# Lectures on differential topology 

Alexander Kupers

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#### Abstract

These are collected lecture notes on differential topology. Starting from the definitions, we discuss the foundational geometric results on smooth manifold. We also give an introduction to intersection theory, de Rham cohomology, and Morse theory.


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## Introduction

These are the collected lecture notes on differential topology. They are based on [BJ82, GP10, BT82, Wal16]. Our reference for multivariable calculus is [DK04a, DK04b].

Differential topology is the study of smooth manifolds; topological spaces on which one can make sense of smooth functions. This is done by providing local coordinates. Through these, many of the results of multivariable calculus can be extended to manifolds. The latter provide a convenient language, the former the technical details: state globally, prove locally.

The motivating goal of differential topology is the classification of smooth manifolds, and maps between smooth manifolds. This is done through numerical invariants extracted from geometric objects living in our manifolds (e.g. submanifolds) or on our manifolds (e.g. differential forms). Particular instances of these ideas are intersection theory and de Rham cohomology.

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## Chapter 1

## Spheres in Euclidean space

In this first chapter we give a taste of differential topology, with a discussion of spheres which are embedded or immersed in $\mathbb{R}^{k}$. The highlight will be Smale's result that the two-dimensional sphere can be everted. Along the way, we meet a significant portion of the cast of this course: smooth manifolds, embeddings, isotopies, orientations, immersions, regular homotopies, winding numbers, and transversality.

### 1.1 Circle eversion

We are all familiar with the circle

$$
S^{1}:=\left\{(x, y) \in \mathbb{R}^{2} \mid x^{2}+y^{2}=1\right\},
$$

which we can thicken to an annulus

$$
\mathbb{A}^{2}:=\left\{(x, y) \in \mathbb{R}^{2} \mid(1+\delta)^{-1}<x^{2}+y^{2}<1+\delta\right\}
$$

for some small $\delta>0$. There is of course a standard inclusion id of $\mathbb{A}^{2}$ into $\mathbb{R}^{2}$, given by sending $(x, y) \in \mathbb{A}^{2}$ to $(x, y) \in \mathbb{R}^{2}$.


Figure 1.1 The circle $S^{1}$ inside the annulus $\mathbb{A}^{2}$.
There are many other inclusions of $\mathbb{A}^{2}$ into $\mathbb{R}^{2}$. We could rotate by $90^{\circ}$ degrees counterclockwise

$$
\begin{aligned}
\operatorname{rot}_{90}: \mathbb{A}^{2} & \longrightarrow \mathbb{R}^{2} \\
(x, y) & \longmapsto(-y, x),
\end{aligned}
$$

reflect in the $x$-axis

$$
\begin{aligned}
\text { refl }: \mathbb{A}^{2} & \longrightarrow \mathbb{R}^{2} \\
(x, y) & \longmapsto(x,-y)
\end{aligned}
$$

or invert the circle

$$
\begin{aligned}
\text { inv }: \mathbb{A}^{2} & \longrightarrow \mathbb{R}^{2} \\
(x, y) & \longmapsto\left(\frac{x}{x^{2}+y^{2}}, \frac{y}{x^{2}+y^{2}}\right),
\end{aligned}
$$

These injective maps are not only continuous, but have three further properties. Firstly, they are smooth: all partial derivatives exist and are continuous at each point in $(x, y) \in \mathbb{A}^{2}$. Secondly, not only does the total derivative exists at each point, but it is injective (in fact, invertible). Thirdly, they are homeomorphisms onto their image.

Definition 1.1.1. A continuous map $\mathbb{A}^{2} \rightarrow \mathbb{R}^{2}$ is called an embedding if it is a smooth map, which is a homeomorphism on its image and whose total derivative is injective everywhere.


Figure 1.2 Three embeddings $\mathbb{A}^{2} \hookrightarrow \mathbb{R}^{2}$.

How different are these embeddings from each other? The maps id and $\operatorname{rot}_{90}$ are closely related to each other: they can be connected by a path of embeddings. This path is given by varying the rotation angle

$$
\begin{aligned}
\operatorname{rot}_{t}:[0,1] \times \mathbb{A}^{2} & \longrightarrow \mathbb{R}^{2} \\
(t,(x, y)) & \longmapsto\left(\cos \left(\frac{\pi}{2} \cdot t\right) x+\sin \left(\frac{\pi}{2} \cdot t\right) y,-\sin \left(\frac{\pi}{2} \cdot t\right) x+\cos \left(\frac{\pi}{2} \cdot t\right) y\right)
\end{aligned}
$$

a path of embeddings. It is called an isotopy because it is also smooth as a map with domain $[0,1] \times \mathbb{A}^{2}$.

However, the cases of reflection and inversion are more subtle.
Proposition 1.1.2. Both refl and inv can not be connected to id (or equivalently rot ${ }_{90}$ ) by such an isotopy.

Proof. The reason is that both refl and inv reverse orientations. The Euclidean space $\mathbb{R}^{2}$ has a so-called orientation, given by a consistent choice of direction of "counterclockwise rotation," and so does $\mathbb{A}^{2}$ as an open subset of $\mathbb{R}^{2}$. As can be seen in Figure 1.3, rotations such as $\operatorname{rot}_{90}$ preserve orientation, but reflection refl and inversion inv do not.


Figure 1.3 The effect of our three embeddings on orientations.

If the identity map id and reflection refl (or inversion inv) were isotopic then the latter would have to preserve orientation, because id does and the embeddings in an isotopy can not switch from being orientation-preserving to being orientation-reversing. (This is the crux of the argument, and making it rigorous is something we will do in these notes.)

However, the composition of reflection and inversion does preserve orientation; reversing orientation twice preserves it. This map

$$
\begin{aligned}
\text { eve }: \mathbb{A}^{2} & \longrightarrow \mathbb{R}^{2} \\
\quad(x, y) & \longmapsto\left(\frac{x}{x^{2}+y^{2}}, \frac{-y}{x^{2}+y^{2}}\right)
\end{aligned}
$$

is called eversion. Can eversion be connected to the identity by an isotopy?

To answer this question, we look at $S^{1} \subset \mathbb{A}^{2}$. This is our first example of a smooth manifold which is not Euclidean space $\mathbb{R}^{n}$ or an open subset thereof. More precisely, it is a one-dimensional smooth manifold; a topological space which locally looks like $\mathbb{R}$ and on which we can make sense of smooth functions. To do the latter, we use local coordinates on $S^{1}$ and our understanding of smooth maps between open subsets of Euclidean space: the two charts ("coordinate patches")

$$
\begin{aligned}
\phi_{0}:(0,2 \pi) & \longrightarrow S^{1} \\
\theta & \longmapsto(\cos (\theta), \sin (\theta)) \\
\phi_{1}:(0,2 \pi) & \longrightarrow S^{1} \\
\theta & \longmapsto(\cos (\theta+\pi), \sin (\theta+\pi))
\end{aligned}
$$

cover all of $S^{1}$, and we say that $f: S^{1} \rightarrow \mathbb{R}^{2}$ is smooth if both $f \circ \phi_{0}$ and $f \circ \phi_{1}$ are smooth. Similarly, it is a embedding if it is a smooth map which is a homeomorphism onto its image and whose total derivative is injective everywhere. It is easy to recognize it is a homeomorphism on its image; when we restrict the target to its image we get a continuous bijection between compact Hausdorff spaces.

If eve: $\mathbb{A}^{2} \rightarrow \mathbb{R}^{2}$ were isotopic to id, then by restricting the isotopy to $S^{1}$ we would be able to prove that eve $\left.\right|_{S^{1}}$ is isotopic to $\left.\mathrm{id}\right|_{S^{1}}=\mathrm{id}$. So, to prove that eve is not isotopic to id, it suffices to show that eve $\left.\right|_{S^{1}}$ is not isotopic to id $\left.\right|_{S^{1}}$.


Figure 1.4 An example of the image of $S^{1}$ under an immersion into $\mathbb{R}^{2}$.

In fact, we will prove something even stronger. We can drop the condition that an embedding is injective. Since the derivative controls the local behavior of smooth maps, that the derivative is everywhere non-zero means it is still locally injective. A smooth $\operatorname{map} S^{1} \rightarrow \mathbb{R}^{2}$ with everywhere non-zero derivative is called an immersion, and a smooth $\operatorname{map}[0,1] \times S^{1} \rightarrow \mathbb{R}^{2}$ consisting of immersions is called a regular homotopy. This is a family of smooth maps where we allow self-intersections to occur, but not the pulling tight of loops (the derivative would blow up there).

Proposition 1.1.3. The embeddings $\mathrm{eve}_{S^{1}}$ and $\left.\mathrm{id}\right|_{S^{1}}$ are not regularly homotopic.
Proof. Suppose a regular homotopy $e_{t}:[0,1] \times S^{1} \rightarrow \mathbb{R}^{2}$ existed between eve $\left.\right|_{S^{1}}$ and id $\left.\right|_{S^{1}}$, then for each $s \in[0,1]$, the map $e_{s}: S^{1} \rightarrow \mathbb{R}^{2}$ is an immersion. Thus, when we take for $\theta \in S^{1}$ the derivative $\frac{d}{d \theta} e_{s}(\cos (\theta), \sin (\theta))$ we get a non-zero vector in $\mathbb{R}^{2}$. If we normalize these to have length 1 , we get a smooth map

$$
\operatorname{gauss}\left(e_{s}\right): S^{1} \longrightarrow S^{1}
$$

Here the domain $S^{1}$ is the circle which is the domain of our immersions, and the target $S^{1}$ is the space of unit length vectors in $\mathbb{R}^{2}$. You can think of the latter as the space of lines through the origin in $\mathbb{R}^{2}$ with a choice of orthonormal basis (in this case just a single vector).

If eve $\left.\right|_{S^{1}}$ and $\left.\mathrm{id}\right|_{S^{1}}$ are regularly homotopic through $e_{t}$, then gauss $\left(\left.\operatorname{eve}\right|_{S^{1}}\right)$ and gauss $\left(\left.\mathrm{id}\right|_{S^{1}}\right)$ can be connected the path gauss $\left(e_{s}\right)$ of maps $S^{1} \rightarrow S^{1}$. In other words, they would be homotopic. But they are not; as gauss $\left(\mathrm{eveve}_{S^{1}}\right)=\operatorname{refl} \circ \operatorname{rot}_{90}$ and $\operatorname{gauss}\left(\left.\mathrm{id}\right|_{S^{1}}\right)=\operatorname{rot}_{90}$ wind around the origin a different number of times; the first once clockwise (so -1 times) and the second once counterclockwise (so 1 times). The difference between these winding numbers implies that gauss(eve $\left.\left.\right|_{S^{1}}\right)$ and gauss $\left(\left.\mathrm{id}\right|_{S^{1}}\right)$ are not homotopic. (Again, this is the crux and we need to rigorously justify this claim.)

### 1.2 Knots

Let us now increase the dimension of the target; instead of looking at circles in $\mathbb{R}^{2}$ we will look at circles in $\mathbb{R}^{3}$. Immersions are significantly easier to study than embeddings; though both are smooth maps with injective total derivative, a local condition, embeddings need to be injective, a global condition. This distinction becomes evident when we try to discern the difference between embeddings $S^{1} \hookrightarrow \mathbb{R}^{3}$ and immersions $S^{1} \leftrightarrow \mathbb{R}^{3}$.

Proposition 1.2.1. Each immersion $S^{1} \leftrightarrow \mathbb{R}^{3}$ is regularly homotopic to an embedding.
Proof. This uses a technique called transversality. Informally, this allows you take smooth maps to be "generic" without loss of generality. This means that by making an arbitrary small change to an immersion $e_{0}: S^{1} \rightarrow \mathbb{R}^{3}$, we can make its self-intersections have the "expected dimension."

Here "arbitrarily small" means that for each $\epsilon>0$, we can find an $e_{1}: S^{1} \rightarrow \mathbb{R}^{3}$ whose values and derivatives are within $\epsilon$ of those for $e_{0}$. By taking $\epsilon$ to be small enough, during a linear interpolation

$$
\begin{aligned}
e_{t}: S^{1} \times[0,1] & \longrightarrow \mathbb{R}^{3} \\
(t, \theta) & \longmapsto(1-t) \cdot e_{0}(\theta)+t \cdot e_{1}(\theta)
\end{aligned}
$$

the derivative never becomes 0 . In particularly, $e_{0}$ is regularly homotopic to $e_{1}$.
The advantage of $e_{1}$ is that its self-intersections have the expected dimension. This expected dimension is that of the intersection of two affine lines $\mathbb{R}^{3}$ with arbitrarily chosen coefficients: two such lines do not intersect, and thus generically the self-intersections are empty as well.

Isotopy classes of embeddings $S^{1} \hookrightarrow \mathbb{R}^{3}$ are called knots. The isotopy class of the standard circle id: $S^{1} \hookrightarrow \mathbb{R}^{3}$ is the unknot, but there are of course many more interesting and complicated knots. At first sight many seem obviously distinct, or at least non-trivial. This is an artifact of our tendency to draw rather simple knots: it is by no means clear to me that Figure 1.5 is not the unknot. It should furthermore not be obvious how to prove that two knots are distinct, as you need to rule out the existence of some extremely complicated isotopy. To do so one uses knot invariants, with such disparate sources as
algebraic topology, combinatorics, number theory, hyperbolic geometry, or quantum field theory [Ada04, Sos02]. We will discuss some of these later in Chapter 16. At any rate, your intuition is correct:


Figure 1.5 Haken's "gordian knot," which is actually unknotted (from [Sos02]).

Proposition 1.2.2. There are infinitely many distinct isotopy classes of embeddings $S^{1} \hookrightarrow \mathbb{R}^{3}$. That is, there are infinitely many knots.

Remark 1.2.3. This doesn't mean distinguishing knots, or recognizing unknots, is easy. Even though there exists an algorithm that says whether a knot is the unknot, these algorithms are not very efficient [HLP99].

