THE RATIONAL HOMOTOPY THEORY OF OPERADS

JUN HOU FUNG

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1. Sullivan's rational homotopy theory

A goal of algebraic topology is to reduce problems about topology and geometry to questions in algebra. This is accomplished by means of assigning to spaces and other homotopical structures functorial algebraic invariants such as homology and homotopy groups. A natural question is to ask how well these algebraic invariants capture the information present in homotopy theory. Most of the time, they are insufficient, but one place where this approach works completely is *rational homotopy theory*.

In fact, there are two approaches to rational homotopy theory, one based on cohomology rings and the other based on homotopy groups. In this section, we focus on the former; the latter is discussed in the next section.

The main theorem is the following.

Theorem 1.1 (Sullivan). *Let $\mathsf{Spaces}^{\mathbb{Q},\mathrm{ft},>1} \subset \mathsf{Spaces}$ be the full subcategory of simply connected spaces X such that each $\pi_i X$ is a finite-dimensional $\mathbb{Q}$-vector space. Then the cochain functor*

$$C^*(-;\mathbb{Q}) : \mathsf{Spaces}^{\mathbb{Q},\mathrm{ft},>1} \to \mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{op}}$$

is a fully faithful embedding.

Remarks 1.2.

(1) We can also identify the essential image of the functor $C^*(-;\mathbb{Q})|_{\mathsf{Spaces}^{\mathbb{Q},\mathrm{ft},>1}}$. It consists of those $A \in \mathsf{CAlg}_{\mathbb{Q}}$ for which each $\pi_i A$ is a finite-dimensional $\mathbb{Q}$-vector space such that $\pi_i A \cong \mathbb{Q}$ if $i = 0$, and is zero if $i = -1$ or $i > 0$.

(2) Instead of working with E_∞-algebras, it is possible over a field of characteristic zero to find an explicit commutative model of cochains in $\mathsf{CAlg}_{\mathbb{Q}}^{\heartsuit}$. This was the original approach of Sullivan, who used the model of PL de Rham forms.

Let us mention the proof of Sullivan's theorem briefly. We will need some facts.

Fact 1.3. If

$$\begin{array}{ccc} Y' \times_Y X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

Date: February 20, 2020.

is a pullback square in Spaces and Y is simply connected, then

$$C^*(Y';\mathbb{Q}) \otimes_{C^*(Y;\mathbb{Q})} C^*(X;\mathbb{Q}) \to C^*(Y' \times_Y X;\mathbb{Q})$$

is an equivalence.

Corollary 1.4 (Künneth)**.** *The functor* $C^*(-;\mathbb{Q}) : (\mathsf{Spaces}^{\mathrm{ft}}, \times) \to (\mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{op}}, \otimes)$ *is symmetric monoidal.*

Fact 1.5. If V is a finite-dimensional $\mathbb{Q}$-vector space, then we have an equivalence

$$C^*(K(V,n);\mathbb{Q}) \simeq \mathrm{Sym}^*(V^\vee[-n]).$$

Proof sketch of (1.1). The functor $C^*(-;\mathbb{Q}) : \mathsf{Spaces} \to \mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{op}}$ admits a right adjoint given by the functor $\mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}}(-,\mathbb{Q})$. To show that $C^*(-;\mathbb{Q})$ is fully faithful, it suffices to show that the unit map

$$u_X : X \to \mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}}(C^*(X;\mathbb{Q}),\mathbb{Q})$$

is an equivalence where X is a simply connected rational space of finite type.

We induct on the Postnikov tower of X[1]. The base case $n = 1$ is obvious, so let $n > 1$. We have a pullback diagram

$$\begin{array}{ccc} \tau_{\le n}X & \longrightarrow & \tau_{\le n-1}X \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & K(\pi_n X, n+1) \end{array}$$

By fact 1.3, we get a pushout square

$$\begin{array}{ccc} C^*(K(\pi_n X, n+1);\mathbb{Q}) & \longrightarrow & C^*(\tau_{\le n-1}X;\mathbb{Q}) \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{Q} & \longrightarrow & C^*(\tau_{\le n}X;\mathbb{Q}). \end{array}$$

To show that $u_{\tau_{\le n}X}$ is an equivalence, it suffices to do so for $u_{\tau_{\le n-1}X}$, which follows from the induction hypothesis, and for $u_{K(\pi_n X, n+1)}$, which follows from fact 1.5. □

Remark 1.6. It is not necessary that X is simply connected for the theorem to hold. Fact 1.3 remains true if $\pi_1(X)$ acts nilpotently on higher homotopy groups/cohomology groups, and if X is nilpotent then we still have a good Postnikov tower for X. Therefore, we may relax the assumption that X is simply connected to just that X is *nilpotent*.

2. Lie models in rational homotopy theory

2.1. **Quillen's approach.** Before Sullivan, Quillen developed another algebraic model of rational homotopy theory in terms of dg Lie algebras. The idea is that instead of looking at the cochains which can be strictified to a cdga, we look at the homotopy groups of a rational space. The extra algebraic structure here is that of a Lie algebra reflected in the Whitehead products.

