Formality of the E_n-operad

Lucy Yang

February 27, 2020

Contents

1 Introduction

Throughout, k will denote a field of characteristic 0 (usually $\mathbb{R}$ or $\mathbb{Q}$).

Definition. A nilpotent, or k-good in the sense of Bousfield-Kan space X is *formal* (resp. *stably formal*) over k if $H^*(X;k) \simeq \mathcal{A}_{PL}(X;k)$ (resp. $H_*(X;k) \simeq C_*(X;k)f$) are quasi-isomorphic *as cdgas* (resp. as chain complexes).

An operad $\{\mathcal{O}(n)\}$ in **Spc** is *formal* (over k) if $\mathcal{A}_{PL}(\mathcal{O}(n)) \otimes k$ and $H^*(\mathcal{O}(n);k)$ are quasi-isomorphic as cooperads of cdgas.

This talk will sketch a proof of the

Theorem (Kontsevich, Lambrechts-Volić)**.** *For $n \geq 3$, the little disks operad E_n is formal over $\mathbb{R}$.*

Outline

$$\mathrm{SemiAlgSets} \hookrightarrow \mathbf{Spc} \xrightarrow{\mathcal{A}_{PL}} \mathbf{cDGA}^{\mathrm{op}}, \qquad \Omega_{PA} : \mathrm{SemiAlgSets} \to \mathbf{cDGA}^{\mathrm{op}}$$

$$\begin{array}{ccccc}
 & & E_n & \longmapsto & \mathcal{A}_{PL}(E_n;\mathbb{Q}) \\
 & & \wr\!\mid & & \wr\!\mid \\
C[-] & = & C[-] & \longmapsto & \Omega_{PA}(C[-];\mathbb{Q}) \\
 & & & & \wr\!\mid \\
 & & & & \mathcal{D}(-) \\
 & & & & \wr\!\mid \\
 & & & & H^*(C[-];\mathbb{Q})
\end{array}$$

So what I am *not* going to do is

- Develop the theory of semialgebraic sets–in a nutshell, they have a good theory of 'piecewise algebraic forms' which allow pushforwards along 'semi-algebraic bundles'. See [3] for more.
- Define Ω_{PA} and the natural equivalence $\mathcal{A}_{PL} \xrightarrow{\simeq} \Omega_{PA}$
- Worry about the Künneth quasi-isomorphism $\mathcal{A}_{PL}(X) \otimes \mathcal{A}_{PL}(Y) \xrightarrow{\sim} \mathcal{A}_{PL}(X \times Y)$ *going the wrong way.*

What I am going to do is

1. Define the Fulton-MacPherson operad $C[-]$ and talk about some of its nice properties.
2. Define Kontsevich's operad $\mathcal{D}(-)$ of admissible diagrams.
3. Show that $\mathcal{D}(-)$ is quasi-isomorphic to $H^*(C[-])$.
4. Define the configuration space integral $I : \mathcal{D}(-) \to \Omega_{PA}(-)$ and show that it is a quasi-isomorphism.

Notation A will always be a finite set, $n \geq 3$ will denote a fixed ambient dimension, k a field.

2 The Fulton-MacPherson operad

Let $C(A) := Conf(A, \mathbb{R}^n)/\mathbb{R}^n \rtimes \mathbb{R}_{>0}$, where the factor of $\mathbb{R}^n$ acts by translation and $\mathbb{R}_{>0}$ acts by dilation. Observe that this space is homeomorphic to the space of configurations with barycenter the origin and radius 1:

$$C(A) \cong \left\{ x : A \to \mathbb{R}^n \mid \bar{x} := \frac{1}{|A|} \sum_a x(a) = 0 \qquad \max_{a \in A} \{x(a) - \bar{x}\} = 1 \right\}.$$

