NRW TALK: E_∞-CELLS AND THE HOMOLOGY OF GENERAL LINEAR GROUPS OVER FINITE FIELDS

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This is a talk about [GKRW18c]. In that paper, our goal is to understand more of the homology groups

$$H_d(\mathrm{GL}_n(\mathbb{F}_q);\mathbb{F}_p) \text{ for } q = p^r.$$

At the end of the talk, a lot of it will still be a mystery—both in terms of what it *is* and what it *means*— but we will give the answer in a much larger range than was previously known.

1. Motivation from algebraic K-theory

One of the foundational results in algebraic K-theory is Quillen's computation of the K-theory of finite fields [Qui72]. Let $q = p^r$, then the algebraic K-theory groups $K_i(\mathbb{F}_q)$ of the finite field $\mathbb{F}_q$ with q elements are by definition the homotopy groups of an infinite loop space space $\Omega^\infty K(\mathbb{F}_q)$. What Quillen used what that the homology groups of $\Omega^\infty K(\mathbb{F}_q)$ can be computed in terms of the group homology of $\mathrm{GL}_n(\mathbb{F}_q)$ in the following manner.

Another way of writing $\mathrm{GL}_n(\mathbb{F}_q)$ is as $\mathrm{GL}(\mathbb{F}_q^n)$, the automorphisms of the n-dimensional vector space $\mathbb{F}_q^n$ over $\mathbb{F}_q$. The inclusion $\mathbb{F}_q^n \to \mathbb{F}_q^{n+1}$ onto the first n component is the inclusion of a direct summand, so we can define a homomorphism

$$\begin{aligned} \sigma\colon \mathrm{GL}_n(\mathbb{F}_q) &\longrightarrow \mathrm{GL}_{n+1}(\mathbb{F}_q) \\ T &\longmapsto T \oplus \mathrm{id}_{\mathbb{F}_q} \end{aligned}$$

Letting $\Omega^\infty_0 K(\mathbb{F}_q)$ denote a path component of $\Omega^\infty K(\mathbb{F}_q)$ (all are homotopy equivalent), there is an isomorphism

$$\operatorname*{colim}_{n\to\infty} H_*(\mathrm{GL}_n(\mathbb{F}_q)) \cong H_*(\Omega^\infty_0 K(\mathbb{F}_q)).$$

To understand the homology groups in the direct system, it suffices to compute them with $\mathbb{Q}$ and $\mathbb{F}_\ell$ coefficients for each prime power ℓ and reassemble them later. In fact, there are really three cases because ℓ coprime to p and $\ell = p$ behave differently.

1.1. **The case $\mathbb{Q}$.** The rational coefficient case is easy: each of the groups $\mathrm{GL}_n(\mathbb{F}_q)$ is finite so has trivial rational homology (by a transfer argument), so $\tilde{H}_*(B\mathrm{GL}_\infty(\mathbb{F}_q);\mathbb{Q}) = 0$.

1.2. **The case ℓ coprime to p.** In this case Quillen computed $H_*(\mathrm{GL}_n(\mathbb{F}_p);\mathbb{F}_\ell)$ for each n. The answer is that there is a ring structure on $\bigoplus_{n\geq 0} H_*(\mathrm{GL}_n(\mathbb{F}_q);\mathbb{F}_\ell)$ (which we will explain later), and as a bigraded ring it is isomorphic to

$$\mathbb{F}_\ell[\sigma,\xi_1,\xi_2,\ldots] \otimes \Lambda[\eta_1,\eta_2,\ldots]$$

Date: November 27, 2018.

for classes of bidegrees $|\sigma| = (1,0)$, $|\xi_i| = (t, 2it)$, and $|\eta_i| = (t, 2it-1)$, where t is the smallest number so that $q^t \equiv 1 \pmod{\ell}$.