Theorem 2.1 (Quillen)**.** *There is an equivalence*

$$\lambda : \mathsf{Spaces}_*^{\mathbb{Q},>1} \to \mathsf{Lie}_{\mathbb{Q}}^{\ge 1}.$$

This is related to the Sullivan theory by means of the Chevalley-Eilenberg construction.

[1]During the talk, Dylan pointed out that one needed to show $C^*(X;\mathbb{Q}) \simeq \operatorname{colim} C^*(\tau_{\le n}X;\mathbb{Q})$. This follows from the fact the map $X \to \tau_{\le n}$ induces an isomorphism on homotopy groups up to degree n, hence an isomorphism of cohomology groups up to n.

2.2. **L_∞-algebras and the Chevalley-Eilenberg construction.** We describe the Chevalley-Eilenberg complex in the more general setting of L_∞-algebras.

Definition 2.2. An *L_∞-algebra* is a graded vector space $\mathfrak{g}$ together with maps

$$\ell_r : \mathfrak{g}^{\otimes r} \to \mathfrak{g}$$

of degree $r-2$ for all $r \geq 1$, satisfying

(i) Graded anti-symmetry: $\ell_r(\ldots, x, y, \ldots) = (-1)^{|x||y|+1}\ell_r(\ldots, y, x, \ldots)$.
(ii) Generalized Jacobi identities: for all $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{\sigma \in \sqcup\!\sqcup(i, n-i+1)} (-1)^{\epsilon} \ell_{n-i}(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0.$$

The sign is given by $\epsilon = i + \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |x_i|\,|x_j| + 1$.

Remark 2.3. Let $d = \ell_1$ and $[-,-] = \ell_2$. Then the first few Jacobi identities are:

$$d^2(x) = 0,$$
$$d([x,y]) = [dx, y] + (-1)^{|x|}[x, dy)],$$
$$[x,[y,z]] - [[x,y],z] - (-1)^{|x||y|}[y,[x,z]] = \pm(d\ell_3 + \ell_3 d)(x,y,z).$$

In other words, $\mathfrak{g}$ has the structure of a chain complex with differential d, such that d is a derivation for the bracket $[-,-]$, which satisfies the usual Jacobi identity up to homotopy, and so on.

Example 2.4. An L_∞-algebra with $\ell_r = 0$ for $r > 2$ is the same as a differential graded Lie algebra.

Conversely, over a field of characteristic zero, every L_∞-algebra is quasi-isomorphic to a dg-Lie algebra.

Remark 2.5. We have opted to give a concrete definition of L_∞-algebras in terms of formulas. Other definitions are possible: for example, an L_∞-structure on a graded vector space $\mathfrak{g}$ is the same as a coderivation differential d on the cofree symmetric coalgebra cogenerated by $\mathfrak{g}[1]$. This perspective will be be useful when we define the Chevalley-Eilenberg construction below.

Definition 2.6. Let $\mathfrak{g}$ be a non-negatively graded L_∞-algebra of finite type. The *Chevalley-Eilenberg construction* $\mathrm{CE}^*(\mathfrak{g})$ is the cdga whose underlying graded vector space is $\mathrm{Sym}^*\,\mathfrak{g}[1]^\vee$. To specify the differential, it suffices to define maps $d_{\mathrm{CE}} : \mathfrak{g}[1]^\vee \to (\mathrm{Sym}^r\,\mathfrak{g}[1])^\vee$ for each r, and we have

$$d_{\mathrm{CE}}(\xi)(x_1[1]\cdots x_r[1]) = \pm\xi(\ell_r(x_1, \ldots, x_r)[1]).$$

Remark 2.7. Note that the canonical isomorphism $(\mathrm{Sym}^r\,\mathfrak{g}[1])^\vee \xrightarrow{\cong} \mathrm{Sym}^r(\mathfrak{g}[1]^\vee)$ is given by multiplication by $\frac{1}{r!}$.

Since CE^* takes the initial object $0 \in \mathsf{Lie}_{\mathbb{Q}}$ to the final object $\mathbb{Q} \in \mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{op}}$, this construction determines a functor

$$\mathrm{CE}^* : \mathsf{Lie}_{\mathbb{Q}} \to (\mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{aug}})^{\mathrm{op}}.$$

Proposition 2.8. *The Chevalley-Eilenberg construction gives an equivalence*

$$\mathrm{CE}^* : \mathsf{Lie}_{\mathbb{Q}}^{\mathrm{ft},\geq 1} \xrightarrow{\simeq} (\mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{aug},\mathrm{ft},>1})^{\mathrm{op}}.$$

Definition 2.9. An L_∞-algebra $\mathfrak{g}$ is an L_∞-model of a simply connected space X if it is quasi-isomorphic to $\lambda(X)$.