For distinct $a, b, c \in A$, we define two functions, normalized direction and relative distance:

$$\begin{aligned} \theta_{ab} : C(A) &\to S^{n-1} \\ x &\mapsto \frac{x(a) - x(b)}{|x(a) - x(b)|} \end{aligned} \qquad \begin{aligned} \delta_{abc} : C(A) &\to [0, \infty] \\ x &\mapsto \frac{|x(a) - x(b)|}{|x(a) - x(c)|}. \end{aligned}$$

Definition. Let $A^n \setminus \Delta$ denote the set of n-tuples with distinct entries and consider the map

$$\begin{aligned} \iota : C(A) &\longrightarrow (S^{n-1})^{A^2 \setminus \Delta} \times ([0, \infty])^{A^3 \setminus \Delta} \\ x &\longmapsto ((\theta_{a,b}), (\delta_{a,b,c})) \end{aligned}$$

The *Fulton-MacPherson compactification* $C[A]$ of $C(A)$ is the closure of the image of ι, i.e. $\overline{\iota(C(A))} =: C[A]$.

Idea. We are allowing points to be 'infinitesimally close' to each other, but by remembering the relative directions between the points labelled by $a \neq b$, the compactification doesn't cause any 'collapse' or change in homotopy type.

We abuse terminology by calling $y \in C[A]$ a 'configuration.'

Proposition. *$C[A]$ is a compact semi-algebraic manifold with interior $C(A)$ and*

$$\dim C[A] = \begin{cases} 0 & |A| \leq 1 \\ n|A| - n - 1 & |A| \geq 2 \end{cases}.$$

Proposition. *For $x \in C[A]$, the following are equivalent: $x \in \partial C[A] \iff \exists a, b, c \in A$ distinct such that $x(a) \simeq x(b)$ rel $x(c)$, i.e. $\delta_{a,b,c}(x) = 0$.*

THE OPERAD STRUCTURE Suppose given a map of sets $\nu : A \to P$ with P ordered[1] and let $A_p = \nu^{-1}(p)$.

$$\begin{gathered} C[P] \times \Pi_{p \in P} C[A_p] \to C[A] \\ (x_0, (x_p)_{p \in P}) \end{gathered}$$

Idea. Replace $x_0(p)$ by the configuration x_p "made infinitesimal."

Note that the unit is given by the unique point $\{*\} \simeq C[1]$.

Proposition. *[5] The Fulton-MacPherson operad $C[-]$ of configurations in $\mathbb{R}^n$ and the little n-disks operad are weakly equivalent as topological operads.*

[1] It has been pointed out that one doesn't need an ordering on P to define the operad structure–this will come in later when we want to put an orientation on the fiber of the structure map.

The canonical projections Given an inclusion $A \subset V$, there is a canonical projection ("forget points labelled by $V \setminus A$"), which is compatible with the induced operad structure map:

$$\begin{array}{ccc} C[V] & \longrightarrow & C[A] \\ \wr\!\! \mid & \nearrow & \\ C[V] \times \Pi_{v \in V} C[A_v] & & \end{array}$$

The Kontsevich configuration space integral is defined via a pushforward of certain semi-algebraic forms along these canonical projections.

Theorem. *Let A finite set and I linearly ordered. Then*

$$\pi \colon C[A \cup I] \to C[A]$$

is an oriented semi-algebraic bundle with fiber of dimension

$$\dim \ \mathit{fib}\ \pi \begin{cases} = n \cdot |I| & |A| \geq 2 \ \mathit{or}\ I = \varnothing \\ < n \cdot |I| & \mathit{otherwise} \end{cases}$$

We can consider the subbundle $C^{\partial}[A \cup I] \to C[A]$ given by, for $x \in C[A]$, $C^{\partial}[A \cup I]_x = \partial(C[A \cup I]_x)$. This is the *fiberwise boundary.*

Decomposition of fiberwise boundary $\partial C[A]$

Idea. Most of the operad structure on $C[-]$ can be understood as an explicit decomposition of the boundary of $C[-]$ as a union of faces which are homeomorphic to products of the form $C[n] \times C[k_1] \times \cdots \times C[k_n]$.