1.3. **The case $\ell = p$.** This case is the topic of the remainder of the talk, and is much more subtle. The strategy to understanding $\mathrm{colim}_{n\to\infty} H_*(\mathrm{GL}_n(\mathbb{F}_q))$ is to first prove a vanishing result. I'm going to explain this in some level of detail, because we will use this type of argument later (it is explained in much more detail in [Spr15]).

As some of the arguments are a bit more stated in terms of cohomology, by the universal coefficient and Künneth theorems we might as well prove it there. Let us illustrate the proof of the vanishing result in the basic case $\mathrm{GL}_2(\mathbb{F}_q)$. Its order is $(q^2-1)(q^2-q)$, so a p-Sylow is given by the upper-triangular matrices U (for unipotent) of the form

$$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}.$$

This contains the upper-diagonal matrices K of the form

$$\begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix},$$

which hence have an index in $\mathrm{GL}_2(\mathbb{F}_q)$ coprime to p. This means that by a transfer argument the map $H^*(\mathrm{GL}_2(\mathbb{F}_q); \mathbb{F}_\ell) \to H^*(K; \mathbb{F}_q)$ is injective and so it suffices to prove that the latter vanishes in a range. In fact, K is a semi-direct product $\mathbb{F}_q^\times \ltimes \mathbb{F}_q$, so we can compute its homology using the Serre spectral sequence

$$E^2_{i,j} = H^i(\mathbb{F}_q^\times; H^j(\mathbb{F}_q; \mathbb{F}_\ell)) \Longrightarrow H^{i+j}(K; \mathbb{F}_\ell).$$

Since the order of $\mathbb{F}_q^\times$ is coprime to p, it has no higher cohomology groups when taking p-torsion coefficients. Thus the spectral sequence degenerates to a row of invariants

$$H^*(K; \mathbb{F}_\ell) \cong H^*(\mathbb{F}_q; \mathbb{F}_q\ell)^{\mathbb{F}_q^\times}.$$

It thus remains to compute $H^*(\mathbb{F}_q; \mathbb{F}_\ell)$ and the coinvariants of the action of $\mathbb{F}_q^\times$.

One case is easy: $q = 2$, as then $\mathbb{F}_q^\times = \{1\}$. We will see later that $q = 2$ behaves quite differently, and this is fundamentally a consequence of the lack of non-trivial units in $\mathbb{F}_2$.

Lemma 1.1. *$H^*(\mathbb{F}_q; \mathbb{F}_\ell)^{\mathbb{F}_q^\times} = 0$ for $0 < * < r(p-1)$.*

Proof. The details depend on whether $p = 2$ or p is odd. For convenience I'll do the former case, the latter being similar (but involving a few more generators), and we will tensoring up the coefficient so that $\ell \geq q$.

The cohomology $H^*(\mathbb{F}_q; \mathbb{F}_\ell)$ is the symmetric algebra on the linear dual of $\mathbb{F}_q \otimes_{\mathbb{F}_2} \mathbb{F}_\ell$, considered as being of degree 1. The reason we passed to $l \geq q$ is that the action of $\mathbb{F}_q^\times$ then diagonalizes (as the $\mathbb{F}_q$ contains the $q-1$st roots of unity); we can find a basis $x_0, \ldots, x_{r-1}$ of $\mathbb{F}_q \otimes_{\mathbb{F}_2} \mathbb{F}_q$ so that $\lambda \in \mathbb{F}_q^\times$ acting on x_i by λ^{2^i} (i.e. apply Frobenius i times to λ first). We see that invariants are pretty rare in positive degree: you need a monomial $x_0^{n_0} \cdots x_{r-1}^{n_{r-1}}$ so that $2^r - 1$ divides $n_0 + 2n_1 + \ldots + 2^{r-1}n_{r-1}$. A little algebra tells us the example of lowest positive degree is $n_0 = \ldots = n_{r-1} = 1$. We see that this vanishes for $* < r-1$. □