Remark 2.10. If X is of finite type, then $\mathfrak{g}$ is an L_∞-model of X iff $\mathrm{CE}^*(\mathfrak{g})$ is a Sullivan model of X.

3. Mapping spaces in rational homotopy theory

Previously, we have identified the mapping spaces in rational homotopy theory with mapping spaces between cdgas. In this section we give a concrete description of these mapping spaces due to Hinich and others.

Here's the basic idea. Let X be a (rational, etc.) space and $A \in \mathsf{CAlg}_{\mathbb{Q}}$. To understand the space of maps $\mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}}(C^*(X;\mathbb{Q}), A)$, we find a Lie model for X, so that we may replace $C^*(X;\mathbb{Q})$ with the quasi-isomorphic algebra $\mathrm{CE}^*(\mathfrak{g})$ for some L_∞-algebra $\mathfrak{g}$. By construction, $\mathrm{CE}^*(\mathfrak{g})$ is freely generated as a cdga, so if we ignore the differential, we have

$$\mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}^{d=0}}(\mathrm{CE}^*(\mathfrak{g}), A) \simeq \mathrm{Map}_{\mathsf{Mod}_{\mathbb{Q}}^{d=0}}(\mathfrak{g}[1]^\vee, A) \simeq (A \otimes \mathfrak{g})_{-1}.$$

The dg-algebra homomorphisms form a subspace of this space consisting of those elements that respect the differential. This condition amounts to the *Maurer-Cartan equation.* We make this more precise below.

Extension of scalars. First we explain what we mean by "$A \otimes \mathfrak{g}$" when A is a cdga and $\mathfrak{g}$ is a L_∞-algebra. This is a new L_∞-algebra whose underlying graded vector space is $A \otimes \mathfrak{g}$, and the differentials and brackets are defined by

$$d_{A\otimes\mathfrak{g}}(a \otimes x) = d_A(a) \otimes x + (-1)^{|a|} a \otimes d_{\mathfrak{g}}(x),$$
$$\ell_r(a_1 \otimes x_1, \dots, a_r \otimes x_r) = (-1)^{\sum_{i<j} |a_j||x_i|} a_1 \cdots a_r \otimes \ell_r(x_1, \dots, x_r).$$

The Maurer-Cartan space. Unlike the Maurer-Cartan equation for dg Lie algebras, the equation for L_∞-algebra involves an infinite sum, so some kind of nilpotence condition is necessary for the equation to converge.

Let $\mathfrak{g}$ be an L_∞-algebra. The *lower central series* of $\mathfrak{g}$ is the filtration

$$\mathfrak{g} = \Gamma^1\mathfrak{g} \supseteq \Gamma^2\mathfrak{g} \supseteq \cdots$$

where $\Gamma^k\mathfrak{g}$ is the sub-L_∞-algebra spanned by bracket expressions formed using at least k elements from $\mathfrak{g}$. In other words, it is the smallest filtration on $\mathfrak{g}$ that is compatible with the L_∞-structure.

Definition 3.1. A L_∞-algebra $\mathfrak{g}$ is *degreewise nilpotent* if for every n there exists k such that $(\Gamma^k\mathfrak{g})_n = 0$.

Remark 3.2. This is a natural condition on L_∞-algebra from the perspective of rational homotopy theory. Under the correspondence of proposition 2.8, degreewise nilpotent connective L_∞-algebra of finite type correspond to connected Sullivan algebras of finite type.

Definition 3.3. Let $\mathfrak{g}$ be a degreewise nilpotent L_∞-algebra.

(a) A *Maurer-Cartan element* of $\mathfrak{g}$ is an element $x \in \mathfrak{g}_{-1}$ such that

$$\sum_{r\geq 1} \frac{1}{r!}\ell_r(x, \dots, x) = 0.$$

(b) The *Maurer-Cartan space* of $\mathfrak{g}$ is the simplicial set

$$\mathrm{MC}_\bullet(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet),$$

where $\Omega_\bullet \cong \Omega_{\mathrm{poly}}(\Delta^\bullet)$ is the simplicial cdga with n-simplices

$$\Omega_n = \mathbb{Q}[t_0, \dots, t, dt_0, \dots, dt_n]/(\sum_i t_i = 1, \sum_i dt_i = 0),$$

where $|t_i| = 0$ and $|dt_i| = 1$.