Let $V = A \cup I$. For each $W \subsetneq V$, we have

$$\Phi_W \colon C[V/W] \times C[W] \to C[V]$$

where the image of Φ_W consists of configurations in which all $w \in W$ are infinitesimally close to one another. When $|W| \geq 2$ and either $W \subseteq A$ or $|W \cap A| \neq 1$, the image of Φ_W lies in the fiberwise boundary $C^{\partial}[V]$. Then we have the decomposition

$$C^{\partial}[A \cup I] = \bigcup_W \mathrm{Im}\ \Phi_W$$

Fact. For distinct W, Z, $\mathrm{Im}\ \Phi_W, \mathrm{Im}\ \Phi_Z$ intersect in strictly smaller dimension.

3 The cdga of admissible diagrams

Strategy Define a cdga cooperad structure on all diagrams $\widetilde{\mathcal{D}}$ and shows it descends to a class of admissible diagrams $\mathcal{D}$ later.

Definition. A *diagram* Γ *on* A is a finite oriented graph with internal I_Γ and external A_Γ vertices and such that the sets of edges E_Γ and internal vertices I_Γ are linearly ordered. We write $s, t : E_\Gamma \to A_\Gamma \sqcup I_\Gamma$ for the source, target respectively.

An edge is

- a *chord* if its endpoints are external
- a *dead-end* if one of its endpoints has only one neighbor[2]
- *contractible* if it is not a chord, dead-end, or a loop

The space of diagrams $\widetilde{\mathcal{D}}(A)$ on A is the free k-module generated by isomorphism classes of diagrams Γ on A modulo the relations

- $\Gamma = (-1)^n\Gamma'$ if Γ, Γ' differ by inversion of one edge (and the ordering on $E_\Gamma = E_{\Gamma'}$ stays the same)
- $\Gamma = (-1)^n\Gamma'$ if Γ, Γ' differ by a transposition in the order of internal vertices
- $\Gamma = (-1)^{n-1}\Gamma'$ if Γ, Γ' differ by a transposition in the order of edges

Note that implicitly, the definition $\tilde{\mathcal{D}}(-)$ depends on n, it's just suppressed from notation. In the following, when a fixed diagram is understood, I write E, I instead of E_Γ, I_Γ for ease of notation.

Definition. The *degree* of a diagram Γ is given by

$$\deg \Gamma = (n-1)|E_\Gamma| - n|I_\Gamma|.$$

Remark. The degree is compatible with the equivalence relation above, hence $\widetilde{\mathcal{D}}(A)$ is a graded k-module.

Definition. The product diagram of $\Gamma_1, \Gamma_2 \in \widetilde{\mathcal{D}}(A)$ is given by "gluing them together along A," i.e.

- $A_{\Gamma_1 \cdot \Gamma_2} = A$
- $E_{\Gamma_1 \cdot \Gamma_2} = E_{\Gamma_1} \otimes E_{\Gamma_2}$, i.e. for all $e_1 \in E_{\Gamma_1}, e_2 \in E_{\Gamma_2}$, $e_1 < e_2$.
- $I_{\Gamma_1 \cdot \Gamma_2} = I_{\Gamma_1} \otimes I_{\Gamma_2}$

and where the source and target maps are given by their restrictions to $E_{\Gamma_1}, E_{\Gamma_2}$.

Remark. The *unit* diagram is the one with no edges or internal vertices.

[2]not the same as univalent! Since there might be loops or double/triple/etc edges.

Proposition. *This extends to a degree 0 linear map*

$$\widetilde{\mathcal{D}}(A) \otimes \widetilde{\mathcal{D}}(A) \to \widetilde{\mathcal{D}}(A)$$

For the following definition, we use the convention that the vertices in a diagram are always ordered such that $a < i$ for $a \in A, i \in I$.

