopy equivalence $\mathrm{Diff}_\partial(D^2) \simeq \mathrm{Diff}_\partial([0,1]^2)$, Here—as always in this talk—the subscript ∂ means that the diffeomorphisms fix a neighborhood of the boundary, so that we don't need to worry about corners.

On the square $[0,1]^2$ there is a vector field $\frac{\partial}{\partial x}$, pointing rightwards. If we start at $(0,y) \in [0,1]^2$ and flow for time t along this vector field, we end up in (t,y). We can push this vector field forward along a diffeomorphism $\phi\colon [0,1]^2 \to [0,1]^2$. By uniqueness of the solutions to ordinary differential equations, if we start at $(0,y) \in [0,1]^2$ and flow for time t along $\phi_* \frac{\partial}{\partial x}$, we end up at $\phi(t,y)$. Thus we can recover ϕ from $\phi_* \frac{\partial}{\partial x}$.

Let X be the space of everywhere non-vanishing vector fields that are equal to $\frac{\partial}{\partial x}$ on a neighborhood of the boundary. Then we have just described an injective continuous map

$$\begin{aligned} \Xi\colon \mathrm{Diff}_\partial([0,1]^2) &\longrightarrow X \\ \phi &\longmapsto \phi_* \frac{\partial}{\partial x}. \end{aligned}$$

In fact, the flow along any $\mathcal{X} \in X$ can be used to produce a diffeomorphism. It is a consequence of the Poincaré–Bendixson theorem that any flowline starting at a point $(0,y) \in \{0\} \times [0,1]$ must exit $[0,1]^2$ through $\{1\} \times [0,1]$ after some finite time $\tau(y) > 0$. The number $\tau(y)$ depends smoothly on y by the smooth dependence of solutions to ordinary differential equations on initial conditions. Now, the flow-line may not exit at height y again, but we can easily adjust for this by a linear interpolation as in dimension 1. Thus there is a smooth map

$$\begin{aligned} \phi_\mathcal{X}\colon [0,1]^2 &\longrightarrow [0,1]^2 \\ (t,y) &\longmapsto \begin{matrix} \text{flow along } \mathcal{X} \text{ starting at } (0,y)\text{, for time} \\ t \cdot \tau(y)\text{, adjusted near } \{1\} \times [0,1] \end{matrix}, \end{aligned}$$

which is a diffeomorphism fixing a neighborhood of $\partial[0,1]^2$. This depends smoothly on $\mathcal{X}$, so we have constructed a continuous map

$$\begin{aligned} \Phi\colon X &\longrightarrow \mathrm{Diff}_\partial([0,1]^2) \\ \mathcal{X} &\longmapsto \phi_\mathcal{X}. \end{aligned}$$

This is a homotopy inverse to Ξ, so $\mathrm{Diff}_\partial([0,1]^2) \simeq X$.

Now we observe that X is contractible, as it is homotopy equivalent to the space of maps from $[0,1]^2$ to S^1 which are the identity on $\partial[0,1]^2$, i.e. $\Omega^2 S^1 \simeq *$. ☐

2.4. **Dimension $d = 3$.** It is a hard result for Hatcher, generalizing a result of Cerf for π_0, that the three-dimensional disk has similar properties [Hat83]:

Theorem 2.3 (Hatcher)**.** $\mathrm{Diff}_\partial(D^3)$ *is contractible.*

This was very recently reproven using parametrized Ricci flower by Balmer and Kleiner.