Remark 3.4. Let $\mathfrak{g}$ be a *complete* L_∞-algebra, i.e., it is equipped with a descending filtration of L_∞-ideals

$$\mathfrak{g} = F^1\mathfrak{g} \supseteq F^2\mathfrak{g} \supseteq \cdots$$

such that each $\mathfrak{g}/F^r\mathfrak{g}$ is a nilpotent L_∞-algebra and $\mathfrak{g} \xrightarrow{\sim} \lim \mathfrak{g}/F^r\mathfrak{g}$. Then $\mathfrak{g}$ is degreewise nilpotent if it is of finite type, and we can define its Maurer-Cartan space as

$$\mathrm{MC}_\bullet(\mathfrak{g}) = \mathrm{MC}(\lim \mathfrak{g}/F^r\mathfrak{g} \otimes \Omega_\bullet).$$

Proposition 3.5. *Let $\mathfrak{g}$ be a non-negatively graded degreewise nilpotent L_∞-algebra of finite type, and A be a cdga. If $\mathfrak{g}$ or A is bounded, then there is an equivalence*

$$\mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}}(\mathrm{CE}^*(\mathfrak{g}), A) \simeq \mathrm{MC}_\bullet(A \otimes \mathfrak{g}).$$

Proof. We prove the π_0 statement. As we discussed, the underlying graded commutative algebra of $\mathrm{CE}^*(\mathfrak{g})$ is free on $\mathfrak{g}[1]^\vee$. Given $\tau \in A \otimes \mathfrak{g}$ of degree -1, we can define a morphism of graded algebras $\mathrm{ev}_\tau : \mathrm{CE}^*(\mathfrak{g}) \to A$ by $\mathrm{ev}_\tau(\xi) = (1 \otimes \xi)(\tau)$. We claim that ev_τ is a chain map iff τ is a Maurer-Cartan element. Let $\xi \in \mathfrak{g}[1]^\vee$; we have:

$$\begin{aligned}(d_A \circ \mathrm{ev}_\tau)(\xi) &= (d_A \otimes \xi)(\tau),\\ (\mathrm{ev}_\tau \circ d_{\mathrm{CE}})(\xi) &= \sum_{r \geq 1} \pm \xi(\ell_r(\tau, \cdots, \tau)).\end{aligned}$$

So ev_τ respects the differentials iff τ satisfies the Maurer-Cartan equation in the L_∞-algebra $A \otimes \mathfrak{g}$.

The verification of this proposition for mapping *spaces* is due to Brown-Szczarba. □

Putting the results from the previous sections all together, we obtain the following theorem[2]

Theorem 3.6. *Let X be connected and Y a nilpotent space of finite type. Let A be a cdga model for X (e.g., $C^*(X; \mathbb{Q})$) and $\mathfrak{g}$ a degreewise nilpotent L_∞-algebra model for Y of finite type. Then there is an equivalence*

$$\mathrm{Map}_{\mathsf{Spaces}}(X, Y_{\mathbb{Q}}) \simeq \mathrm{MC}_\bullet(A \widehat{\otimes} \mathfrak{g}).$$

Proof remarks. Roughly speaking, we have the following chain of equivalences

$$\mathrm{Map}_{\mathsf{Spaces}}(X, Y_{\mathbb{Q}}) \simeq \mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}}(C^*(Y; \mathbb{Q}), C^*(X; \mathbb{Q})) \simeq \mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}}(\mathrm{CE}^*(\mathfrak{g}), A) \simeq \mathrm{MC}_\bullet(A \otimes \mathfrak{g}).$$

Except we need to be a little careful – $\mathfrak{g}$ was not assumed to be bounded. Instead, we need to write $\mathfrak{g}$ as a limit of finite-dimensional nilpotent quotients, so we will have to take a completed tensor product. □

Remark 3.7. The completed tensor product is also a product of expressing "maps" in terms of "tensors".

4. Operadic enhancements

Let V be a symmetric monoidal category, e.g., $\mathsf{V} = \mathsf{Spaces}$ or $\mathsf{CAlg}_{\mathbb{Q}}$. Recall that an *operad* in V is an associative algebra in the category of symmetric sequences in V; a *cooperad* in V is a coassociative coalgebra in the category of symmetric sequences in V.

Theorem 4.1 (Fresse)**.** *There is an equivalence of categories*

$$C^*(-; \mathbb{Q}) : \mathsf{Op}(\mathsf{Spaces}^{\mathbb{Q}, \mathrm{ft}, >1}) \xrightarrow{\sim} \mathsf{CoOp}(\mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{ft}, >1})^{\mathrm{op}}.$$

Proof. Apply $\mathsf{Op}(-)$ to both sides of Sullivan's adjoint equivalence $\mathsf{Spaces}^{\mathbb{Q}, \mathrm{ft}, >1} \simeq (\mathsf{CAlg}_{\mathbb{Q}}^{\mathrm{ft}, >1})^{\mathrm{op}}$ and use the fact that

$$\mathsf{Op}(\mathsf{V}^{\mathrm{op}}) \simeq \mathsf{CoOp}(\mathsf{V})^{\mathrm{op}}.$$

□

Remark 4.2. Fresse spends 100+ pages in his book to prove a version of this theorem. The issue has to do with the fact that the cochain functor is not strong symmetric monoidal on the nose, so doesn't preserve algebra structures on the nose. For example, given the operadic composition map

$$\mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n),$$

applying cochains gives only the zigzag

$$\begin{aligned}C^*(\mathcal{O}(k_1 + \cdots + k_n); \mathbb{Q}) &\to C^*(\mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n); \mathbb{Q})\\ &\xleftarrow{\sim} C^*(\mathcal{O}(n); \mathbb{Q}) \times C^*(\mathcal{O}(k_1); \mathbb{Q}) \times \cdots \times C^*(\mathcal{O}(k_n); \mathbb{Q}).\end{aligned}$$