Definition. Let Γ a diagram and e a contractible edge of Γ. The diagram Γ/e obtained from Γ by *contracting the edge* e is the diagram given by

$$\begin{aligned} V_{\Gamma/e} &= V_\Gamma \setminus \{\max\{s(e), t(e)\}\} \\ E_{\Gamma/e} &= E_\Gamma \setminus \{e\} \end{aligned}$$

We write $\bar{f}$ for the image of $f \in E_\Gamma$ in Γ/e.

Define the *differential* of a diagram to be

$$d\Gamma = \sum_{e \in E_\Gamma^{contr}} \varepsilon(\Gamma, e) \cdot \Gamma/e$$

where

$$\varepsilon(\Gamma, e) = \begin{cases} (-1)^{pos(t(e);I)} & s(e) < t(e) \text{ and } M \text{ odd} \\ -(-1)^{pos(s(e);I)} & s(e) > t(e) \text{ and } M \text{ odd} \\ (-1)^{pos(e;E)} & M \text{ even} \end{cases}.$$

Lemmas.
1. *d defines a linear map $\widetilde{\mathcal{D}}(A) \to \widetilde{\mathcal{D}}(A)$*
2. *d is homogenous of degree $+1$.*
3. *(Liebniz rule)*

$$d(\Gamma \cdot \Gamma') = d(\Gamma) \cdot \Gamma' + (-1)^{\deg \Gamma} \Gamma \cdot d(\Gamma')$$

4. *(chain map) $d^2 = 0$*

Sketch of 4. Note that $\bar{e}_2$ is contractible in $\Gamma/e_1 \iff \bar{e_1}$ is contractible in Γ/e_2, and $(\Gamma/e_1)/\bar{e}_2 \cong (\Gamma/e_2)/\bar{e_1}$. It remains to check that a sign vanishes.

Taken together, the lemmas above imply the

Theorem. $\left(\widetilde{\mathcal{D}}(A), d\right)$ *is a commutative differential graded algebra.*

3.1 Admissible diagrams

Definition. A diagram is *admissible* if it has no loops, double edges, internal vertices of valance ≤ 2, and each internal vertex is connected (in the topological sense, i.e. regarding Γ as a CW complex) to an external vertex.

Remark. An admissible graph does not have dead ends.

Lemma. *The module of non-admissible diagrams is a differential ideal, i.e.*

$$d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$$

Definition. The *cDGA of admissible diagrams* is the quotient

$$\mathcal{D}(A) := \widetilde{\mathcal{D}}(A)/\mathcal{N}(A)$$

A cochain complex is *connected* if it is

- concentrated in nonnegative degrees
- isomorphic to k in degree 0 .

Proposition. *If $n \geq 3$, then $\mathcal{D}_n(A)$ is a connected cDGA.*

Proof. Suppose Γ a nontrivial diagram–want to show that it has positive degree. Consider "half-edges." Since each internal vertex is of valance ≥ 3, there are $\geq 3|I|$ half-edges, and since Γ is nontrivial there is at least one half-edge connecting {internal vertices} to an external vertex.

$$\begin{aligned} |E| &\geq \frac{1}{2}(3|I| + 1) \\ \deg \Gamma = |E|(n-1) - |I|n &\geq \frac{1}{2}(3|I| + 1)(n-1) - |I|n \\ &= \frac{n-1}{2} + |I|\frac{n-3}{2} > 0 \end{aligned}$$

when $n \geq 3$ and $|I| \geq 0$. □

3.2 Cooperad structure

Let $A \subset V$, i.e. $V = A \sqcup I$. Fix a map $\nu : A \to P$ and consider a map $\lambda : V \to P$ which agrees with ν on A[3].

$$\begin{aligned} \widetilde{\Psi}_\nu : \widetilde{\mathcal{D}}(A) &\to \widetilde{\mathcal{D}}(P) \otimes \bigotimes_{p \in P} \widetilde{\mathcal{D}}(A_p) \\ \Gamma &\mapsto \sum_\lambda \pm[p^{th} \text{ clusters collapsed}] \otimes (p^{th} \text{ cluster}) \end{aligned}$$

where the sum is over all λ satisfying the above.