This argument generalizes for $n > 2$ by replacing H with a Borel and filtering the unipotent. This is not the strategy we used above, and if we did, we would have used instead of K the upper-diagonal matrices B of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix},$$

and would have gotten two copies of $\mathbb{F}_p^\times$ to take invariants with respect to. This roughly doubling the vanishing range, a lemma of Friedlander-Parshall [FP86]:

Lemma 1.2. *$\tilde{H}_*(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_\ell) = 0$ for $* < r(2p-3)$.*

To get information about the colimits, one uses any automorphism of $\mathbb{F}_q^n$ gives an automorphism of $\mathbb{F}_{q^s} \otimes_{\mathbb{F}_q} \mathbb{F}_q^n = \mathbb{F}_{q^s}^n$ and by restriction of scalars every automorphism of $\mathbb{F}_{q^s}^n$ gives one of $\mathbb{F}_q^{sn}$. Thus there are homomorphisms

$$\mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathrm{GL}_n(\mathbb{F}_{q^s}) \longrightarrow \mathrm{GL}_{ns}(\mathbb{F}_q).$$

Observe now that by increasing s, the middle term can be made to have $\mathbb{F}_\ell$-homology vanishing in an arbitarily large range.

Their composite is of course not iterated stabilization; it is instead s-fold block sum, which we denote $(-)^{\oplus s}\colon H_*(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_\ell) \to H_*(\mathrm{GL}_{ns}(\mathbb{F}_q); \mathbb{F}_\ell)$. However, a slightly more complicated argument still works:

(thinking of primitive elements in the stable cohomology as additive characteristic classes) the fact that this composition factors over a group whose homology vanishes in an large range—in fact, arbitrarily large by picking s large enough—tells us:

Theorem 1.3 (Quillen)**.** $\mathrm{colim}_{n\to\infty} \tilde{H}_*(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_p) = 0$.

Proof. Take a hypothetical non-zero $\beta \in \lim_{n\to\infty} \tilde{H}^*(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_\ell)$ of lowest positive degree d. This makes it automatically primitive with respect to the coproduct induced by block sum. For any $a \in H_d(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_\ell)$, that β is primitive implies that $\beta(a^{\oplus s}) = s\beta(a)$. Now that s coprime to p large enough so that $H_n(\mathrm{GL}_n(\mathbb{F}_{q^s}); \mathbb{F}_\ell) = 0$. Then $\beta(\alpha)$ is zero if and only $s\beta(\alpha)$ is, and the latter is true because $(-)^{\oplus s}\colon H_n(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_\ell) \to H_n(\mathrm{GL}_{ns}(\mathbb{F}_q); \mathbb{F}_\ell)$ factors over the trivial group $H_n(\mathrm{GL}_n(\mathbb{F}_{q^s}); \mathbb{F}_\ell)$. ☐

1.4. **Finishing the computation.** In some sense this finishes the discussion from the point of view of algebraic K-theory, the final answer being

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/(q^j - 1) & \text{if } i = 2j-1 \\ 0 & \text{if } i = 2j \text{ and non-zero.} \end{cases}$$

However, a big question remains:

Question 1.4. What are the homology groups $H_d(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_p)$?

2. Homological stability

Homological stability provides a strategy for answering this question. It suggest that one prove that the system of groups and homomorphisms

$$\cdots \longrightarrow \mathrm{GL}_n(\mathbb{F}_q) \xrightarrow{\sigma} \mathrm{GL}_{n+1}(\mathbb{F}_q) \xrightarrow{\sigma} \cdots$$

has the property that the relative homology groups $H_d(\mathrm{GL}_n(\mathbb{F}_q), \mathrm{GL}_{n-1}(\mathbb{F}_q); \mathbb{F}_p)$ vanish for $n \gg d$. If this is the case, then the homology of GL_n is independent of n for $n \gg d$, and hence in that range they are zero.