[2] During the talk, Dev observed that (e.g., when $\mathfrak{g}$ is nilpotent), the Maurer-Cartan elements are cut out by polynomial equations, hence form an algebraic variety. The 1-cells of the Maurer-Cartan space are *gauge equivalences*. How should one think of this space of Maurer-Cartan elements up to gauge equivalences? Dev also promoted doing actual computations of mapping spaces (e.g., $S^n \vee S^n \to S^n \times S^n$?) in terms of Maurer-Cartan spaces which sadly the author has not yet done.

The second map is only a quasi-isomorphism but not an isomorphism; this makes defining the coproduct map coherently problematic.

There are two solutions to this. One is to simply assert that these coherence problems are all solved by invoking the yoga of ∞-categories. Fresse takes a different path: he observes that the adjoint functor $\mathrm{Map}_{\mathsf{CAlg}_{\mathbb{Q}}^{\heartsuit}}(-,\mathbb{Q})$ *is* strong symmetric monoidal on the nose, so it extends directly to a functor from cooperads in $\mathsf{CAlg}_{\mathbb{Q}}^{\heartsuit}$ to operads in spaces. Then Fresse shows that this functor admits an adjoint functor, which he takes to be the operadic enhancement of Sullivan's cochain functor. He also proves that for good operads, his operadic cochain functor agrees up to equivalence with the previous Sullivan cochain functor in each arity. Doing this by hand appears to be combinatorially tricky, so Fresse actually proves this theorem three times, in increasing generality on the operad.

Evidently, we may replace "simply-connected" with "nilpotent", etc. in the statement of the theorem. Let us introduce terminology to abbreviate these conditions. We say that a space X is $\mathbb{Q}$-*good* if the induced map $H^*(X_{\mathbb{Q}};\mathbb{Q}) \to H^*(X,\mathbb{Q})$ is an isomorphism. An operad in spaces is $\mathbb{Q}$-*good* if the components spaces in each arity are $\mathbb{Q}$-good.

Example 4.3.

- Nilpotent spaces of finite type are $\mathbb{Q}$-good.
- E_n-operads, for $n \geq 3$, are $\mathbb{Q}$-good.

Corollary 4.4. *Let $\mathcal{O}$ and $\mathcal{P}$ be $\mathbb{Q}$-good operads in spaces. Then*

$$\mathrm{Map}_{\mathsf{Op(Spaces)}}(\mathcal{O},\mathcal{P}_{\mathbb{Q}}) \simeq \mathrm{Map}_{\mathsf{CoOp(CAlg}_{\mathbb{Q}})}(C^*(\mathcal{P};\mathbb{Q}),C^*(\mathcal{O};\mathbb{Q})).$$

It will be convenient in the future to specify the objects of $\mathsf{CoOp(CAlg}_{\mathbb{Q}})$ more precisely.

Definition 4.5. An object of $\mathsf{CoOp(CAlg}_{\mathbb{Q}})$ is a *Hopf cochain dg-cooperad*, which is a cooperad $\mathcal{C}$ in the category of commutative cochain dg-algebras satisfying $\mathcal{C}(0)=\mathcal{C}(1)=\mathbf{1}$.

Remark 4.6. We may also consider cooperads where $\mathcal{C}(1)$ is not $\mathbf{1}$, as long as it satisfies the following conilpotence condition: composition coproducts of a given element after finitely many iterations. This is useful for example when we construct colimits of cooperads.

Finally, we turn to the description of the mapping spaces between cooperads. Fix two cooperads $\mathcal{C},\mathcal{D} \in \mathsf{CoOp(CAlg}_R)$, and consider the (classical) formal moduli problem

$$R \mapsto \mathrm{Map}_{\mathsf{CoOp(CAlg}_R)}(\mathcal{C}\otimes R,\mathcal{D}\otimes R).$$

Slogan 4.7 (Lurie, Pridham, ...)**.** *Formal moduli problems in characteristic zero are controlled by Lie algebras.*

So as before, we expect this moduli problem to be controlled by an L_∞-algebra. More specifically, the mapping spaces between cooperads should be given as the Maurer-Cartan elements of a certain L_∞-algebra.