[pictures]

The sign is given by

$$\begin{aligned} S(I, \lambda) &= \{(v, w) \in I^2 \mid v < w \text{ and } \lambda(v) > \lambda(w)\} \\ S(E, \lambda) &= \{(e, f) \in E^2 \mid e < f \text{ and } \lambda(e) > \lambda(f)\} \\ \varepsilon(\Gamma, \lambda) &= (-1)^{n \cdot |S(I,\lambda)| + (n-1)|S(E,\lambda)|} \end{aligned}$$

[3] [4] refer to these maps as 'condensations'.

Proposition. *This descends to a well-defined linear cooperad structure map*

$$\Psi_\nu : \mathcal{D}(A) \to \mathcal{D}(P) \otimes \bigotimes_{p\in P} \mathcal{D}(A_p).$$

4 Equivalence of the cooperads $\mathcal{D}(-)$ and $H^*(C[-])$

Recall: from Dexter's talk that we have

$H^*(C[A];k) \simeq \bigwedge\{g_{ab}\}_{a\neq b, a,b\in A}/g_{ab}^2 = 0 \quad g_{ab} = (-1)^n g_{ba} \quad g_{ab}g_{bc}+g_{bc}g_{ca}+g_{ca}g_{ab}$

where $g_{ab} = \theta_{ab}^*(\mathrm{vol}) \in H^{n-1}(C[A];k)$.

For $a \neq b$, let $\Gamma\langle a,b\rangle$ denote the diagram with a single chord from a to b with no internal vertices or other edges.

Theorem. *For $n \geq 2$, there is a quasi-isomorphism of cdgas ($\mathbb{Z}$-graded if $n = 2$)*

$$\begin{aligned} J : \mathcal{D}(A) &\to H^*(C[A];k) \\ \Gamma\langle a,b\rangle &\mapsto g_{ab} \\ \Gamma &\mapsto 0 \qquad\qquad \textit{if } \Gamma \textit{ has an internal vertex.} \end{aligned}$$

First, need to show the map is well-defined, i.e. $J(d(-)) = 0$. The Arnold relation comes from the diagram with a single internal vertex which is connected to all external vertices. Since J is surjective on homology, we show that it induces an isomorphism on homology by computing dimension in each degree. The dimension computation involves induction on $|A|$–here it is important to distinguish integer partitions and set partitions.

5 The Kontsevich configuration space integrals

Goal. Construct a cDGA morphism

$$I : \mathcal{D}(A) \to \Omega_{PA}(C[A])$$

which is a quasi-isomorphism and "almost" a morphism of cooperads.

From now on, let Γ be a diagram on A. Let $\mathrm{vol} \in \Omega^{n-1}(S^{n-1})$ be the standard normalized volume form on the sphere $S^{n-1} \subset \mathbb{R}^n$. For every linearly ordered finite set E, let

$$\mathrm{vol}_E = \times_{e\in E}\mathrm{vol}_e \in \Omega^{|E|(n-1)}_{min}((S^{n-1})^{|E|})$$

where the product is taken in the order on e. Recall that given two vertices $v, w \in V$, there is a map $\theta_{v,w} : C[V] \to S^{n-1}$. We use the convention that $\theta_{v,v}$ is the constant map to a basepoint $* \in S^{n-1}$. For an edge $e \in E_\Gamma$, let $\theta_e := \theta_{s(e),t(e)}$. Then define the map

$$\theta_\Gamma = (\theta_e)_{e\in E} : C[V] \to (S^{n-1})^E$$

Furthermore, recall that we have a canonical projection $\pi_\Gamma : C[V_\Gamma] \to C[A]$.