Armed with this knowledge, Proposition 1.2 .1 seems rather useless. All we have shown is that immersions of a circle into $\mathbb{R}^{3}$ can be represented by knots. However, we can use that this representation is not unique. In particular, if we are interested in immersions we are allowed to make the strands of a knot self-intersect! Using this, it is not hard to give an informal proof of the following:

Proposition 1.2.4. All immersions $S^{1} \leftrightarrow \mathbb{R}^{3}$ are regularly homotopic.

Proof sketch. By another application of transversality, it is possible to draw each knot as you are used to; a circle in the plane with some crossings, which never occur at the same point. As we just explained, you can change any crossing using a regular homotopy. Let us explain through an example a procedure to change crossings to end up with an
unknot. Suppose our starting point is:


We fix a point $p_{0}$ in the knot, and start moving along it in an arbitrary direction. When we cross under a strand, we (a) keep it as it is if we haven't seen the crossing yet, but (ii) if we have seen it we change the crossing. For example, the first crossing clockwise from $p_{0}$ is not changed but the second one is. The result will be:


I'll leave it to the reader to understand why this procedure always produces an unknot (hint: look at the height of the strands).

### 1.3 Sphere eversion

Let us now increase the dimension of the domain. There is a two-dimensional sphere $S^{2}:=\left\{(x, y, z) \in \mathbb{R}^{3} \mid x^{2}+y^{2}+z^{2}=1\right\}$. This is a two-dimensional smooth manifold which is a subset of the thickened sphere $\mathbb{A}^{3}:=\left\{(x, y, z) \in \mathbb{R}^{3} \mid(1+\delta)^{-1}<x^{2}+y^{2}+z^{2}<1+\delta\right\}$ for some small $\delta>0$.

Again, in addition to the identity map id: $\mathbb{A}^{3} \hookrightarrow \mathbb{R}^{3}$ there are many other inclusions; we could rotate by applying an element $A \in \mathrm{SO}(3)$ (the group of rotations around some axis through the origin in $\left.\mathbb{R}^{3}\right)$, reflect in the $(x, y)$-plane

$$
\begin{aligned}
& \text { refl: } \mathbb{A}^{3} \longrightarrow \mathbb{R}^{3} \\
& (x, y, z) \longmapsto(x, y,-z)
\end{aligned}
$$

or invert it

$$
\begin{aligned}
& \text { inv: } \mathbb{A}^{3} \longrightarrow \mathbb{R}^{3} \\
& (x, y, z) \longmapsto\left(\frac{x}{x^{2}+y^{2}+z^{2}}, \frac{y}{x^{2}+y^{2}+z^{2}}, \frac{z}{x^{2}+y^{2}+z^{2}}\right) .
\end{aligned}
$$

All of these are smooth maps, and in fact embeddings. The rotation by $A$ is isotopic to the identity because the group $\mathrm{SO}(3)$ is path-connected (move the rotation angle to 0 ), while both refl and inv are not isotopic to the identity because they do not preserve the orientation.

However, the eversion

$$
\begin{aligned}
& \text { eve }:=\text { refl } \circ \text { inv }: \mathbb{A}^{3} \longrightarrow \mathbb{R}^{3} \\
& \qquad(x, y, z) \longmapsto\left(\frac{x}{x^{2}+y^{2}+z^{2}}, \frac{y}{x^{2}+y^{2}+z^{2}}, \frac{-z}{x^{2}+y^{2}+z^{2}}\right)
\end{aligned}
$$

does preserve the orientation. Is it isotopic to the identity? The answer turns out to be negative; in fact, eve $\left.\right|_{S^{2}}$ is already not isotopic to id $\left.\right|_{S^{2}}$. If it were, we could "drag along" the disk $D^{3} \subset \mathbb{R}^{3}$ that bounds the image of $\left.\mathrm{id}\right|_{S^{2}}$ on the inside along an isotopy of embeddings - a result called isotopy extension (which requires the embeddings and isotopy are proper) - and would have to end up with a disk that bounds the image of eve $\left.\right|_{S^{2}}$ on the outside, which is clearly impossible. (This requires justification.)

However, it is a surprising result of Smale that eve $\left.\right|_{S^{2}}$ is regularly homotopic to id $\left.\right|_{S^{2}}$ [Sma58]. That is, these two embeddings can be connected by a family of immersions; self-intersections are allowed to form, but not the pulling tight of the fabric of $S^{2}$. The procedure is rather complicated, but you can watch a video of it called Outside In online. The reason this works is that the two-dimensional versions of the Gauss maps, gauss $\left(\right.$ eve $\left.\left.\right|_{S^{2}}\right)$ and gauss $\left(\left.\mathrm{id}\right|_{S^{2}}\right)$, which are maps from $S^{2}$ to the space $V_{2}\left(\mathbb{R}^{3}\right)$ of two-dimensional planes though origin with a choice of orthonormal basis, are homotopic. This homotopy can then be approximated by a regular homotopy using holonomic approximation, a instance of general philosophy called an h-principle [EM02]. Explicitly implementing this approximation gives the video referred to above.

### 1.4 Problems

Problem 1.4.1. Is the following knot trivial (i.e. isotopic to the unknot)?


## Chapter 2

## Smooth manifolds

In this chapter we give the modern definition of a smooth manifold, which is the one we will use throughout this course. It is given in [BJ82, Chapter 1], but unfortunately not in [GP10]. References for further reading are [Tu11, Chapter 5] or [Wal16, Section 1.1]. We also give a number of examples (you need to know $S^{n}, \mathbb{R} P^{n}$, and $\mathbb{C} P^{n}$, but not the examples of moduli spaces).

### 2.1 Topological manifolds

Underlying every smooth manifold is a topological manifold. This is a topological space which locally looks like Euclidean space, though we will ask it safisfies some point-set topological conditions to make it more well-behaved.

A local property of a topological space is one which concerns sufficiently small open subsets. For a $k$-dimensional topological manifold the relevant local condition is "being homeomorphic to an open subset of $\mathbb{R}^{k}:$ "

Definition 2.1.1. A topological space $X$ is locally Euclidean of dimension $k$ if each point $x \in X$ has an open neighborhood $V_{x} \subset X$ which is homeomorphic to an open subset $U_{x} \subset \mathbb{R}^{k}$.

This models a "world" which, for a tiny creature living in it, is indistinguishable from $\mathbb{R}^{k}$. This intuition is not compatible with certain pathological examples. The "world" is not supposed to "split into two points" somewhere, as occurs in a plane with doubled origin $[\mathrm{SS95}, \S 74]$. This is ruled out by demanding $X$ is Hausdorff (any two distinct points have distinct open neighborhoods). Furthermore, the "world" should admit a notion of distance, i.e. a metric. For a Hausdorff locally Euclidean topological space, being metrizable is equivalent to being second-countable (admitting a countable basis for its topology) [Gau09], and hence we demand that $X$ is also second-countable. An example of a locally Euclidean space which is Hausdorff but not second-countable is the long line, created by "concatenating" uncountably many real lines [SS95, §45].

Definition 2.1.2. A $k$-dimensional topological manifold is a second-countable Hausdorff space $X$ which is locally Euclidean of dimension $k$.

This definition only involves properties of $X$. We can rephrase the property that it is locally Euclidean as data instead, which will be necessary to define smooth manifolds.


Figure 2.1 A chart.

Definition 2.1.3. A triple $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of open subsets $U_{\alpha} \subset \mathbb{R}^{k}, V_{\alpha} \subset X$, and a homeomorphism $\phi_{\alpha}: U_{\alpha} \rightarrow V_{\alpha}$ is called a chart or a local parametrization.

Definition 2.1.4. A collection of charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ such that $\bigcup_{\alpha} V_{\alpha}=X$ is a $k$ dimensional atlas for $X$.

Two local parametrizations $\phi_{\alpha}: \mathbb{R}^{k} \supset U_{\alpha} \rightarrow V_{\alpha} \subset X$ and $\phi_{\beta}: \mathbb{R}^{k} \supset U_{\beta} \rightarrow V_{\beta} \subset X$ give two competing identifications of $V_{\alpha} \cap V_{\beta} \subset X$ with an open subset of $\mathbb{R}^{k}$, which we can compare by the transition function

$$
\begin{equation*}
\psi_{\alpha \beta}:=\phi_{\beta}^{-1} \circ \phi_{\alpha}: \mathbb{R}^{k} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \xrightarrow{\phi_{\alpha}} V_{\alpha} \cap V_{\beta} \xrightarrow{\phi_{\beta}^{-1}} \phi_{\beta}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \subset \mathbb{R}^{k} \tag{2.1}
\end{equation*}
$$

(It would be better to use the notation $\left.\phi_{\beta}^{-1} \circ \phi_{\alpha}\right|_{\phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right)}$ to point out we are restricting the domain, but this notation would quickly become unwieldy.)

An atlas for a topological manifold $X$ is not unique, but it turns out there is a unique maximal one. We shall not discuss this in detail now, saving a discussion of maximal atlases for smooth manifolds (where there is no longer a unique one, i.e. there are exotic smooth structures). An alternative equivalent definition of a $k$-dimensional topological manifold is then:

Definition 2.1.5. A $k$-dimensional topological manifold is a second-countable Hausdorff space $X$ with a maximal $k$-dimensional atlas.

### 2.2 Smooth manifolds

On a topological manifold, as on any topological space $X$, we can make sense of continuous function $X \rightarrow \mathbb{R}^{m}$. A smooth manifold is a refinement of a topological manifold which allows us to make sense of smooth functions $X \rightarrow \mathbb{R}^{m}$. This will use that we know what a smooth function $\mathbb{R}^{k} \rightarrow \mathbb{R}^{m}$ is: a map which has partial derivatives of arbitrary degree, in other words, an infinitely-many times differentiable function.

As the domain of a chart is an open subset of $\mathbb{R}^{k}$, we know what it means for a continuous function to be smooth with respect to the local coordinates provided by a chart. To make guarantee consistency between charts, we require that the transition functions $\psi_{\alpha \beta}$ are smooth.

Definition 2.2.1. A $k$-dimensional smooth atlas for a topological space $X$ is a collection of triples $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ consisting of

- an open subset $U_{\alpha} \subset \mathbb{R}^{k}$,
- an open subset $V_{\alpha} \subset X$, and
- a homeomorphism $\phi_{\alpha}: U_{\alpha} \rightarrow V_{\alpha}$,
so that $\cup V_{\alpha}=X$ and all maps

$$
\psi_{\alpha \beta}=\phi_{\beta}^{-1} \circ \phi_{\alpha}: \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \longrightarrow \phi_{\beta}^{-1}\left(V_{\alpha} \cap V_{\beta}\right)
$$

are smooth maps between open subsets of $\mathbb{R}^{k}$. The triples $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ are called charts and the maps $\phi_{\beta}^{-1} \circ \phi_{\alpha}$ are called transition functions.

Observe that these transition function have the following properties:

$$
\psi_{\alpha \alpha}=\mathrm{id} \quad \text { and } \quad \psi_{\alpha \beta} \circ \psi_{\beta \gamma}=\psi_{\alpha \gamma}
$$

Taking $\gamma=\alpha$, this gives

$$
\psi_{\alpha \beta} \circ \psi_{\beta \alpha}=\mathrm{id}
$$

as smooth maps $\mathbb{R}^{k} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \rightarrow \phi_{\beta}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \subset \mathbb{R}^{k}$. This shows that $\psi_{\alpha \beta}$ is a smooth bijection with smooth inverse, and hence is what we call a diffeomorphism. Thus, in a smooth atlas the transition functions are always diffeomorphisms.


Figure 2.2 A transition function.

Two atlases for $X$ are said to be compatible if their union is an atlas. A maximal atlas is one with the property that every atlas compatible with it, is in fact contained in it.

Lemma 2.2.2. Every $k$-dimensional smooth atlas is contained in a unique maximal $k$-dimensional smooth atlas.

Proof. For uniqueness, it suffices to prove that every two $k$-dimensional smooth atlases $\mathcal{A}^{\prime}=\left\{\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)\right\}$ and $\left.\mathcal{A}^{\prime \prime}=\left\{U_{\gamma}^{\prime \prime}, V_{\gamma}^{\prime \prime}, \phi_{\gamma}^{\prime \prime}\right)\right\}$ containing a given one $\mathcal{A}=\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ are compatible. That is, we must prove that every map

$$
\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ \phi_{\beta}^{\prime}:\left(\phi_{\beta}^{\prime}\right)^{-1}\left(V_{\beta}^{\prime} \cap V_{\gamma}^{\prime \prime}\right) \rightarrow\left(\phi_{\gamma}^{\prime \prime}\right)^{-1}\left(V_{\beta}^{\prime} \cap V_{\gamma}^{\prime \prime}\right)
$$

is smooth. Since being smooth is a local property, it is enough to prove that each $x \in\left(\phi_{\beta}^{\prime}\right)^{-1}\left(V_{\beta}^{\prime} \cap V_{\gamma}^{\prime \prime}\right)$ has an open neighborhood such that the restriction of $\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ \phi_{\beta}^{\prime}$ to this open neighborhood is smooth. Let us pick a chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right) \in \mathcal{A}$ so that $\phi_{\beta}^{\prime}(x) \in V_{\alpha}$. Then we can write the restriction of $\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ \phi_{\beta}^{\prime}$ to $\left(\phi_{\beta}^{\prime}\right)^{-1}\left(V_{\alpha} \cap V_{\beta}^{\prime} \cap V_{\gamma}^{\prime \prime}\right)$ as

$$
\left(\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ \phi_{\alpha}\right) \circ\left(\phi_{\alpha}^{-1} \circ \phi_{\beta}^{\prime}\right),
$$

which is a composition of two smooth functions because both $\mathcal{A}^{\prime}$ and $\mathcal{A}^{\prime \prime}$ are compatible with $\mathcal{A}$. Hence it is smooth, and hene so is $\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ \phi_{\beta}^{\prime}$. Thus $\mathcal{A}^{\prime}$ and $\mathcal{A}^{\prime \prime}$ are compatible.