3. Exotic spheres

The examples discussed so far might lead you to believe that $\mathrm{Diff}_\partial(D^d)$ is contractible for all d. This is *not true* for all $d \geq 5$. Let see this is the case for $d = 6$, using a famous result of Milnor [Mil56]:

Theorem 3.1 (Milnor)**.** *There are smooth 7-dimensional manifolds which are homeomorphic to S^7 but not diffeomorphic to it.*

How did Milnor prove that these M's are homeomorphic to S^7? It turns out that they admit a Morse function $f\colon M \to \mathbb{R}$ with two critical points, necessarily a minimum and a maximum. The Morse lemma gives coordinates around the minimum and maximum in which f looks like $f(\min) + \sum_{i=1}^7 x_i^2$ and $f(\max) - \sum_{i=1}^7 x_i^2$. In particular, we can find little 7-dimensional disks $D_{\min}$ and $D_{\max}$ around the minimum and maximum. Outside of these disks the gradient vector field of f (depending on a Riemannian metric) is non-vanishing and flowing along it gives a diffeomorphism

$$M \setminus \mathrm{int}(D_{\min} \sqcup D_{\max}) \cong \partial D_{\min} \times [0,1],$$

relative to $\partial D_{\min}$. Gluing this cylinder to $D_{\min}$ to get a larger disk $\tilde{D}_{\min}$, we see that M is obtained by gluing two together two disks $D^7 \cup_\phi D^7$ along a diffeomorphism (without loss of generality orientation-preserving)

$$\phi\colon \partial\tilde{D}_{\min} = S^6 \longrightarrow S^6 = \partial D_{\max}.$$

Since ϕ extends over $D_{\max}$ by a homeomorphism, using the Alexander trick, we can use this as a different identification of $D_{\max}$ and get that $D^7 \cup_\phi D^7$ is homeomorphic $D^7 \cup_{\mathrm{id}} D^7 = S^7$. That a compact manifold with a Morse function with two critical points is homeomorphic to a sphere is known as Reeb's theorem [Ree46].

Similarly, if ϕ extended over $D_{\max}$, we could use this diffeomorphism as a different identification of $D_{\max}$ with D^7 and get $D^7 \cup_\phi D^7 \cong D^7 \cup_{\mathrm{id}} D^7 = S^7$. By Milnor's result we know this is impossible, so ϕ does not extend over D^7. This means it can't be isotopic to the identity: otherwise we could combine the trace $S^6 \times [0,1] \to S^6 \times [0,1]$ of this isotopy by the identity on D^6 to obtain an extension of ϕ over D^7. We have thus found an orientation-preserving diffeomorphism of S^6 which is not isotopic to the identity.

This in fact arises from a diffeomorphism of D^6 which is not isotopic to the identity. Acting by an orientation-preserving diffeomorphism of S^6 on an orientation-preserving embedding of D^6 into S^6, isotopy extension gives a fiber sequence

$$\mathrm{Diff}_\partial(D^6) \longrightarrow \mathrm{Diff}^+(S^6) \longrightarrow \mathrm{Emb}^+(D^6, S^6).$$

Take the derivative at the origin of D^6 gives a homotopy equivalence $\mathrm{Emb}^+(D^6, S^6) \simeq \mathrm{Fr}^+(TS^6)$. The oriented frame bundle $\mathrm{Fr}^+(TS^6)$ is easily seen to be path-connected, so by the long exact sequence of homotopy groups the map $\pi_0(\mathrm{Diff}_\partial(D^6)) \to \pi_0(\mathrm{Diff}^+(S^6))$ must be surjective. We conclude that:

Theorem 3.2. *$\pi_0(\mathrm{Diff}_\partial(D^6))$ is not trivial.*

Remark 3.3. In fact $\mathrm{Fr}^+(TS^d) \cong SO(d+1)$ and the rotations $SO(d+1) \to \mathrm{Diff}^+(S^d)$ split the map $\mathrm{Diff}^+(S^d) \to \mathrm{Fr}^+(TS^d)$. Thus there is a homotopy equivalence

$$\mathrm{Diff}^+(S^d) \simeq \mathrm{Diff}_\partial(D^d) \times SO(d+1),$$

and there is in fact an isomorphism $\pi_0(\mathrm{Diff}_\partial(D^6)) \cong \pi_0(\mathrm{Diff}^+(S^7))$.

This relationship between diffeomorphisms of disks and exotic spheres is not particular to dimension $d = 6$ [Cer70]:

Definition 3.4. Let Θ_{d+1} denote the group of oriented homotopy $(d+1)$-sphere, up to oriented diffeomorphism, with addition given by connected sum.

