THURSDAY SEMINAR: AUTOMORPHISMS OF E_n-OPERADS

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Abstract. The thursday seminar in Spring 2020 was about the automorphisms of E_n-operads. Our main goal is to understand the computation of Fresse, Turchin and Willwacher of $\mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})$ for $m \leq n$ in terms of graph homology. These notes collect three talks: an introduction, and two talks explaining the main computation.

Warning 0.1. These notes are likely to contain many mistakes, and are sloppy in the following particular ways (and possibly many more):

- The distinction between non-unital and unital operads
- The distinction between working derived and non-derived.
- Eschewing any discussion of model structures, and checking cofibrancy or fibrancy.

My excuse for this is that these points, though interesting, only serve to cloud the answer to the core question:

> What do graph complexes have to do with the automorphisms of rationalized E_n-operads?

Warning 0.2. I followed [FTW17] in working cohomologically. In retrospect I think this is not an optimal choice, and it is clearer to work with homological grading to avoid some counter-intuitive grading conventions (e.g. a n-Poisson operad concentrated in non-positive degrees).

Part 1. Introduction

1. E_n-operads and maps between them

1.1. E_n-operads. An operad is a mathematical object that encodes an algebraic structure which has n-to-one operations for all n, as well as a way to compose such operations. The example of interest to us is the little n-disks operad D_n, which encodes an algebraic structure inspired by the following geometric operations on n-fold loop spaces [May72]:

Example 1.1. Let X be a based space, and $\Omega^n X$ the topological space of pointed map $(D^n, \partial D^n) \to (X, x_0)$ with the compact-open topology. Suppose we are given three maps $e_1, e_2, e_3 \colon D^n \to D^n$, each of which is a compositions of scaling and translation, and all of

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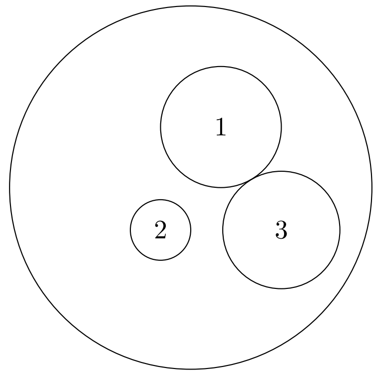
which have disjoint interior, e.g.



Then we can take a triple of maps $f_1, f_2, f_3 \colon (D^n, \partial D^n) \to (X, x_0)$ and produce a new map $f \colon (D^n, \partial D^n) \to (X, x_0)$ as follows: insert f_i suitably reparametrized on the image of e_i, and extend to the remainder of D^n by the constant map with x_0 to get a new map $f \colon (D^n, \partial D^n) \to (X, x_0)$. More precisely, it is given by

$$f(x) \coloneqq \begin{cases} f_i \circ e_i^{-1}(x) & \text{if } x \in e_i(D^n), \\ x_0 & \text{otherwise.} \end{cases}$$

This construction is continuous in the e_i's and f_i's. The former assemble to a topological space $D_n(3)$, in fact a finite-dimensional manifold homotopy equivalent to the configuration of three distinct ordered points in $\mathrm{int}(D^n) \cong \mathbb{R}^n$.

Of course, the choice of the number 3 is irrelevant for this construction: it works for any number k of little disks e_i and maps f_i. The collection of these constructions for all $k \geq 0$ is then compatible with composition: if one of the inputs f_i is obtained from this construction from maps e'_j and f'_j, we can compute the output by replacing f_i by the f'_j and e_i by the composition $e_i \circ e'_j$ (that is, compose the little n-disks).

Definition 1.2. The *little n-disks operad* is the collection of topological spaces given by

$$D_n(k) \coloneqq \left\{ \begin{matrix} k \text{ maps } e_1, \ldots, e_k \colon D_n \to D_n \\ \text{composition of scaling and translation, with disjoint interior} \end{matrix} \right\},$$

with the composition maps

$$D_n(\ell) \times (D_n(k_1) \times \cdots \times D_n(k_\ell)) \longrightarrow D_n(k_1 + \cdots + k_n)$$

given by composition of maps, and unit $1 \in D_n(1)$ given by the identity map $D^n \to D^n$.

Warning 1.3. In these notes I will not distinguish between the version of E_n given above, and the one with $E_n(0) = \varnothing$. This distinction is important for some technical steps, but will not serve to clarify the main points.

The composition operation is associative, and the identity map $\mathrm{id} \in D_n(1)$ serves as a unit for it. There is an action of the symmetric group Σ_k on $D_n(k)$, permuting the maps, and the composition is equivariant for this.

This leads to the general definition of an *operad* $\mathcal{O}$ in a symmetric monoidal category $(\mathsf{C}, \otimes, \mathrm{id})$ as a collection of objects $\mathcal{O}(k)$ for $k \geq 0$ with Σ_k-action, an associative and equivariant composition operations

$$\mathcal{O}(\ell) \otimes (\mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_\ell)) \longrightarrow \mathcal{O}(k_1 + \cdots + k_\ell) \qquad \text{for } \ell \geq 0,\ k_1, \ldots, k_\ell \geq 0,$$

and a morphism $1 \to \mathcal{O}(1)$ which serves as a unit for the composition. A *morphism of operads* is a collection of morphisms $\mathcal{O}(k) \to \mathcal{O}'(k)$ compatible with the composition, the identity, and the symmetric group actions.

The operad $\mathcal{O}$ encodes the following algebraic structure:

Definition 1.4. An $\mathcal{O}$-*algebra* is an object A of C with morphisms

$$\mathcal{O}(k) \otimes A^{\otimes k} \longrightarrow A$$

which are compatible with the composition, the identity, and the symmetric group actions.

Example 1.5. $\Omega^n X$ is an D_n-algebra. In fact, the recognition principle says that any path-connected D_n-algebra is weakly equivalent to an n-fold loop space (in fact, you only need to be group-like).

If C has a notion of weak equivalence, we say that a morphism of operads is a weak equivalence of all $\mathcal{O}(k) \to \mathcal{O}'(k)$ are. This leads to an ∞-category $\mathsf{Op}(\mathsf{C})$ of operads in C.

Remark 1.6. For technical reasons, if you want a model for this ∞-category, it is better to use as a starting point of a reformulation of the above definition of an operad. This leads to such models as ∞-operads, dendroidal sets, or dendroidal Segal spaces.

The E_n-operad is the object of $\mathsf{Op}(\mathsf{S})$ (S is our notation for a category of spaces, e.g. S or sSet) provided by the little n-disks operad:

Definition 1.7. An E_n-*operad* is any operad in $(\mathsf{S}, \times, *)$ weakly equivalent to D_n.

1.2. **Maps between E_n-operads.** In these notes we will interested in the following object:

Definition 1.8. $\mathrm{Map}^h(E_m, E_n)$ is the mapping space from E_m to E_n in $\mathsf{Op}(\mathsf{S})$.

In other words, it is a "derived" mapping space, encoding all ways to naturally consider an E_n-algebra as an E_m-algebra. One canonical way is through the operad map $i\colon E_m \to E_n$ induced by the inclusion $\mathbb{R}^m \to \mathbb{R}^n$. The following question is almost completely open, with only a few low-dimensional cases known:

Question 1.9. What is the homotopy type of $\mathrm{Map}^h(E_m, E_n)$?

Remark 1.10. In fact, preparing this talk, I realized I don't even know what $\mathrm{Map}^h(E_m, E_n)$ is when $m > n$. It is presumably empty? As evidence, since $BP_{(2)}$ is E_4 but can't be E_{12}, $\mathrm{Map}^h(E_m, E_n)$ is empty when $n \leq 4$ and $m \geq 12$.

Of particular interest is the case $m = n$. In this case we let $\mathrm{Aut}^h(E_n) \subset \mathrm{Map}^h(E_n, E_n)$ denote the homotopy-invertible path components. This might not matter:

Question 1.11. Is $\mathrm{Map}^h(E_n, E_n)$ group-like?

Example 1.12. $\mathrm{Aut}^h(E_1) \simeq O(1)$, and $\mathrm{Aut}^h(E_2) \simeq O(2)$ by a result of Horel [Hor17].

The identifications in the previous example are as group-like topological monoids. Of course, $\mathrm{Aut}^h(E_n)$ is in general such a monoid, and hence we can take its classifying space. It would be even better to know about the homotopy type of this space:

Question 1.13. What is the homotopy type of $B\mathrm{Aut}^h(E_n)$?

There is a conjecture about the case $n = 3$ [AFT17], which uses the non-obvious fact that there is a map $\mathrm{Top}(n) \to \mathrm{Aut}^h(E_n)$, where $\mathrm{Top}(n)$ is the topological group of homeomorphisms of $\mathbb{R}^n$ fixing the origin. This is compatible with the previous example, because the inclusion $O(n) \hookrightarrow \mathrm{Top}(n)$ is a weak equivalence for $n \leq 3$.

Conjecture 1.14 (Ayala–Francis–Tanaka)**.** *The map*

$$\mathrm{Top}(n) \longrightarrow \mathrm{Aut}^h(E_n)$$

is a weak equivalence if and only if $n \leq 3$.

Even the π_0-case is open and interesting:

Question 1.15. Does the map $\mathrm{Top}(n) \to \mathrm{Aut}^h(E_n)$ induce an isomorphism on π_0? It is well-known that $\pi_0 \mathrm{Top}(n) = \mathbb{Z}/2$ for all $n \geq 4$.

2. Rationalized E_n-operads and maps between them

A first instinct of a topologist is to separate the primes, and study the rational and p-local cases separately. The rational case should be easiest. One reason is that in general rational homotopy theory is easier, but a more convincing one is that the E_n-operad simplifies a lot rationally. Before going into this, a part of the proof of statements to come, we will give those statements:

2.1. **Statements of results.**

Definition 2.1. $E_n^{\mathbb{Q}}$ is the operad obtained by rationalizing each of its spaces of operations.

Warning 2.2. This is one of those constructions that takes a lot of care in some models for operad (you need sufficiently functorial rationalization functors), but less so in other models.

The universal property of rationalization then implies that precomposition with the rationalization map $E_m \to E_m^{\mathbb{Q}}$ induces a weak equivalence

$$\mathrm{Map}^h(E_m^{\mathbb{Q}}, E_n^{\mathbb{Q}}) \longrightarrow \mathrm{Map}^h(E_m, E_n^{\mathbb{Q}}).$$

Question 2.3. What is the homotopy type of $\mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})$?

Question 2.4. What is the homotopy type of $B\mathrm{Aut}^h(E_n^{\mathbb{Q}})$?

Since post-composition with rationalization gives a map $\mathrm{Map}^h(E_m, E_n) \to \mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})$ and the target is a rational space, there is an induced map

$$\mathrm{Map}^h(E_m, E_n)_{\mathbb{Q}} \longrightarrow \mathrm{Map}^h(E_m, E_n^{\mathbb{Q}}).$$

Like for ordinary mapping spaces, it is in general not a weak equivalence. However, in good circumstances it does induce a weak equivalence when we fix the path components. For example, if $m < n-2$, then $\mathrm{Map}^h(E_m, E_n; i)_\mathbb{Q} \to \mathrm{Map}^h(E_m, E_n^\mathbb{Q}; i_\mathbb{Q})$ is a weak equivalence, with $i\colon E_m \to E_n$ the standard inclusion.

It turns out that the answer to the first of the previous two questions is known, and these results the first goal of this seminar. I'll start with the formulation of the case $n = m$, which is most striking. To start with, we start with the observation that $D_n(2) \simeq S^{n-1}$, two points moving around each other. Thus an automorphism $E_n^\mathbb{Q} \to E_n^\mathbb{Q}$ induces a homotopy equivalence $S^{n-1}_\mathbb{Q} \to S^{n-1}_\mathbb{Q}$, which is classified by a unique $\lambda \in \mathbb{Q}^\times$. This gives a surjective function

$$F\colon \pi_0(\mathrm{Aut}(E_n^\mathbb{Q})) \longrightarrow \mathbb{Q}^\times.$$

This amounts to recording the effect of the automorphism on the Browder bracket. The following appears in [FTW17]:

Theorem 2.5 (Fresse–Turchin–Willwacher)**.** *Suppose $n \geq 2$ and $\lambda \in \mathbb{Q}^\times$*

- *If $n > 2$ and $n \not\equiv 1 \pmod 4$, then $F^{-1}(\lambda)$ is path-connected.*
- *If $n > 2$ and $n \equiv 1 \pmod 4$, then $F^{-1}(\lambda)$ has path-components indexed by $\mathbb{Q}$.*
- *If $n = 2$, then $F^{-1}(\lambda)$ has group of path-components given by the elements of the Grothendieck–Teichmüller group $\mathrm{GT}_1(\mathbb{Q}) \subset \mathrm{GT}(\mathbb{Q})$ whose cyclotomic character is 1.*

In all these cases, letting $\mathrm{Aut}^h(E_n^\mathbb{Q})_c$ denote the path-component of c, we have for $* > 0$,

$$\pi_*(\mathrm{Aut}^h(E_n^\mathbb{Q})_c) \cong H_k(\mathrm{GC}_n^2).$$

Remark 2.6. In fact, the extension $1 \to \mathrm{GT}_1(\mathbb{Q}) \to \pi_0(\mathrm{Aut}^h(E_n^\mathbb{Q})) \to \mathbb{Q}^\times \to 1$ in the case $n = 2$ is as expect: the middle term is $\mathrm{GT}(\mathbb{Q})$.

Remark 2.7. There are also path components of $\mathrm{Map}(E_n, E_n^\mathbb{Q})$ which are not homotopy-invertible. It is not known what their rational homotopy groups are.

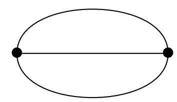
Leaving the case $n = 2$ for later discussion, let us focus on the latter statement. Suppose that $\lambda = 1$ for convenience (all components are homotopy equivalent to each other anyway). Then $\mathrm{Aut}^h(E_n^\mathbb{Q})_1$ is a path-connected rational H-space, so formal. This means that its rational homotopy groups determine its homotopy type. These homotopy groups are given by the homology of a certain chain complex of at least 2-valent graphs. We will discuss this again in more detail in Section 7.2.

Definition 2.8. The *(dual) graph complex* GC_n^2 is the chain complex generated over $\mathbb{Q}$ by connected finite graphs whose vertices are at least bivalent. A graph G with $|V|$ vertices and $|E|$ edges contributes to degree $(n-1)|E| - n|V| + n$. The edges and vertices are presumed ordered and the edges direct, and an equivalence relation takes care of an orientation induced by this: if n is even then we orient the set of edges, if n is odd we orient the set of vertices, and orient each edge individually. The differential is given by expanding vertices into an edges in all possible ways, with appropriate signs.

Remark 2.9. As we'll see, this is in fact the linear dual of a complex G_n^2. So in fact one ought to think of a graph Γ in GC_n^2 as an indicator function.

Since the differential replaces a vertex (of degree $-n$) by two vertices and an edges (of total degree $-2n + (n-1) = -n-1$), the differential decreases degree.

Example 2.10. The Θ-graph



vanish when n is even (there is a symmetry which induces an odd permuting of edges). When n is odd, it does not vanish: interchanging the vertices induces an odd permutation of vertices and changing the direction all three edges. Its differential vanishes; the reason is that each edge gets doubled twice an edge expansion, but with opposite sign.