Now that we have proven that $\mathcal{A} \subset \mathcal{A}^{\prime}$ and $\mathcal{A} \subset \mathcal{A}^{\prime \prime}$ implies that $\mathcal{A}^{\prime}$ and $\mathcal{A}^{\prime \prime}$ are compatible, we can just define

$$
\mathcal{A}_{\max }:=\bigcup_{\mathcal{A} \subset \mathcal{A}^{\prime}} \mathcal{A}^{\prime}
$$

Definition 2.2.3. A $k$-dimensional smooth manifold is a Hausdorff second-countable topological space $X$ with a maximal $k$-dimensional smooth atlas.

In essence, it is a $k$-dimensional topological manifold where all transition functions are smooth. Some questions and answers about this definition:
(a) How should I think of the smooth atlas? The interpretation that follows directly from the definition is that it provides local coordinates, via the maps $\phi_{\alpha}^{-1}$, and the transition between two of these coordinate systems is smooth.
A different perspective on the role of an atlas is that it tells you when a continuous function $f: X \rightarrow \mathbb{R}$ is smooth:
Definition 2.2.4. A continuous function $f: X \rightarrow \mathbb{R}$ is smooth when

$$
f \circ \phi_{\alpha}: \mathbb{R}^{k} \supset U_{\alpha} \longrightarrow \mathbb{R}
$$

is smooth for all charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$.
This definition generalizes with ease to the case where the target is $\mathbb{R}^{m}$. To generalize to the case that the target is another smooth manifold, we involve the charts of the target. We discuss the following definition in more detail in Chapter 4:
Definition 2.2.5. Let $X$ and $Y$ be manifolds with atlases $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ and $\left\{\left(U_{\alpha^{\prime}}^{\prime}, V_{\alpha^{\prime}}^{\prime}, \phi_{\alpha^{\prime}}^{\prime}\right)\right\}$ respectively. A continuous map $f: X \rightarrow Y$ is smooth if

$$
\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}: \mathbb{R}^{k} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\alpha^{\prime}}^{\prime}\right)\right) \longrightarrow U_{\alpha^{\prime}}^{\prime} \subset \mathbb{R}^{k^{\prime}}
$$

is smooth for all charts.
(b) When are two manifolds "the same"? Saying when two manifolds are equivalent involves Definition 2.2.5:

Definition 2.2.6. A smooth map $f: X \rightarrow Y$ is a diffeomorphism if it is a bijection with smooth inverse.

Two manifolds $X$ and $Y$ are to be considered equivalent when they are diffeomorphic.
(c) Why demand $X$ is Hausdorff and second-countable? We mentioned that this "fits our intuitions," but there is a more utilitarian answer. First, atlases on topological spaces without these properties rarely arise in practice. Second, having these properties is helpful, as they imply the existence of certain smooth functions $X \rightarrow \mathbb{R}$. For example, Hausdorffness will mean we can construct a smooth function which separates two distinct points, and both properties are used to construct partitions of unity.
(d) Why demand that the atlas is maximal? If we did not, then $S^{2}$ with two charts would be a different smooth manifold than $S^{2}$ with three charts. This would be absurd. Furthermore, we often want certain nice charts to exist. If our atlas has few charts this may not be the case.
However, in practice we will want to specify a smooth manifold with an atlas that is as small as possible; a finite amount of data is easier to comprehend that an infinite amount. Then Lemma 2.2.2 generates for us a unique maximal atlas.
(e) Can a topological space $X$ have more than one maximal atlas? The answer is yes, as you can always change the charts by a homeomorphism $X \rightarrow X$. Many of these will result in a diffeomorphic manifold. Problem 2.3.8 gives an example of this.
However, even up to diffeomorphism a topological space $X$ can have more than one maximal atlas. Another term for a maximal atlas is a smooth structure. Milnor surprised the mathematical community when he proved that $S^{7}$ admits more than one smooth structure up to diffeomorphism [Mil56a]; there are in fact 15. ${ }^{1}$ This is a global phenomenon except when $n=4$, as $\mathbb{R}^{n}$ admits a unique smooth structure up to diffeomorphism when $n \neq 4$. On the other hand, $\mathbb{R}^{4}$ admits uncountable many smooth structures up to diffeomorphism [Sco05, Section 5.4], and yes, you should be surprised by that. ${ }^{2}$
(f) Can a topological space $X$ have atlases of different dimensions? This is not possible by a famous result of algebraic topology due to Brouwer called invariance of domain, which says that any injective map from an open subset of $\mathbb{R}^{k}$ to $\mathbb{R}^{k}$ has image given by an open subset [Hat02, Theorem 2.B.3]. (A weaker smooth version is Problem 4.4.1.)

If two such atlases did exist, charts from them would give a homeomorphism $f: \mathbb{R}^{k} \supset$ $U \rightarrow V \subset \mathbb{R}^{k^{\prime}}$ between open subsets of $\mathbb{R}^{k}$ and $\mathbb{R}^{k^{\prime}}$ for say $k>k^{\prime}$. But invariance of domain implies that the composition of $f$ with the inclusion

$$
i \circ f: \mathbb{R}^{k} \supset U \longrightarrow V \subset \mathbb{R}^{k^{\prime}} \longrightarrow \mathbb{R}^{k}
$$

[^0]has image both an open subset of $\mathbb{R}^{k}$ and contained in the subset $\mathbb{R}^{k^{\prime}} \subset \mathbb{R}^{k}$, which is impossible.

### 2.2.1 Examples of manifolds

Example 2.2.7 (Euclidean spaces). The prototypical example of a $k$-dimensional smooth manifold is $\mathbb{R}^{k}$ itself. It has second-countable and Hausdorff, and has an atlas with a single chart: $(U, V, \phi)=\left(\mathbb{R}^{k}, \mathbb{R}^{k}, \mathrm{id}\right)$.
Example 2.2.8 (Spheres). Recall that the $k$-sphere is the subspace of $\mathbb{R}^{k+1}$ defined by

$$
S^{k}:=\left\{\left(x_{0}, \ldots, x_{k}\right) \mid \sum_{i=0}^{k} x_{i}^{2}=1\right\}
$$

As a subspace of a second-countable Hausdorff topological space, it is second-countable and Hausdorff, Problem 2.3.1. We will now describe a $k$-dimensional smooth atlas on it, making it a $k$-dimensional smooth manifold, in terms of $2(k+1)$ different hemispheres. It suffices to describe the $\phi^{-1}$ 's (and then we can of course recover the $\phi$ 's as their inverse). For $0 \leq j \leq k$ and $i \in\{0,1\}$, we have a chart given by

$$
\begin{aligned}
\phi_{i j}^{-1}: S^{k} \supset V_{i j}=\left\{x \in S^{k} \mid(-1)^{i} x_{j}>0\right\} & \longrightarrow U_{i j}=\operatorname{int}\left(D^{k}\right) \subset \mathbb{R}^{k} \\
\left(x_{0}, \ldots, x_{k}\right) & \longmapsto\left(x_{0}, \cdots, \widehat{x_{j}}, \cdots, x_{k}\right)
\end{aligned}
$$

The transition functions have most entries of the form $x_{i}$, except that one has the form $\sqrt{1-\sum_{i \neq j} x_{i}^{2}}$. These are clearly smooth.

A different but compatible $k$-dimensional smooth atlas with only two charts is given by stereographic projection. As before, we describe the $\phi^{-1}$ 's: if $C_{N}, C_{S} \subset S^{k}$ denote small closed neighborhoods of the north and south pole $( \pm 1,0, \ldots, 0)$, then $\phi^{-1}$ is given by casting rays form $N$ through $S^{k} \backslash C_{N}$ onto a plane below the sphere, see Figure 2.3.


Figure 2.3 To obtain $\phi^{-1}: V \rightarrow U$, the inverse of a local parametrization of $S^{k} \subset \mathbb{R}^{k+1}$, follow the rays.

Example 2.2.9 (Real projective spaces). The real projective space $\mathbb{R} P^{k}$ is the space of lines through the origin in $\mathbb{R}^{k+1}$. Such a line is specified by a unit vector, up to multiplication by $\pm 1$. That is, it is the quotient space

$$
\mathbb{R} P^{k}=S^{k} / \sim
$$

with $\sim$ the equivalence relation generated by $\left(x_{0}, \ldots, x_{k}\right) \sim\left(-x_{0}, \ldots,-x_{k}\right)$. We will denote an example class as $\left[x_{0}: \cdots: x_{k}\right]$.

The first of the atlases for $S^{n}$ given in the previous example induces a $k$-dimensional smooth atlas on $\mathbb{R} P^{k}$. It has $(k+1)$ charts given as follows: for $0 \leq j \leq k$ it is

$$
\begin{aligned}
\bar{\phi}_{j}^{-1}: \mathbb{R} P^{k} \supset V_{j}=\left\{x \in \mathbb{R} P^{k} \mid x_{j} \neq 0\right\} & \longrightarrow U_{j}=\operatorname{int}\left(D^{k}\right) \subset \mathbb{R}^{k} \\
{\left[x_{0}: \ldots: x_{k}\right] } & \longmapsto \operatorname{sign}\left(x_{j}\right)\left(x_{0}, \cdots, \hat{x}_{j}, \cdots, x_{k}\right) .
\end{aligned}
$$

Example 2.2.10 (Surfaces of genus $g$ ). We will not describe atlases for these yet, but for each $g \geq 0$ there is a compact surface of genus $g$. It looks like a sphere with $g$ handles added to it:


Figure 2.4 A surface of genus $g=2$.

The classification of surfaces say that all compact orientable two-dimensional smooth manifolds (we will define "orientable manifolds" in Chapter 17) are diffeomorphic to $\Sigma_{g}$ for some $g$.

### 2.2.2 Constructions of manifolds

Example 2.2.11 (Open subsets). Suppose $U \subset X$ is an open subset of a $k$-dimensional smooth manifold. If $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ is an atlas of $X$, then the maps

$$
\left.\phi_{\alpha}\right|_{\phi_{\alpha}^{-1}\left(V_{\alpha} \cap U\right)}: \mathbb{R}^{k} \supset U_{\alpha} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap U\right) \longrightarrow V_{\alpha} \cap U \subset U
$$

endow $U$ with a $k$-dimensional smooth atlas. If the atlas of $X$ is maximal, so is this atlas of $U$.
Example 2.2.12 (Disjoint unions). Let $M$ and $N$ be smooth manifolds with smooth atlases $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ and $\left\{\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)\right\}$, of same dimension $m=n$. Then their union is an atlas for the disjoint union $M \sqcup N$, though it is in general not maximal even if the atlases on $M$ and $N$ were. This is the disjoint union of smooth manifolds.
Example 2.2.13 (Products). Now let $M$ and $N$ be smooth manifolds with smooth atlases $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ and $\left\{\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)\right\}$, of dimension $m$ and $n$ respectively. Then the maps

$$
\phi_{\alpha} \times \phi_{\beta}^{\prime}: \mathbb{R}^{m} \times \mathbb{R}^{n} \supset U_{\alpha} \times U_{\beta}^{\prime} \longrightarrow V_{\alpha} \times V_{\beta}^{\prime} \subset M \times N
$$

endow the cartesian product $M \times N$ with an $(m+n)$-dimensional smooth structure. This is the product of smooth manifolds.
Example 2.2.14 (Pre-manifolds). A $k$-dimensional smooth pre-manifold is a set $X$ together with a collection $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ of $U_{\alpha} \subset \mathbb{R}^{k}$ an open subset, $V_{\alpha} \subset X$ a subset, and $\phi_{\alpha}: U_{\alpha} \rightarrow V_{\alpha}$ a bijection. We require that all maps

$$
\psi_{\alpha \beta}=\phi_{\beta}^{-1} \circ \phi_{\alpha}: \mathbb{R}^{k} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \longrightarrow \phi_{\beta}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \subset \mathbb{R}^{k}
$$

are smooth.
Then we can give $X$ the smallest topology such that all $\phi_{\alpha}$ are continuous. If this is Hausdorff and second countable, then $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ serves as a $k$-dimensional smooth atlas on $X$ and hence makes it into a $k$-dimensional smooth manifold.

### 2.2.3 Riemann's vision

In this more advanced section, we recall some historical context. You should not be surprised if much of this material is unfamiliar to you.

## One-dimensional complex manifolds

If you have studied complex analysis, the following example may illuminate the definition of a $k$-dimensional smooth manifold.

We will define complex manifolds by replacing $\mathbb{R}$ by $\mathbb{C}$ and smooth maps by holomorphic maps: a 1-dimensional complex atlas for topological space $X$ is a collection of triples ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) of an open subset $U_{\alpha} \subset \mathbb{C}$, an open subset $V_{\alpha} \subset X$, and a homeomorphism $\phi_{\alpha}: U_{\alpha} \rightarrow V_{\alpha}$, so that $\bigcup V_{\alpha}=X$ and all maps

$$
\phi_{\beta}^{-1} \circ \phi_{\alpha}: \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \longrightarrow \phi_{\beta}^{-1}\left(V_{\alpha} \cap V_{\beta}\right)
$$

are holomorphic maps between open subsets of $\mathbb{C}$. A 1-dimensional complex manifold is then a second-countable Hausdorff topological $X$ with a maximal 1-dimensional complex atlas.

Since $\mathbb{C}$ can be identified with $\mathbb{R}^{2}$ and all holomorphic maps are smooth, any 1dimensional complex manifold is a 2 -dimensional smooth manifold. However, since it is much harder for a function to be holomorphic than for it to be smooth, it is harder to produce 1-dimensional complex manifolds than 2-dimensional smooth manifold.
Remark 2.2.15. By replacing $\mathbb{C}$ by $\mathbb{C}^{k}$, this definition generalizes to that of a $k$-dimensional complex manifold. Such a complex manifold always gives rise to a $2 k$-dimensional smooth manifold.

