A SHORT PROOF THAT $K_8(\mathbb{Z}) \cong 0$.

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Abstract. We give a short proof of the result announced by Dutour-Sikirić–Elbaz–Vincent-Martinet that $K_8(\mathbb{Z}) \cong 0$, by combining results of Lee-Szczarba, Elbaz-Vincent–Gangl–Soulé and Church–Putman.

1. Introduction

The orders of the algebraic K-theory groups of the integers have been determined to a large extent. The remaining open cases are $K_{4i}(\mathbb{Z})$, with the exception of $K_0(\mathbb{Z}) \cong \mathbb{Z}$, $K_4(\mathbb{Z}) \cong 0$ due to Rognes [Rog00], and $K_8(\mathbb{Z}) \cong 0$ announced by Dutour-Sikirić, Elbaz-Vincent and Martinet [EV16]. The proof of the last statement recently appeared [DSEVKM19] and here we give a more economical proof, which was found independently by the author in 2017 and is referenced in several papers and preprints.

The groups $K_{4i}(\mathbb{Z})$ are known to be finitely-generated torsion groups at irregular primes $\ell > 10^8$, see Chapter VI.10 of [Wei13]. Thus $K_8(\mathbb{Z}) = 0$ follows from $K_8(\mathbb{Z}) \otimes \mathbb{Z}[1/7!]$, for which we give a short proof combining [LS76, LS78, EVGS13] with [CP17]. Unlike [DSEVKM19], we do not require additional computer calculations.

Theorem A. *$K_8(\mathbb{Z}) \cong 0$.*

2. The proof

Work of Quillen gives a link between the algebraic K-theory groups of $\mathbb{Z}$ and the homology of $\mathrm{GL}_n(\mathbb{Z})$ with coefficients in the Steinberg module St_n [Qui73]. Filtering the Q-construction by rank, he constructed spaces $BQ(n)$ and $BQ \coloneqq \mathrm{colim}_{n \to \infty} BQ(n)$, such that (i) $BQ(0) \simeq *$, (ii) $BQ \simeq \Omega^{\infty-1} K(\mathbb{Z})$ with $K(\mathbb{Z})$ the algebraic K-theory spectrum, and (iii) for all $n \geq 1$ there is a long exact sequence

$$\cdots \longrightarrow H_m(BQ(n);\mathbb{Z}) \longrightarrow H_{m-n}(GL_n(\mathbb{Z});\mathrm{St}_n) \longrightarrow H_{m-1}(BQ(n-1);\mathbb{Z}) \longrightarrow H_{m-1}(BQ(n);\mathbb{Z}) \longrightarrow \cdots.$$

These long exact sequences give rise to an exact couple and hence a spectral sequence

$$E^1_{p,q} = H_q(\mathrm{GL}_p(\mathbb{Z});\mathrm{St}_p) \Longrightarrow H_{p+q}(BQ;\mathbb{Z}). \tag{1}$$

It is possible to determine the E^1-page of this spectral sequence for most of the range $p \leq 9$, as long as we invert 7!. The entries $E^1_{p,q} = H_q(\mathrm{GL}_p(\mathbb{Z}); \mathrm{St}_p \otimes \mathbb{Z}[1/7!])$ are given in the table below (with p on the horizontal axis and empty entries denoting 0). We will explain how the table was obtained from [LS78], [EVGS13] and [CP17].