Here are two observations regarding graph homology:

- Graph homology admits a direct sum decomposition by loop order (equivalently, genus), which is independent but related to the decomposition by degree. In particular, the 1-loop parts splits off and what remains is quasi-isomorphic to the subcomplex with all vertices of valence ≥ 3:
$$H_*(\mathrm{GC}_n^2) \cong H_*(\mathrm{GC}_n^{\text{1-loop}}) \oplus H_*(\mathrm{GC}_n^3),$$
and for $n > 2$, $H_*(\mathrm{GC}_n^{\text{1-loop}})$ is given by a single $\mathbb{Q}$ every four degrees represented by cycles with only bivalent vertices (as far as is known, no relation to algebraic K-theory or BO). This is helpful, because GC_n^3 is degree-wise finite-dimensional.
- Up to regrading GC_n^2 depends only on the parity on n. Thus it is sensible to talk about "even" and "odd" graph homology.
- Exploiting the previous point and a connection to number theory for $n = 2$, we know that $H_*(\mathrm{GC}_n^3)$ contains a free Lie algebra on infinitely many generators when n is even (the degrees of the generators depends on n, but in terms of graphs their "leading terms" are wheels with odd numbers of spokes) [Wil15].
- Computer calculations of $H_*(\mathrm{GC}_n^3)$ have been performed [BNM01], and give for example that for n odd the contributions of trivalent graphs (these are all cycles, but quotient by boundaries imposes some relations between them) to g-loop part has dimensions given by This may seem small, but it is known to generally grow

g	1	2	3	4	5	6
dim	1	1	1	2	2	3

super-exponentially.

2.2. The cases $n = 2$ and $m < n$.

Before going into the proof, I have two more things to explain. Firstly, I didn't explain the statement for $n = 2$. Secondly, I didn't state the result for $m < n$.

2.2.1. *The case* $n = 2$. Recall that for $n = 2$, after we fix $\lambda \in \mathbb{Q}^\times$ the path components are given by elements of the Grothendieck–Teichmüller group [Fre17a, Fre17b]. This group $\mathrm{GT}(\mathbb{Q})$ was defined by Drinfel'd, and has a free and transitive action on the set of rational associators. It plays a role in many parts of mathematics, including deformation quantization, representation theory, the study of multiple zeta values, and number theory.

In fact, there are many Grothendieck–Teichmüller groups, depending on a notion of completion. The one we are interested in is the rational one, arising from Malcev completion, but there are also ones arising from p-completion or profinite completion. The quickest definition using a model for the little 2-disks operad in the category of groupoids, i.e. it is an operad in the category of groupoids which whose nerve is an E_2-operad. This model the *parenthesized braids operad* PaB.

Definition 2.11. The *Grothendieck–Teichmüller group* $\mathrm{GT}(\mathbb{Q})$ is the group of automorphisms of the Malcev completion $\mathrm{PaB}^{\wedge}_{\mathbb{Q}}$.

There is a different $\mathrm{GRT}(\mathbb{Q})$, which is the associated graded of a filtration on $\mathrm{GT}(\mathbb{Q})$ and has the advantage that it is pro-unipotent. That the map $\mathrm{GT}(\mathbb{Q}) \to \mathrm{GRT}(\mathbb{Q})$ is an isomorphism follows from the existence of a rational associator. Since $\mathrm{GRT}(\mathbb{Q})$ is pro-unipotent, it is determined by Lie algebra $\mathfrak{grt}$.

One of the striking links between number theory and the Grothendieck–Teichmüller is the existence of an injective homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GT}^{\wedge}$, the profinite version. The analogous result for the rational version is a theorem of F. Brown [Bro12]:

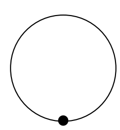
$$\mathrm{Gal}_{MT(\mathbb{Z})} \hookrightarrow \mathrm{GT}(\mathbb{Q}).$$

Here the right hand side is the motivic Galois group of the category of integral mixed Tate motives. This implies there is an inclusion of a free Lie algebra $\mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \ldots) \hookrightarrow \mathfrak{grt}$. Thus $\mathrm{GT}(\mathbb{Q})$ is a very large group. In particular, is significantly larger than $\pi_0\, O(2) = \mathbb{Z}/2$, and thus provides a striking example of the map $\mathrm{Aut}(E_n) \to \mathrm{Aut}(E_n^{\mathbb{Q}})$ not being a weak equivalence.

Are there non-trivial positive degree homotopy groups in $\mathrm{Aut}(E_2^{\mathbb{Q}})$? When $n = 2$, the grading on graphs in GC_n^2 is given by

$$|E| - 2|V| + 2 = g + 1 - |V|.$$

Thus loops live in degree $2 - |V|$. Thus the only such graph in positive degree is the following:



This is non-zero because for even n, the set of edges is oriented. It is also a cycle. This gives a contribution $\mathbb{Q} \subset \pi_1 \mathrm{Aut}(E_2^{\mathbb{Q}})$, which is in fact hit by $\mathbb{Q} = \pi_1\, O(2)_{\mathbb{Q}}$.

To understand the contributions of GC_2^3, we give the "bottom" and "top" degrees of the g-loop contributions. The "top" of the graph complex are the graphs with a single vertex, which live in degree g. The "bottom" of the graph complex are the trivalent graphs. Such a graph of genus g has $2g - 2$ vertices, and hence live in degree $-g - 1$. So at first, we think that there could be many non-trivial higher-degree homotopy groups, contributing from the top half of the graph complex. These all vanish, by the following result of Willwacher [Wil15].

Theorem 2.12 (Willwacher)**.** *$H_*(\mathrm{GC}_2^3) = 0$ for $* > 0$ and $H_0(\mathrm{GC}_2^3) = \mathfrak{grt}$.*

Remark 2.13. A different proof was given by Chan–Galatius–Payne, using the virtual cohomological dimension of moduli spaces of curves [CGP18].

This is also the source of the many graph homology classes for n even: $H_0(\mathrm{GC}_2^3)$ contains $\mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \ldots)$ by F. Brown's theorem, and

$$H_0(\mathrm{GC}_2^{3,g\text{-loop}}) \cong H_{g(n-2)}(\mathrm{GC}_2^{3,g\text{-loop}}).$$

Conjecture 2.14 (Ihara–Deligne)**.** *$H_{-1}(\mathrm{GC}_2^3) = 0$.*

2.2.2. *The case $m < n$.* There is also a version of Theorem 2.5 for the case

$$\mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})$$

when $m < n$. The answer is given in terms of a version of GC_n^2 called $\mathrm{HGC}_{m,n}^2$, a complex of hairy graphs:

Definition 2.15. The *(dual) hairy graph complex* $\mathrm{HGC}_{m,n}^2$ is the chain complex generated by connected finite graphs. The vertices can have arity 1, in which case we call that vertex external, and the edge adjacent to it a hair. The degree is determined as follows: an internal edges contribute $n-1$, internal vertices $-n$, each hair $n-1$, and external vertex $-m$. The orientations on internal vertices and edges are as in GC_n^2 and when m is odd there is an orientation on the external vertices. The differential is as in GC_n^2.

Theorem 2.16. *If $n-2 \geq m \geq 1$, then $\mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})$ is simply-connected with homotopy groups given by*

$$\pi_*(\mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})) \cong H_*(\mathrm{HGC}_{m,n}^2).$$

When $m = n-1$, then there is a surjective function $F\colon \pi_0(\mathrm{Map}^h(E_{n-1}, E_n^{\mathbb{Q}})) \to \mathbb{Q}^\times$. For $\lambda \in \mathbb{Q}^\times$, $F^{-1}(\lambda)$ is path-connected and we have

$$\pi_*(\mathrm{Map}^h(E_{n-1}, E_n^{\mathbb{Q}})_\lambda) \cong H_*(\mathrm{HGC}_{n-1,n}).$$

Remark 2.17. It is helpful to observe at this point that not only is $\mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})$ simply-connected when $n - m \geq 2$, but so is $\mathrm{Map}^h(E_m, E_n)$. Thus $\mathrm{Map}^h(E_m, E_n)_{\mathbb{Q}}$ has the same rational homotopy groups as $\mathrm{Map}^h(E_m, E_n)$. With a little bit of work, the arguments that establish this also tell one that

$$\mathrm{Map}^h(E_m, E_n)_{\mathbb{Q}} \longrightarrow \mathrm{Map}^h(E_m, E_n^{\mathbb{Q}})$$

is a weak equivalence.

2.3. **An outline of the proof.** Finally, I will explain why rationalizing helps a lot when considering E_n-operads. The reason is formality.

2.3.1. *Rationalized operads.* To understand spaces rationally, particular algebraic models can be useful. This will also be true of our rational operads. We need to do a particular computation, and this should make it permissible to use a particular model for operads of rational operads.

To every topological space we can assign a rational commutative dg-algebra (CDGA) $C^*(X; \mathbb{Q})$ of cochains.

Remark 2.18. If you want it be strictly commutative instead of up to coherent homotopy, you need to use a functor $A^*_{\mathrm{PL}}(X)$ of polynomial de Rham forms, a version of de Rham forms but over the rationals and valid for general spaces:

$$\begin{aligned} \mathsf{S}^{\mathrm{op}} &\longrightarrow \mathsf{CDGA} \\ X &\longmapsto A^*_{\mathrm{PL}}(X) \coloneqq \mathrm{Sing}(X) \otimes_\Delta A^*_{\mathrm{PL}}(\Delta^\bullet). \end{aligned}$$

We can localize S at the rational equivalence, and study rational spaces instead. This retains all information about rational cohomology and rational homotopy groups, at least for 1-connected spaces (in fact, nilpotent suffices). For 1-connected spaces, all this can be recovered from $C^*(X;\mathbb{Q})$, in the following precise sense:

Theorem 2.19 (Sullivan)**.** *Let $\mathsf{S}^{\mathbb{Q}}_{f,>1}$ be the ∞-category of finite type rational 1-connected spaces, and $\mathsf{CDGA}_{f,>1}$ the ∞-category of finite type 1-connected CDGA's. Then rational cochains induces an equivalence of ∞-categories*

$$\mathsf{S}^{\mathbb{Q}}_{f,>1} \xrightarrow{\sim} \mathsf{CDGA}_{f,>1}.$$

Given an operad $\mathcal{O}$ in spaces, we can apply $C^*(X;\mathbb{Q})$ to get a cooperad in CDGA. This requires a bit care to make precise, and is best handled by ∞-categories:

$$\begin{aligned} \mathsf{Op}(\mathsf{S})^{\mathrm{op}} &\longrightarrow \mathsf{CoOp}(\mathsf{CDGA}) \\ \{\mathcal{O}(n)\} &\longmapsto \{C^*(X;\mathbb{Q})\}. \end{aligned}$$

We could ask for an analogue of Sullivan's theorem. This exists by work of Fresse [Fre18a]:

Theorem 2.20 (Fresse)**.** *Let $\mathsf{Op}(\mathsf{S}^{\mathbb{Q}})_{f,>1}$ be the ∞-category of operads $\mathcal{O}$ with each $\mathcal{O}(n)$ a finite type rational 1-connected space and such that $\mathcal{O}(0) = \varnothing$. Then $\mathsf{CoOp}(\mathsf{CDGA})_{f,>1}$ denote the ∞-category of cooperads $\mathcal{C}$ with each $\mathcal{C}^*(n)$ a finite type 1-connected CDGA's and $\mathcal{C}^*(0) = 0$. Then $X \mapsto C^*(X;\mathbb{Q})$ induces an equivalence of ∞-categories*

$$\mathsf{Op}(\mathsf{S}^{\mathbb{Q}})_{f,>1} \xrightarrow{\sim} \mathsf{CoOp}(\mathsf{CDGA})_{f,>1}.$$

In particular, this says that there is a weak equivalence of derived mapping spaces:

$$\mathrm{Map}^h(E_m, E_n^{\mathbb{Q}}) \simeq \mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(C^*(E_n;\mathbb{Q}), C^*(E_m;\mathbb{Q})).$$

2.3.2. *Formality.* Now we finally make precise in what sense $E_n^{\mathbb{Q}}$ is simpler than E_n; it is a formal operad.

You may be familiar with formality for spaces. There is of course another natural CDGA associated to X in addition to $C^*(X;\mathbb{Q})$: the rational cohomology ring $H^*(X;\mathbb{Q})$, with zero differential.

Definition 2.21. A space X is *formal* if there is a zig-zag of weak equivalences $C^*(X;\mathbb{Q}) \simeq H^*(X;\mathbb{Q})$.

Example 2.22. Spheres S^n are formal, as is any H-space or compact Kähler manifold. More importantly for us, the configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$ is formal for $n \geq 2$.

We could similarly ask for whether the cooperad of CDGA's $C^*(\mathcal{O};\mathbb{Q})$ is weakly equivalent to the cohomology cooperad $H^*(\mathcal{O};\mathbb{Q})$. If this is the case we say that $\mathcal{O}$ is formal.

Remark 2.23. This is to be distinguished from $C_*(\mathcal{O};\mathbb{Q})$ being weakly equivalent to $H_*(\mathcal{O};\mathbb{Q})$, a different notion of formality for operads.

The following is proven in [LV14, Tam03]:

Theorem 2.24 (Kontsevich, Tamarkin, Lambrechts–Volic, Fresse–Willwacher)**.** *E_n is formal, and its cohomology operad is the n-Poisson cooperad* Pois_n^c*.*

Remark 2.25. Slightly subtle is that the Kontsevich proof (with details filled in by Lambrechts and Volic) only works over the reals. Hence we need to resort to the other proofs if we want a rational result.

This formality allows us to replace the cochains of E_n by its cohomology cooperad, which is the n-Poisson cooperad. We obtain that:

$$\mathrm{Map}^h_{\mathsf{CoOp}}(C^*(E_n;\mathbb{Q}), C^*(E_m;\mathbb{Q})) \simeq \mathrm{Map}^h_{\mathsf{CoOp}}(\mathrm{Pois}_n^c, \mathrm{Pois}_m^c).$$

The proof by Kontsevich and Lambrechts–Volic uses an intermediate cooperad of graphs. Related constructions will play an important role in the computation, and are in fact the source of the graphs in the graph complexes.

Remark 2.26. Kontsevich's original motivation was deformation quantization, see [Kon03] or [Fre18b].

2.3.3. *Computing derived mapping spaces.* Let us outline the computation of the derived mapping spaces

$$\mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_\mathbb{Q})}(\mathrm{Pois}_n^c, \mathrm{Pois}_m^c),$$

many details to be given in the next two talks. We shall mostly follow [FW20].

(I) The first step is to pick particular models. For the left hand side we pick a cooperad $\mathrm{Graphs}_n^{c,2}$ of graphs closely related to the one that appears in Kontsevich's proof of formality (it happens to be cofibrant). For the right hand side [FTW17] picks a cooperadic W-construction $W^c(\mathrm{Pois}_m^c)$ but we will use the bar-cobar construction (it happens to be fibrant):

$$\mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_\mathbb{Q})}(\mathrm{Pois}_n^c, \mathrm{Pois}_m^c) \simeq \mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_\mathbb{Q})}(\mathrm{Graphs}_n^{c,2}, B\Omega\mathrm{Pois}_m^c).$$

(II) The first reason for picking these larger models is that they are sufficiently free, resp. cofree, that we can compute the right hand side in terms of maps from the generators IG_n^2 of the domain to the cogenerators $\Omega\mathrm{Pois}_m^c$ of the target, modeled by a Maurer–Cartan space.

$$\mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_\mathbb{Q})}(\mathrm{Graphs}_n^{c,2}, B\Omega\mathrm{Pois}_m^c) \simeq \mathrm{MC}_\bullet(\mathrm{Map}_{\mathsf{SymSeq}(\mathsf{CoCh}_\mathbb{Q})}(\mathrm{IG}_n^2, \Omega\mathrm{Pois}_m^c)).$$

(III) Koszul duality for Poisson cooperads gives a weak equivalence $\Omega\mathrm{Pois}_m^c \longrightarrow \Lambda^m\mathrm{Pois}_m$, so we simplify this to

$$\mathrm{MC}_\bullet(\mathrm{Map}^h_{\mathsf{SymSeq}(\mathsf{CoCh}_\mathbb{Q})}(\mathrm{IG}_n^2, \Omega\mathrm{Pois}_m^c)) \simeq \mathrm{MC}_\bullet(\mathrm{Map}^h_{\mathsf{SymSeq}(\mathsf{CoCh}_\mathbb{Q})}(\mathrm{IG}_n^2, \overline{\Lambda^m\mathrm{Pois}_m}[1])).$$

(IV) The second reason for picking these larger models is that we can easily map in a hairy graph complex HCG_n^2, and then prove by a spectral sequence comparison argument that this is an equivalence.