## The moduli spaces of Riemann surfaces

It is in Riemann's Habilitationsvortrag that the general concept of a manifold first appeared [Rie13]. ${ }^{3}$ He proposed that geometry should study "extended magnitude or

[^1]quantity," objects made of points with a continuous transition from one to another. To be mathematically useful, these objects should have sufficiently many functions so that it is possible to find coordinate functions which specify points uniquely, at least locally. One example he had in mind is quite advanced even from our modern point of view: the moduli space of Riemann surfaces of genus $g$ with $n$ marked points.

A Riemann surface is a compact one-dimensional complex manifold, as above. It is a rather deep result that all of these are algebraic, that is, cut out by polynomial equations in a complex projective space. Riemann's idea was that deformations of a Riemann surface structure on a fixed surface of genus $g$ with $n$ marked points as pictured in Figure 2.5 are uniquely specified (up to isomorphism) by $3 g-3+n$ complex parameters. He wanted to use this to show that one can organize all such Riemannn surfaces into (something like) a $(6 g-6+2 n)$-dimensional smooth manifold, each complex parameter giving rise to two dimensions [Loo00], so that you could study all Riemann surfaces at the same time. This has proven wildly successful, with entire fields doing dynamics and geometry on such moduli spaces.

We are far from having the theory to make this precise, but this example holds an important lesson: unlike spheres, many examples of smooth manifolds do not arise as subsets of some Euclidean space.


Figure 2.5 A surface of genus $g=2$ with $n=3$ marked points.

### 2.3 Problems

Problem 2.3.1 (Point-set topology of subspaces).
(a) Prove that every subspace of a Hausdorff space is Hausdorff.
(b) Prove that every subspace of a second-countable space is second-countable.

Problem 2.3.2 (Connected vs. path components). Prove that for a topological manifold, connected components coincide with path components.

Problem 2.3.3 (Gluing smooth structures). Suppose that if $U, V \subset X$ is an open cover of a second-countable Hausdorff space, and that we are given smooth atlases on $U$ and
$V$ which agree on $U \cap V$. Prove that there exists a unique smooth maximal atlas on $X$ which is compatible with the given ones on $U$ and $V$.

Problem 2.3.4 (Some examples). Draw three 2-dimensional smooth manifolds that are pairwise non-diffeomorphic. You do not need to prove that they are not diffeomorphic to each other.

Problem 2.3.5 (Real projective plane revisited). An alternative definition of the real projective plane is as the quotient

$$
\left(\mathbb{R}^{k+1} \backslash\{0\}\right) / \sim
$$

where $\sim$ is the equivalence relation generated by $\left(x_{0}, \ldots, x_{k}\right) \sim\left(\lambda x_{0}, \ldots, \lambda x_{k}\right)$ for $\lambda \in$ $\mathbb{R} \backslash\{0\}$. We will denote an equivalence class by $\left[x_{0}: \ldots: x_{k}\right]$.
(a) Show that $\left(\mathbb{R}^{k+1} \backslash\{0\}\right) / \sim$ is homeomorphic to $\mathbb{R} P^{k}$ as in Example 2.2.9.
(b) Explain why we can think of $\mathbb{R} P^{k}$ as the set of lines through the origin in $\mathbb{R}^{k+1}$. There is a smooth atlas on $\mathbb{R} P^{k}$ provided by homogeneous coordinates, which is diffeomorphic to that in Example 2.2.9: for $0 \leq j \leq k$ these are given by

$$
\begin{aligned}
\phi_{j}: \mathbb{R}^{k} & \longrightarrow \mathbb{R} P^{k} \\
\left(x_{1}, \ldots, x_{k}\right) & \longmapsto\left[x_{1}: \cdots: x_{j-1}: 1: x_{j}: \cdots: x_{k}\right]
\end{aligned}
$$

(c) Compute explicitly the transition functions and verify they are smooth.

Problem 2.3.6 (Complex projective plane). There is a complex analogue of the real projective plane $\mathbb{R} P^{k}$. The complex projective plane $\mathbb{C} P^{k}$ has points given by complex lines in $\mathbb{C}^{k+1}$, or equivalently by the quotient

$$
\left(\mathbb{C}^{k+1} \backslash\{0\}\right) / \sim
$$

where $\sim$ is the equivalence relation generated by $\left(z_{0}, \ldots, z_{k}\right) \sim\left(\lambda z_{0}, \ldots, \lambda z_{n}\right)$ for $\lambda \in$ $\mathbb{C} \backslash\{0\}$. Give $\mathbb{C} P^{k}$ a $2 k$-dimensional smooth atlas.

Problem 2.3.7. Recall that the quaternions $\mathbb{H}$ are the 4 -dimensional non-commutative $\mathbb{R}$-algebra with generators $i, j, k$ and relations

$$
\begin{gathered}
i^{2}=j^{2}=k^{2}=-1, \quad i j=-j i, \quad i k=-k i, \quad j k=-k j \\
i j=k, j k=i, k i=j
\end{gathered}
$$

In analogy with the previous exercise, construct the quaternionic projective plane $\mathbb{H} P^{k} .{ }^{4}$ What is its dimension?

Problem 2.3.8 (Different atlases on the same topological manifold). Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be the homeomorphism $x \mapsto x^{3}$. Let $\mathcal{A}=\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ be the usual maximal smooth atlas on $\mathbb{R}$, that is, the unique one containing $(\mathbb{R}, \mathbb{R}, \mathrm{id})$. Let $g(\mathcal{A})$ be the maximal smooth atlas $\left\{\left(U_{\alpha}, g\left(V_{\alpha}\right), g \circ \phi_{\alpha}\right)\right\}$, that is, the unique one containing $(\mathbb{R}, \mathbb{R}, g)$.

[^2](i) Prove that the atlases $\mathcal{A}$ and $g(\mathcal{A})$ are not compatible.
(ii) Prove that $(\mathbb{R}, \mathcal{A})$ and $(\mathbb{R}, g(\mathcal{A})$ ) are diffeomorphic (Hint: $g$ is the diffeomorphism!).

## Chapter 3

## Submanifolds and tori

In the previous chapter we defined smooth manifold, and we now discuss smooth submanifolds. We will use some calculus to produce examples of submanifolds of Euclidean spaces. I will assume you know the relevant results, but if you do not you can find these in Chapters $3 \& 4$ of [DK04a]. After that we will give five constructions of the 2-torus.

### 3.1 Submanifolds

A loop of string in $\mathbb{R}^{3}$ can be thought of as a subset $S$ of $\mathbb{R}^{3}$. Which subsets $S$ describe such loops of string? Let us abstract the situation by declaring that the string is infinitely thin and bendable, but can not make sharp corners. Certainly an ordinary circle $\left\{(x, y, z) \mid x^{2}+y^{2}+z^{2}=r^{2}\right\} \subset \mathbb{R}^{3}$ describes a loop of string, but so do many other subsets. Some differ from the circle by being more wiggly, and some by being knotted, see Figure 3.1.


Figure 3.1 Some subsets of $\mathbb{R}^{3}$ which describe strings.

However, in spite of their complicated global behavior all locally look like smooth line segments: they are one-dimensional smooth submanifolds of $\mathbb{R}^{3}$, subsets of $\mathbb{R}^{3}$ that locally looks like $\mathbb{R}$. This illustrates why the study of smooth manifolds is so interesting: they have a straightforward local structure, but a rich global structure.

Of course, we need not restrict ourselves to one-dimensional objects: spheres, tori and the surface of a coffee mug locally look like $\mathbb{R}^{2}$. Indeed, for any $r \geq 0$ we can define $r$-dimensional smooth submanifolds as subsets of $\mathbb{R}^{k}$ that locally look like $\mathbb{R}^{r}$. More generally, we use charts to replace the ambient space $\mathbb{R}^{k}$ by a $k$-dimensional smooth manifold $N$.

### 3.1.1 The definition

To give a precise definition of a submanifold of a manifold, we recall the definitions in Chapter 2. A $k$-dimensional topological manifold is a second-countable Hausdorff space $X$ which is locally homeomorphic to an open subset of $\mathbb{R}^{k}$. To make this into a $k$-dimensional smooth manifold, we need to give the additional data of a maximal $k$-dimensional smooth atlas. This is a collection $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of homeomorphisms $\phi_{\alpha}: \mathbb{R}^{k} \supset U_{\alpha} \rightarrow V_{\alpha} \subset X$ such that (i) $\bigcup_{\alpha} V_{\alpha}=X$, and (ii) all transition functions $\phi_{\beta}^{-1} \circ \phi_{\alpha}$ are smooth maps between open subsets of $\mathbb{R}^{k}$.

Intuitively, a submanifold is a manifold which lives inside another manifold. This is made precise by demanding it looks like a linear subspace of Euclidean space with respect to the atlas.

Definition 3.1.1. Let $N$ be a $k$-dimensional smooth manifold. A subset $X \subset N$ is an $r$-dimensional submanifold if for each $p \in X$ there is a chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of $N$ around $p$ such that $\phi_{\alpha}^{-1}(X)=U_{\alpha} \cap \mathbb{R}^{r}$.

If $X$ is a submanifold, it comes with a canonical structure of an $r$-dimensional smooth manifold. Firstly, $X$ with the subspace topology is second countable and Hausdorff by Problem 2.3.1. We produce an atlas on this by taking a chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ for $N$ as above, and creating from it a chart $\left(U_{\alpha}^{\prime}, V_{\alpha}^{\prime}, \phi_{\alpha}^{\prime}\right)$ for $X$ as follows:

$$
U_{\alpha}^{\prime}:=U_{\alpha} \cap \mathbb{R}^{r}, \quad V_{\alpha}^{\prime}:=X \cap V_{\alpha}, \quad \text { and } \quad \phi_{\alpha}^{\prime}:=\left.\phi_{\alpha}\right|_{U_{\alpha}^{\prime}}
$$

### 3.2 Examples of submanifolds using calculus

We now concentrate on submanifolds of Euclidean space, so that we can apply tools from multivariable calculus. We will eventually generalize these tools to manifolds, the philosophy being that differential topology is globalized multivariable analysis.

### 3.2.1 $S^{n}$ by equations

Last chapter we defined the $n$-sphere by equations

$$
S^{n}=\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_{i}^{2}=1\right\}
$$

and by hand gave an atlas for it.
However, when you define a manifold by equations, it is much easier to obtain the atlas using a result from multivariable calculus; the inverse function theorem. This uses the notion of a total derivative of a map $g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ (or between open subsets thereof)
[DK04a, Section 4.5]: at $x \in \mathbb{R}^{n}$, the total derivative $D g_{x}$ of $g$ at $x$ is the linear map described by the $(p \times n)$-matrix of partial derivatives

$$
\left[\begin{array}{cccc}
\frac{\partial g_{1}}{\partial x_{1}}(x) & \frac{\partial g_{1}}{\partial x_{2}}(x) & \cdots & \frac{\partial g_{1}}{\partial x_{n}}(x) \\
\frac{\partial g_{2}}{\partial x_{1}}(x) & \frac{\partial g_{2}}{\partial x_{2}}(x) & \cdots & \frac{\partial g_{2}}{\partial x_{n}}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{p}}{\partial x_{1}}(x) & \frac{\partial g_{p}}{\partial x_{2}}(x) & \cdots & \frac{\partial g_{p}}{\partial x_{n}}(x)
\end{array}\right] .
$$

The local version of the inverse function theorem then says [DK04a, Theorem 3.2.4]:
Theorem 3.2.1 (Inverse function theorem). Let $U_{0} \subset \mathbb{R}^{n}$ be open and $a \in U_{0}$. Suppose $g: U_{0} \rightarrow \mathbb{R}^{n}$ is a smooth map whose total derivative $D g_{a}$ at $a$ is an invertible linear map. Then there exists an open neighborhood $U \subset U_{0}$ of a such that $g(U)$ is open and

$$
\left.g\right|_{U}: U \longrightarrow g(U)
$$

is a diffeomorphism onto this open subset.
By adding dummy variables, you can deduce the implicit function theorem [DK04a, Theorem 3.5.1] from this. The following is a consequence of that result [DK04a, Section 4.5]:

Theorem 3.2.2 (Submersion theorem). Let $U_{0} \subset \mathbb{R}^{n}$ be open and $a \in U_{0}$. Suppose $g: U_{0} \rightarrow \mathbb{R}^{p}, p \leq n$ is a smooth map whose total derivative $D g_{a}$ of $g$ at $a$ is a surjective linear map. Then there exist open neighborhoods $U \subset U_{0}$ of a and $V \subset \mathbb{R}^{p}$ of $g(a)$, and diffeomorphisms $\psi: \mathbb{R}^{n} \rightarrow U$ and $\varphi: \mathbb{R}^{p} \rightarrow V$, such that
(i) $\psi(0)=a$,
(ii) $\varphi(0)=g(a)$, and
(iii) the following diagram commutes

with $\pi_{p}$ the projection $\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(x_{1}, \ldots, x_{p}\right)$. That is,

$$
g\left(\psi\left(x_{1}, \ldots, x_{n}\right)\right)=\varphi\left(x_{1}, \ldots, x_{p}\right) .
$$

Remark 3.2.3. A stronger version of this theorem, which is the one stated as [DK04a, Theorem 4.5.2(iv)], says that $\varphi$ can be taken to be translation near 0 .

Parts (i) and (ii) are just normalizations, part (iii) is where the magic happens: the diffeomorphism $\psi$ restricted to $\{0\} \times \mathbb{R}^{n-p} \subset \mathbb{R}^{n}$ gives a local parametrization of the inverse image $g^{-1}(g(x))$ around $x$, identifying it with an open subset of the origin in $\mathbb{R}^{n-p}$. We conclude that the subset $g^{-1}(c)$ for $c \in \mathbb{R}^{p}$ is an $(n-p)$-dimensional smooth submanifold of $\mathbb{R}^{n}$ when each of the total derivatives $D g_{x}$ for $x \in g^{-1}(c)$ is surjective.