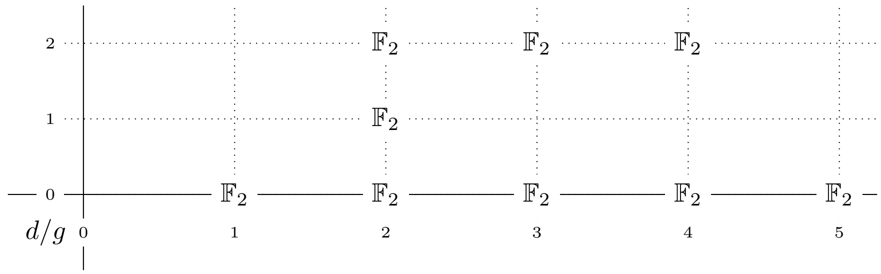

FIGURE 1. The additive generators of the $\mathbb{F}_2$-homology of $\mathrm{GL}_n(\mathbb{F}_2)$ for low degree and low rank. Empty entries are 0.

This raises two questions:

- Do these general linear groups exhibit homological stability? If so, what is the exact range of vanishing?
- What is the interpretation of the non-trivial unstable homology?

To see that these question are not vacuous, Figure 1 provides a table for $q = 2$.

In our paper we focused on the first question, as it is almost certainly a prerequisite to answering the latter. Nesterenko-Suslin have provided a beautiful answer to both question in the case of infinite fields [NS89]: if $\mathbb{F}$ is an infinite field, then $H_d(\mathrm{GL}_n(\mathbb{F}), \mathrm{GL}_{n-1}(\mathbb{F})) = 0$ for $d < n$ and the last non-vanishing relative group in degree d is $H_d(\mathrm{GL}_d(\mathbb{F}), \mathrm{GL}_{d-1}(\mathbb{F})) \cong K_d^M(\mathbb{F})$, the latter being the dth Milnor K-theory group. This is the associative algebra presented by generators $\mathbb{F}^\times$ and relations given by the Steinberg relations.

Their techniques do *not* apply to finite fields, so what happens in that case? As Quillen completely computed the homology with coefficients away from the characteristic, these question are all about $H_d(\mathrm{GL}_n(\mathbb{F}_q), \mathrm{GL}_{n-1}(\mathbb{F}_q); \mathbb{F}_p)$. The best vanishing range you can find in the literature is $d < n/2$ (due to Maazen), but in Quillen's unpublished notes you can find a proof for $d < n$ when $q \neq 2$ (unfortunately, the first pages are bleached, but Sprehn & Wahl are writing an exposition of that argument).

However, Nesterenko-Suslin predict we should be able to do better: the Milnor K-theory groups $K_*^M(\mathbb{F}_q)$ of $\mathbb{F}_q$ are $\mathbb{Z}$ if $* = 0$, $\mathbb{F}_q^\times$ is $* = 1$ and vanish otherwise. (You can deduce this from the fact that Milnor K-theory is generated by the units in degree 1 with the Steinberg relation in degree 2, which kills everything because we saw above that $K_2(\mathbb{F}_q)$ always vanishes.) In particular $K_*^M(\mathbb{F}_q) \otimes \mathbb{F}_p = 0$ for $* > 0$.

In fact, we will be able to get an even better result, *except when* $q = 2$.

Theorem 2.1 (Galatius-K.-Randal-Williams)**.** *If* $q = p^r \neq 2$*, then*

$$H_d(\mathrm{GL}_n(\mathbb{F}_q), \mathrm{GL}_{n-1}(\mathbb{F}_q); \mathbb{F}_p) = 0 \text{ for } d < n + r(p-1) - 2.$$

Remark 2.2. This result is optimal in some sense; if $r = 1$, then it is known that $H_{2p-2}(\mathrm{GL}_{p+1}(\mathbb{F}_p), \mathrm{GL}_p(\mathbb{F}_p); \mathbb{F}_p) = \mathbb{F}_p$, and this has $d = 2p-2$ and $n + r(p-1) - 2 = p + 1 + p - 1 - 2 = 2p - 2$, so there is exactly an equality here.