(V) We then prove that the hairy graph complex HCG_n^2 is equivalent to GC_n^2.

3. Applications to spaces of embeddings

An important motivation for studying maps between E_n-operads is their appearance in embedding calculus.

3.1. **A quick introduction to embedding calculus.** Embedding calculus is the study of embeddings by their restrictions to little disks in the domain. (So the appearance of E_n, as a local situation, is not so surprising.)

The object of interest is the following: the ∞-category Mfd_n of n-dimensional manifolds and embeddings contains a full subcategory Disk_n whose objects are disjoint unions of $\mathbb{R}^n$'s (open disks). By restriction $\mathrm{Emb}(-, M)$ is a presheaf on Disk_n, and it's in terms of these presheaves that embedding calculus is defined (for convenience we will take $\dim M = n = \dim N$, which can always be arranged by replacing M by a thickening) [BdBW13]:

Definition 3.1. The *limit of the Taylor tower* $T_\infty \mathrm{Emb}(M, N)$ is defined as the derived mapping space

$$\mathrm{Map}^h(\mathrm{Emb}(-, M), \mathrm{Emb}(-, N))$$

in $\mathsf{PreSh}(\mathsf{Disk}_n)$.

There is a natural map

$$\mathrm{Emb}(M, N) \longrightarrow T_\infty \mathrm{Emb}(M, N),$$

and the convergence results of Goodwillie–Klein–Weiss say that this is an equivalence if the handle dimension of M is $\leq n - 3$ [GW99, GK15].

Remark 3.2. The Taylor tower is obtained by filtering Disk_n by cardinality.

3.2. **Configuration categories.** Though there are several equivalent setups, the one that makes the connection between embedding calculus and E_n-operads most clear is that of configuration categories [BdBW18].

The starting point here is the Ran space

$$\mathrm{Ran}(M) = \operatorname*{colim}_{\underline{k} \in \mathsf{FinSet}_{\mathrm{surj}}^{\mathrm{op}}} M^{\underline{k}}.$$

We didn't specify the category in which we are taking the colimit. If we had just used S, the result is just the infinite symmetric product. However, $M^{\underline{k}}$ is naturally a stratified topological space, with stratification encoded by the map from $M^{\underline{k}}$ to the poset of partitions of $\underline{k}$ recording which particles are in the same location. This map should be continuous if we put on the partition poset the topology where open subsets are those collections U of partitions closed under passing to refinements. We will then take this colimit in the category $\mathsf{StratTop}$ of stratified topological spaces.

Any stratified topological spaces X has a *exit path* ∞-category $\mathsf{Exit}(X)$. Its space of objects is the disjoint union of the strata X_α of X, and its space of morphisms from X_α to X_β are continuous paths $\gamma\colon [0, 1] \to X$ such that (i) $\gamma(0) \in X_\alpha$ and $\gamma(1) \in X_\beta$, and (ii) for all $s \leq t$, the stratum containing $\gamma(s)$ is contained in the closure of the stratum containing $\gamma(t)$. The object we will want to consider is a variation on the exit path category of $\mathrm{Ran}(M)$: its strata are unordered configuration spaces and its morphisms are paths of configurations where once points move apart they can't collide again.

This variation makes two modifications: (i) we will remember the identification of the points with a finite set, (ii) we will allow ourselves to add new points. That is,

Definition 3.3. The *configuration category* $\mathsf{Con}(M)$ is the ∞-category over FinSet with objects

$$\bigsqcup_S \mathrm{Emb}(S, M),$$

and morphisms

$$\bigsqcup_{f\colon S\to T} \left\{ (x_S, x_T, \gamma) \;\middle|\; \begin{array}{c} \text{reverse exit path } \gamma \text{ from} \\ x_S \in \mathrm{Emb}(S,M) \text{ to } \vec{x}_T \circ f \text{ with} \\ x_T \in \mathrm{Emb}(T,M). \end{array} \right\}$$

That is, a morphism from x_S to x_T is a "sticky" path from x_S to a subset of the point in x_T. The function $f\colon S \to T$ keeps track of which points in x_T are hit and which points in x_S collide.

Remark 3.4. Observe that $\mathsf{con}(M)$ only depends on M as a topological manifold.

These configuration categories will serve to produce approximations to spaces of embeddings: the space of functors

$$\mathrm{Map}^h_{/\mathsf{FinSet}}(\mathsf{con}(M), \mathsf{con}(N))$$

receives a map from $\mathrm{Emb}(M, N)$, and embedding calculus aims to understand to what extent this is an equivalence. When $M = N$ we can further consider the automorphisms

$$\mathrm{Aut}^h_{/\mathsf{FinSet}}(\mathsf{con}(M)).$$

It is these mapping spaces and automorphisms that are related to E_n-operads:

Lemma 3.5. $\mathrm{Map}^h_{/\mathsf{FinSet}}(\mathsf{con}(\mathbb{R}^m), \mathsf{con}(\mathbb{R}^n)) \simeq \mathrm{Map}^h(E_m, E_n)$ *and* $\mathrm{Aut}^h_{/\mathsf{FinSet}}(\mathsf{con}(\mathbb{R}^n)) \simeq \mathrm{Aut}^h(E_n)$.

Proof indication. $\mathsf{con}(\mathbb{R}^n)$ is essentially the PROP built from E_n, and when the 0-ary and 1-ary operations of $\mathcal{O}$ and $\mathcal{P}$ are contractible then maps between the PROPs associated to $\mathcal{O}$ and $\mathcal{P}$ are the same as the maps between the operad $\mathcal{O}$ and $\mathcal{P}$. $\square$

Remark 3.6. It should now be evident that there is a map $\mathrm{Top}(n) \to \mathrm{Aut}^h(E_n)$: obviously the homeomorphisms of $\mathbb{R}^n$ acts on the configuration category of $\mathbb{R}^n$.

One thing that the space of functor

$$\mathrm{Map}^h_{/\mathsf{FinSet}}(\mathsf{con}(M), \mathsf{con}(N))$$

doesn't capture, is the derivative of an embedding. We can add it in to get an even better approximation to embeddings:

Theorem 3.7 (Boavida de Brito–Weiss)**.** *There is a homotopy pullback square*

$$\begin{tikzcd} T_\infty \mathrm{Emb}(M,N) \arrow[r] \arrow[d] & \mathrm{Map}^h_{/\mathsf{FinSet}}(\mathsf{con}(M), \mathsf{con}(N)) \arrow[d] \\ \mathrm{Bun}(TM, TN) \arrow[r] & \mathrm{Map}^h_{/\mathsf{FinSet}}(\mathsf{con}^{\mathrm{loc}}(M), \mathsf{con}^{loc}(N)). \end{tikzcd}$$

One obtained in this theorem hasn't appeared yet: $\mathsf{con}^{\mathrm{loc}}(M)$. It is the "comma category" whose objects are morphisms $\{x_S, x_{\{1\}}, \gamma\}$ for $f \colon S \to \{1\}$. Through the equivalence of the previous lemma, this is the space of invertible maps from the family of E_n-operads indexed by TM from the family indexed by TN.

Thus, just as there is a fiber sequence

$$\mathrm{Lin}^{\mathrm{inj}}(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow \mathrm{Bun}(TM, TN) \longrightarrow \mathrm{Map}(M, N),$$

there is a fiber sequence

$$\mathrm{Map}^h(E_m, E_n) \longrightarrow \mathrm{Map}^h_{/\mathsf{FinSet}}(\mathsf{con}^{\mathrm{loc}}(M), \mathsf{con}^{loc}(N)) \longrightarrow \mathrm{Map}(M, N).$$

Example 3.8. If $M = N = D^n$ and we work relative to the boundary, the Alexander trick for configuration categories tells us that $\mathrm{Map}^h_{\partial,/\mathsf{FinSet}}(\mathsf{con}(D^n), \mathsf{con}(D^n)) \simeq *$, and we get a fiber sequence

$$T_\infty \mathrm{Emb}_\partial(D^n, D^n) \longrightarrow \Omega^n O(n) \longrightarrow \Omega^n \mathrm{Aut}^h(E_n).$$

Example 3.9. If instead we take $m \leq n - 3$, convergence of the embedding calculus tower gives us a fiber sequence

$$\mathrm{Emb}_\partial(D^m, D^n) \longrightarrow \Omega^m \mathrm{InjLin}(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow \Omega^m \mathrm{Map}^h(E_m, E_n).$$

That is, up to some easy contributions from a Stiefel manifold, the homotopy groups of automorphisms of the mapping spaces $\mathrm{Map}^h(E_m, E_n)$ are those of the spaces of "long knots."

4. Further open problems

Question 4.1. What happens in the p-complete or profinite setting?

The case $n = 2$ is treated in work of Horel [Hor17], and the answer is the direct analogue of the rational case.

Willwacher has recently announced some results related to the following question:

Question 4.2. What happens for other manifolds? That is, we can associate to a framed manifold a right E_n-module E_M, and one may wonder about $\mathrm{Aut}^h(E_M)$, $\mathrm{Aut}^h(E_M^{\mathbb{Q}})$, or various mapping spaces between such modules.

Graph homology appeared in the study of embedding spaces and diffeomorphisms in a different but tantalizingly similar way: through configuration space integrals. To make this more concrete, if we let G_n^3 denote the predual cochain complex (so the differential is given by collapsing edges), then Kontsevich constructed for every disk bundle $E \to B$ with B a manifold (and which admits a propagator),[1] a map of cochain complexes

$$G_n^3 \longrightarrow \Omega^*_{dR}(B).$$

That is, he constructed characteric classes of disk bundles (which admit a propagator). Watanabe has given a number of examples of disk bundles on which these evaluate non-trivial, in high odd dimensions and dimension 4 [Wat09, Wat18].

Question 4.3. What is the precise relationship between the appearence of graph cohomology here and the graph homology which appears in the automorphisms of E_n-operads?

[1] This is parenthical because we nowadays know a better way of phrasing these results that removes the need for this condition.

Part 2. Graph complexes

The next part picks up after a few talks in the seminar, and we recap what has been discussed in those talks. From now on, we specialize to $n = m$, so automorphisms of $E_n^{\mathbb{Q}}$ instead of mapping spaces, and $n \geq 3$

Recall that we want compute the homotopy type of the monoid

$$\mathrm{Aut}(E_n^{\mathbb{Q}}) \coloneqq \mathrm{Map}^h_{\mathsf{Op}(\mathsf{S})}(E_n^{\mathbb{Q}}, E_n^{\mathbb{Q}})^\times$$

of homotopy-invertible (derived) self-maps of the rationalized E_n-operad, for $n \geq 3$. In Jun Hou's talk we learned that for $n \geq 3$, we can replace topological operads with cooperads in $\mathsf{CDGA}_{\mathbb{Q}}$ (E_∞-algebras in rational chain complexes or equivalently CDGA's):

$$\mathrm{Aut}(E_n^{\mathbb{Q}}) \simeq \mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(C^*(E_n; \mathbb{Q}), C^*(E_n; \mathbb{Q}))^\times.$$

In Lucy's talk, we learned that E_n-operad was formal over the reals: the Kontsevich admissible graphs cooperad $\mathrm{Graphs}^c_{n,\mathbb{R}}$ is both weakly equivalent to the the cooperad of real cochains on E_n, via configuration space integrals, and weakly equivalent to its cohomology:

$$C^*(E_n; \mathbb{R}) \xleftarrow{\simeq} \mathrm{Graphs}^c_{n,\mathbb{R}} \xrightarrow{\simeq} H^*(E_n; \mathbb{R}) \xrightarrow{\cong} \mathrm{Pois}^c_{n,\mathbb{R}},$$

where I have used to that in Dexter's talk we saw the cohomology operad of E_n is the n-Poisson cooperad. In fact, Lucy stated an even stronger result: that E_n is intrinsically formal. This is proven by obstruction theory, and since the obstruction groups over $\mathbb{R}$ are just tensored up from those over $\mathbb{Q}$, this in particular implies that we have weak equivalences as above over the rationals. So for the sake of simplicity, I'll use rational coefficients instead of real ones.

We are going to use the full range of these weak equivalences to our advantage: we may replace the $C^*(E_n; \mathbb{Q})$ in the domain and target of $\mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(C^*(E_n; \mathbb{Q}), C^*(E_n; \mathbb{Q}))^\times$ with our favorite weakly equivalent model. It'll turn out be convenient to *not* take the smallest possible choice, Pois^c_n, but instead take choices inspired by two considerations. Both tend to bias "larger" models:

Large strict automorphisms: If we have a strict action of a simplicial group G on $\mathcal{C}$, then for each $f \in \mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D})^\times$ we get a map

$$\begin{aligned} G &\longrightarrow \mathrm{Map}^h_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D})^\times \\ g &\longmapsto f \circ g. \end{aligned}$$

It would be great if we could pick a model for $\mathcal{C}$ nice enough that it admits an action of a large enough G so that the above map is weak equivalence.

Thus our first objective will be to find a model for $C^*(E_n; \mathbb{Q})$ with many strict automorphisms.

Freeness and cofreeness: The structure of a cooperad is quite involved, and we would like to trade away some of it at the cost of making $\mathcal{C}$ and $\mathcal{D}$ larger.

Jun Hou also explained that morphisms of cooperads can be computed as Maurer–Cartan elements

$$\mathrm{Hom}^h_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{MC}(\mathrm{Def}(\mathcal{C}, \mathcal{D})),$$

in an L_∞-algebra $\mathrm{Def}(\mathcal{C}, \mathcal{D})$ of biderivations.

This L_∞-algebra takes a simple form in the following special case. Recall that an object is refered to as *quasi-free* or *quasi-cofree* if it is so after forgetting the differentials.[2] Suppose $\mathcal{C}$ is a cooperad that is arity-wise quasi-free on a symmetric sequence $V \in \mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})$ of graded vector spaces, and suppose that $\mathcal{D}$ is a cooperad which is quasi-cofree cogenerated by a symmetric sequence $W \in \mathsf{SymSeq}(\mathsf{GrAlg}_{\mathbb{Q}})$ of graded-commutative algebras, then the underlying graded-vector space of $\mathrm{Def}(\mathcal{C}, \mathcal{D})$ is given by

$$\mathrm{Def}(\mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(V, W).$$

As symmetric sequences are much easier to understand than cooperads, our second objective will hence be to find models for $C^*(E_n; \mathbb{Q})$ which are arity-wise quasi-free, resp. quasi-cofree.

Example 4.4. Eventually, we will take the domain

$$\mathcal{C} = \mathrm{Graphs}_n^{c,2}$$

very similar to $\mathcal{D}_n$ from the previous lecture, which has an action of the graph complexes GC_n^2 we introduced in the first lecture. We will take the target to be the bar-cobar-construction for cooperads

$$\mathcal{D} = B\Omega(\mathrm{Pois}_n^c).$$

Convention 4.5. I will not try to be precise about signs, since it is difficult to parse the formula's anyway. In general, all should arise from the following:

(i) the Koszul sign rule,
(ii) the orientations of graphs,
(iii) any operation that is performed on graph involves moving the vertices and edges involved to the front/back of the order (depending on which side of a tensor product those vertices and edges appear).