Theorem 4.8 (Fresse-Willwacher)**.** *Suppose $\mathcal{C},\mathcal{D} \in \mathsf{CoOp(CAlg}_{\mathbb{Q}})$ be two cooperads. Let $\mathcal{C}'$ be a model of $\mathcal{C}$ that is aritywise quasi-free generated by $V \in \mathsf{SSeq(Mod}_{\mathbb{Q}}^{d=0})$, and let $\mathcal{D}'$ be a model of $\mathcal{D}$ that is quasi-cofree cogenerated by $W \in \mathsf{SSeq(CAlg}_{\mathbb{Q}}^{d=0})$.*

There is an L_∞-algebra $\mathrm{Def}(\mathcal{C}',\mathcal{D}')$ such that

$$\mathrm{Hom}_{\mathsf{CoOp(CAlg}_{\mathbb{Q}})}(\mathcal{C},\mathcal{D}) \cong \mathrm{MC}(\mathrm{Def}(\mathcal{C}',\mathcal{D}')).$$

Proof. As in the non-operadic case, there are two steps in the proof.

(1) Ignoring the differential, $\mathrm{Hom}_{\mathsf{CoOp(CAlg}_{\mathbb{Q}}^{d=0})}(\mathcal{C}',\mathcal{D}') \cong \mathrm{Hom}_{\mathsf{SSeq(Mod}_{\mathbb{Q}}^{d=0})}(V,W)$.

(2) There is an L_∞-structure on $\mathrm{Hom}_{\mathsf{SSeq(Mod}_{\mathbb{Q}})}(V,W)$ such that the Maurer-Cartan equation encodes the condition of being a chain map.

The first step follows from the quasi-freeness, quasi-cofreeness, and other conditions we demand on $\mathcal{C}'$. Recall that an operad in cdgas is quasi-(co)free if its underlying operad in graded vector spaces is (co)free.

We have:

$$\begin{aligned}\Psi : &\mathrm{Hom}_{\mathsf{CoOp}(\mathsf{CAlg}_{\mathbb{Q}}^{d=0})}(\mathcal{C}', \mathcal{D}') \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{SSeq}(\mathsf{CAlg}_{\mathbb{Q}}^{d=0})}(\mathcal{C}', W) && \text{since } \mathcal{D}' \text{ is quasi-cofree} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{SSeq}(\mathsf{Mod}_{\mathbb{Q}}^{d=0})}(V, W) && \text{by conditions on } \mathcal{C}'\end{aligned}$$

Remarks 4.9.

(1) In the language of model categories, $\mathcal{C}'$ and $\mathcal{D}'$ are respectively cofibrant and fibrant replacements for $\mathcal{C}$ and $\mathcal{D}$ in some model structure on cooperads.

(2) The conditions on $\mathcal{C}'$ to make step (1) work are a little subtle. Fresse and Willwacher assume moreover that

  (i) the generating symmetric sequence V is equipped with an exhaustive bounded below filtration
$$F^1V \subset F^2V \subseteq \cdots$$
  compatible with the cooperad structure such that $\dim F^pV(r)/F^{p+1}V(r) < \infty$ for each $p, r \geq 1$.

  (ii) $\mathcal{C}'$ is equipped with an augmentation from $\mathcal{C}'$ to the commutative cooperad $\mathcal{C}om^c$.

  These properties are needed to build a map of cooperads from a map between generators and cogenerators by inducting along the filtration.

Let
$$\Phi : \mathrm{Hom}_{\mathsf{SSeq}(\mathsf{Mod}_{\mathbb{Q}})}(V, W) \to \mathrm{Hom}_{\mathsf{CoOp}(\mathsf{CAlg}_{\mathbb{Q}})}(\mathcal{C}', \mathcal{D}')$$
be the inverse map to Ψ. We want to consider encode those morphisms $x : V \to W$ for which $\Phi(x) : \mathcal{C}' \to \mathcal{D}'$ preserves the differential structure, and show that they can be cut out by the Maurer-Cartan equation for some L_∞-structure on $\mathrm{Def}(\mathcal{C}', \mathcal{D}') := \mathrm{Hom}_{\mathsf{SSeq}(\mathsf{Mod}_{\mathbb{Q}})}(V, W)$. To do this we first describe a criterion for defining L_∞-structures with a specified Maurer-Cartan equation.

Lemma 4.10. *Let V be a graded vector space. A functorial power series $M : R \mapsto M^R(x) : (R \otimes V)_{-1} \to (R \otimes V)_{-2}$ is the Maurer-Cartan series for some L_∞-structure on V iff $DM^R(x)(M^R(x)) = 0$.*

Proof. Given an L_∞-structure on V, we get a functorial Maurer-Cartan series by extension of scalars. For example, if $R = \mathbb{Q}[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1^2, \ldots, \epsilon_n^2)$, then we have

$$\begin{aligned}M^R(\sum_{i=1}^n \epsilon_i x_i) &= \sum_{r=1}^\infty \frac{1}{r!} \ell_r(\sum_{i=1}^n \epsilon_i x_i, \ldots, \sum_{i=1}^n \epsilon_i x_i) \\ &= \sum_{r=1}^\infty \frac{1}{r!} \sum_{\sigma:\{1,\ldots,r\}\hookrightarrow\{1,\ldots,n\}} \pm \epsilon_{\sigma(1)} \cdots \epsilon_{\sigma(r)} \ell_r(x_{\sigma(1)}, \ldots, x_{\sigma(r)}).\end{aligned}$$