What happens when $q = 2$? This case is special, because $\mathbb{F}_2^\times = \{1\}$ implies that a number of groups of invariants/coinvariants that vanish for $q > 2$ do *not* vanish in this case. We already saw an example of this special behavior of $\mathbb{F}_2$ when we discussed Quillen's vanishing argument. This also explains the deviation from the Suslin-Nesterenko prediction, as in their argument for infinite fields such vanishing results play an important role. However, with some additional work we were still able to improve on Maazen:

Theorem 2.3 (Galatius-K.-Randal-Williams)**.** *$H_d(\mathrm{GL}_n(\mathbb{F}_2), \mathrm{GL}_{n-1}(\mathbb{F}_2); \mathbb{F}_2) = 0$ for $d < \frac{2(n-1)}{3}$.*

Though not as good as for the case $q > 2$, it has a consequence for a proposed method to find non-zero classes in $H^*(\mathrm{GL}_n(\mathbb{F}_2); \mathbb{F}_2)$. Milgram-Priddy observed that the determinant in

$$H^n(M_{n,n}(\mathbb{F}_2); \mathbb{F}_2)^{\mathrm{GL}_n(\mathbb{F}_2)\times \mathrm{GL}_n(\mathbb{F}_2)}$$

is in the image of restriction from $\mathrm{GL}_{2n}(\mathbb{F}_2)$ for $n = 1, 2$ and asked whether it is for all $n \geq 3$ [MP87]. Suppose this to be true, the groups $H_n(\mathrm{GL}_{2n}(\mathbb{F}_2); \mathbb{F}_2)$ lying on a line of slope $1/2$ would have to be non-zero. However, our stability result says that $H_d(\mathrm{GL}_n(\mathbb{F}_2); \mathbb{F}_2)$ attains its stable value 0 below a line of slope $2/3$. Thus the Milgram-Priddy question has a negative answer for n sufficiently large. In fact, the third determinant class is still non-zero, but further ones are 0.

3. The proof using E_∞-cells

We shall now explain the proof of the above two theorems, which relies on the techniques developed in [GKRW18a] and applied to mapping class groups in [GKRW18b].

3.1. E_∞-algebras.

Let us consider the space

$$\bigsqcup_{n\geq 0} B\mathrm{GL}_n(\mathbb{F}_q).$$

It arises as the classifying space of the nerve of the groupoid of finite-dimensional $\mathbb{F}_q$-vector spaces and their isomorphisms. Direct sum makes this into a symmetric monoidal groupoid, and this endows $\bigsqcup_{n\geq 0} B\mathrm{GL}_n(\mathbb{F}_q)$ with the structure of an E_∞-algebra, which we shall denote $\mathbf{R}$. This is the homotopy-theoretic version of a commutative monoid.

The homotopy-theoretic versions of abelian groups are infinite loop spaces, or equivalently connective spectra. Just like a commutative monoid can be group completed to an abelian group, an E_∞-algebra can be group-completed to a infinite loop space (this is denoted $\Omega B(-)$). Using this group completion construction gives the space $\Omega^\infty K(\mathbb{F}_q)$ mentioned before:

$$\Omega B(\mathbf{R}) \simeq \Omega^\infty K(\mathbb{F}_q).$$

What is the advantage of knowing that $\mathbf{R}$ has the additional algebraic structure of E_∞-algebra? Firstly, we can define stabilization in terms of it: picking an point σ in $B\mathrm{GL}_1(\mathbb{F}_q)$ and multiplying with it using the product of the E_∞-algebra structure give rise to a map

$$\sigma \cdot - \colon B\mathrm{GL}_n(\mathbb{F}_q) \longrightarrow B\mathrm{GL}_{n+1}(\mathbb{F}_q)$$

homotopic to the stabilization map. From this point of view, it is helpful to keep track of the components of $\mathbf{R}$, that is, considering it as an E_∞-algebra over $\mathbb{N}$, or equivalently as an E_∞-algebra in $\mathsf{sSet}^{\mathbb{N}}$.