5. Mapping spaces of cooperads and actions

From an action of a DGLA $\mathfrak{g}$ in cochain complexes on a cooperad $\mathcal{C}$ in CDGA by biderivations, we construct a simplicial group $Z_\bullet(\mathfrak{g})$ with map to the simplicial monoid $\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{C})^\times$ of the homotopy-invertible endomorphisms, and more generally, given a weak equivalence $h \colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ we give a map

$$Z_\bullet(\mathfrak{g}) \longrightarrow \mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D})^\times. \tag{1}$$

We will then give a criterion to determine whether this is a weak equivalence, in the case that $\mathcal{C}$ is arity-wise quasi-free and $\mathcal{D}$ is quasi-cofree.

5.1. The construction of the map (1).

5.1.1. *A model for mapping spaces of cooperads.* We have avoided talking about a model for the (derived) mapping spaces

$$\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D})$$

[2]Warning: these notions are not homotopy-invariant, so in general on should probably work with filtered objects whose associated graded is free (the filtration here being simply by degree).

so far. However, its construction is quite straightforward. For any cooperad $\mathcal{C}$ and CDGA A, $\mathcal{C} \otimes A$ with r-ary cooperations given by CDGA $\mathcal{C}(r) \otimes A$ is also a cooperad *in* CDGA_A, the CDGA's over A. The cooperad structure is given by A-linearly extending the cooperations.

Definition 5.1. We define a mapping space

$$\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D}) \in \mathsf{sSet}$$

by taking its p-simplices to be

$$\mathrm{Hom}_{\mathsf{CoOp}(\mathsf{CDGA}_{\Omega^*(\Delta^p)})}(\mathcal{C} \otimes \Omega^*(\Delta^p), \mathcal{D} \otimes \Omega^*(\Delta^p)).$$

The following is a consequence of some model-categorical considerations:

Lemma 5.2 (Fresse–Willwacher)**.** *The simplicial set* $\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D})$ *computes the derived mapping space when* $\mathcal{C}$ *is arity-wise cofree and* $\mathcal{D} = W(\mathcal{D}')$*.*

5.1.2. *Simplicial groups from DGLA's.* Suppose that $\mathfrak{g}$ is a pro-nilpotent DGLA in cochain complexes. That is, its differential increases degree, it comes with a complete descending filtration

$$\mathfrak{g} = F_1\mathfrak{g} \supset F_2\mathfrak{g} \supset F_3\mathfrak{g} \supset \cdots,$$

by DGLA's, such that $[F_p\mathfrak{g}, F_q\mathfrak{g}] \subset F_{p+q}\mathfrak{g}$. This implies that the associated graded is nilpotent. Then we assign to it a group

$$\begin{aligned} Z\colon \mathsf{LieAlg}(\mathsf{CoCh}_{\mathbb{Q}})_{\text{pro-nilp}} &\longrightarrow \mathsf{Grp} \\ \mathfrak{g} &\longmapsto \left(\{\gamma \in \mathfrak{g}^0 \mid d\gamma = 0\}, *\right), \end{aligned}$$

with product $*$ given by the Baker–Campbell–Hausdorff formula

$$\gamma * \eta = \mathrm{BCH}(\gamma, \eta) = \gamma + \eta + \frac{1}{2}[\gamma, \eta] + \cdots.$$

For example, the unit is 0 and inverse of γ is just $-\gamma$. That $\mathfrak{g}$ has a complete filtration compatible with the bracket is necessary for the convergence of this series.

A DGLA can be tensored with a CDGA; the result $\mathfrak{g} \otimes A$ is again a DGLA with differential as usual and $[\gamma \otimes a, \eta \otimes b] = (-1)^{|a||\eta|}[\gamma, \eta] \otimes ab$. If $\mathfrak{g}$ comes with a filtration as above, this may not be complete and we should instead take the completion $\mathfrak{g} \hat{\otimes} A$ of the filtered object $(F_p\mathfrak{g} \otimes A)_{p \geq 1}$. Thus we can use the standard simplicial CDGA $\Omega^*(\Delta^\bullet)$ to built a simplicial group

$$\begin{aligned} Z\colon \mathsf{LieAlg}(\mathsf{CoCh}_{\mathbb{Q}})_{\text{pro-nilp}} &\longrightarrow \mathsf{sGrp} \\ \mathfrak{g} &\longmapsto Z_\bullet(\mathfrak{g}) = Z(\mathfrak{g} \hat{\otimes} \Omega^*(\Delta^\bullet)). \end{aligned}$$

5.1.3. *DGLA's acting on cooperads.* Next suppose that the pro-nilpotent DGLA $\mathfrak{g}$ acts nilpotently on a cooperad $\mathcal{C} \in \mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})$ by biderivations, in the strong sense that for all $c \in \mathcal{C}$ we have $\mathrm{ad}(F_p\mathfrak{g})(c) = 0$ for some $p \geq 1$.

This already uses our notation for the action: the action of $x \in \mathfrak{g}$ on $c \in \mathcal{C}$ is denoted $\mathrm{ad}(\gamma)(c)$; this is the value of the map

$$\mathrm{ad}\colon \mathfrak{g} \otimes \mathcal{C}(r) \longrightarrow \mathcal{C}(r)$$

on the elements $\gamma \otimes c$.

Recall that a biderivation is a map which is a derivation for the commutative algebra structure arity-wise, and a coderivation for the cooperad structure. That is, we have

$$\mathrm{ad}(\gamma)(c \cdot c') = \mathrm{ad}(\gamma)(c) \cdot c + (-1)^{|\gamma||c|} c \cdot \mathrm{ad}(\gamma)(c'),$$

and if $\Delta(c) = \sum_j c_j \otimes (c_{r_1} \otimes \cdots \otimes c_{r_j})$, then

$$\begin{aligned}\Delta(\mathrm{ad}(\gamma)(c)) = &\sum_j \mathrm{ad}(\gamma)(c_j) \otimes (c_{r_1} \otimes \cdots \otimes c_{r_j}) \\ &+ \sum_j \sum_i (-1)^{|\gamma|(|c_j|+|c_{r_1}|+\ldots+|c_{r_{i-1}}|)} c_j \otimes (c_{r_1} \otimes \cdots \otimes \mathrm{ad}(\gamma)(c_{r_i}) \otimes \cdots \otimes c_{r_j}).\end{aligned}$$

That this is an action means that ad is linear, compatible with the differential, and $[\mathrm{ad}(\gamma), \mathrm{ad}(\eta)] = \mathrm{ad}([\gamma, \eta])$, where the left hand side is the commutator of linear maps $\mathcal{C} \to \mathcal{C}$.

Lemma 5.3. *Assigning $\gamma \in Z(\mathfrak{g})$ the automorphism of $\mathcal{C}$ given by*

$$c \longmapsto \exp(\mathrm{ad}(\gamma))(c) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}(\gamma)^n (c)$$

gives an homomorphism

$$Z(\mathfrak{g}) \longrightarrow \mathrm{Hom}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{C})^{\times}.$$

Proof sketch. The nilpotency of the action makes the sum converge. The BCH formula is defined such that

$$\exp(\mathrm{ad}(\gamma)) \exp(\mathrm{ad}(\eta)) = \exp(\mathrm{BCH}(\mathrm{ad}(\gamma), \mathrm{ad}(\eta)),$$

and since $\mathrm{ad}(-)$ is an action we have

$$\mathrm{BCH}(\mathrm{ad}(\gamma), \mathrm{ad}(\eta)) = \mathrm{ad}(\mathrm{BCH}(\gamma, \eta)) = \mathrm{ad}(\gamma * \eta).$$

This implies that $\exp(\mathrm{ad}(-))$ is compatible with composition and thus lands in the invertible endomorphisms.

Let's check that if $\mathrm{ad}(\gamma)$ is a derivation of the commutative algebra structure, then $\exp(\mathrm{ad}(\gamma))$ is a homomorphism. That it is a map of cooperads if $\mathrm{ad}(\gamma)$ is a coderivation of the cooperad structure is the dual argument: since $\mathrm{ad}(\gamma)$ is a derivation, we have that $\mathrm{ad}(\gamma)^k(c \cdot c') = \sum_{i+j=k} \binom{k}{i} \mathrm{ad}^i(c)\mathrm{ad}^j(c')$ (there are no signs since $|\gamma| = 0$). Dividing by $k!$ we get $\sum_{i+j=k} \frac{1}{i!j!} \mathrm{ad}^i(c)\mathrm{ad}^j(c')$ and it is now evident that

$$\exp(\mathrm{ad}(\gamma))(c \cdot c') = \exp(\mathrm{ad}(\gamma))(c) \cdot \exp(\mathrm{ad}(\gamma))(c'). \qquad \square$$

If $\mathfrak{g}$ acts on $\mathcal{C}$, then $\mathfrak{g} \hat{\otimes} A$ acts on $\mathcal{C} \hat{\otimes} A$ by

$$\mathrm{ad}(\gamma \otimes a)(c \otimes a') = (-1)^{|a||c|} \mathrm{ad}(\gamma)(c) \otimes aa'.$$

This is evidently a map of A-algebras, so by tensoring with the simplicial CDGA $\Omega^*(\Delta^\bullet)$ we get a map

$$\begin{aligned}Z_\bullet(\mathfrak{g}) = Z(\mathfrak{g} \hat{\otimes} \Omega^*(\Delta^\bullet)) &\longrightarrow \mathrm{Hom}_{\mathsf{CoOp}(\mathsf{CDGA}_{\Omega^*(\Delta^\bullet)})}(\mathcal{C} \otimes \Omega^*(\Delta^p), \mathcal{C} \otimes \Omega^*(\Delta^\bullet))^{\times} \\ &= \mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{C})^{\times}.\end{aligned}$$

Finally, given a weak equivalence $h\colon \mathcal{C} \to \mathcal{D}$ we get a map

$$\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C},\mathcal{C})^{\times} \longrightarrow \mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C},\mathcal{D})^{\times}.$$

Composing by it gives (1):

$$Z_{\bullet}(\mathfrak{g}) \longrightarrow \mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C},\mathcal{D})^{\times}.$$

5.2. **Return to Maurer–Cartan spaces.** We now return to the setting discussed by Jun Hou. Firstly, we assume that $\mathcal{C}$ is arity-wise quasi-free, that is, forgetting the differential $\mathcal{C}(r)$ is a free graded-commutative algebra $S(V(r))$ for some symmetric sequence V in graded vector spaces. Secondly, we assume that $\mathcal{C}$ is quasi-cofree, that is, forgetting the differential $\mathcal{D}$ is a cofree cooperad $\mathbb{F}^c(W)$ for some symmetric sequence W in graded vector spaces.

If we let $(-)^{\flat}$ denote forgetting the differential, then these assumptions imply that

$$\mathrm{Hom}_{\mathsf{CoOp}(\mathsf{GrAlg}_{\mathbb{Q}})}(\mathcal{C}^{\flat},\mathcal{D}^{\flat}) = \mathrm{Hom}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(V,W).$$

We then explained that respecting the differential on the left hand side, on the right hand side corresponds to satisfying the Maurer–Cartan equation for a certain L_∞-algebra structure. The right hand side with this L_∞-structure is denoted

$$\mathrm{Def}(0\colon \mathcal{C} \to \mathcal{D}).$$

That is,

$$\mathrm{Hom}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C},\mathcal{D}) \cong \mathrm{MC}(\mathrm{Def}(0\colon \mathcal{C} \to \mathcal{D})).$$

This justifies the notation 0: MC has a specified basepoint given by the zero element, corresponding to the trivial cooperad map, the one given by the zero map of symmetric sequences. Indeed, we could recover the Maurer–Cartan element x corresponding to f by

$$x = \pi f \iota,$$

where $\iota\colon V \to \mathcal{C}$ is the inclusion of generators and $\pi\colon \mathcal{D} \to W$ is the projection to the cogenerators.

We can pick a Maurer-Cartan element x corresponding to a morphism h of cooperads and twist the L_∞-structure with this (if it were a DGLA, we would replace d with $d + [x,-]$, in general there are higher brackets):

$$\mathrm{Def}(f\colon \mathcal{C} \to \mathcal{D}) \coloneqq \mathrm{Def}(0\colon \mathcal{C} \to \mathcal{D})^{x}.$$

This doesn't really affect the set of Maurer–Cartan elements: if y was a Maurer–Cartan elements for the original L_∞-structure, then $y - x$ is for the twisted one. Instead, you should think of this as rechosing the basepoint. For us, 0 is not a good choice of basepoint, and we would rather pick a specified weak equivalence $h\colon \mathcal{C} \to \mathcal{D}$. It is then equally valid to write

$$\mathrm{Hom}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C},\mathcal{D}) \cong \mathrm{MC}(\mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D})).$$

Example 5.4. If $\mathcal{D} = \mathcal{C}$, we'd take $h = \mathrm{id}$. Indeed, intuitively $\mathfrak{g}$ gives a deformation of the identity rather than the zero map.

Let us upgrade this to a statement on the level of spaces:

Definition 5.5. For an L_∞-algebra L, we define the Maurer–Cartan space $\mathrm{MC}_{\bullet}(L)$ by

$$[p] \longmapsto \mathrm{MC}_p(L) \coloneqq \mathrm{MC}_{\bullet}(L \otimes \Omega^*(\Delta^p)).$$

Here we have used that just like we can tensor a DGLA with a CDGA, we can tensor an L_∞-algebra with a CDGA.

Example 5.6. $Z_\bullet(\mathfrak{g}) = \mathrm{MC}_\bullet(\mathfrak{g}[-1])$ with $\mathfrak{g}[-1]$ considered as an abelian L_∞-algebra, i.e. only the 1-ary bracket/differential is non-zero.

Tensoring $\mathcal{C}$ and $\mathcal{D}$ with $\Omega^*(\Delta^\bullet)$, we obtain an isomorphism of simplicial sets

$$\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_\mathbb{Q})}(\mathcal{C}, \mathcal{D}) \cong \mathrm{MC}_\bullet(\mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D})),$$

and the previous construction affords a map of Maurer spaces

$$\mathrm{MC}_\bullet(\mathfrak{g}[-1]) \longrightarrow \mathrm{MC}_\bullet(\mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D})).$$

We claim this arises from an L_∞-morphism of L_∞-algebras

$$\mathfrak{g}[-1] \longrightarrow \mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D}).$$

Recalling that as a graded-vector space, $\mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D}) = \mathrm{Hom}_{\mathsf{SymSeq}(\mathsf{GrVect}_\mathbb{Q})}(V, W)$, this is given by

$$\gamma \longmapsto \pi\,(h \circ (\exp(\gamma) - 1))\,\iota.$$

I won't verify the L_∞-properties for this, though it is not so hard.

5.3. **The Goldman–Millson theorem and variations.** What is the point of L_∞-algebras? These types of algebras often arise from the homotopy transfer theorem; this answers the question whether there is a (presumably very complicated) algebraic structure which we can put on the (presumably very simple) graded vector spaces $H^*(A)$ to remember the homotopy type of an algebraic structure on the cochain complex A. That is, passing to L_∞-algebras trades *simple algebraic structures on complicated objects* to *complicated algebraic structures on simple objects*.

In our case we traded a question about the simplicial monoids $Z_\bullet(\mathfrak{g})$ and $\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_\mathbb{Q})}(\mathcal{C}, \mathcal{D})$ to the question about the L_∞-algebras $\mathfrak{g}[-1]$ and $\mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D})$. We now hope to forget most of the algebraic structure on these L_∞-algebras. This uses the following variation of the Goldman–Millson theorem. Suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are complete filtered L_∞-algebras, and $U\colon \mathfrak{g} \to \mathfrak{h}$ an L_∞-morphism. Every L_∞-algebra is in particular a cochain complex,[3] so we get a map of filtered cochain complexes and hence an associated spectral sequence.

Theorem 5.7 (Schwarz)**.** *Suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are complete filtered L_∞-algebras, and $U\colon \mathfrak{g} \to \mathfrak{h}$ an L_∞-morphism that induces an isomorphism on the E_2-page of the aforementioned spectral sequences. Then the induced map*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h})$$

is a weak equivalence when $H^1(\mathfrak{g}/F_2\mathfrak{g})$, $H^1(\mathfrak{h}/F^2\mathfrak{h})$ and $H^0(\mathfrak{h}/F_2\mathfrak{h})$ all vanish.