Definition. The *Kontsevich configuration space integral* $\tilde{I}$ is given by: if $|A| \geq 2$,

$$\tilde{I}(\Gamma) = \pi_{\Gamma *}\theta_\Gamma^*(\mathrm{vol}_E) \in \Omega_{PA}(C[A])$$

and if $|A| \leq 1$, then

$$\tilde{I}(\Gamma) = \begin{cases} 1 & \Gamma \text{ a unit} \\ 0 & \text{otherwise} \end{cases}.$$

Lemma. *$\tilde{I}$ defines, for any finite set A, a degree zero linear map*

$$\tilde{I} : \tilde{D}(A) \to \Omega_{PA}(C[A])$$

5.1 I is a map of cdgas

Proposition. *$\tilde{I}$ is a morphism of algebras, i.e. $\tilde{I}(\Gamma_1 \cdot \Gamma_2) = \tilde{I}(\Gamma_1)\tilde{I}(\Gamma_2)$.*

Proposition. *$\tilde{I}(\mathcal{N}(A))$, i.e. $\tilde{I}$ vanishes on nonadmissible diagrams.*

Corollary. *$\tilde{I}$ descends to a map of algebras*

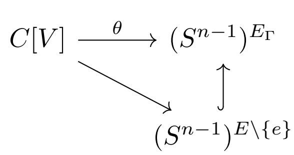
$$I : \mathcal{D}(A) \to \Omega_{PA}(C[A])$$

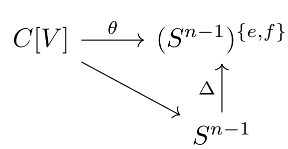
The latter proposition follows from

Lemmas. *$\tilde{I}$ vanishes on diagrams*

1. *with loops*
2. *with double edges*
3. *containing an internal vertex not connected to any external vertices*
4. *containing a univalent internal vertex*
5. *containing a bivalent internal vertex.*

Sketch of 1, 2. These both follow from noticing that θ_Γ factors through a lower dimensional space, e.g.

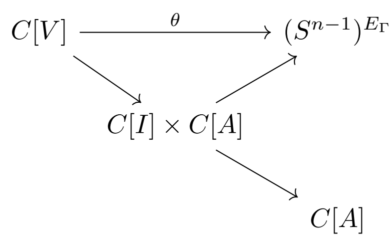




and the

Fact. Let $\alpha \in \Omega_{PA}(X)$. If $\deg \alpha > \dim X$, then $\alpha = 0$.

Sketch of 3, 4. These follow from recognizing that θ factors through a canonical projection, e.g. for 3, wlog assume (by multiplicativity) that no internal vertices are connected to external vertices. Then



and we have the

Fact. Let $\pi : E \to B$ an oriented semialgebraic bundle which factors as $E \xrightarrow{\rho} Z \to qB$. Suppose that $\dim q^{-1}(b) < \dim \pi^{-1}(b)$ pointwise. Then for any $\mu \in \Omega_{min}(Z)$, we have $\pi_*(\rho^*(\mu)) = 0$.

The result follows by comparing the dimensions of the fibers of $C[V] \to C[A]$ and $C[I] \times C[A] \twoheadrightarrow C[A]$. The proof of 4. is similar.

Sketch of 5. Suppose i the internal vertex of valance 2. The idea is to consider the automorphism of $C[V_\Gamma]$ which replaces the point labeled by i by a point symmetric to it with respect to the barycenters of the points v, w. For simplicity, suppose that the only vertices in our diagram are i, v, w. Consider the aforementioned automorphism of $C[\{i, v, w\}]$:

- this 'antipode about $\frac{v+w}{2}$' picks up a sign of $(-1)^n$
- swapping the edges (i, v) and (i, w) picks up a sign of $(-1)^{n-1}$.

hence $I(\Gamma) = -I(\Gamma) \implies I(\Gamma) = 0$.

The argument is similar in the case of more vertices in the diagram.