Conversely, given M^R, if we let $n = r$ and look at the coefficient of $\epsilon_1 \cdots \epsilon_n$, we see that we can recover a bracket from M^R:

$$\ell_r(x_1, \ldots, x_r) := \pm[\epsilon_1 \cdots \epsilon_r] M^R(\sum_{i=1}^r \epsilon_i x_i).$$

It remains to show that the brackets $\{\ell_r\}$ obtained this way satisfy the generalized Jacobi identities. Let us abbreviate $x := \sum_{i=1}^n \epsilon_i x_i$ and $\epsilon_\sigma := \epsilon_{\sigma(1)} \cdots \epsilon_{\sigma(n)}$. Then from the previous equation we see that the differential of M^R at x is given by

$$DM^R(x)(h) = \sum_{r=1}^\infty \frac{1}{r!} \sum_\sigma \pm \epsilon_\sigma \ell_r(h, x_{\sigma(2)}, \ldots, x_{\sigma(r)}),$$

so that

$$DM^R(x)(M^R(x)) = \sum_{r=1}^\infty \frac{1}{(r-1)!} \sum_{s=1}^\infty \frac{1}{s!} \sum_{\substack{\sigma,\tau \\ \sigma\cap\tau=\emptyset}} \pm \epsilon_{\sigma\cup\tau} \ell_r(\ell_s(x_{\tau(1)}, \ldots, x_{\tau(s)}), x_{\sigma(2)}, \ldots, x_{\sigma(r)}).$$

Setting $DM^R(x)(M^R(x)) = 0$, $n = r+s-1$, and looking at the coefficient of $\epsilon_1 \cdots \epsilon_n$, we see that

$$\sum_{r+s=n+1} \sum_{(\sigma,\tau)\in\sqcup(r,s)} \pm\ell_r(\ell_s(x_{\tau(1)},\dots,x_{\tau(s)}), x_{\sigma(2)},\dots,x_{\sigma(r)}) = 0,$$

which is the generalized Jacobi identity. □

Proof. For $x \in \mathrm{Def}(\mathcal{C}',\mathcal{D}')$, we set $M(x) := \Psi(d_{\mathcal{D}'}\Phi(x) \pm \Phi(x)d_{\mathcal{C}'})$. By the lemma, once we show that $DM(x)(M(x)) = 0$, we would obtain a L_∞-structure on $\mathrm{Hom}_{\mathsf{SSeq}(\mathsf{Mod}_{\mathbb{Q}})}(V,W)$ such that the Maurer-Cartan elements are precisely the chain maps.

We claim that $D\Phi(x)(M(x)) = d_{\mathcal{D}'}\Phi(x) \pm \Phi(x)d_{\mathcal{C}'}$. This follows[3] from:

(i) $d_{\mathcal{D}'}\Phi(x) - \Phi(x) - \Phi(x)d_{\mathcal{C}'}$ is a *biderivation* (see following remark) of $\Phi(x)$.
(ii) If θ is any biderivation of $\Phi(x)$, then $D\Phi(x)(\Psi(\theta)) = \theta$.

Finally, we have

$$\begin{aligned} DM(x)(M(x)) &= \Psi(d_{\mathcal{D}'}D\Phi(x)(M(x)) \pm D\Phi(x)(M(x))d_{\mathcal{C}'}) && \text{by the chain rule} \\ &= \Psi(d_{\mathcal{D}'}(d_{\mathcal{D}'}\Phi(x) - \Phi(x)d_{\mathcal{C}'}) \pm (d_{\mathcal{D}'}\Phi(x) - \Phi(x)d_{\mathcal{C}'})d_{\mathcal{C}'}) && \text{by the claim} \\ &= \Psi(d_{\mathcal{D}'}\Phi(x)d_{\mathcal{C}'} - d_{\mathcal{D}'}\Phi(x)d_{\mathcal{C}'}) && \text{since } d^2 = 0 \\ &= 0. \end{aligned}$$

This completes the proof. □

Remark 4.11. This L_∞-algebra $\mathrm{Def}(\mathcal{C}',\mathcal{D}')$ is called the *deformation complex*, and its underlying graded vector space can be identified with a certain complex of *biderivations* of the canonical morphism $\mathcal{C}' \to \mathcal{C}om^c \to \mathcal{D}'$. Consequently, $\mathrm{Def}(\mathcal{C}',\mathcal{D}')$ is independent of the choice of V and W. (Recall that a degree k biderivation of a map $f : \mathcal{C} \to \mathcal{D}$ is a degree k map $\theta : \mathcal{C} \to \mathcal{D}$ of symmetric sequences such that $f + \epsilon\theta$ is a morphism of cooperads in algebras, where $\epsilon^2 = 0$. This graded vector space of biderivations inherits a differential where $d\theta = d_{\mathcal{D}} \circ \theta - (-1)^k \theta \circ d_{\mathcal{C}}$.)