In other words, we have given conditions under which the map

$$\mathrm{MC}_\bullet(\mathfrak{g}[-1]) \longrightarrow \mathrm{MC}_\bullet(\mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D}))$$

[3] Recall we use a cohomological grading convention, so the r-ary bracket is an operation of degree $2 - r$.

being a weak equivalence amounts to some homology computations, involving only the original differential of the L_∞-algebras and the connecting homomorphism in

$$\cdots \longrightarrow H^*(F_{k+1}\mathfrak{g}/F_{k+2}\mathfrak{g}) \longrightarrow H^*(F_k\mathfrak{g}/F_{k+2}\mathfrak{g}) \longrightarrow H^*(F_k\mathfrak{g}/F_{k+1}\mathfrak{g}) \longrightarrow \cdots,$$

and similarly for $\mathfrak{h}$.

6. Intermezzo: specializing to E_n-operads

Recall that we were attempting to compute $\mathrm{Aut}(E_n^{\mathbb{Q}})$ by picking $\mathcal{C}, \mathcal{D} \in \mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})$ weakly equivalent to $C^*(E_n;\mathbb{Q})$ such that (i) $\mathcal{C}$ is arity-wise quasi-free, (ii) $\mathcal{C}$ has an action of a large DGLA $\mathfrak{g}$ by biderivations, (iii) $\mathcal{D}$ is quasi-cofree. Our hope is that we have chosen $\mathfrak{g}$ correctly, so that

$$\mathfrak{g}[-1] \longrightarrow \mathrm{Def}(h\colon \mathcal{C} \to \mathcal{D})$$

satisfies the conditions in Theorem 5.7. In this lecture, we will construct $\mathcal{C}$, $\mathcal{D}$ and $\mathfrak{g}$. In the next lecture, we will prove that the above map satisfies the conditions in Theorem 5.7. There will be a small wrinkle to the story, as the construction in terms of graphs misses some of the path components.

Remark 6.1. This gives the computation of the rational homotopy groups, the path components of $\mathrm{MC}_\bullet(\mathfrak{g}[-1])$ are just $H^0(\mathfrak{g}[-1])$ and given a Maurer–Cartan elements γ, we can compute the homotopy groups by $\pi_*(\mathrm{MC}_\bullet(\mathfrak{g}[-1]),\gamma) = H^*(\mathfrak{g}[-1]^\gamma)$.

7. The graphs cooperad and the action of the graph complex

We will take the cooperad $\mathcal{C}$ to be $\mathrm{Graphs}_n^{2,c}$, closely related to the $\mathcal{D}_n$ from last lecture. We will verify this satisfies the two desired properties: (i) it is arity-wise quasi-free, and (ii) it has an action of the graph complex GC_n^2.

7.1. The graphs cooperad.
Let me define the cooperad $\mathrm{Graphs}_n^{c,2}$.

Definition 7.1. $\mathrm{Graphs}_n^{c,2} \in \mathsf{Op}(\mathsf{CDGA}_{\mathbb{Q}})$ is given as follows:[4]

- The r-ary operations $\mathrm{Graphs}_n^{c,2}$ is the CDGA with underlying vector space $\mathrm{Graphs}_n^{c,2}(r)$ generated by oriented finite graphs Γ; these have r "external vertices" labeled $1,\ldots,r$, additional ordered "internal vertices." We set to zero those graphs with (i) connected components not containing an external vertex, (ii) each internal vertex has valence ≥ 2, as well as say that changing the orientation induces a sign:
- For n even an orientation is an ordering of the set of edges, and for n odd an orientation is an ordering of the set of internal vertices and a direction on each of the edges. If Γ differs from Γ' by changing the orientation by a (pair of) permutation, $\Gamma \sim \mathrm{sign}\,\Gamma'$ with sign (the product of) the sign(s) of the permutation(s).
- The grading on $\mathrm{Graphs}_n^{c,2}(r)$ is given by grading the graphs: $\deg(\Gamma) = (n-1)\#\text{edges} - n\#\text{internal vertices}$.
- The differential on $\mathrm{Graphs}_n^{c,2}(r)$ is given by a sum over edge collapses, with sign given by moving the edge and vertices to the front of the order before collapsing. It increases degree by 1 (we lose an edge of edge $(n-1)$ and a vertex of edge $-n$, so gain 1 in total).

[4]The signs should be taken with a grain of salt, I'm not confident of them.

It is helpful to think of the linear dual differential: this is a signed sum over edge expansions, moving vertices to the front of the order and adding new edges and vertices at the front of the order.

- The product $\Gamma_1 \cdot \Gamma_2$ is given by union along the external vertices, with vertices and edges in lexicographic order.

  It is helpful to think of the linear dual coproduct: this is a signed sum over decomposition along the external vertices, with sign given by moving the edges and vertices in the first component to the front before decomposing the graph.
- The cocomposition of $\Gamma \in \mathrm{Graphs}_n(r)$ is given by a signed sum over all "microscopic divisions"/"subgraph collapses" $\Gamma_1, \ldots, \Gamma_s$ of the external vertices $1, \ldots, r$ and some edges, rearranging the edges and vertices so that the remaining ones appear first and then Γ_1, etc., and taking $\bar{\Gamma} \otimes \Gamma_1 \otimes \cdots \otimes \Gamma_s \in \mathrm{Graphs}_n^{c,2}(s) \otimes \bigotimes_i \mathrm{Graphs}_n^{c,2}(r_i)$. Here $\bar{\Gamma}$ is obtained by collapsing the Γ_i's in Γ.

  It is helpful to think of the linear dual operation: this is inserting the $\Gamma_1, \ldots, \Gamma_r$ into $\bar{\Gamma}$ and connecting the edges in all possible ways, with edges and vertices lexicographically ordered.

Remark 7.2. Setting all graphs with bivalent internal vertices to zero gives another cooperad $\mathrm{Graphs}_n^c \in \mathsf{Op}(\mathsf{CDGA}_{\mathbb{Q}})$. The quotient map $\mathrm{Graphs}_n^{c,2} \to \mathrm{Graphs}_n^c$ is a weak equivalence of cooperads. Most of the results below have an analogue without the decoration by a superscript 2, but for the sake of brevity I will not mention these.

Remark 7.3. Graphs_n^c is not exactly what Lucy defined; she also set to zero graphs with loops and double edges to get $\mathcal{D}_n$. Loops are ruled for n odd by the orientations, and double edges for n even. In either case, allowing them does not affect the homotopy type as $\mathrm{Graphs}_n^c \to \mathcal{D}_n$ is a weak equivalence of cooperads.

Observe that every graph is a uniquely a product of its internally connected pieces: let $\mathrm{IG}_n^2(r) \subset \mathrm{Graphs}_n^{2,c}(r)$ denote the symmetric sequence in graded vector spaces spanned by internally connected oriented graphs as above.

Lemma 7.4. *If we forget the differential,*

$$\mathrm{Graphs}_n^{2,c}(r) = S(\mathrm{IG}_n^2(r)).$$

That is, these cooperads are quasi-free.

7.2. **The graph complex.** The graph complex is roughly $\mathrm{Graphs}_n^{2,c}$ but without the external vertices:

Definition 7.5. $\mathrm{G}_n^2 \in \mathsf{CoCh}_{\mathbb{Q}}$ is given as follows:[5]

- The underlying vector space is generated by oriented finite graphs γ. We set to zero those graphs which (i) are not connected, (ii) can be split into two components by deleting a vertex, (iii) have univalent vertices, as well as say that changing the orientation induces a sign:
- For n even an orientation is an ordering of the set of edges, and for n odd an orientation is an ordering of the set of vertices and a direction on each of the edges. If

[5]Again the signs should be taken with a grain of salt

γ differs from γ' by changing the orientation by a (pair of) permutation, $\gamma \sim \operatorname{sign}\gamma'$ with sign (the product of) the sign(s) of the permutation(s).

- The grading on G_n^2 is given by grading the graphs: $\deg(\gamma) = (n-1)\#\text{edges} - n\#\text{vertices} - n$.
- The differential on G_n^2 is given by a sum over edge collapses. It is helpful to think of the linear dual differential: this is a signed sum over edge expansions.

Remark 7.6. Setting all graphs with bivalent vertices to zero gives another cochain complex G_n. As mentioned in the first lecture, the quotient map $\mathrm{G}_n^2 \to \mathrm{G}_n$ induces an isomorphism on homology.

Remark 7.7. This is not exactly the complex we mentioned in the first lecture. Firstly, there is a harmless dualization. Secondly, we didn't have condition (ii) of vertex 1-irreducibility. However, this does not affect the homology.

This complex in fact has a Lie cobracket. This is constructed as follows: in $\Delta(\gamma)$ there is a contribution for each subgraph γ' of γ. This is declared to be "microscopic", moving the edges and vertices in γ' to the back, and letting $\bar{\gamma}$ be obtained from Γ by collapsing γ'. Then the contribution to $\Delta(\gamma)$ is given by

$$\bar{\gamma} \otimes \gamma + (-1)^{|\gamma||\gamma'|}\gamma \otimes \bar{\gamma}.$$

This is another instance of subgraph collapsing.

The graph complex GC_n^2 is the linear dual of G_n^2. By duality, this has a Lie bracket. This is constructed by anti-symmetrizing a "pre-bracket" $\bar{\gamma} \circ \gamma'$ given by inserting γ' in all vertices of $\bar{\gamma}$ and reconnecting in all possible ways, with edges and vertices lexicographically ordered.

7.3. **The graph complex acts on the graphs cooperad.** We claim that GC_n^2 acts on the cooperad $\mathrm{Graphs}_n^{2,c}$ by biderivations, i.e. derivations of the commutative algebra structure and coderivations with respect to the cooperad structure. This will be obtained by dualizing a co-action of G_n, that is, it is given by

$$\mathrm{GC}_n^2 \otimes \mathrm{Graphs}_n^{2,c} \xrightarrow{\mathrm{id}\otimes\mathrm{coact}} \mathrm{GC}_n^2 \otimes \mathrm{G}_n \otimes \mathrm{Graphs}_n^{2,c} \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} \mathrm{Graphs}_n^{2,c}.$$

More precisely, a graph γ in GC_n^2 is an indicator function on G_n and the above construction picks out the contributions with first term γ from the coaction. (Note that for this to be reasonable, it is better to think of GC_n^2 as a cochain complex and hence reverse the grading in comparison to Definition 2.8.)

The coaction is given by subgraph collapses; each subgraph γ of Γ which lies in G_n and has ≤ 1 external vertex has a contribution $\pm\gamma \otimes \Gamma/\gamma$. You should verify as an exercise that this is compatible with the differential (i.e. edge collapses), at least up to signs. Condition (ii) (that γ can't be split into two components by deleting a vertex) then guarantees that $\mathrm{coact}(\Gamma \cdot \Gamma') = \mathrm{coact}(\Gamma) \cdot (1 \otimes \Gamma') \pm (1 \otimes \Gamma) \cdot \mathrm{coact}(\Gamma')$. Thus dualizing, the action of GC_n^2 is by derivations of the commutative algebra structure.

7.4. **The loop grading.** We claimed in the introduction that $\pi_0 \operatorname{Aut}(E_n^{\mathbb{Q}})$ is either one or two copies of $\mathbb{Q}^\times$. However, if $n \geq 3$ usually GC_n^2 will vanish in low degrees. The reason is that GC_n^2 is missing some of the automorphisms of $\mathrm{Graphs}_n^{2,c}$.

Let the abelian Lie-algebra $\mathbb{Q} \cdot L$ act on GC_n^2 by sending γ to $\#\mathrm{loops} \cdot \gamma$. This gives a semi-direct product Lie algebra $\mathbb{Q} \cdot L \ltimes \mathrm{GC}_n^2$. The action of GC_n^2 to $\mathrm{Graphs}_n^{c,2}$ extends to this semi-direct product by letting L send Γ to $(\#\mathrm{edges} - \#\mathrm{internal\ vertices})\Gamma$.

From this we obtain a map

$$\mathbb{Q}^\times \ltimes Z_\bullet(\mathrm{GC}_n^2) \longrightarrow \mathrm{Map}_{\mathsf{CoOp(CDGA}_{\mathbb{Q}})}(\mathrm{Graphs}_n^{c,2}, \mathrm{Graphs}_n^{c,2})^\times,$$

where we had to cheat a bit becauase $\mathbb{Q}L \ltimes \mathrm{GC}_n^2$ is not pro-nilpotent. Similarly, for a weak equivalence $h\colon \mathrm{Graphs}_n^{c,2} \to \mathcal{D}$ we get a map

$$\mathbb{Q}^\times \ltimes Z_\bullet(\mathrm{GC}_n^2) \longrightarrow \mathrm{Map}_{\mathsf{CoOp(CDGA}_{\mathbb{Q}})}(\mathrm{Graphs}_n^{c,2}, \mathcal{D})^\times.$$

Part 3. Finishing the computation

We just constructed the map (1):

$$Z_\bullet(\mathfrak{g}) \longrightarrow \mathrm{Map}_{\mathsf{CoOp(CDGA}_{\mathbb{Q}})}(\mathcal{C}, \mathcal{D})^\times.$$

Our intention is to apply to this morphism the variation of the Goldman–Millson theorem described in Theorem 5.7. The loop order action can be handled separately, at the cost of restricting our attention to the subspace $\mathrm{Def}^1 \subset \mathrm{Def}$ consisting of those deformations which induces the identity on cohomology (equivalently are the identity on the Browder cobracket). Filtering by number of edges will endow both

$$\mathfrak{g} \coloneqq \mathrm{GC}_n^2 \qquad \text{and} \qquad \mathfrak{h} \coloneqq \mathrm{Def}^1(h\colon \mathrm{Graphs}_r^{c,2} \to B\Omega(\mathrm{Pois}_n^c))$$

with the structure of complete filtered L_∞-algebras. As every L_∞-algebra is in particular a chain complex, we get a map of filtered chain complexes and hence an associated spectral sequence. The result is then that if this morphism induces an isomorphism on the E_2-page of this spectral sequences, the induced map

$$\mathbb{Q} \times Z_\bullet(\mathrm{GC}_n^2) = \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h}) = \mathrm{Map}_{\mathsf{CoOp(CDGA}_{\mathbb{Q}})}(\mathrm{Graphs}_n^{c,2}, B\Omega(\mathrm{Pois}_n^c))^\times$$

is a weak equivalence as long as $H_1(\mathfrak{g}/F_2\mathfrak{g})$, $H_1(\mathfrak{h}/F^2\mathfrak{h})$ and $H_0(\mathfrak{h}/F_2\mathfrak{h})$ all vanish. Verifying these conditions is the goal of this talk:

Theorem 7.8. *Writing $\mathfrak{g} = \mathrm{GC}_n^2$ and $\mathfrak{h} = \mathrm{Def}^1(h\colon \mathrm{Graphs}_r^{c,2} \to B\Omega(\mathrm{Pois}_n^c))$, the L_∞-morphism $U\colon \mathfrak{g} \to \mathfrak{h}$ induces an isomorphism on the E^2-page of the associated spectral sequences and $H^1(\mathfrak{g}/F_2\mathfrak{g})$, $H^1(\mathfrak{h}/F^2\mathfrak{h})$ and $H^0(\mathfrak{h}/F_2\mathfrak{h})$ all vanish.*

This theorem is the final step in the computation of the automorphisms of the rationalized E_n-operads, for $n \geq 3$ at least, as given in Theorem 5.7.