The second advantage is that free E_∞-algebras exhibited homological stability, and in fact all their homology with $\mathbb{F}_p$-coefficients is known due to a computation by F. Cohen [CLM76]; they are free Dyer-Lashof algebras. In particular, $H_*(F^{E_\infty}(X); \mathbb{F}_p)$ is the free graded-commutative algebra on iterated Dyer-Lashof operations on $H_*(X; \mathbb{F}_p)$. From this you can easily read off that free E_∞-algebras (of path-connected X) exhibit homological stability.

This suggests the following strategy for proving our theorems: build $\mathbf{R}$ out of free E_∞-algebra in a controlled manner. Here "building" means constructing $\mathbf{R}$ by iterated E_∞-cell attachments. The free E_∞-algebra functor F^{E_∞} is the left adjoint to the functor U^{E_∞} forgetting the E_∞-algebra structure, and given a diagram

$$\begin{array}{ccc} \partial D^{n,d} & \xrightarrow{e} & U^{E_k}(\mathbf{R}) \\ \big\downarrow & & \\ D^{n,d} & & \end{array}$$

with $D^{n,d}$ denote the d-disk in grading $n \in \mathbb{N}$, we get an adjoint diagram in $\mathsf{Alg}_{E_\infty}(\mathsf{sSet}^{\mathbb{N}})$ of which we can take the pushout

$$\begin{array}{ccc} F^{E_\infty}(\partial D^{n,d}) & \longrightarrow & \mathbf{R} \\ \big\downarrow & & \big\downarrow \\ F^{E_\infty}(D^{n,d}) & \longrightarrow & \mathbf{R} \cup_e^{E_\infty} D^{n,d}. \end{array}$$

This is what we call an *E_∞-cell attachment*.

3.2. **Using E_∞-cells.** We need to able to do two things to make this useful:

- Calculate the effect of such a E_∞-cell attachment on homology.
- Calculate which E_∞-cells we need to build $\mathbf{R}$.

For the former, there is a number of spectral sequences. Let us instead focus on the former, which is where most of the action happens. How do we recover X from $F^{E_\infty}(X)$? Here we must switch to the non-unital (or augmented) setting, which in our examples mean throwing out the rank 0 component. If we additionally decide to work in a pointed setting, there is another construction of a non-unital E_∞-algebra from X, in addition to the free non-unital E_∞-algebra construction. This is the *trivial algebra* $Z^{E_\infty^{\mathrm{nu}}}(X)$, with underlying object given by X and all (≥ 2)-ary operations map to the basepoint. It has a left adjoint $Q^{E_\infty^{\mathrm{nu}}} \colon \mathsf{Alg}_{E_\infty^{\mathrm{nu}}}(\mathsf{C}) \to \mathsf{C}$ of E_∞-indecomposables. Since $U^{E_\infty^{\mathrm{nu}}} Z^{E_\infty^{\mathrm{nu}}} = \mathrm{id}$, so is $Q^{E_\infty^{\mathrm{nu}}} F^{E_\infty^{\mathrm{nu}}}$.

Definition 3.1. Suppose that $\mathbf{R}$ is a non-unital E_∞-algebra in $\mathsf{sSet}_*^{\mathbb{N}}$, then *$E_\infty$-homology groups* $H_{n,d}^{E_\infty}(\mathbf{R})$ of $\mathbf{R}$ are the homology groups of the derived functor of $Q^{E_\infty^{\mathrm{nu}}}$ applied $\mathbf{R}$.

Just like for ordinary homology, in a 1-connected or additive setting homology detects minimal cell decompositions. As we are interested in homology in the end, we might as well look at $\mathbf{R}' \coloneqq \mathbb{F}_p[\mathbf{R}] \in \mathsf{Alg}_{E_\infty}(\mathsf{sMod}_{\mathbb{F}_p}^{\mathbb{N}})$ and apply this observation.