Example 3.2.4. If we take

$$
\begin{aligned}
g: \mathbb{R}^{n+1} & \longrightarrow \mathbb{R} \\
\left(x_{0}, \ldots, x_{n}\right) & \longmapsto x_{0}^{2}+\ldots+x_{n}^{2},
\end{aligned}
$$

and $c \neq 0 \in \mathbb{R}$, then the total derivative at $x=\left(x_{0}, \ldots, x_{n}\right)$ satisfying $x_{0}^{2}+\ldots+x_{n}^{2}=c$ is given by the $(1 \times n)$-matrix

$$
\left[\begin{array}{llll}
2 x_{0} & 2 x_{1} & \cdots & 2 x_{n}
\end{array}\right]
$$

with not all $x_{i}$ zero. If $c \neq 0$, then not all entries can vanish at the same time and this matrix is surjective. In particular, we can take $c=1$ to obtain another proof that the $n$-sphere is a smooth manifold.
Example 3.2.5. Let $p, q$ be positive integers, and take

$$
\begin{aligned}
g: \mathbb{C}^{2} & \longrightarrow \mathbb{C} \times \mathbb{R} \\
\left(z_{1}, z_{2}\right) & \longmapsto\left(z_{1}^{p}+z_{2}^{q},\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}\right) .
\end{aligned}
$$

This is smooth, and its total derivative is surjective at all points $g^{-1}(0, \epsilon)$ for $(0, \epsilon) \in$ $\mathbb{C} \times \mathbb{R}$ with $\epsilon>0$ small enough. Thus the inverse image $g^{-1}(0, \epsilon)$ is a one-dimensional submanifold of $\mathbb{C}^{2}$, which lies inside of $S^{3}$. It is in fact also a one-dimensional submanifold of $S^{3}$, and if we remove a point from $S^{3}$ and identify this with $\mathbb{R}^{3}$, the result is a so-called ( $p, q$ )-torus link. See Figure 3.2 for an example.


Figure 3.2 A (3,7)-torus knot (since 3 and 7 are coprime, there is only a single component).

### 3.2.2 $S^{n}$ by parametrizations

One can often parametrize solution sets of equations, e.g. $S^{1}$ is the image of

$$
\begin{aligned}
h: \mathbb{R} & \longrightarrow \mathbb{R}^{2} \\
\theta & \longmapsto(\cos (\theta), \sin (\theta)) .
\end{aligned}
$$

This map is not a bijection, but it is locally a bijection. It seems quite plausible that it is in fact a local diffeomorphism of $\mathbb{R}$ onto $S^{1}$, though giving an explicit formula may be hard. However, the difficulty of finding explicit formula's can be avoided by using the inverse function theorem again, in a slightly different guise [DK04a, Section 4.3].

Theorem 3.2.6 (Immersion theorem). Let $U_{0} \subset \mathbb{R}^{p}$ be an open subset and $a \in U_{0}$. Suppose $h: U_{0} \rightarrow \mathbb{R}^{n}, p \leq n$, is a smooth map whose total derivative $D h_{a}$ of $h$ at $a$ is injective. Then there exist open neighborhoods $U \subset U_{0}$ of a and $V \subset \mathbb{R}^{n}$ of $h(a)$, and diffeomorphisms $\psi: \mathbb{R}^{p} \rightarrow U$ and $\varphi: \mathbb{R}^{n} \rightarrow V$, such that
(i) $\psi(0)=a$,
(ii) $\varphi(0)=h(a)$, and
(iii) the following diagram commutes

with $\iota_{p}$ the inclusion $\left(x_{1}, \ldots, x_{p}\right) \mapsto\left(x_{1}, \ldots, x_{p}, 0, \ldots, 0\right)$. That is,

$$
h\left(\psi\left(x_{1}, \ldots, x_{p}\right)\right)=\varphi\left(x_{1}, \ldots, x_{p}, 0, \ldots, 0\right)
$$

Remark 3.2.7. As before, there is a stronger version stated as [DK04a, Theorem 4.3.1] which says that $\psi$ can be taken to be translation near 0 .

Again part (iii) is the important part: it provides a chart for $h(U)$ as in the definition of a submanifold. We will later see that the image of $h$ is a submanifold if we not only suppose that its derivative is injective everywhere but also that the map $h$ is a homeomorphism onto its image.
Example 3.2.8. If we want to parametrize the $n$-sphere $S^{n}$, we will need more than one function $h_{i}$. For example, we can use $2(n+1)$ ones indexed by $0 \leq i \leq n$ and a sign $\pm 1$ :

$$
\begin{aligned}
h_{i}^{ \pm}:\left\{y \in \mathbb{R}^{n} \mid\|y\|<1\right\} & \longrightarrow \mathbb{R}^{n+1} \\
\left(y_{1}, \ldots, y_{n}\right) & \longmapsto\left(y_{1}, \ldots, y_{i-1}, \pm \sqrt{1-\|y\|^{2}}, y_{i}, \ldots, y_{n}\right)
\end{aligned}
$$

Each covers one of the two hemispheres in each of the $n+1$ directions of $\mathbb{R}^{n+1}$.

### 3.3 Five constructions of the 2-torus

Another important example of a smooth manifold is the 2-torus, one of the basic surfaces. We will now give five constructions of the torus,
(1) By specifying it as a submanifold of $\mathbb{R}^{3}$ using equations.
(2) By parametrizing it as a submanifold of $\mathbb{R}^{3}$.
(3) As a product of two circles.
(4) By gluing edges of a square $[0,1]^{2}$.
(5) As a quotient $\mathbb{R}^{2} / \mathbb{Z}^{2}$.

All these constructions give us diffeomorphic smooth manifolds, but we will not prove this. The first three can be thought of as naturally being subsets of some Euclidean spaces, but the underlying topological space of a smooth manifold obtained by gluing or quotients is not naturally a subset of a Euclidean space. This is one of the reasons we gave an abstract definition of manifold in the last chapter.

### 3.3.1 The 2-torus specified by equations

Our first construction of a 2 -torus is as those points that are distance 1 from a circle of radius $\sqrt{2}$ : it consists of those points $(x, y, z)$ in $\mathbb{R}^{3}$ satisfying the equation $\left(2-\sqrt{x^{2}+y^{2}}\right)^{2}+z^{2}=1$. More precisely, define

$$
\begin{aligned}
g: \mathbb{R}^{3} & \longrightarrow \mathbb{R} \\
(x, y, z) & \longmapsto\left(2-\sqrt{x^{2}+y^{2}}\right)^{2}+z^{2}
\end{aligned}
$$

This is smooth and has surjective total derivative at all points in the pre-image of 1 . Thus the submersion theorem tells us that $g^{-1}(1)$ is a two-dimensional smooth submanifold of $\mathbb{R}^{3}$ :

$$
\mathbb{T}^{2}=g^{-1}(1)
$$

### 3.3.2 The 2-torus parametrized

This is a incomplete construction, and we will revisit it in XXX. We can parametrize the 2 -torus, defined as $g^{-1}(1) \subset \mathbb{R}^{3}$, as the image of

$$
\begin{aligned}
h: \mathbb{R}^{2} & \longrightarrow \mathbb{R}^{3} \\
(\theta, \phi) & \longmapsto[(2+\cos (\theta)) \cos (\phi),(2+\cos (\theta)) \sin (\phi), \sin (\theta)]
\end{aligned}
$$

This is smooth and has injective total derivative at all points in its domain. Thus the immersion theorem provides local charts for the image of $h$. These exhibit the image of $h$ as a submanifold of $\mathbb{R}^{3}$, and give another description of the 2 -torus as a two-dimensional smooth submanifold of $\mathbb{R}^{3}$ :

$$
\mathbb{T}^{2}=\operatorname{im}(h)
$$

Some care is require now, as $h$ is not a homeomorphism onto its image. Trying to amend this leads one to the definition of the 2-torus as a quotient.

### 3.3.3 The 2-torus as a product

There is a general method to produce new submanifolds out of old ones.
Lemma 3.3.1. Suppose that $X \subset \mathbb{R}^{n}$ and $Y \subset \mathbb{R}^{m}$ are submanifolds of dimensions $p$ and $q$ respectively. Then $X \times Y \subset \mathbb{R}^{n} \times \mathbb{R}^{m}=\mathbb{R}^{n+m}$ is a $(p+q)$-dimensional submanifold of $\mathbb{R}^{n+m}$.

Sketch of proof. Local parametrizations of $X$ near $x$ and $Y$ near $y$ combine a local parametrization of $X \times Y$ near $(x, y)$.

This gives a different construction of $\mathbb{T}^{2}$ as a submanifold of $\mathbb{R}^{4}$ : take the product of $S^{1} \subset \mathbb{R}^{2}$ with itself. Of course, we can forget that $S^{1}$ is a submanifold of $\mathbb{R}^{2}$, and instead take the abstract product of manifolds discussed in the previous chapter, Example 2.2.13:

$$
\mathbb{T}^{2}=S^{1} \times S^{1}
$$

### 3.3.4 The 2 -torus by gluing

Let us take a square $[0,1]^{2}$ and make identifications along its boundary $\partial[0,1]^{2}=$ $\left\{(x, y) \in[0,1]^{2} \mid x \in\{0,1\}\right.$ or $\left.y \in\{0,1\}\right\}$ as in Figure 3.3: take $[0,1]^{2} / \sim$ with $\sim$ the equivalence relation generated by

$$
(0, y) \sim(1, y) \quad \text { and } \quad(x, 0) \sim(x, 1)
$$

That is, the left edge $\{0\} \times[0,1]$ gets identified with right edge $\{1\} \times[0,1]$ and the bottom edge $[0,1] \times\{0\}$ with the top edge $[0,1] \times\{1\}$ Such a gluing of the square produces a torus.


Figure 3.3 The 2-torus is obtained by identifying edges of $[0,1]^{2}$.

We now give a 2 -dimensional smooth atlas on $[0,1]^{2} / \sim$, see Figure 3.4. It is easy to give charts for a point represented by $(x, y) \in(0,1)^{2}$; just use a small open disk $B_{\epsilon}(x, y)$ contained in $(0,1)^{2}$. For equivalence classes $[(x, 0)]$ represented by $(x, 0)$ with $x \in(0,1)$ we use the chart determined by

$$
\begin{aligned}
\phi: B_{\epsilon}(x, 0) & \longrightarrow[0,1]^{2} / \sim \\
\left(x^{\prime}, y^{\prime}\right) & \longmapsto \begin{cases}{\left[\left(x^{\prime}, y^{\prime}+1\right)\right]} & \text { if } y^{\prime}<0 \\
{\left[\left(x^{\prime}, y^{\prime}\right)\right]} & \text { if } y^{\prime} \geq 0\end{cases}
\end{aligned}
$$

and similarly for the element represented by $(0, y)$ with $y \in(0,1)$. For the equivalence
class $[(0,0)]$ we use the chart determined by

$$
\begin{aligned}
& \phi: B_{\epsilon}(0,0) \longrightarrow[0,1]^{2} / \sim \\
& \quad\left(x^{\prime}, y^{\prime}\right) \longmapsto \begin{cases}{\left[\left(x^{\prime}+1, y^{\prime}+1\right)\right]} & \text { if } y^{\prime}<0, x^{\prime}<0, \\
{\left[\left(x^{\prime}+1, y^{\prime}\right)\right]} & \text { if } x^{\prime}<0, \\
{\left[\left(x^{\prime}, y^{\prime}+1\right)\right]} & \text { if } y^{\prime}<0, \\
{\left[\left(x^{\prime}, y^{\prime}\right)\right]} & \text { otherwise. }\end{cases}
\end{aligned}
$$



Figure 3.4 The open subsets $V_{\alpha}$ for three charts, one of each type.

The transition functions are mostly given by the identity map which is obviously smooth, but sometimes by a translation which is also obviously smooth. See Figure 3.5 for the hardest case. We conclude that

$$
\mathbb{T}^{2}=[0,1]^{2} / \sim
$$



Figure 3.5 A transition function.
The lesson is, using terms we have not defined yet: a sufficiently nice gluing of a $k$ dimensional manifold with corners along its boundary is again a $k$-dimensional manifold. In the above example $k=2$, the manifold with corners is $[0,1]^{2}$ and the boundary is $\partial[0,1]^{2}$.
Example 3.3.2. Changing the identifications to those in Figure 3.6 and using similar charts we can endow the Klein bottle and real projective plane with a 2 -dimensional smooth structure.


Figure 3.6 Two more 2-dimensional smooth manifolds obtained by identifying edges of $[0,1]^{2}$.

### 3.3.5 The 2-torus as a quotient

Let us recast this definition in terms of group theory. If you are not familiar with group theory, you should take a look at a textbook on it, e.g. [Arm88].

We can add elements of $\mathbb{R}^{2}$ from which we obtain an action of the abelian group $\mathbb{Z}^{2}$ on $\mathbb{R}^{2}$ : the element $(n, m) \in \mathbb{Z}^{2}$ acts on $(x, y)$ by sending it to its translate $(n, m) \cdot(x, y):=$ $(x+n, y+m)$. Let us look at the set

$$
\mathbb{R}^{2} / \mathbb{Z}^{2}:=\mathbb{R}^{2} / \sim \text { with }(x, y) \sim\left(x^{\prime}, y^{\prime}\right) \text { if }(n, m) \cdot(x, y)=\left(x^{\prime}, y^{\prime}\right) \text { for }(m, n) \in \mathbb{Z}^{2}
$$

with the quotient topology. This is still Hausdorff and second countable.
We claim that $\mathbb{R}^{2} / \mathbb{Z}^{2}$ inherits from $\mathbb{R}^{2}$ the structure of a 2 -dimensional smooth manifold. To do so we describe a 2 -dimensional smooth atlas on $\mathbb{R}^{2} / \mathbb{Z}^{2}$ : for a point $(x, y) \in \mathbb{R}^{2}$ we can consider the open disks $B_{\epsilon}(x, y)$ for $\epsilon<\frac{1}{4}$. The composition of the inclusion with the quotient map

$$
B_{\epsilon}(x, y) \hookrightarrow \mathbb{R}^{2} \xrightarrow{q} \mathbb{R}^{2} / \mathbb{Z}^{2}
$$

is injective as $\epsilon<\frac{1}{4}$. We denote its image by $V_{(x, y)}^{\epsilon}$ and resulting map by

$$
\phi_{(x, y)}^{\epsilon}: B_{\epsilon}(x, y) \longrightarrow V_{(x, y)}^{\epsilon} .
$$

We claim these chart give an atlas. Since the map $q: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} / \mathbb{Z}^{2}$ is surjective, the $V_{(x, y)}^{\epsilon}$ cover. For any two open subsets $V_{(x, y)}^{\epsilon}, V_{\left(x^{\prime}, y^{\prime}\right)}^{\epsilon^{\prime}}$, the transition function is just given by translation and hence is smooth.