Theorem 3.5 (Cerf)**.** *For $d \geq 5$, $\pi_0(\mathrm{Diff}_\partial(D^d)) \cong \Theta_{d+1}$.*

The map $\pi_0(\mathrm{Diff}_\partial(D^d)) \to \Theta_{d+1}$ is easy to describe: we extend a diffeomorphism of D^d to S^d by the identity, and using this to glue two copies of D^{d+1} along their boundary.

Remark 3.6. The groups Θ_{d+1} are closely related the stable homotopy groups of spheres. For example, this can be used to show that Θ_{d+1} is finite abelian as long as $d \neq 4$.

4. Algebraic K-theory

What can we say about the higher homotopy groups of $\mathrm{Diff}_\partial(D^d)$? Let's look at a technical tool used in the proof of Theorem 3.5.

Definition 4.1. Two diffeomorphisms f_0, f_1 of D^d are said to be *pseudo-isotopic* if there is a diffeomorphism F of $D^d \times [0,1]$ fixing a neighborhood of $\partial D^d \times [0,1]$ pointwise and satisfying $F(d,0) = (f_0(d), 0)$ and $F(d,1) = (f_1(d), 1)$.

Theorem 3.5 relies on the fact that pseudo-isotopy implies isotopy when $d \geq 5$. Since isotopy classes of diffeomorphisms of D^d are a group of path components, $\pi_0(\mathrm{Diff}_\partial(D^d))$, we would like a similar description for pseudo-isotopy classes. This is provided by the topological group $\widetilde{\mathrm{Diff}}(D^d)$ of *block diffeomorphisms*. Let me not define these, but just tell you that its definition is such that its path components $\pi_0(\widetilde{\mathrm{Diff}}(D^d))$ are equal to the pseudo-isotopy classes of diffeomorphisms of D^d.

To make this precise, one uses that there is a map $\mathrm{Diff}_\partial(D^d) \to \widetilde{\mathrm{Diff}}(D^d)$ of topological groups which is an isomorphism on π_0. It is a fact that $\widetilde{\mathrm{Diff}}(D^d)$ has finite homotopy groups, and thus the map

$$\frac{\widetilde{\mathrm{Diff}}(D^d)}{\mathrm{Diff}(D^d)} \longrightarrow B\mathrm{Diff}(D^d)$$

induces an isomorphism on rational homotopy groups.

Thus to understand the higher rational homotopy groups of $B\mathrm{Diff}(D^d)$, the right hand side, we need to understand the left hand side. The resemblance of a pseudo-isotopy to an h-cobordism suggests that this is related to algebraic K-theory. Let $A(*)$ be the algebraic K-theory spectrum of the sphere spectrum $\mathbb{S}$, also known as Waldhausen's algebraic K-theory of spaces. This splits as $A(*) = \mathbb{S} \times \mathrm{Wh}^{\mathrm{Diff}}(*)$, and Weiss–Williams proved that there is map

$$\frac{\widetilde{\mathrm{Diff}}(D^d)}{\mathrm{Diff}(D^d)} \longrightarrow \Omega^\infty(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)_{hC_2}),$$

which induces an isomorphism on π_i for $i \leq d/3$ approximately [WW88]. It may seem that the right-hand side is independent of d, *but the C_2-action depends on the parity of d.*[1]

Since $\mathbb{S}$ has finite higher homotopy groups and $\pi_0(\mathbb{S}) = \mathbb{Z}$, we have isomorphisms of rational homotopy groups for $i \geq 0$:

$$\pi_i(\Omega^\infty(\Omega \mathrm{Wh}^{\mathrm{Diff}}(*)_{hC_2})) \otimes \mathbb{Q} = \pi_{i+1}(\mathrm{Wh}^{\mathrm{Diff}}(*)_{hC_2}) \otimes \mathbb{Q} = (K_{i+1}(\mathbb{Z}) \otimes \mathbb{Q})^{C_2}.$$

It turns out that the action of C_2 by -1 when d is odd and by -1 when d is even. We conclude that [FH78]

[1]The determination of this C_2-action has a fraught history. In fact, it seems it was never carefully proven, and only very recently were completely arguments provided.