So to understand how to build $\mathbf{R}'$ out of free non-unital E_∞-algebras, we need to be able to compute $H^{E_\infty}_{n,d}(\mathbf{R}')$. This is done by computing $H^{E_1}_{n,d}(\mathbf{R}')$ instead. Indeed, every E_∞-algebra is an E_1-algebra, just like every commutative algebra is an associative algebra. The suspension of E_1-homology can be computed by a bar construction, and the suspension spectrum of E_∞-homology by an infinite bar spectrum. Thus a vanishing line for E_1-homology can be transfered to one for E_∞-homology.

Finally, taking advantages of the fact that both bar constructions and the construction of $\mathbf{R}$ are homotopy colimits, one can compute the bar construction $\mathbf{R}$ in terms of our symmetric monoidal groupoid. Hence the bar construction in each rank n takes the form of the homotopy quotient of some $\mathrm{GL}_n(\mathbb{F}_q)$-space by a $\mathrm{GL}_n(\mathbb{F}_q)$.

Proposition 3.2. *$H^{E_1}_{n,d}(\mathbf{R}') \cong \tilde{H}_{d-1}(S^{E_1}(\mathbb{F}_q^n) /\!\!/ \mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_p)$, where $S^{E_1}(\mathbb{F}_q^n)$ is the semisimplicial set k-simplices given by ordered direct sum decomposition $V_0 \oplus \cdots \oplus V_k$ of $\mathbb{F}_q^n$ with all V_i's non-zero, and face maps merging adjacent terms.*

We call $S^{E_1}(\mathbb{F}_q^n)$ the E_1*-split building*, in analogy with the *Tits building* $\mathcal{T}(\mathbb{F}_q^n)$ which has k-simplices given by flags decomposition $0 \subsetneq F_0 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq \mathbb{F}_q^n$. The Tits building is well-known to have two nice properties:

- it is $(n-2)$-spherical (i.e. homotopy equivalent to a wedge of such spheres) and,
- for $n \geq 2$ its reduced top homology group with $\mathbb{F}_p$-coefficients $\mathrm{St}(\mathbb{F}_q^n) \coloneqq \tilde{H}_{n-2}(\mathcal{T}(\mathbb{F}_q^n); \mathbb{F}_p)$ is an irreducible projective $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_q)]$-module called the *Steinberg module*. In particular $H_*(\mathrm{GL}_n(\mathbb{F}_q); \mathrm{St}(\mathbb{F}_q^n)) = 0$.

We can compare the E_1-split building to the Tits building using the map

$$\begin{aligned} \mathcal{S}^{E_1}(\mathbb{F}_q^n) &\longrightarrow \mathcal{T}^{E_1}(\mathbb{F}_q^n) \\ V_0 \oplus \cdots \oplus V_k &\longmapsto 0 \subsetneq V_0 \subsetneq V_0 \oplus V_1 \subsetneq \ldots \subsetneq \mathbb{F}_q^n. \end{aligned}$$

Using one can prove that

- $S^{E_1}(\mathbb{F}_q^n)$ is also $(n-2)$-spherical, and
- for $n \geq 2$ its reduced top homology group with $\mathbb{F}_p$-coefficients $\mathrm{St}^{E_1}(\mathbb{F}_q^n) \coloneqq \tilde{H}_{n-2}(\mathcal{S}^{E_1}(\mathbb{F}_q^n); \mathbb{F}_p)$ satisfies $H_*(\mathrm{GL}_n(\mathbb{F}_q); \mathrm{St}^{E_1}(\mathbb{F}_q^n)) = 0$ for $* < r(p-1)-1$.

The latter has two ingredients: (i) the corresponding result for the Steinberg module kills of one contribution, (ii) the type of vanishing results that Quillen used kills of the others.