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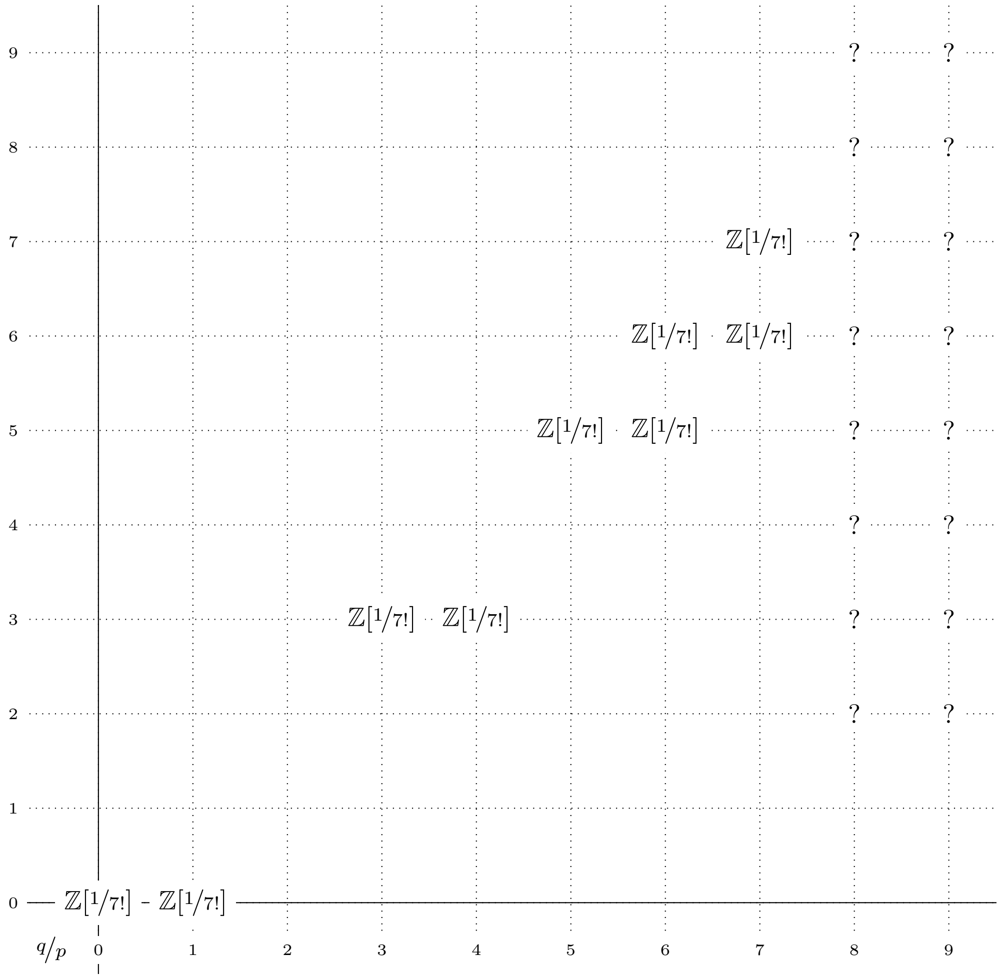

FIGURE 1. Some known entries $E^1_{p,q} = H_q(\mathrm{GL}_p(\mathbb{Z}); \mathrm{St}_p \otimes \mathbb{Z}[1/7!])$. Empty entries are 0.

In [LS78] and [EVGS13], Voronoi's reduction theory is used to compute the homology of $\mathrm{GL}_n(\mathbb{Z})$ with coefficients in St_n by constructing a CW-complex with $\mathrm{GL}_n(\mathbb{Z})$-action, so that the chain complex $\mathrm{Vor}(n)$ given by the cellular chains of the quotient with $\mathbb{Z}[1/(n+1)!]$-coefficients satisfies

$$H_{*-1}(\mathrm{Vor}(n)) \cong H_{*-n}(\mathrm{GL}_n(\mathbb{Z}), \mathrm{St}_n \otimes \mathbb{Z}[1/(n+1)!]).$$

The computations in the cases $n \leq 4$ are due to Lee–Szczarba, see in particular page 104 of [LS78]. The computations in the cases $n = 5, 6, 7$ are due to Elbaz-Vincent–Gangl–Soulé, see Theorem 4.3 of [EVGS13]. This gives the pth columns for $p \leq 7$. To obtain the first two rows for $n > 7$, we use Theorem 1.3 of [LS76] and Theorem A of [CP17].

Lemma 2.1 (Lee–Szczarba)**.** *We have that* $H_0(\mathrm{GL}_n(\mathbb{Z}), \mathrm{St}_n) = 0$ *for all* $n \geq 2$.

Lemma 2.2 (Church–Putman)**.** *We have that* $H_1(\mathrm{GL}_n(\mathbb{Z}), \mathrm{St}_n \otimes \mathbb{Z}[1/n!]) = 0$ *for all* $n \geq 3$.

Proof. Theorem A of [CP17] proves this rationally. To extend their result to $\mathbb{Z}[1/n!]$ with trivial action instead of the trivial rational representation of $\mathrm{GL}_n(\mathbb{Z})$, one modifies the proof as follows.

The first modification occurs when they apply their Lemma 3.2. The following stronger version of that lemma applies in this case: Suppose one has a group G, X a simplicial complex on which G acts simplicially and Y a subcomplex of X which is preserved by the G-action. Furthermore, for some $p \geq 0$, assume that the setwise stabilizer subgroup G_σ has order that is invertible in $\mathbb{Z}[1/n!]$ for every p-simplex σ of X that is not contained in Y. Then the G-module $C_p(X, Y; \mathbb{Z}[1/n!])$ of relative simplicial p-chains is a flat $\mathbb{Z}[1/n!][G]$-module. The second and final modification is when they use their Lemma 3.1, which in fact only requires in their notation that N is $\mathbb{Z}$-torsion free. In our case N is $\mathbb{Z}[1/n!]$. ☐

Remark 2.3. A version of Lemma 2.2 was also proven in [MNP18]: $H_1(\mathrm{GL}_n(\mathbb{Z}), \mathrm{St}_n) = 0$ for $n \geq 6$.

Remark 2.4. Lemma 2.2 can be replaced by further computations in the Voronoi complex for $n = 8$. This is the approach taken by Dutour-Sikirić, Elbaz-Vincent and Martinet in [EV16] and the relevant computations already appeared in [DSHS15].