Theorem 4.2 (Farrell–Hsiang)**.** *For $d \geq 5$ and $* \leq d/3$ approximately, there are isomorphisms*

$$\pi_*(B\mathrm{Diff}_\partial(D^d)) = \begin{cases} K_{i+1}(\mathbb{Z}) \otimes \mathbb{Q} & \text{if } d \text{ is odd,} \\ 0 & \text{if } d \text{ is even.} \end{cases}$$

Remark 4.3. There is a Hatcher–Waldhausen map $G/O \to \Omega \mathrm{Wh}^{\mathrm{Diff}}(*)$ which is a rational equivalence [BW87]. Geometrically this is leads to a construction of smooth disk bundles built by gluing two handles along varying attaching maps.

5. Moduli spaces of manifolds and embedding calculus

Let's recap what we have learned about diffeomorphisms of disks so far:

- For $d \leq 3$, $\mathrm{Diff}_\partial(D^d) \simeq *$.
- For $d \geq 5$, $\pi_0(\mathrm{Diff}_\partial(D^d)) = \Theta_{d+1}$.
- For $d \geq 5$, we can compute the higher rational homotopy groups in a range $i \leq d/3$ in terms of algebraic K-theory.

A lot of open questions remain, and the most pressing one is probably the following:

Question 5.1. What are the rational homotopy groups of $\mathrm{Diff}_\partial(D^d)$ outside the range $* \leq d/3$?

Let me know describe a new approach to study these groups, pioneered by Weiss and applied in my thesis and in forthcoming joint work in Oscar Randal–Williams.

5.1. **Moduli spaces of manifolds.** The first major application of moduli spaces of manifolds was the proof of the Mumford conjecture. This concerns the stable homology of mapping class groups, and it is proven by combining homological stability for mapping class groups with the computation of the homotopy groups of the two-dimensional oriented cobordism category, work of Harer [Har85] and Madsen–Weiss [MW07] respectively. Both these inputs were generalized by Galatius and Randal–Williams to higher dimensions $d = 2n \geq 6$ [GRW18, GRW17, GRW14] Here is a particular instance of much more general result obtained by them concerning certain high-dimensional analogues of surfaces:

Theorem 5.2 (Galatius–Randal-Williams)**.** *Let $2n \geq 6$, and $W_{g,1} := D^{2n} \# S^n \times S^n$. Then for $* \leq \frac{g-3}{2}$, there is an isomorphism*

$$H_*(B\mathrm{Diff}_\partial(W_{g,1})) \cong H_*(\Omega^\infty_0 MT\theta_{2n}),$$

with $MT\theta_{2n}$ the Thom spectrum of $-\theta^*\gamma$ with θ the n-connective cover $BO(2n)\langle n \rangle \to BO(2n)$ and $\gamma \to BO(2n)$ the universal bundle.

Two observations about this result:

(i) we get information about diffeomorphisms in a range which only depends on g, not on the dimension $2n$,
(ii) $\Omega^\infty_0 MT\theta_{2n}$ is a homotopy-theoretic object and hence much more amenable to computation, e.g. rationally its cohomology is a polynomial ring in generators κ_c with c in a basis $\mathcal{B}$ of monomials in the Euler class and certain Pontryagin classes.

5.2. **The Weiss fiber sequence.** However, it seems like we have just traded a dependence on the dimension $2n$ for a dependence on the genus g. It turns out that we can strip away the stabilization by connected sum with $S^n \times S^n$ and return to the even-dimensional disk D^{2n} that we are interested in.