8. Understanding $\mathfrak{h}$

Our first goal is to explain the target in $\mathfrak{h} = \mathrm{Def}(h\colon \mathrm{Graphs}_n^{c,2} \to B\Omega(\mathrm{Pois}_n^c))$, and how to obtain the differential of the L_∞-structure on this deformation complex.

8.1. **Bar-cobar duality for (co)operads.** Let $\mathbb{1}$ be the operad/cooperad given by $\mathbb{1}(1) = \mathbb{Q}$ and 0 otherwise. Indeed, as the unit for the composition product in symmetric sequences, it has a canonical algebra and coalgebra structure.

An operad $\mathcal{O}$ is a unital algebra in symmetric sequences, so comes with a map $\mathbb{1} \to \mathcal{O}$. An *augmented* operad is an operad with an augmentation $\mathcal{C} \to \mathbb{1}$, and is *reduced* if this is a weak equivalence in arities 0 and 1. Any augmented operad has a augmentation ideal $\bar{\mathcal{O}} = \ker(\mathcal{O} \to \mathbb{1})$.

Dually, a cooperad is counital coalgebra in symmetric sequences, so comes with a map $\mathcal{C} \to \mathbb{1}$. A *coaugmented* cooperad comes with a coaugmentation $\mathbb{1} \to \mathcal{C}$, and is *reduced* if this is a weak equivalence in arities 0 and 1. Any coaugmented cooperad has a coaugmentation ideal $\bar{\mathcal{C}} = \mathrm{coker}(\mathbb{1} \to \mathcal{C})$.

For an coaugmented cooperad, we can take the cobar construction

$$\Omega \colon \mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})_{\mathrm{coaugm}} \longrightarrow \mathsf{Op}(\mathsf{CDGA}_{\mathbb{Q}})_{\mathrm{augm}},$$

and dually, for an augmented operad, we can take the bar construction

$$B \colon \mathsf{Op}(\mathsf{CDGA}_{\mathbb{Q}})_{\mathrm{augm}} \longrightarrow \mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})_{\mathrm{coaugm}}.$$

Theorem 8.1 (Bar-cobar duality for (co)operads)**.** *The bar and cobar construction participate in an adjunction and the unit and counit*

$$\mathcal{C} \longrightarrow B\Omega\mathcal{C}, \qquad \Omega B\mathcal{O} \to \mathcal{O}$$

are weak equivalences when $\mathcal{C}$ or $\mathcal{O}$ are reduced.

We have not defined B and Ω yet, but shall give a construction of these that is most convenient for our purposes. In particular, we would like $B\mathcal{O}$ and $\Omega\mathcal{C}$ to be quasi-(co)free, i.e. (co)free (co)operads after forgetting the differentials. Let us hence just define them this way (below we shall explain a general equivalent construction):

$$\Omega\mathcal{C} = (\mathbb{F}(\bar{\mathcal{C}}[1]), d),$$

the free operad on the desuspension of the coaugmentation ideal of $\mathcal{C}$, with a differential mixing the internal differential of $\mathcal{C}$ and one coming from the cooperad structure.

$$B\mathcal{O} = (\mathbb{F}^c(\bar{\mathcal{O}}[-1]), d),$$

the free operad on the suspension of the augmentation ideal of $\mathcal{O}$, with a differential mixing the internal differential of $\mathcal{O}$ and one coming from the operad structure.

Let us describe the differential in the first case, the second one being dual: it has two terms

$$d = d_{\mathrm{int}} + d_{\mathrm{cooperad}}.$$

Forgetting the cooperad structure, $\mathbb{F}^c(\overline{\mathcal{C}}[1])$ is just a direct sum over trees of tensor product of $\bar{\mathcal{C}}[1]$ and thus has an internal differential d_{int} coming from the differential of $\mathcal{C}$ (this increases degree by our conventions). The second part d_{cooperad} will be defined by demanding that it is a coderivation of cooperads. Since the domain is free, we can describe it by specifying its value on $\mathcal{C}[1]$; there it is given by mapping $c \in \overline{\mathcal{C}}[1]$ to its cocomposition in $\overline{\mathcal{C}}[1] \circ \overline{\mathcal{C}}[1]$ (which increases degree by 1 due to the suspension). Since the internal differential of $\mathcal{C}$ is compatible with cocomposition, $[d_{\mathrm{int}}, d_{\mathrm{cooperad}}] = 0$ and since the cocomposition is coassociative $d_{\mathrm{cooperad}}^2 = 0$. This implies that $(d_{\mathrm{int}} + d_{\mathrm{cooperad}})^2 = 0$.

Remark 8.2. There are more general equivalent construction of B and Ω, by literally taking the (co)bar construction for (co)augmented (co)algebras in symmetric sequences. That is, if

$\circ$ denotes the composition product,

$$\Omega(\mathcal{C}) \simeq \mathrm{CoBar}(\mathbb{1}, \mathcal{C}, \mathbb{1})$$

is the totalization of the cosimplicial object $[p] \longmapsto \mathcal{C}^{\circ p}$ (the coaugmentation is used for the outer coface maps). Similarly,

$$B(\mathcal{O}) \simeq \mathrm{Bar}(\mathbb{1}, \mathcal{O}, \mathbb{1})$$

he realization of the simplicial object $[p] \mapsto \mathcal{O}^{\circ p}$ (the augmentation is used for the outer face maps).

(In other words, $B\mathcal{O}$ is the derived indecomposables of $\mathcal{O}$ as operad and $\Omega\mathcal{C}$ is the dual of that. This makes clear that $B\mathcal{O}(r)$ still admits the structure of a bimodule over $\mathcal{O}(r)$, and dually for $\Omega\mathcal{C}(r)$.)

8.2. **Maps of cooperads into ΩB.** We describe how to compute the differential on

$$\mathrm{Def}(\psi\colon \mathcal{C} \to B\Omega\mathcal{D})$$

when ψ arises for a morphism of cooperads $\mathcal{C} \to \mathcal{D}$ followed by the unit of the bar-cobar adjunction, and $\mathcal{C}$ is arity-wise quasi-free. That is, we are assuming that $\mathcal{C}(r)^\flat = S^*(V(r))$ for some symmetric sequence $V \in \mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})$, where $(-)^\flat$ means we forget the differentials. By definition $B\Omega\mathcal{D}$ is quasi-cofree, i.e. $(B\Omega\mathcal{D})^\flat = \mathbb{F}^c(\Sigma\overline{\Omega\mathcal{C}})$. For simplicity we will introduce the notation $W := \Sigma\overline{\Omega\mathcal{C}}$.

8.2.1. *Recalling the deformation complexes.* Let us recall the definition of the above deformation complex, as an L_∞-algebra. Recall that we obtained it by first forgetting the differentials, and then seeing what was needed to restrict attention to cooperad maps which preserve the differentials. In particular, the above quasi-(co)freeness assumptions imply that

$$\begin{aligned} \Psi\colon \mathrm{Hom}_{\mathsf{CoOp}(\mathsf{GrVect}_{\mathbb{Q}})}(\mathcal{C}^\flat, (B\Omega\mathcal{D})^\flat) &\longrightarrow \mathrm{Hom}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(V, W) \\ \theta &\longmapsto \pi \circ \theta \circ \iota \end{aligned}$$

where $\iota\colon V \to \mathcal{C}$ is the inclusion of generators and $\pi\colon (B\Omega\mathcal{D})^\flat \to W$ is the projection to the cogenerators. Let us denote $\Phi := \Psi^{-1}$.

We then observed that the subspace of the left hand side of maps which preserves the differentials, can be identified with the subspace of the right hand side of Maurer–Cartan elements of a particular L_∞-structure. To describe this L_∞-structure, it suffices to give the power series

$$M(x) = \sum_{n \geq 1} \frac{1}{n!}\mu_n(x, \ldots, x),$$

as the multi-linear operation $\mu_n(x_1, \ldots, x_n)$ may be recovered from it by extending scalar to the algebra $\mathbb{Q}[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1^2, \ldots, \epsilon_n^2)$ and looking at the $\epsilon_1 \cdots \epsilon_n$-component of $M(x_1\epsilon_1 + \cdots + x_n\epsilon_n)$. If $M(m) = 0$, we say that m is a Maurer–Cartan element. If we want Maurer–Cartan elements to correspond to cooperad maps compatible with the differentials this forces us to take

$$M(f) = \pi(d_{\Omega B\mathcal{D}}\Phi(f) - \Phi(f)d_{\mathcal{C}})\iota.$$

We then defined $\mathrm{Def}(0\colon \mathcal{C} \to B\Omega\mathcal{D})$ to be $\mathrm{Hom}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(V, W)$ with this L_∞-structure.

To get $\mathrm{Def}(\psi\colon \mathcal{C} \to B\Omega\mathcal{D})$ we take ψ to the Maurer–Cartan element $\mu = \Phi(\psi)$, and twist the L_∞-structure by it. By tensoring with $\Omega^*_{\mathrm{PL}}(\Delta^\bullet)$, we then upgrade the map Ψ from a bijection of morphisms to an isomorphism of simplicial sets

$$\mathrm{Map}_{\mathsf{CoOp}(\mathsf{CDGA}_{\mathbb{Q}})}(\mathcal{C}, B\Omega\mathcal{D}) \xrightarrow{\cong} \mathrm{MC}_\bullet(\mathrm{Def}(\psi\colon \mathcal{C} \to B\Omega\mathcal{D})).$$

Since $\mathcal{C}$, resp. $B\Omega\mathcal{C}$, are cofibrant and fibrant the right hand side is the derived mapping space. For particular models $\mathcal{C}$ and $\mathcal{D}$ of the cochains on E_n, this is what we set out to compute.

8.2.2. *Describing the differential.* Our goal is now to describe the differential in $\mathrm{Def}(\psi\colon \mathcal{C} \to B\Omega\mathcal{D})$ explicitly, using that the map $\psi\colon \mathcal{C} \to \Omega B\mathcal{D}$ factors over $\mathcal{D}$. This is the linear part of $M(x)$. To obtain it, we note that Φ can formally be written as a sum $\sum_{n=1}^\infty \Phi_n(x, \ldots, x)$ with $\Phi_n(x_1, \ldots, x_n)$ a multilinear function (again, you can obtain this by adding nilpotents). The linear part $\Phi_1(f)$ of an element $f \in \mathrm{Def}(\psi\colon \mathcal{C} \to B\Omega\mathcal{D})$ is then a biderivation $\theta_f\colon \mathcal{C} \to \Omega B\mathcal{D}$ of ψ, i.e. derivation with respect to the multiplicative structure and a coderivation with respect to the cooperad structure. The latter only makes sense relative to a fixed cooperad map, in our case ψ.

In these terms, the differential on $\mathrm{Def}(\psi\colon \mathcal{C} \to \Omega B\mathcal{D})$ is given by

$$d(f) = \pi(d_{B\Omega\mathcal{D}}\theta_f - \theta_f d_{\mathcal{C}})\iota.$$

We shall write

$$d'(f) \coloneqq \pi(d_{B\Omega\mathcal{D}}\theta_f)\iota, \qquad d''(f) \coloneqq \pi(\theta_f d_{\mathcal{C}})\iota,$$

and compute both separately. The first we call the operadic part, the second the internal part.

Lemma 8.3. *For a generator $x \in V$ we have that the operadic part of the differential is given by*

$$\begin{aligned} d'(f)(x) = d_{\overline{\Omega\mathcal{D}}}f(x) \\ &+ \sum_{rth\ cocomposition} \sum_i \pm(\psi(x'_1)\cdots f(x'_i)\cdots\psi(x'_m)) \circ_r \pi(\psi(x''_1\cdots x''_k)) \\ &+ \sum_{rth\ cocomposition} \sum_j \pm\pi(\psi(x'_1\cdots x'_m)) \circ_r \psi(x''_1)\cdots f(x''_j)\cdots\psi(x''_k) \end{aligned}$$

where inner sum uses the rth cocomposition of x into $\sum x'_1\cdots x'_m \otimes x''_1\cdots x''_k$ with $x'_i, x''_j \in V$ generators.

Proof idea. The difficulty is determining $\Phi(f)$. This is made easier as the map factors over $\mathcal{D} \to B\Omega\mathcal{D}$: this implies that $\pi(\psi(a)\cdot b) = \psi(a)\cdot\pi(b)$. It is this fact that makes the computation possible. □

To make sense of these formulas, we use that $\overline{\Omega\mathcal{D}}[1]$ is an operad, as well as its r-ary operations form a bimodule over $\mathcal{D}$.

Lemma 8.4. *For a generator $x \in V$ we have that the internal part of the differential is given by*

$$d''(f)(x) = \sum\sum_j \pm\psi(x_1)\cdots f(x_j)\cdots\psi(x_m)$$

where the outer sum is over $d_{\mathcal{C}}(x) = \sum x_1\cdots x_m$.

9. Hairy graph complexes

Our next goal is to understand

$$\mathrm{Def}(*\colon \mathrm{Graphs}_n^{c,2} \to B\Omega\mathrm{Pois}_n^c),$$

where $*$ is the composition

$$\mathrm{Graphs}_n^{c,2} \to \mathrm{Com}^c \to \mathrm{Pois}_n^c \to B\Omega\mathrm{Pois}_n^c$$

where the first map determined by sending graphs with internal vertices to 0 and sending two external vertices to $\mu_2^\vee$ and an edge connecting two external vertices to 0. The middle expresses the fact that every n-Poisson coalgebra is a coalgebra. The last one is the unit of the bar-cobar adjunction. This is not the complex we are interested in at first, but it is something that will appear as an associated graded in the next section.

Recall that $\mathrm{Graphs}_n^{c,2}$ is arity-wise quasi-free on the symmetric sequence IG_n^2 of the internally connected graphs, so that we can identify the chain complex underlying the L_∞-algebra $\mathrm{Def}(h\colon \mathrm{Graphs}_n^{c,2} \to \Omega B\mathrm{Pois}_n^c)$ with

$$\left(\mathrm{Map}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(*\colon \mathrm{IG}_n^2 \to \overline{\Omega\mathrm{Pois}_n^c}[1]), d\right)$$

with differential d computed as above. Here Map denotes the graded vector space of morphisms of symmetric sequences of varying degrees.

9.1. **Koszul duality for the n-Poisson cooperad.**

The (co)bar construction come with natural filtrations; In the construction we gave in terms of (co)free (co)operads with twisted differential, these are given by the filtration by number of vertices in trees.

Let us focus on the case of the cobar construction $\Omega\mathcal{C} = (\mathbb{F}(\overline{\mathcal{C}}[1]), d)$ and make this more concrete. Additively, $\mathbb{F}(\overline{\mathcal{C}}[1])$ is a direct sum over rooted trees labels of the vertices in $\overline{\mathcal{C}}[1]$. In fixed arity r, since $d = d_{\mathrm{int}} + d_{\mathrm{cooperad}}$ where d_{int} preserves the number of vertices while d_{cooperad} increases the number of vertices by 1, this is the total cochain complex of a double complex

$$\cdots \longrightarrow \bigoplus_{T,\, \leq r-2 \text{ vertices}} (\mathbb{F}(\overline{\mathcal{C}}[1])_T, d_{\mathrm{int}}) \xrightarrow{d_{\mathrm{cooperad}}} \bigoplus_{T,\, \leq r-1 \text{ vertices}} (\mathbb{F}(\overline{\mathcal{C}}[1])_T, d_{\mathrm{int}}) \longrightarrow 0,$$

since $\overline{\mathcal{C}}$ is concentrated in arity ≥ 2 and a bivalent tree with r leaves has $r-1$ vertices.

We see from this that $\Omega\mathcal{C}$ may be projected to the last filtration quotient of the associated graded, which we shall denote $\mathcal{C}^\perp$.

Definition 9.1. $\mathcal{C}$ is *Koszul* if the map of cooperads

$$\Omega\mathcal{C} \longrightarrow \mathcal{C}^\perp$$

is an equivalence.