Example 3.3. Let us look at the case $n = 2$. Let $\tilde{\mathbb{F}}_p[X]$ denote the reduced free abelian group on a set (so the sum of coefficient is 0), we have

$$\begin{aligned} \mathrm{St}(\mathbb{F}_q^2) &= \tilde{\mathbb{F}}_p[\text{lines in } \mathbb{F}_q^2], \\ \mathrm{St}^{E_1}(\mathbb{F}_q^2) &= \tilde{\mathbb{F}}_p[\text{splittings of } \mathbb{F}_q^2]. \end{aligned}$$

There is a short exact sequence

$$0 \longrightarrow \bigoplus_{\text{lines } L} \tilde{\mathbb{F}}_p[\text{complements to } L] \longrightarrow \tilde{\mathbb{F}}_p[\text{lines}] \longrightarrow \tilde{\mathbb{F}}_p[\text{splittings}] \longrightarrow 0.$$

Since the homology of $\mathrm{GL}_2(\mathbb{F}_q)$ with coefficients in the Steinberg vanishes, an application of Shapiro's lemma says

$$H_*(\mathrm{GL}_2(\mathbb{F}_q);\tilde{\mathbb{F}}_p[\text{splittings}]) \cong H_*(\mathrm{Stab}(L);\tilde{\mathbb{F}}_p[\text{complements to } L]),$$

where $\mathrm{Stab}(L)$ is ismorphic to the Borel of matrices

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

we encountered before.

Let us use another long exact sequence

$$0 \longrightarrow \tilde{\mathbb{F}}_p[\text{complements to } L] \longrightarrow \mathbb{F}_p[\text{complements to } L] \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

Another application of Shapiro tells us that $H_*(\mathrm{Stab}(L);\mathbb{F}_p[\text{complements to } L]) \cong H_*((\mathbb{F}_q^\times)^2;\mathbb{F}_p)$, which vanishes for $* > 0$. On the other hand, we saw that $H_*(\mathrm{Stab}(L);\mathbb{F}_p) = 0$ for $0 < * < r(2p-3)$. This proves that

$$H_*(\mathrm{Stab}(L);\tilde{\mathbb{F}}_p[\text{complements to } L]) = 0 \text{ for } * < r(2p-3)-1.$$

This is a larger range than claimed above.

Returning to E_1-homology, we conclude that:

Corollary 3.4. *For $n \geq 2$*

$$H^{E_1}_{n,d}(\mathbf{R}') \cong H_{d-n+1}(\mathrm{GL}_n(\mathbb{F}_q);\mathrm{St}^{E_1}(\mathbb{F}_q^n)),$$

which vanishes for $d < r(p-1)+n-2$.

This tells us all E_1-cells with the exception of a single cell σ in bidegree $(1,0)$ live on or above the line $d = r(p-1)+n-2$. It may seen we are done now, but unfortunately this single σ makes it hard to do the bar spectral sequences to get E_∞-homology.

Instead it is more convenient to do the following. There is a non-unital E_∞-algebra $\mathbf{N}'$, which is $\mathbb{F}_p$ in rank > 0 and 0 in rank 0 and a unique map

$$f\colon \mathbf{R}' \longrightarrow \mathbf{N}'$$

of E_∞-algebras.

It is easy to prove its relative E_1-homology groups vanish for $d < r(p-1)+n-1$ *without the restriction $n \geq 2$*. The vanishing line transfers without issues and we get $H^{E_\infty}_{d,n}(f)$ vanishes in the same range. These groups control the E_∞-cells needed for a relative CW-approximation of f: it has we only need to use cells on or above the line $d = r(p-1)+n-1$. They turned out to affect relative homology at most a single degree below this line and thus we get that

$$H_d(\mathbf{R}(n),\mathbf{R}(n-1);\mathbb{F}_p) \cong H_d(\mathbb{N}(n),\mathbb{N}(n-1);\mathbb{F}_p)$$

for $d < r(p-1)+n-2$. The latter of course vanishes, and this gives our improved homological stability result when $q \neq 2$. The case $q = 2$ requires a different custom-built E_∞-algebra to compare $\mathbf{R}'$ to.

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