In some sense this is the more natural route to take. The input to the Lurie-Pridham machine relating formal moduli problems to Lie algebras is a calculation of the tangent space of the moduli problem. In this case, just as the tangent space in commutative algebra maps are derivations, the tangent space in maps of cooperads of algebras are "by definition" biderivations.

5. Applications to automorphisms of E_n-operads

Let $n \geq 3$. The spaces in E_n are simply connected, and therefore $\mathbb{Q}$-good, so corollary 4.4 gives an equivalence

$$\mathrm{End}((E_n)_{\mathbb{Q}}) \simeq \mathrm{End}(C^*(E_n;\mathbb{Q})).$$

[3]These statements are much easier to check in the non-cooperadic context. For (i), let $f : (A, m_A, d_A) \to (B, m_B, d_B)$ be a map of cdgas when we forget about the differential. We want to show that $d_B f - f d_A$ is a derivation of f, i.e., $f + \epsilon(d_B f - f d_A)$ is also a map of algebras. For example, multiplication is respected:

$$\begin{aligned} (f + \epsilon(d_B f - f d_A))m_A &= f m_A + \epsilon(d_B f m_A - f d_A m_A) \\ & f m_A + \epsilon(d_B m_B(f\otimes f) - f(m_A(d_A \otimes 1 + 1 \otimes d_A))) \\ &= m_B(f\otimes f) + \epsilon(m_B(d_B \otimes 1 + 1 \otimes d_B)(f\otimes f) - m_B(f\otimes f)(d_A \otimes 1 + 1 \otimes d_A)) \\ &= m_B(f\otimes f + \epsilon(d_B f \otimes f + f \otimes d_B f - f d_A \otimes f - f \otimes f d_A)) \\ &= m_B(f + \epsilon(d_B f - f d_A) \otimes f + \epsilon(d_B f - f d_A)). \end{aligned}$$

For (ii), we have a bijection

$$\mathrm{Hom}_{\mathsf{CAlg}}(\mathrm{Sym}\, V, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Mod}}(V, A)$$

sending an algebra homomorphism f to $f \circ i$, where i is the inclusion of generators. Denote the inverse map by Φ; if $x \in \mathrm{Hom}_{\mathsf{Mod}}(V,A)$, $\Phi(x)$ is the unique extension of x to a map of algebras, e.g., $\Phi(x)(v_1 + v_2 v_3) = x(v_1) + x(v_2)x(v_3)$.

Now let $h \in \mathrm{Hom}_{\mathsf{Mod}}(\epsilon \otimes V, \epsilon \otimes A)$. The differential $D\Phi$ is defined as

$$D\Phi(x)(h) = \left.\frac{\Phi(x + \epsilon h) - \Phi(x)}{\epsilon}\right|_{\epsilon=0}.$$

It is straightforward to check that $D\Phi(x)(h)$ is the unique derivation on the free symmetric algebra extending h which is obtained by the Leibniz rule. Consequently, if h is the restriction of a derivation θ to begin with, then $D\Phi(x)(h) = \theta$.

In fact, this is an equivalence of monoids (under composition of endomorphisms), so passing to the path components that are invertible on π_0, we deduce:

Proposition 5.1. *For* $n > 2$, $\mathrm{Aut}((E_n)_\mathbb{Q}) \simeq \mathrm{Aut}(C^*(E_n;\mathbb{Q}))$.

The result is also true when $n = 2$, but we don't know whether E_2 is $\mathbb{Q}$-good. Instead, we proceed by manual computation.

Proposition 5.2. $\mathrm{Aut}((E_2)_\mathbb{Q}) \simeq \mathrm{Aut}(C^*(E_2;\mathbb{Q}))$.

Really rough sketch. In the case $n = 2$, the spaces $E_2(r)$ can be identified with the classifying space BP_r of the pure braid group on r strands. Moreover, the rationalization $E_2(r)_\mathbb{Q}$ of this space can be identified with the classifying space $B\hat{P}_r$ of the "Malcev completion" of P_r. To identify their mapping spaces, we consider the homotopy spectral sequence for maps from E_2 and from $(E_2)_\mathbb{Q}$ to certain Eilenberg-Mac Lane spaces, and show that their E^2-pages are isomorphic. It turns out that E^2-pages only depend on $H_*(P_r;\mathbb{Q})$ and $H_*(\hat{P}_r;\mathbb{Q})$ for $* \leq 1$. This is a direct computation: we have $H_0 \cong \mathbb{Q}$ for both groups, and for H_1 we compute the abelianizations of P_r and $\hat{P}_r$ and show that they are isomorphic.

□

Later in this seminar, we shall construct explicit Lie models for the E_n-operads and use them to compute these mapping spaces.

References

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[5] Benoit Fresse and Thomas Willwacher. "The Bousfield-Guggenheim-Hinich model of mapping spaces for Hopf cooperads and automorphisms of the rationalization of E_n-operads."