Proposition. *The Kontsevich configuration space integrals commute with the differential, i.e.*

$$\tilde{I} \circ d = d \circ \tilde{I} \qquad I \circ d = d \circ I. \tag{1}$$

Proposition (fiberwise Stokes formula)**.** *Let $\pi : E \to B$ an oriented SA bundle with k-dimensional fiber and $\pi^\partial : E^\partial \to B$ its fiberwise boundary. Then for $\mu \in \Omega_{min}(E)$,*

$$d(\pi_*(\mu)) = \pi_*(d\mu) + (-1)^{|\mu|-k} \pi_*^\partial(\mu|_{E^\partial})$$

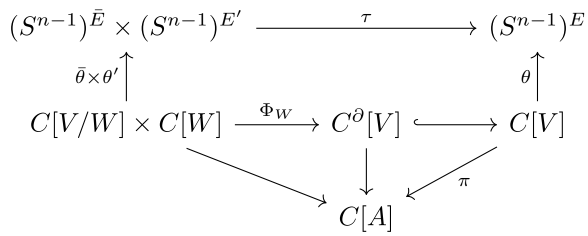
Fact. If π as above and $E = \bigcup E_\lambda$ where $E_\lambda \cap E_\eta$ has fiber dimension $< k$ for all $\lambda \neq \eta$, then the pushforward satisfies $\pi_* = \sum \pi_{\lambda *}(-)|_{E_\lambda}$.

Applying these to (1), we get that the RHS is equal to

$$d(\tilde{I}(\Gamma)) = d\pi_*(\theta^*(\mathrm{vol})) = \pi_*(d\theta^*(\mathrm{vol})) + (-1)^{|\mathrm{vol}_{E_\Gamma}|-|I|\cdot n}\pi_*^{\partial}(\theta^*(\mathrm{vol}_{E_\Gamma}|_{C^\partial[V]}))$$
$$= \sum_W \left(\pi^\partial|_{\mathrm{Im}\ \Phi_W}\right)_* (\mu)$$

Claim. All terms in the sum on the RHS above vanish *except* when W is the pair of endpoints of a contractible edge e of Γ.

The rough idea is to use the following diagram to reduce the question (push-forward of a form from $C[V/W]$) and (evaluating/integrating a form on $C[W]$) and vice versa.



Vanishing occurs for degree reasons and proof follows from casework. Then we have to check that the signs agree.

6 Proof of formality

$n \geq 3$: We only need to check that I defines a quasi-isomorphism. It is surjective on cohomology because $\Gamma\langle a, b\rangle \mapsto g_{ab}$, and we know that $\mathcal{D}(-) \xrightarrow{\sim} \Omega_{PA}(C[-])$ because Ω_{PA} is quasi-isomorphic to (singular) cochains.

7 Intrinsic formality

(Follows [2].)

Remark. Fresse-Willwacher consider *nonunital* operads with extra structure they call 'λ-operations,' which capture the structure of a unit. More precisely, there is an equivalence of theories {unital operads} $\simeq$ {nonunital Λ-operads}.

Recall from Jun Hou's talk that the inverse to the embedding of 'nice' rational spaces into **cDGA** is given by[4]

$$G := \mathrm{Hom}_{\mathbf{cDGA}}(-, \mathbb{Q}) : \mathbf{cDGA} \to \mathbf{sSet}$$

Fact. $H^*(E_n) \simeq \mathrm{Pois}_n^c$ admits a cofibrant resolution in Hopf dg cooperads given by $CE^*(\mathfrak{p}_n)$ where $\mathfrak{p}_n$ is a graded version of Lie algebras of infinitesimal braids.

[4]In the following, I make no distinction between G and FW's 'derived' version $G_\bullet$.

Note that there is an orientation-reversing involution on the little n-disks operad which induces an involution on its homology $J : H_*(E_n) \simeq \mathrm{Pois}_n \xrightarrow{\simeq} \mathrm{Pois}_n$.