One way to visualize the result is to give a fundamental domain: an open subset $U \subset \mathbb{R}^{n}$ such that $U \rightarrow \mathbb{R}^{2} / \mathbb{Z}^{2}$ is injective and $\bar{U} \rightarrow \mathbb{R}^{2} / \mathbb{Z}^{2}$ is surjective. Then you can think of $\mathbb{R}^{2} / \mathbb{Z}^{2}$ as being obtained from $\bar{U}$ by making identifications along $\partial U$. In this case a moment's reflection produces $(0,1)^{2} \subset \mathbb{R}^{2}$ as a candidate; no two elements differ by translation by $(m, n) \in \mathbb{Z}^{2}$ so $(0,1)^{2} \rightarrow \mathbb{R}^{2} / \mathbb{Z}^{2}$ is injective, but $(x, y) \sim(x-\lfloor x\rfloor, y-\lfloor y\rfloor) \in$ $[0,1]^{2}$ so $[0,1]^{2} \rightarrow \mathbb{R}^{2} / \mathbb{Z}^{2}$ is surjective. Thus $\mathbb{R}^{2} / \mathbb{Z}^{2}$ is homeomorphic to $[0,1]^{2} / \sim$ as in the previous section, and thus we have produced another description of the 2 -torus. Under this identification, the charts we have described to go the charts in the previous section. We get

$$
\mathbb{T}^{2}=\mathbb{R}^{2} / \mathbb{Z}^{2}
$$

There is a general lesson here: a quotient of a $k$-dimensional smooth manifold by a sufficiently nice action of a discrete group $G$ is again a $k$-dimensional smooth manifold. In the above example $k=2$, the manifold is $\mathbb{R}^{2}$ and $G=\mathbb{Z}^{2}$.

Example 3.3.3. Can we come up with other examples? One idea would be to use with some subgroup $G$ of $\mathbb{Z}^{2}$, and take

$$
\mathbb{R}^{2} / G:=\mathbb{R}^{2} / \sim \text { with }(x, y) \sim\left(x^{\prime}, y^{\prime}\right) \text { if } g \cdot(x, y)=\left(x^{\prime}, y^{\prime}\right) \text { for } g \in G
$$

instead of $\mathbb{R}^{2} / \mathbb{Z}^{2}$. Most of these seem to give variations on the 2-torus, but the subgroup $\mathbb{Z} \times\{0\} \subset \mathbb{Z}^{2}$ does not. In this case a fundamental domain is given by $(0,1) \times \mathbb{R}$, and $\mathbb{R}^{2} /(\mathbb{Z} \times\{0\})$ is given by identifying the left edge $\{0\} \times \mathbb{R}$ of the infinite strip $[0,1] \times \mathbb{R}$ with the right edge $\{1\} \times \mathbb{R}$; an infinite cylinder.

### 3.4 Quotients of the upper half plane

In Section 2.2.3, we discussed moduli spaces of Riemann surfaces with marked points as manifolds. Let me explain how this works in the case of genus 1 surfaces with a single marked point. Again, this material may be unfamiliar to you.

### 3.4.1 The moduli space of genus 1 Riemann surfaces with a marked point

Let us ignore some technical issues, and consider the "coarse" moduli space of genus 1 Riemann surfaces with a marked point. Here "genus 1" means that they look like a 2-torus, i.e. have a single "handle", and "with a marked point" means that we have decided we have declared a single point to be special. Riemann surfaces were defined in Section 2.2.3, but you can get along fine without knowing the details by thinking of everything we do here as studying "lattices in $\mathbb{C}$ up to the action of multiplying with non-zero complex numbers."

Recall that Riemann tells us that we can parametrize genus $g$ Riemann surfaces with $n$ marked points at least locally with $6 g-6+2 n$ real parameters. Specializing to $g=1$ and $n=1$, we expect that there is a moduli space of genus 1 Riemann surfaces with a marked point, which should ideally be a $6 g-6+2 n=2$-dimensional smooth manifold.

It turns out that each genus 1 Riemann surfaces with a marked point arises as a quotient space $\mathbb{C} /(\tau \mathbb{Z}+\mathbb{Z})$ with $\tau \in \mathbb{H}^{2}=\{z \in \mathbb{C} \mid \operatorname{im}(z)>0\}$, with marked point given by the image of 0 . That is, they are given as a quotient space ${ }^{1}$

$$
\mathbb{C} /(\tau \mathbb{Z}+\mathbb{Z})=\mathbb{C} / \sim \text { with } z \sim z^{\prime} \text { if } z-z^{\prime}=\tau p+q \text { with } p, q \in \mathbb{Z}
$$

By the same argument as for the 2-torus, these are 2-dimensional smooth manifolds.

[^3]Many of these Riemann surfaces with a marked point are the same; Riemann would consider $\mathbb{C} /(\tau \mathbb{Z}+\mathbb{Z})$ to be isomorphic to $\mathbb{C} /\left(\tau^{\prime} \mathbb{Z}+\mathbb{Z}\right)$ if there is a non-zero complex number $\lambda \in \mathbb{C}$ such that $\lambda\left(\tau^{\prime} \mathbb{Z}+\mathbb{Z}\right)=\tau \mathbb{Z}+\mathbb{Z}$. When is this the case? For the image of 1 to lie in $\tau \mathbb{Z}+\mathbb{Z}, \lambda$ should be some $c \tau+d$ with $c, d \in \mathbb{Z}$. Similarly, for the image of $\tau$ to lie in $\tau \mathbb{Z}+\mathbb{Z}, \lambda \tau^{\prime}$ should be $a \tau+b$ with $a, b \in \mathbb{Z}$. In particular, we can recover $\tau^{\prime}$ as $\lambda \tau^{\prime} / \lambda=\frac{a \tau+b}{c \tau+d}$. It turns out that $a \tau+b$ and $c \tau+d$ are generators of $\tau \mathbb{Z}+\mathbb{Z}$ if and only if the matrix

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

has determinant $\pm 1$. However, note that

$$
\begin{aligned}
2 \cdot \operatorname{im}\left(\tau^{\prime}\right) & =2 \cdot \operatorname{im}\left(\frac{a \tau+b}{c \tau+d}\right) \\
& =\frac{a \tau+b}{c \tau+d}-\frac{\overline{a \tau+b}}{\overline{c \tau+d}} \\
& =\frac{(a \tau+b)(c \bar{\tau}+d)-(c \tau+d)(a \bar{\tau}+b)}{|c \tau+d|^{2}} \\
& =\frac{2(a d-b c) \operatorname{im}(\tau)}{|c \tau+d|^{2}} \\
& =\operatorname{det}\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \frac{2 \operatorname{im}(\tau)}{|c \tau+d|^{2}} .
\end{aligned}
$$

So im $\left(\tau^{\prime}\right)>0$ means the determinant should be +1 ; the group $(2 \times 2)$-matrices with integer entries and determinant +1 is called $\mathrm{SL}_{2}(\mathbb{Z})$.

The upshot is that we get the isomorphic Riemann surfaces with marked point if we replace the parameter $\tau$ by

$$
\tau \longmapsto \frac{a \tau+b}{c \tau+d} \quad \text { for } \quad\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in \mathrm{SL}_{2}(\mathbb{Z})
$$

As two matrices differing by $\pm 1$ in all entries act in the same manner, a genus 1 Riemann surface is described by an element $\tau \in \mathbb{H}^{2}$ up to the action by quotient group $\operatorname{PSL}_{2}(\mathbb{Z}):=\mathrm{SL}_{2}(\mathbb{Z}) /\{ \pm \mathrm{id}\}$. Thus we conclude that this moduli space is (roughly) given by the quotient space $\mathbb{H}^{2} / \mathrm{PSL}_{2}(\mathbb{Z})$. We will prove later in this lecture that this is indeed a 2-dimensional manifold, but given its description as being obtained from the upper-half plane by making some identifications this should not be surprising.

### 3.4.2 Fuchsian groups acting on $\mathbb{H}^{2}$

The formula

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \cdot z=\frac{a z+b}{c z+d}
$$

extends to an action of $\mathrm{PSL}_{2}(\mathbb{R}):=\mathrm{SL}_{2}(\mathbb{R}) /\{ \pm \mathrm{id}\}$ on $\mathbb{H}^{2}$, and these are in fact all the isometries of $\mathbb{H}^{2}$ with the hyperbolic metric. A subgroup $G \subset \operatorname{PSL}_{2}(\mathbb{R})$ acts properly
discontinuously on $\mathbb{H}^{2}$ if only and if it is discrete, in which we call it a Fuchsian group [Fun04, Chapter 1]. If so, its stabilizers have to be finite.

As suggested by our lesson about group quotients, for any freely-acting Fuchsian group $G \subset \mathrm{PSL}_{2}(\mathbb{R})$ the quotient space $\mathbb{H}^{2} / G$ inherits from $\mathbb{H}^{2}$ the structure of a 2-dimensional smooth manifold. It is not so hard to recognize a freely-acting Fuchsian group: because stabilizers are finite any torsion-free Fuchsian group acts freely. It is a consequence of the uniformization theorem that the 2-dimensional compact smooth manifolds obtained as a quotient of $\mathbb{H}^{2}$ by a torsion-free Fuchsian group are exactly the surfaces of genus $\geq 2$.

Because $\mathrm{PSL}_{2}(\mathbb{Z}) \subset \mathrm{PSL}_{2}(\mathbb{R})$ is a discrete subgroup, the (coarse) moduli space of genus 1 Riemann surfaces with marked point $\mathbb{H}^{2} / \mathrm{PSL}_{2}(\mathbb{Z})$ should be a 2-dimensional smooth manifold apart from issues with stabilizers. However, these issues can be resolved: there are only two orbits with stabilizers $\mathbb{Z} / 3$ and $\mathbb{Z} / 2$ respectively and the quotient map looks like $z \mapsto z^{2}: \mathbb{C} \rightarrow \mathbb{C}$ or $z \mapsto z^{3}: \mathbb{C} \rightarrow \mathbb{C}$ is suitable coordinates around points in these orbits. Since the target $\mathbb{C}$ is still a manifold, this means that even though our lesson does not directly apply, the quotient $\mathbb{H}^{2} / \operatorname{PSL}_{2}(\mathbb{Z})$ still inherits from $\mathbb{H}^{2}$ the structure of a 2-dimensional smooth manifold. It is in fact diffeomorphic to $\mathbb{C}$.

Let us at least describe the $\operatorname{map} \mathbb{H}^{2} / \mathrm{PSL}_{2}(\mathbb{Z}) \rightarrow \mathbb{C}$, though we shall not prove that it is a diffeomorphism. For $k \geq 1$ there is an Eisenstein series

$$
G_{2 k}(\tau)=\sum_{(m, n) \in \mathbb{Z}^{2} \backslash(0,0)} \frac{1}{(m \tau+n)^{2 k}}
$$

which converges on $\mathbb{H}^{2}$ to a holomorphic function. Note that given $\tau$, its value $G_{2 k}(\tau)$ is given by a sum over the function $z \mapsto 1 / z^{2 k}$ applied to the points in the lattice $\tau \mathbb{Z}+\mathbb{Z}$. Suppose we apply an element of $\mathrm{SL}_{2}(\mathbb{Z})$ to $\tau$, we get

$$
\begin{aligned}
G_{2 k}\left(\frac{a \tau+b}{c \tau+d}\right) & =\sum_{(m, n) \in \mathbb{Z}^{2} \backslash(0,0)} \frac{1}{\left(m \frac{a \tau+b}{c \tau+d}+n\right)^{2 k}} \\
& =\sum_{(m, n) \in \mathbb{Z}^{2} \backslash(0,0)} \frac{(c \tau+d)^{2 k}}{(m(a \tau+b)+n(c \tau+d))^{2 k}}
\end{aligned}
$$

In our derivation of the action above, we saw that $a \tau+b$ and $c \tau+d$ just give a different set of generators for $\tau \mathbb{Z}+\mathbb{Z}$ than $\tau$ and 1 , so the latter is equal to the sum

$$
\sum_{(m, n) \in \mathbb{Z}^{2} \backslash(0,0)} \frac{(c \tau+d)^{2 k}}{(m \tau+n)^{2 k}}=(c \tau+d)^{2 k} G_{2 k}(z)
$$

A holomorphic function $f: \mathbb{H}^{2} \rightarrow \mathbb{C}$ satisfying $f\left(\frac{a z+b}{c z+d}\right)=(c z+d)^{2 k} f(z)$ (as well as a condition on its behavior as $z \rightarrow i \infty)$ is called a modular form of weight $2 k .{ }^{2}$

It turns out that the $G_{2 k}$ are modular forms of weight $2 k$ (that is, they satisfy the aforementioned condition as $z \rightarrow i \infty)$. Using them we can define the following modular forms of weight 4,6 and 12 respectively

$$
g_{2}(\tau)=60 \sum_{(m, n) \in \mathbb{Z}^{2} \backslash(0,0)} \frac{1}{(m \tau+n)^{4}}, \quad g_{3}(\tau)=140 \sum_{(m, n) \in \mathbb{Z}^{2} \backslash(0,0)} \frac{1}{(m \tau+n)^{6}}
$$

[^4]$$
\Delta(\tau)=g_{2}(\tau)^{3}-27 g_{3}(\tau)^{2}
$$

If we then define the $j$-invariant by $j(\tau)=1728 \frac{g_{2}(\tau)^{3}}{\Delta(\tau)}$, we get a function which satisfies

$$
j\left(\frac{a z+b}{c z+d}\right)=j(z)
$$

and hence induces a function

$$
\begin{aligned}
j: \mathbb{H}^{2} / \mathrm{PSL}_{2}(\mathbb{Z}) & \longrightarrow \mathbb{C} \\
{[\tau] } & \longmapsto 1728 \frac{g_{2}(\tau)^{3}}{\Delta(\tau)} .
\end{aligned}
$$

Of course we must be careful that this is well-defined, as we are dividing by something that may vanish, but it is not only well-defined but in fact a diffeomorphism (you can see its values in Figure 3.7). ${ }^{3}$

[^5]

Figure 3.7 A plot of of the function $j$ (norm is greyscale, argument is hue). Can you guess a fundamental domain of the action of $\mathrm{PSL}_{2}(\mathbb{Z})$ on $\mathbb{H}^{2}$ based on this image? Source: https: //en.wikipedia.org/wiki/J-invariant\#/media/File:KleinInvariantJ.jpg.