Dually, there is a map $\mathcal{O}^\perp \to B\mathcal{O}$ and $\mathcal{O}$ is Koszul if this is an equivalence. If $\mathcal{O}$ is quadratically presented then $\mathcal{O}^\perp$ is the quadratic dual cooperad $\mathcal{O}^!$. The following will be helpful: an operad $\mathcal{O}$ is Koszul if and only if quadratic dual $\mathcal{O}^\perp$ is, and in this case $\mathcal{O}$ is the Koszul dual of $\mathcal{O}^\perp$ and vice versa.

Theorem 9.2 (Getzler–Jones [GJ94]). *Pois_n is Koszul.*

Remark 9.3. Before we get confused about gradings: Pois_n^c arises as the cohomology of the E_n-operad, while Pois_n arises as its homology. By our conventions to make cohomological gradings positive, Pois_n is non-positively graded!

What is its Koszul dual? The operad Pois_n is cogenerated by a product μ_n of degree 0 and a bracket λ_n of degree $-(n-1)$, with relations given by associativity, commutativity, (anti-)symmetry, the Jacobi identity, and that the bracket is a derivation of the product. I'm not going to compute the quadratic dual, but the claim is that this is n-fold operadic desuspension $\Lambda^{-n}\mathrm{Pois}_n^c$. Then dually, Pois_n^c is Koszul with Koszul dual $\Lambda^n\mathrm{Pois}_n$, with now explicitly the operadic suspension given by $\Lambda^n\mathrm{Pois}_n(r) \coloneqq \mathrm{Pois}_n(r)[(1-r)n] \otimes \mathrm{sign}^{\otimes n}$; thus it has product $\tilde{\mu}_n$ of degree n and a bracket $\tilde{\lambda}_n$ of degree 1.

In conclusion, we have an equivalence

$$\rho\colon \Omega\mathrm{Pois}_n^c \longrightarrow \Lambda^n\mathrm{Pois}_n$$

of operads. To describe its underlying morphism, it suffices to give this on the generators of $(\Omega\mathrm{Pois}_n^c)^\flat = \mathbb{F}(\overline{\mathrm{Pois}_n^c}[1])$: it is determined by mapping the coproduct $\mu_n^\vee$ to $\tilde{\lambda}_n$ and the cobracket $\lambda_n^\vee$ to $\tilde{\mu}_n$, and sending all other generators to 0. (The suspension makes the degrees work out correctly.)

We obtain from this a weak equivalence

$$\begin{gathered}\left(\mathrm{Map}_{\mathsf{SymSeq}(\mathsf{GrVect}_\mathbb{Q})}(*\colon \mathrm{IG}_n^2 \to \overline{\Omega\mathrm{Pois}_n^c}[1]), d\right) \\ \Big\downarrow \simeq \\ \left(\mathrm{Map}_{\mathsf{SymSeq}(\mathsf{GrVect}_\mathbb{Q})}(*\colon \mathrm{IG}_n^2 \to \overline{\Lambda^n\mathrm{Pois}_n}[1]), d\right).\end{gathered}$$

The differential d on the right hand side is defined to make this a map of cochain complexes. That is, one uses Lemma 8.3 and Lemma 8.4 and projects from $\Omega\mathrm{Pois}_n^c$ to $\overline{\Lambda^n\mathrm{Pois}_n}$.

In those formula's ψ here is given as follows. Firstly, we map a graph $\Gamma \in \mathrm{Graphs}_n^{2,c}(r)$, which is multiplicatively generated by $\mathrm{IG}_n^2(r)$, to 0 unless it has no internal vertices. If it has no internal vertices we first map to Pois_n^c, a map which is determined by sending an edge between two vertices to 0 and two disjoint vertices to a comultiplication $\mu_n^\vee$. These terms can then be included in $B\Omega\mathrm{Pois}_n^c$.

Precomposing with $\iota\colon \mathrm{IG}_n^2(r) \to \mathrm{Graphs}_n^{2,c}(r)$ is easy, and postcomposition with $\rho \circ \pi\colon B\Omega\mathrm{Pois}_n^c \to \Lambda^n\mathrm{Pois}_n$ is the projection determined by $\lambda_n^\vee \mapsto \tilde{\mu}_n$ and $\mu_n^\vee \mapsto \tilde{\lambda}_n$. Thus concretely, since $\Lambda^n\mathrm{Pois}_n$ has a zero internal differential and $\rho \circ \pi \circ \psi$ vanishes unless we evaluate on a graph without edges, the cooperadic part $d'(f)$ is given by taking the cocompositions of $\Gamma \in \mathrm{IG}_n^2$, collecting those terms $\sum s$ vertices $\otimes\, \Gamma'$ in the ith cocomposition or $\sum \Gamma'' \otimes s$ vertices in the jth cocomposition, and taking

$$d'(f)(\Gamma) = \sum_{i=1}^2 \pm\tilde{\lambda}_n^{s-1} \circ_i f(\Gamma') + \sum_{j=1}^r \pm f(\Gamma'') \circ_j \tilde{\lambda}_n^{s-1}.$$

The internal part $d''(f)$ is given on Γ by $f(d\Gamma)$.

9.2. **The hairy graph complex.** Our next goal is to construct the hairy graph complex $\mathrm{HCG}_{n,n}$, map it into the above version of the deformation complex, and show this is an equivalence:

$$\mathrm{HGC}_{n,n} \xrightarrow{\simeq} \left(\mathrm{Map}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(*\colon \mathrm{IG}_n^2 \to \overline{\Lambda^n \mathrm{Pois}_n}[1]), d\right).$$

9.2.1. *Defining the hairy graph complex.* Recall the CDGA's $\mathrm{Graphs}_n^{c,2}(r)$ which appeared in Definition 7.1. They are free graded-commutative algebras on $\mathrm{IG}_n^2(r)$. The hairy graph complex is essentially obtained by dualizing the latter, and regarding the r external vertices as indistinguishable hairs, with a sign if n is odd and a grading shift:

Definition 9.4. Let $\mathrm{IG}_n^{2,=1}(r)$ be the quotient of $\mathrm{IG}_n^2(r)$ where we set to 0 all graphs where some external vertices has valence > 1. The *hairy graph complex* is the cochain complex

$$\mathrm{HCG}_{n,n}^2 = \prod_{r \geq 1} (\mathrm{IG}_n^{2,=1}(r))^\vee \otimes_{\Sigma_r} \mathrm{sign}[n]^{\otimes n}[-n],$$

with differential induced from that on $\mathrm{Graphs}_n^{2,c}(r)$.

This is to be regarded as a complete filtered cochain complex, given by functions that are invariant up to a sign under rearranging the external vertices. A function Ξ of degree $-s$ is non-vanishing on graphs of degree s.

9.2.2. *Mapping hairy graphs to deformations.* We now write a cochain map

$$\omega\colon \mathrm{HCG}_{n,n}^2 \longrightarrow \left(\mathrm{Map}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(*\colon \mathrm{IG}_n^2 \to \overline{\Lambda^n \mathrm{Pois}_n}), d\right).$$

The idea is to send a function Ξ to the map $\omega(\Xi)$ which sends an internally connected graph $\Gamma \in \mathrm{IG}_n^2(r)$ to 0 if it has external vertices of valence > 1 and $\Xi(\Gamma) \cdot \tilde{\mu}_n^r$ otherwise, where $\tilde{\mu}_n^r$ is the r-fold multiplication in $\Lambda^n \mathrm{Pois}_n$, which lives in degree $(r-1)n$. The degree shifts and signs in the definition of $\mathrm{HCG}_{n,n}^2$ are designed to make this work.

We next need to check that this is compatible with the differentials: since the internal part is given by $d''(\Xi)(\Gamma) = \Xi(d\Gamma)$, we need to verify that the cooperadic part $d'(\Xi)$ vanishes. This is given by (a) a term $\Xi(\Gamma')\tilde{\lambda}_n \circ_i \tilde{\mu}_n^r$ if Γ can be obtained by inserting Γ' into the ith vertex of two disjoint vertices, and (b) a term $\Xi(\Gamma'')\tilde{\mu}_n^{r-1} \circ_j \tilde{\lambda}_n$ if Γ can be obtained by inserting the two disjoint vertices into the ith one of Γ''. We claim these cancel. This uses two observations: for (a) to appear we need Γ to have a disjoint external vertex, for (b) to appear we need Γ to have a disjoint external vertex and the reconnecting to be performed solely to one of them (because Γ'' would otherwise have an external vertex of valence 2, and Ξ would vanish on it). For the sake of simplicity, let's thus assume that Γ has 1 as a unique disjoint external vertex; then (a) will give $\Xi(\Gamma')\tilde{\lambda}_n \circ_2 \tilde{\mu}_n^r$ and (b) will give $\sum \pm \Xi(\Gamma')\tilde{\mu}_n^{r-1} \circ_j \tilde{\lambda}_n$. That these cancel is the fact that the bracket is a derivation for the product.

Theorem 9.5. *The map*

$$\omega\colon \mathrm{HCG}_{n,n}^2 \longrightarrow \left(\mathrm{Map}_{\mathsf{SymSeq}(\mathsf{GrVect}_{\mathbb{Q}})}(*\colon \mathrm{IG}_n^2 \to \overline{\Lambda^n \mathrm{Pois}_n}[1]), d\right)$$

is an equivalence.

Sketch of proof. We filter both by the number of edges and hairs, observe that ω is compatible with these filtrations, and get from it a map on the corresponding spectral sequence. It suffices to prove that this is an isomorphism on E^1-pages.

Let us first investigate the associated graded, i.e. the E^0-pages. Since the differential on the hairy graph complex involves edges expansions, on the left hand side it is filtered away and we just get the underlying graded vector space $(\mathrm{HCG}^2_{n,n})^\flat$. On the right hand side, the internal part d'' vanishes for a similar reason. what remains is the cooperadic part d'. This differential sends f to $\sum \tilde{\lambda}_n \circ_i f(\Gamma') + \sum f(\Gamma'') \circ_j \tilde{\lambda}_n$. Thinking of a function in terms of its coefficients for each graph (let's call them cographs), this is a product of complexes for each cograph "character," obtained by deleting the external vertices and leaving half-edges behind. This is because we are only looking at composition with two disjoint vertices. These complexes are rather simple, as they don't depend the graph except through its number of half-edges: they always have homology concentrated in a single degree, which is exactly given by case where the labels in $\Lambda^n \mathrm{Pois}_n$ is $\tilde{\mu}_n$. □

10. Verifying the conditions in Theorem 7.8

Recall the graph complex $\mathfrak{g} = \mathrm{GC}^2_n[-1]$ as the linear dual of G^2_n from 7.5. The map $\mathfrak{g} \to \mathfrak{h}$ was obtained through an action $\mathrm{ad}(\gamma)$ of the functions γ in GC^2_n on $\mathrm{Graphs}^{c,2}_n$; you do a subgraph collapse to obtain graphs without external edges from $\Gamma \in \mathrm{Graphs}^{c,2}_n$ and evaluate γ on these. The L_∞-morphism $U\colon \mathfrak{g} \to \mathfrak{h}$ encoding the action of $\mathrm{GC}^2_n[-1]$ on $\mathrm{Def}^1(h\colon \mathrm{Graphs}^{c,2}_n \to B\Omega\mathrm{Pois}^c_n)$ is then given by the formula

$$U(\gamma) = \pi h(\exp(\mathrm{ad}(\gamma)) - 1)\iota.$$

Recall that both $\mathfrak{g}$ and $\mathfrak{h}$ have complete descending filtrations by number of edges, and that to prove that U induces an equivalence

$$\mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h})$$

is suffices to verify that its linear part induces an isomorphism on the E^2-pages of the corresponding spectral sequences, as well as that $H^1(\mathfrak{g}/F_2\mathfrak{g})$, $H^1(\mathfrak{h}/F_2\mathfrak{h})$ and $H^0(\mathfrak{h}/F_2\mathfrak{h})$ all vanish (Theorem 5.7).

Firstly we study the associated graded: since the internal differentials change the number of edges as well as cocomposition involving anything but disjoint unions of external vertices, we may as well replace in the deformation complexes the homotopy equivalence

$$h\colon \mathrm{Graphs}^c_n \longrightarrow \mathrm{Pois}^c_n \longrightarrow B\Omega\mathrm{Pois}^c_n$$

with the trivial map

$$*\colon \mathrm{Graphs}^c_n \longrightarrow \mathrm{Com}^c \longrightarrow \mathrm{Pois}^c_n \longrightarrow B\Omega\mathrm{Pois}^c_n.$$

Thus on the E^0-page we are looking at a map

$$\mathrm{gr}(\mathrm{GC}^2_n[-1]) \longrightarrow \mathrm{gr}\,\mathrm{Def}^1(h\colon \mathrm{Graphs}^{c,2}_n \to B\Omega\mathrm{Pois}^c_n) = \mathrm{gr}\,\mathrm{Def}^1(*\colon \mathrm{Graphs}^{c,2}_n \to B\Omega\mathrm{Pois}^c_n).$$

This fits into a larger diagram

$$\begin{tikzcd} \mathrm{gr}(\mathrm{GC}_n^2[-1]) \arrow[r] & \mathrm{gr}\,\mathrm{Def}^1(*\colon \mathrm{Graphs}_n^{c,2} \to B\Omega\mathrm{Pois}_n^c) \arrow[d] \\ & \mathrm{gr}\,\mathrm{Map}^1(*\colon \mathrm{IG}_n^2 \to \overline{\Lambda^n\mathrm{Pois}_n}) \\ & \mathrm{gr}\,\mathrm{HCG}_{n,n}^{2,1}. \arrow[u, "\omega"] \end{tikzcd}$$

with $\mathrm{HCG}_{n,n}^{2,1}$ obtained by deleting the summand spanned by a single hair (so no vertices). Upon passing to E^1-pages, the bottom vertical map becomes an isomorphism by considering the case $*$ instead of h. We also know that eventually the top vertical map becomes an equivalence.

At this point I am going to drop my conviction to distinguish between graphs and functionals on graphs, as it becomes hard to talk in the latter language. If we had used $*$ instead of h, then on E^1-pages the differentials of both $\mathrm{GC}_n^2[-1]$ and $\mathrm{HCG}_{n,n}^{2,1}$ already have their usual differentials, as these differentials change the number of edges by 1. However, since we are interested in the case h instead of $*$ this requires a modification to the differential $\mathrm{HCG}_{n,n}^2$: an additional term d_{attach} is added which adds a hair in possible locations, because in the formula for the cooperadic part of the differential, we now allow a single cocomposition involving an edge. We shall call this cochain complex with modified differential

$$(\mathrm{HCG}_{n,n}^2, d + d_{\mathrm{attach}}).$$

Tracing through the definitions, the map on E^1-pages can be extended to a commutative diagram

$$\begin{tikzcd} \mathrm{GC}_n^2[-1] \arrow[r] \arrow[ddr, "\varpi"'] & E^1\,\mathrm{Def}^1(h\colon \mathrm{Graphs}_n^{c,2} \to B\Omega\mathrm{Pois}_n^c) \arrow[d] \\ & E^1\,\mathrm{Map}^1(\mathrm{IG}_n^2, \overline{\Lambda^n\mathrm{Pois}_n}) \arrow[d, equal, "\omega"'] \\ & (\mathrm{HCG}_{n,n}^{2,1}, d + d_{\mathrm{attach}}). \end{tikzcd}$$

by the cochain map

$$\varpi\colon \mathrm{GC}_n^2[-1] \longrightarrow \mathrm{HCG}_{n,n}^{2,1}$$

given by adding a single hair to a graph in all possible ways. That is, it is the linear dual to subgraph extraction on $\mathrm{IG}_n^{2,=1}(1)$ of a single edge to the external vertex. Recalling that GC_n^2 was shifted in degree by n relative $\mathrm{Graphs}_n^{2,c}$ and observing this adds an edge but no internal vertex, this is compatible with the degrees. This is compatible with the differentials since the $d_{\mathrm{attach}}(\varpi(\Gamma)) = 0$ since hairs cancel.