Theorem. *[2] Let $\mathcal{P}$ a Λ-operad in* **sSet**. *Suppose each $\mathcal{P}(r)$ is $\mathbb{Q}$-good in the sense of Bousfield-Kan. Assume that we have an isomorphism of Λ-operads (in coalgebras over $\mathbb{Q}$): $H_*(\mathcal{P};\mathbb{Q}) \simeq Pois_n$ for $n \geq 3$. If $4|n$, assume that $\mathcal{P}$ is equipped with an involution reflecting the involution of the Poisson operad. Then $\mathcal{P}$ is* rationally *weakly equivalent to $G(Pois_n)$ as an operad in* **sSet**.

Sketch. Suppose given $\chi : \mathrm{Pois}_n^c \to H^*(\mathcal{P})$ a homology isomorphism.

1. Take resolutions $\mathrm{Res}_\bullet \mathcal{P}$ and $\mathrm{Res}^\bullet \mathrm{Pois}_n$. Note that these are the quasi-(co)free conditions Jun Hou talked about last week. These resolutions come from a cofree (coalg)-forgetful adjunction.
2. Consider the (bicosimplicial) mapping space

$$X^{\bullet,\bullet} := Map_{Op^\Lambda_{dg}}(\mathrm{Res}_\bullet \mathrm{Pois}_n^c, \mathrm{Res}^\bullet H^*(\mathcal{P}))$$

   Note that it is sufficient to consider the totalization of the diagonal cosimplicial object $X^{n,n}$ because Δ is cosifted.
   Furthermore, if $\mathcal{P}$ has an involution, then the involutions on it and Pois_n^c act on the mapping space X.
3. Compute $\pi^0\pi_0(X) = Map_{Op^{Hopf,\Lambda}_{dg}}(\mathrm{Pois}_n^c, H^*(\mathcal{P}))$. Then χ is an element on the RHS, and we want to lift this to an element of $\mathrm{Tot}\, X^\bullet$.
4. [1] The obstruction to lifting χ to the ℓth stage lies in $\pi^{\ell+1}\pi_\ell(X)$. The obstruction to uniqueness of lifting lives in $\pi^\ell\pi_\ell(X)$.
   Note that if $4|n$, need to do this obstruction theory equivariantly.
5. Computation of cohomotopy groups goes through a series of simplifications:
   (a) Have an isomorphism of normalized cochain complexes

$$N\pi_*(X,*) \simeq N\mathrm{BiDer}_{Op^\Lambda_{dg}}(\mathrm{Res}_\bullet \mathrm{Pois}_n^c, \mathrm{Res}^\bullet H^*(\mathcal{P}))$$

   (b) Use Koszul duality to relate the biderivation complex to a deformation bicomplex.
   (c) The deformation bicomplex is equivalent to a twisted end complex, which is defined in terms of graph complexes.
   (d) Compute the homology of some graph complexes. This step looks somewhat like the proof of formality which we saw earlier.

Remark. A similar technique can also be used to analyze the mapping space $Map(E_n^{\mathbb{Q}}, E_m^{\mathbb{Q}})$ for n, m distinct.

References

[1] A. K. Bousfield. Homotopy spectral sequences and obstructions. *Israel J. Math.*, 66(1-3):54–104, 1989.

[2] Benoit Fresse and Thomas Willwacher. The intrinsic formality of e_n-operads. *To appear in J. Eur. Math. Soc.*, 2018.

[3] Robert Hardt, Pascal Lambrechts, Victor Turchin, and Ismar Volić. Real homotopy theory of semi-algebraic sets. *Algebr. Geom. Topol.*, 11(5):2477–2545, 2011.

[4] Pascal Lambrechts and Ismar Volić. Formality of the little N-disks operad. *Mem. Amer. Math. Soc.*, 230(1079):viii+116, 2014.

[5] Paolo Salvatore. Configuration spaces with summable labels. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 375–395. Birkhäuser, Basel, 2001.