Lemma 2.5. *We have that* $H_9(BQ; \mathbb{Z}[1/7!]) \cong 0$.

Proof. The E^1-page of the spectral sequence (1) vanishes on the line $p + q = 9$. ☐

Proof of Theorem A. Table VI.10.1.1 of [Wei13] implies

$$K_*(\mathbb{Z}) \otimes \mathbb{Z}[1/7!] \cong \begin{cases} \mathbb{Z}[1/7!] & \text{if } * = 0, 5, \\ 0 & \text{if } * = 1, 2, 3, 4, 6, 7. \end{cases}$$

The non-zero groups can be represented by a map of spectra $\mathbb{S} \vee \Sigma^5 \mathbb{S} \to K(\mathbb{Z})$. The homotopy groups of $\mathbb{S}$ are the stable homotopy groups of spheres, which only have p-torsion in degrees $\geq 2p - 3$. Thus they vanish after inverting 7! in degrees $* \leq 10$ with the exception of $\pi_0(\mathbb{S}) \otimes \mathbb{Z}[1/7!] \cong \mathbb{Z}[1/7!]$. We conclude that the homotopy fiber

$$F \coloneqq \mathrm{hofib}(\mathbb{S} \vee \Sigma^5 \mathbb{S} \to K(\mathbb{Z}))$$

is 0-connected, 7!-locally 6-connected, and $\pi_7(F) \otimes \mathbb{Z}[1/7!] \cong K_8(\mathbb{Z}) \otimes \mathbb{Z}[1/7!]$. There is a fiber sequence

$$\Omega^{\infty-1} F \longrightarrow \Omega^{\infty-1} \mathbb{S} \times \Omega^{\infty-6} \mathbb{S} \xrightarrow{p} \Omega^{\infty-1} K(\mathbb{Z}) \simeq BQ$$

and thus a Serre spectral sequence for homology with $\mathbb{Z}[1/7!]$-coefficients

$$E^2_{p,q} = H_p(BQ; H_q(\Omega^{\infty-1} F; \mathbb{Z}[1/7!])) \Longrightarrow H_{p+q}(\Omega^{\infty-1} \mathbb{S} \times \Omega^{\infty-6} \mathbb{S}; \mathbb{Z}[1/7!])$$

Let us describe this spectral sequence. As F is 7!-locally 6-connected, we have that $\Omega^{\infty-1} F$ is 7!-locally 7-connected. Thus for $0 < q < 8$ the qth row vanishes. Furthermore, since p is an infinite loop map the action of $\pi_1(BQ)$ on $H_*(\Omega^{\infty-1} F; \mathbb{Z}[1/7!])$ is trivial and the 0th column is given by $E^2_{0,q} = H_q(\Omega^{\infty-1} F; \mathbb{Z}[1/7!])$.

Furthermore $H_8(\Omega^{\infty-1} \mathbb{S} \times \Omega^{\infty-6} \mathbb{S}; \mathbb{Z}[1/7!]) \cong 0$ by an Postnikov tower computation. This uses that $\Omega^{\infty-1} \mathbb{S} \times \Omega^{\infty-6} \mathbb{S}$ is simple, that $K(\mathbb{Z}, 1) \simeq S^1$, and that $S^6 \to K(\mathbb{Z}, 6)$ is 7!-locally 10-connected using the relative 7!-local Hurewicz theorem, the Freudenthal suspension theorem and bounds on torsion in the stable homotopy groups of spheres. Hence the spectral sequence converges to zero on the diagonal $p + q = 8$. So there must be a d_8-differential from $E^8_{9,0} = H_9(BQ; \mathbb{Z}[1/7!])$ to $E^8_{0,8} = H_8(\Omega^{\infty-1} F; \mathbb{Z}[1/7!])$ which is a surjection. Then $H_9(BQ;\mathbb{Z}[1/7!]) \cong 0$ implies $H_8(\Omega^{\infty-1}F;\mathbb{Z}[1/7!]) \cong 0$. As $\Omega^{\infty-1}F$ is simply-connected and 7!-locally 7-connected, by the 7!-local Hurewicz theorem $\pi_8(\Omega^{\infty-1}F)\otimes\mathbb{Z}[1/7!] \cong H_8(\Omega^{\infty-1}F;\mathbb{Z}[1/7!]) \cong 0$. We finish by recalling $\pi_8(\Omega^{\infty-1}F) \otimes \mathbb{Z}[1/7!] \cong \pi_7(F) \otimes \mathbb{Z}[1/7!] \cong K_8(\mathbb{Z}) \otimes \mathbb{Z}[1/7!]$. □

Remark 2.6. If Conjecture 2 of [CFP14] were true with coefficients in $\mathbb{Z}[1/(n+1)!]$ instead of $\mathbb{Q}$, it could be used to prove that $K_{12}(\mathbb{Z}) \cong 0$ in a similar manner.

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