Hence it suffices to prove that ϖ is equivalence, *at least* if we remember that we needed to add a copy of $\mathbb{Q}$ for the automorphism that changes the loop order. Alternatively, we may subtract the corresponding copy of $\mathbb{Q}$ from the target.

Theorem 10.1. *The map* $\varpi\colon \mathrm{GC}_n^2[-1] \to (\mathrm{HCG}_{n,n}^{2,1}, d + d_{\mathrm{attach}})$ *is an equivalence.*

Proof. The mapping cone of ϖ (with codomain restricted as above) is equivalent to the complex $\mathrm{HCG}^{2,1}_{n,n}$ of hairy graphs with at least one vertices and one hair, and it suffices to prove that this is acyclic. We filter by number of edges once more, and get an associated graded with only d_{attach} as differential. This complex splits as a direct sum over "characters" of graphs, now obtained by removing all hairs. These complexes only depends on the number of vertices. If we ignore the symmetries, they are given by the tensor product of terms $(\mathbb{Q}\cdot \text{no hair} \to \mathbb{Q}\cdot \text{one hair})$, each of which is acyclic. If we remember the symmetries, they are the invariants in such a tensor product under some group and hence are still acyclic. □

This implies that the map $\mathfrak{g} \to \mathfrak{h}$ is an isomorphism on E^2-pages. It remains to verify that $H^1(\mathfrak{g}/F_2\mathfrak{g})$, $H^1(\mathfrak{h}/F_2\mathfrak{h})$ and $H^0(\mathfrak{h}/F_2\mathfrak{h})$ all vanish:

- The quotient $\mathfrak{g}/F_2\mathfrak{g}$ is generated by graphs with exactly one edge, which is only the tadpole living in degree $(n-1)-n-n = -1-n \neq 1$. Hence its first cohomology vanishes.
- We may compute $H^1(\mathfrak{h}/F_2\mathfrak{h})$ and $H^0(\mathfrak{h}/F_2\mathfrak{h})$ using the hairy graph complex. Since the relevant quotient of the hairy graph complex consists of hairy graphs with exactly one edge or hair and we removed this to get $\mathrm{HCG}^{2,1}_{n,n}$, these vanish as well.

Appendix A. The cooperadic W-construction

The paper [FTW17] uses a cooperadic W-construction, which I did not use. However, I wrote some notes trying to understand it which are reproduced below:

A.1. The operadic W-construction in spaces.

The W construction was first given as cofibrant replacement of operads in topological spaces. Informally, the points in $W\mathcal{O}$ are metric rooted trees whose internal vertices are labeled by elements of $\mathcal{O}$ topologized such that if the length of an edge goes to 0, i.e. collapses, we compose the operations at its two endpoints. Let us give its definition, just to get used to its features, before doing the less familiar case of cooperads in CDGA's.

Let us recall some terminology: a *tree with root* is a finite graph with a designated 1-valent vertex called the *root*; the remaining 1-valent vertices are called *leaves*. For a finite set S (eventually to be inputs for operations), let Tree_S be the category whose objects are trees with root whose leaves are indexed by S. The morphisms are generated by isomorphisms and inner edge collapses. Operads in S are canonically augmented, and to make sense of the augmentation ideal we suppose that $\mathcal{O}(0) = \varnothing$ and $\mathcal{O}(1) = *$ ($\mathcal{O}(1) \simeq *$ is fine as well, and at any rate we can replace $\mathcal{O}(1)$ by the image of the unit): $\bar{\mathcal{O}}(1)$ is then obtained by replacing $\mathcal{O}(1)$ with $\varnothing$. Every such operad gives rise to a functor

$$\begin{aligned} \mathcal{O}_S \colon \mathsf{Tree}_S &\longrightarrow \mathsf{S} \\ T &\longmapsto \prod_{\text{internal vertices } v} \bar{\mathcal{O}}(\mathrm{in}(v)), \end{aligned}$$

where $\mathrm{in}(v)$ is the set of incoming edges at v (those not going towards the root). A morphism given by an edge collapse is sent by composition in the operad.

On the other hand, there is a functor

$$\begin{aligned} \mathrm{Length}_S \colon \mathsf{Tree}_S^{\mathrm{op}} &\longrightarrow \mathsf{S} \\ T &\longmapsto \prod_{\text{internal edges } e} [0,1], \end{aligned}$$

where for a morphism given by an edge expansion (opposite of edge collapse) we add a new number with value 0. This functor should be thought as assigning length in an interval $[0,1]$ to an edge.

The W-construction has k-ary operations given by the coend

$$W(\mathcal{O})(S) = \int^{T \in \mathsf{Tree}_S} \mathrm{Length}_S(T) \times \mathcal{O}_S(T) = \mathrm{Length}_S \times_{\mathrm{Tree}(S)} \mathcal{O}_S.$$

Concretely, this is given by metric trees with operations at each vertex. If an edge has length going to 0, one uses the operad composition to compose the vertex labels. This makes clear that there is an operad structure given by grafting of trees. Observe that edges adjacent to the leaves or root have no length; when a new edge is created by grafting it is given length 1.

Remark A.1. To get this operad structure, it is convenient to consider a category $\mathsf{Forests}$ of finite forests of trees with roots. Its morphisms are generated by isomorphisms, edge collapses and cutting open edges into a root and a leaf. This is symmetric monoidal under disjoint union. Then a topological operad is a symmetric monoidal functor $\mathsf{Forests} \to \mathsf{S}$ which sends cutting morphisms to isomorphism (a Segal-like condition). We can then upgrade a functor Length to a functor

$$\mathsf{Length} \colon \mathsf{Forests} \times \mathsf{Forests}^{\mathrm{op}} \longrightarrow \mathsf{S}$$

sending pair (F_1, F_2) to disjoint union over the morphisms $f \colon F_2 \to F_1$ which preserve connected components of

$$\prod_{\text{edges collapsed by } f} [0,1] \times \prod_{\text{edges cut by } f} \{1\}.$$

A morphism $(f_1 \colon F_1 \to F_1', f_2 \colon F_2' \to F_2)$ sends f to $f_2 \circ f \circ f_1$; if this collapses an additional edges we add 0 in the corresponding term, if this cut an additional edge we add a 1 in the corresponding term. It is symmetric monoidal under disjoint union, and satisfies the Segal-like condition in the first entry. Thinking of this as a convolution kernel we can form the functor

$$\begin{aligned} W\mathcal{O} \colon \mathsf{Forests} &\longrightarrow \mathsf{S} \\ F &\longmapsto \mathrm{Length}(F, -) \times_{\mathsf{Forests}} \mathcal{O}. \end{aligned}$$

This is then again symmetric monoidal and satisfies the Segal-like condition. In particular, to understand its value we only need to evaluate it on a corolla $F = *_S$ with leaves in indexed by S; this is the coend over $\mathrm{Forests}^{/S}$ of $\mathrm{Length}(S, -) \times \mathcal{O}_S$. Since $\mathrm{Forests}^{/S} = \mathrm{Tree}_S$ since cutting edges increases the number of connected components, we see we get exactly $W\mathcal{O}(S)$.

The natural transformation $\mathrm{Length}_S \to *_S$, the terminal functor assigning a point to all trees, induces a map

$$W\mathcal{O}(S) = \mathrm{Length}_S \times_{\mathrm{Tree}(S)} \mathcal{O}_S \longrightarrow *_S \times_{\mathrm{Tree}(S)} \mathcal{O}_S.$$

The latter is nothing but the colimit of $\mathrm{Tree}(S)$ of $\mathcal{O}(S)$; since $\mathrm{Tree}(S)$ has a terminal object this is $\mathcal{O}(S)$. This natural transformation can be upgraded as in the previous remark to give a map of operads

$$W\mathcal{O} \longrightarrow \mathcal{O}.$$

Lemma A.2 (Boardman–Vogt)**.** *The map $W\mathcal{O} \to \mathcal{O}$ is a weak equivalence.*

Proof. The natural transformation $\mathrm{Length}_S \to *_S$ is a natural weak equivalence. □

Lemma A.3. *The operad of sets underlying $W\mathcal{O}$ is a free operad.*

Proof. Let ${}^\circ W\mathcal{O}$ be the symmetric sequence of sets given by those labeled metric trees whose edges have length < 1. There is an inclusion ${}^\circ W\mathcal{O} \to W\mathcal{O}^\delta$ of symmetric sequences of sets, which induces a map

$$\mathbb{F}({}^\circ W\mathcal{O}) \longrightarrow W\mathcal{O}^\delta$$

of operads of sets. It is easy to see that this is a bijection. □

Remark A.4. By the construction of the coend there are maps $\mathcal{O}(S) \to W\mathcal{O}(S)$, which of course do not assemble to a map of operads. However, it is does induce a map $\mathbb{F}\mathcal{O}(S) \to W\mathcal{O}(S)$.

A.2. **The cooperadic W-construction.** We now dualize the foregoing construction, adapted to cooperads in $\mathsf{CDGA}_\mathbb{Q}$. Every augmented cooperad gives rise to a functor

$$\begin{aligned} \mathcal{C}_S \colon \mathsf{Tree}_S^{\mathrm{op}} &\longrightarrow \mathsf{CDGA}_\mathbb{Q} \\ T &\longmapsto \bigotimes_{\text{internal vertices } v} \bar{\mathcal{C}}(\mathrm{in}(v)), \end{aligned}$$

where $\mathrm{in}(v)$ is the set of incoming vertex at v (those not going towards the root). A morphism, corresponding to an edge expansion, is given by cocomposition in the cooperad.

On the other hand, there is a functor

$$\begin{aligned} \mathrm{Length}_S^\vee \colon \mathsf{Tree}_S &\longrightarrow \mathsf{S} \\ T &\longmapsto \bigotimes_{\text{internal edges } e} (\mathbb{Q}[t,dt], d(t) = dt). \end{aligned}$$

One should think of these functions on length of edges, and a morphism amounts to setting the corresponding generators t, dt equal to 0.

The W-construction then has k-ary cooperations given by the end

$$W^c\mathcal{C}(S) = \int_{T \in \mathsf{Tree}_S} \mathcal{C}_S(T) \otimes \mathrm{Length}_S^\vee = \mathcal{S}_S \otimes^{\mathrm{Tree}_S} \mathrm{Length}_S^\vee.$$

Intuitively, this is given by functions ξ assigning to trees with values S a collection of decorations by polynomial differential forms on internal edges and labels in $\mathcal{C}$ to internal vertices. This has a cooperad structure given by degrafting: given a pair of trees T', T'' such that $T = T' \circ_i T''$, the component $\xi(T') \otimes \xi(T'')$ in the cocomposition of $\xi \in \mathcal{C}(T)$ is given by evaluating the form on the grafted edges at $t = 1$. Maybe more concretely, it takes the same decorations on edges and vertices of subtrees, and evaluates the polynomial differential forms on the deleted edge at $t = 1$. More precisely, I think one may proceed as in the remark.

There is an inclusion $\mathbb{Q} \to (\mathbb{Q}[t, dt], d(t) = dt)$ as constant functions. This is upgrades to a natural transformation of functors $\mathsf{Tree}_S \to \mathsf{S}$ from the constant functor $\mathbb{Q}_S$ to $\mathrm{Length}_S^\vee$. This induces a map of ends

$$\mathcal{C}(S) = \mathcal{C}_S \otimes^{\mathrm{Tree}_S} \mathbb{Q}_S \longrightarrow \mathcal{S}_S \otimes^{\mathrm{Tree}_S} \mathrm{Length}_S^\vee = W^c\mathcal{C}(S).$$

Lemma A.5. *The map $\mathcal{C} \to W^c\mathcal{C}$ is a weak equivalence.*

Proof. The natural transformation $\mathbb{Q}_S \to \mathrm{Length}_S^\vee$ is a natural weak equivalence. ☐

Lemma A.6. *The cooperad of graded vector spaces underlying $W^c\mathcal{C}$ is cofree, i.e. $W^c\mathcal{C}$ is cofree.*

Proof. We start by constructing the map $W^c\mathcal{C}^\flat \to {}^\circ W^c\mathcal{C}$ which will be projection to the cogenerators. This induces a map

$$W^c\mathcal{C}^\flat \longrightarrow \mathbb{F}^c({}^\circ W^c\mathcal{C})$$

of cooperads in graded vector spaces, which we will verify to be an isomorphism.

The maps $(\mathbb{Q}[t, dt], d(t) = dt)^\flat = \mathbb{Q}[t, dt] \to \ker(\mathrm{ev}_1 \colon \mathbb{Q}[t, dt] \to \mathbb{Q})$ given by $p(t, dt) \mapsto p(t, dt) - tp(1, 0)$ induces a natural transformation

$$\mathrm{Length}_S^\vee \longrightarrow {}^\circ\mathrm{Length}_S^\vee \coloneqq \bigotimes_{\text{internal edges}} \ker(\mathrm{ev}_1 \colon \mathbb{Q}[t, dt] \to \mathbb{Q}),$$

and hence a map of ends

$$\mathcal{S}_S \otimes^{\mathrm{Tree}_S} \mathrm{Length}_S^\vee \longrightarrow \mathcal{S}_S \otimes^{\mathrm{Tree}_S} {}^\circ\mathrm{Length}_S^\vee \eqqcolon {}^\circ W\mathcal{C}(S).$$

In other words, ${}^\circ W\mathcal{C}(S)$ consists of those functions on trees with leaves S which have the property that the evaluation at 1 of the polynomial differential forms associated to an edge vanish.

At this point we have to prove that the map $W^c\mathcal{C}^\flat \to \mathbb{F}^c({}^\circ W^c\mathcal{C})$ is an isomorphism. It suffices to prove it is surjective, since it is injective on cogenerators and the target is cofree. I'll leave surjectivity to [FTW17, Lemma 5.3]. ☐

A.3. **Relation to the bar construction.** It is not the case that $W^c\mathcal{C}$ is cofree.. To understand better the isomorphism $W^c\mathcal{C}^\flat = \mathbb{F}^c({}^\circ W^c\mathcal{C})$, we shall upgrade this to an isomorphism that takes into account the differential.

We start with two observations. Firstly, ${}^\circ W^c\mathcal{C}$ is actually a symmetric sequence in CDGA's, as the condition that the evaluation at 1 of edge labels vanish is compatible with the product and differential. Secondly, ${}^\circ W^c\mathcal{C}$ can be made into an operad in CDGA's. We need to describe how to assign to functions ξ', ξ'' on trees with leaves S' or S'' a function on trees with leaves $S = S' \circ_i S''$ (i.e. delete $i \in S'$ and insert S'' there): a tree T with leaves S has at most one decomposition $T = T' \circ_i T''$ into trees T' and T'' with leaves S' and S'' respectively. If it does not, $\xi' \circ_i \xi''$ assigns to it the value 0. If it does decompose, we assign to it the values $\xi(T') \otimes dt \otimes \xi(T'')$, i.e. label the new edge dt.

Let us now give a different perspective on this:

Lemma A.7. *There is isomorphism $W^c\mathcal{C} \cong B({}^\circ W^c\mathcal{C})$ of cooperads in CDGA's.*

As a consequence, we can up to weak equivalence identify the cogenerators with the cobar construction:

Corollary A.8. $\Omega\mathcal{C} \simeq {}^{\circ}W^c\mathcal{C}$.

In other words, we have been writing a smaller bar-cobar construction.

Remark A.9. I would like to say the following, but I haven't checked it: just like the W-construction for operads, the W-construction for cooperads depends on a choice of interval. If it had use $\Delta^1_\bullet$ and the copowering over simplicial sets, this would actually give $B\Omega\mathcal{C} = W^c\mathcal{C}$, but we choose a smaller interval instead.

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