## Chapter 4

## Smooth maps and their derivatives

In this chapter we will define smooth maps, and their derivatives. This material appears at the end of Section 1 of [BJ82], as well as Section 2. For more details, see [Tu11, Chapters 6, 8].

### 4.1 Smooth maps and diffeomorphisms

Let us recall some definitions from Chapter 2, on which we shall elaborate now:
Definition 4.1.1. Let $M$ and $N$ be smooth manifolds of dimension $m$ and $n$, with smooth atlases $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ and $\left\{\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)\right\}$. A continuous map $f: M \rightarrow N$ is said to be smooth if for all charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of $M$ and $\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)$ of $N$, the map

$$
\begin{equation*}
\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}: \mathbb{R}^{m} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \longrightarrow\left(\phi_{\beta}^{\prime}\right)^{-1}\left(V_{\beta}^{\prime}\right)=U_{\beta^{\prime}}^{\prime} \subset \mathbb{R}^{n} \tag{4.1}
\end{equation*}
$$

between open subsets of Euclidean spaces is smooth.
It may be helpful to expand (4.1) into a commutative diagram

$$
\begin{aligned}
& \mathbb{R}^{m} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \xrightarrow{\phi_{\alpha}} \cong V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right) \subset M \\
& \quad\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha} \downarrow \\
& \mathbb{R}^{n} \supset U_{\beta^{\prime}}^{\prime}=\left(\phi_{\beta}^{\prime}\right)^{-1}\left(V_{\beta}^{\prime}\right) \xrightarrow[\phi_{\beta}^{\prime}]{\cong} V_{\beta}^{\prime} \subset N .
\end{aligned}
$$

Using this definition we can say when we consider two smooth manifolds to be the same:

Definition 4.1.2. A smooth map $g: M \rightarrow N$ between smooth manifolds is a diffeomorphism if it is bijective with smooth inverse.

We say $M$ and $N$ are diffeomorphic if there is a diffeomorphism between them. This is an equivalence relation.
Example 4.1.3. The real projective space $\mathbb{R} P^{1}$ is diffeomorphic to $S^{1}$.
Example 4.1.4. The complex projective plane $\mathbb{C} P^{1}$ is diffeomorphic to $S^{2}$.

Example 4.1.5. All five definitions of $\mathbb{T}^{2}$ that we gave - by equations, by parametrization, as a product, by gluing, and a quotient-are diffeomorphic.
Example 4.1.6. $\mathbb{R}^{k}$ is diffeomorphic to $\mathbb{R}^{l}$ if and only if $k=l$, see Problem 4.4.1.

### 4.1.1 Properties of smooth maps

Definition 4.1.1 at first sight involves a condition that is hard to check, as both atlases will in general have infinitely many charts. However, it suffices to only verify the condition on a smaller collection of charts; all these need to do is cover the entire domain $M$, as well as the image $f(M) \subset N$ in the target.

Lemma 4.1.7. Let $\left\{\left(U_{i}, V_{i}, \psi_{i}\right)\right\}_{i \in I}$ and $\left\{\left(U_{j}^{\prime}, V_{j}^{\prime}, \psi_{j}^{\prime}\right)\right\}_{j \in J}$ be collections of charts of $M$ and $N$ respectively, such that $\bigcup_{i \in I} V_{i}=M$ and $f(M) \subset \bigcup_{j \in J} V_{j}^{\prime}$. If for all $i \in I$ and $j \in J$, the map

$$
\left(\psi_{j}^{\prime}\right)^{-1} \circ f \circ \psi_{i}: \mathbb{R}^{m} \supset \psi_{i}^{-1}\left(V_{i} \cap f^{-1}\left(V_{j}^{\prime}\right)\right) \longrightarrow\left(\psi_{j}^{\prime}\right)^{-1}\left(V_{j}^{\prime}\right)=U_{j}^{\prime} \subset \mathbb{R}^{n}
$$

between open subsets of a Euclidean space is smooth, then $f$ is smooth.
Sometimes you can pick a few charts particularly well-suited to your situation:
Example 4.1.8. A map $f: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ is smooth in the above sense if and only if it is smooth in the sense of multivariable calculus, since we may use the identity as a single chart for both $\mathbb{R}^{m}$ and $\mathbb{R}^{n}$.

Example 4.1.9. If $M$ and $N$ are spheres $S^{m}$ and $S^{n}$, we know that each of them can be covered by two charts using stereographic projection and hence we can get away with checking only four cases.

You do not need to explicitly pick a collection of charts beforehand:
Corollary 4.1.10. A map $f: M \rightarrow N$ is smooth if and only if for all $m \in M$ there is $a$ chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ around $m$ in $M$ and a chart $\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)$ around $f(m)$ in $N$, such that the map

$$
\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}: \mathbb{R}^{m} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \longrightarrow\left(\phi_{\beta}^{\prime}\right)^{-1}\left(V_{\beta}^{\prime}\right)=U_{\beta}^{\prime} \subset \mathbb{R}^{n}
$$

between open subsets of Euclidean spaces, is smooth at m.
Proof. Pick charts as in the hypothesis for each $m \in M$ and apply Lemma 4.1.7 to this collection.

Example 4.1.11. The diagonal map

$$
\begin{aligned}
\Delta: M & \longrightarrow M \times M \\
p & \longmapsto(p, p)
\end{aligned}
$$

is smooth, where the target is made into a smooth manifold as in Example 2.2.13. Indeed, we can verify this using charts ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) on the domain and charts of the form ( $U_{\alpha} \times U_{\alpha}, V_{\alpha} \times V_{\alpha}, \phi_{\alpha} \times \phi_{\alpha}$ ) on the target. The result then amounts to verifying that the diagonal $\mathbb{R}^{k} \rightarrow \mathbb{R}^{k} \times \mathbb{R}^{k}$ given by $x \mapsto(x, x)$ is smooth.

In practice, one often constructs new smooth maps out of old ones using one of the following tools. Parts (iii) and (iv) use the construction, from the Example 2.2.11, of a smooth structure on an open subset of a smooth manifold.

## Lemma 4.1.12.

(i) For every smooth manifold $M$, the identity map $\mathrm{id}_{M}$ is smooth.
(ii) If $\left\{U_{i}\right\}$ is an open cover of $M$ and each $\left.f\right|_{U_{i}}: U_{i} \rightarrow N$ is smooth, then $f: M \rightarrow N$ is smooth.
(iii) If $f: M \rightarrow N$ and $g: N \rightarrow P$ are smooth, then so is $g \circ f: M \rightarrow P$.
(iv) If $f: M \rightarrow N$ is smooth and $U \subset M$ is open, then $\left.f\right|_{U}: U \rightarrow N$ is smooth.

Note that (iv) gives the converse to (ii), so we can replace "if" by "if and only if" there.

Proof. (i) If $f=\mathrm{id}_{M}$, then (4.1) becomes

$$
\phi_{\beta}^{-1} \circ \phi_{\alpha}: \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \longrightarrow \phi_{\beta}^{-1}\left(V_{\beta}\right)
$$

which is smooth by definition of an atlas, as it is a transition function followed by the inclusion of an open subset.
(ii) By Lemma 4.1.7, it is enough to verify smoothness with respect to the collection of charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ with the property that $U_{\alpha} \subset U_{i}$ for some $i$. In that case, we can replace in (4.1) the map $f$ by $\left.f\right|_{U_{i}}$ and smoothness follows from the hypothesis that $\left.f\right|_{U_{i}}$ is smooth.
(iii) We write out (4.1) as

$$
\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ g \circ f \circ \phi_{\alpha}: \mathbb{R}^{m} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap(g \circ f)^{-1}\left(V_{\gamma}^{\prime \prime}\right)\right) \longrightarrow\left(\phi_{\gamma}^{\prime \prime}\right)^{-1}\left(V_{\gamma}^{\prime \prime}\right)=U_{\gamma}^{\prime \prime} \subset \mathbb{R}^{p}
$$

Then for each chart $\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)$ we can write $\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ g \circ f \circ \phi_{\alpha}$ as

$$
\left(\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ g \circ \phi_{\beta}^{\prime}\right) \circ\left(\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)
$$

when restricting to $\phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right) \cap(g \circ f)^{-1}\left(V_{\gamma}^{\prime \prime}\right)\right)$. This is a composition of a smooth map between open subsets of $\mathbb{R}^{m}$ and $\mathbb{R}^{n}$ with a smooth map between open subsets of $\mathbb{R}^{n}$ and $\mathbb{R}^{p}$, and hence is smooth. Since the open subsets $\phi_{\alpha}^{-1}\left(V_{\alpha} \cap\right.$ $\left.f^{-1}\left(V_{\beta}^{\prime}\right) \cap(g \circ f)^{-1}\left(V_{\gamma}^{\prime \prime}\right)\right)$ give an open cover of $\phi_{\alpha}^{-1}\left(V_{\alpha} \cap(g \circ f)^{-1}\left(V_{\gamma}^{\prime \prime}\right)\right)$ and smoothness is a local property, this tells us that $\left(\left(\phi_{\gamma}^{\prime \prime}\right)^{-1} \circ g \circ f \circ \phi_{\alpha}\right)$ is smooth.
(iv) It suffices to prove that the inclusion $i_{U}: U \rightarrow M$ is smooth, as then $\left.f\right|_{U}$ is the composition $f \circ i_{U}$ of two smooth maps. Using the chart on $U$ obtained by restricting those on $M$, (4.1) becomes

$$
\phi_{\beta}^{-1} \circ \phi_{\alpha}: \phi_{\alpha}^{-1}\left(U \cap V_{\alpha} \cap V_{\beta}\right) \longrightarrow \phi_{\beta}^{-1}\left(V_{\alpha} \cap V_{\beta}\right),
$$

which is just the restriction of the smooth map $\phi_{\beta}^{-1} \circ \phi_{\alpha}$ to an open subset.

Remark 4.1.13. Using part (i) and (iii) we can define a category Mfd of smooth manifolds; its objects are smooth manifolds and morphisms from $M$ to $N$ are smooth maps. Part (i) then implies that this category has identity morphisms and part (iii) implies that composition is well-defined. We takes this up again in Section 20.1.4.

Category theory is a useful language for studying topology and related fields, as many objects of interest can be defined in terms of universal properties saying how other objects should map to them or receive maps from them. Let us give two examples.

Recall that in Example 2.2.12 we defined the disjoint union of $M \sqcup N$ of two manifolds of the same dimension. It is a consequence of parts (ii) and (iv) that a map $f: M \sqcup N \rightarrow P$ is smooth if and only if $\left.f\right|_{M}$ and $\left.f\right|_{N}$ are. This exhibits $M \sqcup N$ as the (categorical) coproduct in Mfd.

We also defined the product $M \times N$ of two smooth manifolds, in Example 2.2.13. By Problem 4.4.2 the projection $\pi_{1}: M \times N \rightarrow M$ and $\pi_{2}: M \times N \rightarrow N$ are smooth. Thus by (iii) if $f: P \rightarrow M \times N$ is smooth so are its components $\pi_{1} \circ f$ and $\pi_{2} \circ f$. Note that we can recover $f$ as

$$
P \xrightarrow{\Delta} P \times P \xrightarrow{\left(\pi_{1} \circ f\right) \times\left(\pi_{2} \circ f\right)} M \times N,
$$

which is smooth as a consequence of (iii), Example 4.1.11, and Problem 4.4.2 (d). We conclude that $f: P \rightarrow M \times N$ is smooth if and only if its components $\pi_{1} \circ f: P \rightarrow M$ and $\pi_{2} \circ f: P \rightarrow N$ are. Thus $M \times N$ is the (categorical) product in Mfd.

It is particularly easy to construct smooth maps into or out of submanifolds.
Lemma 4.1.14. Suppose that $X \subset M$ is a submanifold.
(i) The inclusion $i: X \rightarrow M$ is a smooth map.
(ii) If $f: X \rightarrow N$ extends to a smooth map $\tilde{f}: M \rightarrow N$, then $f$ is smooth.
(iii) If $g: N \rightarrow X$ is such that $i \circ g$ is smooth, then $g$ is smooth.