What is the strategy for this? Any compact smooth manifold of dimension can be built from finitely many i-handles $D^i \times D^{2n-i}$ with $0 \leq i \leq 2n$; for $W_{g,1}$ you need a 0-handles and $2g$ n-handles. A diffeomorphism of $W_{g,1}$ fixing a neighborhood its boundary gives an embedding of the 0- and n-handles into $W_{g,1}$. This is invertible up to isotopy, and in particular fixes a neighborhood of the $(2n-1)$-disk where the 0-handle meets the boundary. The remaining data is an identification of the complement with a disk [Kup17]:

Theorem 5.3 (K.)**.** *Let* $\mathrm{Emb}^{\cong}_{1/2\partial}(W_{g,1})$ *denote the group-like topological monoid of embeddings* $W_{g,1} \hookrightarrow W_{g,1}$ *fixing a neighborhood of* $D^{2n-1} \subset S^{2n-1} = \partial W_{g,1}$ *which are isotopic to a diffeomorphism. Then there is a fiber sequence*

$$B\mathrm{Diff}_\partial(W_{g,1}) \longrightarrow B\mathrm{Emb}^{\cong}_{1/2\partial}(W_{g,1}) \longrightarrow B^2\mathrm{Diff}_\partial(D^{2n}).$$

The Galatius–Randal-Williams results allow one to understand the cohomology of the left term in a range depending on g. A technique called embedding calculus allows one to understand the homotopy groups of the middle term for any g [Wei99].

The base is independent of g, so we may let $g \to \infty$ and knowledge of the $B\mathrm{Diff}_\partial(D^{2n})$ follows without restrictions on the dimension $2n$ or the genus g (except $2n \geq 6$ necessary for the results of Galatius–Randal-Williams and for the embedding calculus tower to converge).

5.3. **Finite generation.** For example, it is easy to compute that $\Omega^\infty_0 MT\theta_{2n}$ has finitely-generated homology groups and hence so does $B\mathrm{Diff}_\partial(W_{g,1})$ in a range depending on g. Similarly, we can combine calculations in embedding calculus with the fact that arithmetic groups have classifying spaces which are finite CW-complexes (at least after passing to finite index subgroups) to prove that $B\mathrm{Emb}^{\cong}_{1/2\partial}(W_{g,1})$ also has finitely-generated homology groups. From this we can conclude that $B^2\mathrm{Diff}_\partial(D^{2n})$ has finitely-generated homology groups and with a little bit of work we get (as well as similar tricks for odd dimensions) [Kup17]:

Theorem 5.4 (K.)**.** *Suppose* $d \neq 4, 5, 7$*, then all homotopy groups* $\pi_*(\mathrm{Diff}_\partial(D^d))$ *are finitely generated.*

5.4. **Progress on rational homotopy groups.** The goal of the joint work with Oscar Randal–Williams is to compute these homotopy groups, at least rationally, using the Weiss fiber sequence and much finer information about $B\mathrm{Diff}_\partial(W_{g,1})$ and $B\mathrm{Emb}^{\cong}_{1/2\partial}(W_{g,1})$ that just finite generation of their homology groups.

Theorem 5.5 (Work-in-progress, K.–Randal-Williams)**.** *Suppose that* $2n \geq 6$*, and let* $\epsilon = 1$ *if* n *is odd and* $\epsilon = 3$ *if* n *is even. Then the rational homotopy of* $\mathrm{Diff}_\partial(D^{2n})$ *is given by*

$$\pi_*\left(B\mathrm{Diff}_\partial(D^{2n})\right) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } * \leq 2n-2, \\ \mathbb{Q} & \text{if } * = 4k+\epsilon \geq 2n-1, \text{ and } * \notin \bigcup_{r \geq 2}[2r(n-2), 2rn], \\ 0 & \text{if } * \neq 4k+\epsilon \geq 2n-1, \text{ and } * \notin \bigcup_{r \geq 2}[2r(n-2), 2rn]. \end{cases}$$

That is, in low degrees we see the vanishing as in the Farrell–Hsiang theorem. Then up to degree $4n-8$, all we see is a $\mathbb{Q}$ every four degrees. This is a reflection of the fact, due to

Weiss, that the Pontryagin class p_i for $i > 2n$ doesn't need to vanish in $H^{4i}(B\mathrm{Top}(2n);\mathbb{Q})$. After this mysterious "bands" start to appear: these are conjectured to only contain graph homology.

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