Proof. (i) Since $X$ is a submanifold, we can find charts ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) of $M$ covering $X$ such that $\phi_{\alpha}^{-1}\left(V_{\alpha} \cap X\right)=U_{\alpha} \cap \mathbb{R}^{k}$. In fact, it is these charts that generate the atlas on $X$. By Lemma 4.1.12 (ii), it suffices to prove that $\left.i\right|_{V_{\alpha} \cap X}: V_{\alpha} \cap X \rightarrow M$ is smooth. Since we can cover $V_{\alpha} \cap X$ by the single chart ( $\left.U_{\alpha} \cap \mathbb{R}^{k}, V_{\alpha} \cap X,\left.\phi_{\alpha}\right|_{U_{\alpha} \cap \mathbb{R}^{k}}\right)$ and its image by the chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$, by Lemma 4.1.7 it suffices to prove that

$$
\left.\left.\phi_{\alpha}^{-1} \circ i\right|_{V_{\alpha} \cap X} \circ \phi_{\alpha}\right|_{U_{\alpha} \cap \mathbb{R}^{k}}: U_{\alpha} \cap \mathbb{R}^{k} \longrightarrow U_{\alpha}
$$

is smooth. But it is just the inclusion of those points with last $m-k$ coordinates equal to 0 , which is clearly smooth!
(ii) Since $f=\tilde{f} \circ i$, this follows from (i) and Lemma 4.1.12 (iii).
(iii) We again use charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of $M$ covering $X$ such that $\phi_{\alpha}^{-1}\left(V_{\alpha} \cap X\right)=U_{\alpha} \cap \mathbb{R}^{k}$. By Lemma 4.1.7, $g$ is smooth if and only if

$$
\left(\left.\phi_{\alpha}\right|_{U_{\alpha} \cap \mathbb{R}^{k}}\right)^{-1} \circ g \circ \phi_{\beta}^{\prime}
$$

is smooth for all charts $\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)$ of $N$. However, we are guaranteed that all maps

$$
\left(\phi_{\alpha}\right)^{-1} \circ g \circ \phi_{\beta}^{\prime}
$$

are smooth, which differ from the previous maps by composition with the standard inclusion $\mathbb{R}^{k} \rightarrow \mathbb{R}^{m}, m \geq k$. Thus the proof amounts to the observation that a map from an open subset of $\mathbb{R}^{n}$ to an open subset of $\mathbb{R}^{k}$ is smooth when its composition with this standard inclusion is smooth.

Remark 4.1.15. We will later be able to prove that (ii) is actually an "if and only if" (see Problem 13.4.3).
Example 4.1.16 (Rotations as diffeomorphisms of $S^{n}$ ). By Lemma 4.1 .14 (ii) and (iii), a $\operatorname{map} S^{n} \rightarrow S^{n}$ is smooth if it extends to a smooth map $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$. We will use this to construct diffeomorphisms of $S^{n}$. Let us take a matrix $A \in O(n+1)$, the group of orthogonal $(n+1) \times(n+1)$-matrices. By definition an orthogonal matrix preserves the Euclidean norm $\|x\|$, and hence $x \mapsto A x$ sends $S^{n}$ to $S^{n}$. Furthermore, each entry of $A x$ is just a linear combination of the entries of $x$ so is easily seen to be smooth. Thus $x \mapsto A x$ gives an example of a smooth map $S^{n} \rightarrow S^{n}$. It has an evident smooth inverse given by $x \mapsto A^{-1} x$.

We have thus just produced a map $O(n+1) \rightarrow \operatorname{Diff}\left(S^{n}\right)$, the latter the group of diffeomorphisms of $S^{n}$. The latter can be endowed with a natural topology which makes this map continuous. If $n \leq 3$, it is a homotopy equivalence by work of Smale and Hatcher [Sma59, Hat83] (see Problem 25.4.2). If $n \geq 4$, it is not a homotopy equivalence; the case $n=4$ was only proven recently [Wat18].
Example 4.1.17 (General linear groups). The set $M_{n}(\mathbb{R})$ of $(n \times n)$-matrices with real entries can be identified with $\mathbb{R}^{n^{2}}$, and through this identification can be made into a smooth $n^{2}$-dimensional manifold. Matrix multiplication gives a map

$$
\begin{aligned}
\mu: M_{n}(\mathbb{R}) \times M_{n}(\mathbb{R}) & \longrightarrow M_{n}(\mathbb{R}) \\
(A, B) & \longmapsto A B
\end{aligned}
$$

which we claim is smooth. To check this, we use that there is a single chart covering $M_{n}(\mathbb{R})$, the standard identification, and similarly a single chart covering $M_{n}(\mathbb{R}) \times M_{n}(\mathbb{R})$, a product of two standard identifications. By Lemma 4.1.7 it suffices to prove that matrix multiplication is smooth with respect to these charts only; this is true because it is a polynomial in the entries of the matrices and hence smooth.

The open subset $\mathrm{GL}_{n}(\mathbb{R}) \subset M_{n}(\mathbb{R})$ of invertible matrices, which can be described as the complement of the closed subset determined by the equation det $=0$, is hence also a smooth $n^{2}$-dimensional manifold. Since a composition of invertible matrices is again invertible, Lemma 4.1.14 implies that matrix multiplication restricts to a smooth map

$$
\mu: \mathrm{GL}_{n}(\mathbb{R}) \times \mathrm{GL}_{n}(\mathbb{R}) \longrightarrow \mathrm{GL}_{n}(\mathbb{R})
$$

We can also take the inverse of an invertible matrix, giving a map

$$
\begin{aligned}
\iota: \mathrm{GL}_{n}(\mathbb{R}) & \longrightarrow \mathrm{GL}_{n}(\mathbb{R}) \\
A & \longmapsto A^{-1},
\end{aligned}
$$

which is also smooth. Indeed, using again the standard identifications as charts, we can use Cramer's rule:

$$
A^{-1}=\frac{1}{\operatorname{det}(A)} C^{T}
$$

with $C$ the cofactor matrix; its $(i, j)$ th entry is given by $(-1)^{i+j} \operatorname{det}\left(\hat{A}_{i j}\right)$ where $\hat{A}_{i j}$ is obtained from $A$ by deleting the $i$ th row and $j$ th column. The details are not important, only that it is a smooth function of the entries of an invertible matrix.

An example of a group which compatibly is a smooth manifold deserves a name:
Definition 4.1.18. A Lie group is a smooth manifold $G$ which is also a group, such that multiplication $\mu: G \times G \rightarrow G$ and inverse $\iota: G \rightarrow G$ are both smooth.

### 4.2 Derivatives and tangent spaces

We want to extend the notion of a derivative of a smooth map between two open subsets of Euclidean space, to a smooth map between manifolds. This is useful because the derivative determines the local behavior of smooth maps. Using it, we will be able to formulate and prove global versions of the submersion and immersion theorem.

If you are unfamiliar with the total derivative of smooth maps between open subsets of Euclidean spaces, take a look at Chapter 2 of [DK04a]. For each $x \in \mathbb{R}^{k}$, we can think of $\mathbb{R}^{k}$ as a space of vectors based at $x$. It has a standard basis. A smooth map $g: \mathbb{R}^{k} \supset U \rightarrow \mathbb{R}^{k^{\prime}}$ has a total derivative at $x$ given by the linear map, whose matrix with respect to the standard bases is the $\left(k^{\prime} \times k\right)$-matrix of partial derivatives

$$
\left[\begin{array}{cccc}
\frac{\partial g_{1}}{\partial x_{1}}(x) & \frac{\partial g_{1}}{\partial x_{2}}(x) & \cdots & \frac{\partial g_{1}}{\partial x_{k}}(x) \\
\frac{\partial g_{2}}{\partial x_{1}}(x) & \frac{\partial g_{2}}{\partial x_{2}}(x) & \cdots & \frac{\partial g_{2}}{\partial x_{k}}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{k^{\prime}}}{\partial x_{1}}(x) & \frac{\partial g_{k^{\prime}}}{\partial x_{2}}(x) & \cdots & \frac{\partial g_{k^{\prime}}}{\partial x_{k}}(x)
\end{array}\right],
$$

with $g_{j}: U \rightarrow \mathbb{R}$ the $j$ th component of $g$.
Our goal will be to construct for each point $m$ in a $k$-dimensional manifold $M$ a tangent space $T_{m} M$, as well as for each smooth map $f: M \rightarrow N$ a derivative $d_{m} f: T_{m} M \rightarrow$ $T_{f(m)} N$. The tangent space should satisfy the following properties:
(I) Each tangent space $T_{m} M$ is a $k$-dimensional $\mathbb{R}$-vector space.
(II) In local coordinates it can be identified with $\mathbb{R}^{k}$ in a natural manner.

The derivative should satisfy similar properties:
(I') Each derivative $d_{m} f$ is a linear map.
(II') It satisfies $d_{m}\left(\operatorname{id}_{M}\right)=\mathrm{id}_{T_{m} M}$, and the chain rule $d_{m}(g \circ f)=d_{f(m)} g \circ d_{m} f$.
(III') In local coordinates it can be identified with the total derivative in a natural manner.
We haven't explained what "in a natural manner" means here. It is intended informally, but can be given some content by demanding that the identifications are compatible with changing coordinates.

There is a number of perspectives on tangent spaces and derivatives, leading to different but equivalent definitions. Which is most useful depends on your setting, so we will discuss all five of them eventually. In the end, the "stating globally" part of our philosophy to state globally and prove locally, will allow us to dispense with the details of the definitions.

### 4.2.1 The algebraicists' definition

Intuitively, the tangent space to a $k$-dimensional submanifold of Euclidean space at some point is the $k$-dimensional affine linear subspace that best approximates it. However, we do not know (yet) that every smooth manifold is a submanifold of some Euclidean space, nor do we want to verify that the resulting definition is independent of the choice of such an embedding. So instead, we give a definition of $T_{m} M$ that only refers to $M$ and its maximal atlas. The first definition we will give, the algebraicists' one, does so, and will be our official one. However, you are free to use one of the definitions in the next section, if those are more convenient for solving the problem at hand.

## Germs of smooth maps and smooth functions

We start with the observation that the derivative of $f: M \rightarrow N$ at $m \in M$ should only depend on the behavior of $f$ in a small neighborhood of $m$. Let us define an equivalence relation $\sim$ on the set

$$
\{f: U \rightarrow N \mid U \subset M \text { an open neighborhood of } m, f \text { smooth }\}
$$

by saying that
$f \sim g$ if there exists an open neighborhood $V$ of $m$ such that $\left.f\right|_{V}=\left.g\right|_{V}$.
Definition 4.2.1. The equivalence class of $f: U \rightarrow N$ under $\sim$ is called germ of $f$ at $m$, and denoted $\bar{f}:(M, m) \rightarrow N$. If we like to stress that $f(m)=n$, we will use the notation $\bar{f}:(M, n) \rightarrow(N, n)$.

We can compose germs: given $\bar{f}:(M, m) \rightarrow(N, n)$ and $\bar{g}:(N, n) \rightarrow(P, p)$, their composition is

$$
\bar{g} \circ \bar{f}:=\overline{g \circ f} .
$$

I'll leave it to you check this is well-defined, i.e. independent of the choice of representatives.
Definition 4.2.2. A function germ is a germ $\bar{\alpha}:(M, m) \rightarrow \mathbb{R}$. The set of function germs is denoted $\mathcal{E}(M, m)$.

Pointwise addition, scaling, and multiplication of functions induces on $\mathcal{E}(M, m)$ the structure of an $\mathbb{R}$-algebra: this means it has addition, scaling, and multiplication operations

$$
\bar{f}+\bar{g}:=\overline{f+g}, \quad \lambda \bar{f}=\overline{\lambda f}, \quad \text { and } \quad \bar{f} \bar{g}:=\overline{f g}
$$

These should behave in the usual manner, i.e. satisfy commutativity, associativity, unitality, and distributivity axioms. I'll leave it to you to check these operations are well-defined, and satisfy the required properties (which will follow directly from the corresponding properties of the real numbers)
Example 4.2.3. Evaluation at $m \in M$ induces a map

$$
\begin{aligned}
\mathcal{E}(M, m) & \longrightarrow \mathbb{R} \\
\bar{f} & \longmapsto \bar{f}(m) .
\end{aligned}
$$

This is an $\mathbb{R}$-algebra homomorphism, i.e. preserves addition, scaling, and multiplication.

We can precompose function germs in $\mathcal{E}(M, m)$ by a germ $f:(Q, q) \rightarrow(M, m)$, and thus get an $\mathbb{R}$-algebra homomorphism

$$
\begin{aligned}
f^{*}: \mathcal{E}(M, m) & \longrightarrow \mathcal{E}(Q, q) \\
\bar{\alpha} & \longmapsto \bar{\alpha} \circ f=\overline{\alpha \circ f} .
\end{aligned}
$$

The usual properties of composition of functions imply:

## Lemma 4.2.4.

- $f^{*}$ is an $\mathbb{R}$-algebra homomorphism,
- $\mathrm{id}^{*}=\mathrm{id}$, and
- $(g \circ f)^{*}=f^{*} \circ g^{*}$.

In particular, if $\phi$ is a diffeomorphism then $\phi^{*}$ is an isomorphism of $\mathbb{R}$-algebras; its inverse is given by $\left(\phi^{-1}\right)^{*}$. Furthermore, since germs only involve small open neighborhoods of $m$, it suffices that $\phi$ is a local diffeomorphism, since then its restriction to an open neighborhood around the point at which we take germs is a diffeomorphism.

We can apply this observation to a chart ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) with $m \in V_{\alpha}$. By translation, we may assume without loss of generality that $\phi_{\alpha}(0)=m$. This is a local diffeomorphism and hence induces an isomorphism

$$
\left(\phi_{\alpha}\right)^{*}: \mathcal{E}(M, m) \longrightarrow \mathcal{E}_{k},
$$

of $\mathcal{E}(M, m)$ with $\mathcal{E}_{k}:=\mathcal{E}\left(\mathbb{R}^{k}, 0\right)$, the $\mathbb{R}$-algebra of functions germs $\left(\mathbb{R}^{k}, 0\right) \rightarrow \mathbb{R}$. Any two such identifications differ by an isomorphism $\left(\psi_{\beta \alpha}\right)^{*}$ induced by a transition function.

## From germs to the algebraicists' definition of tangent spaces

The idea behind this definition is that a vector $\vec{v}$ based at $m \in M$ induces a directional derivative of functions $f: M \rightarrow \mathbb{R}$, which we can imprecisely write as

$$
\begin{equation*}
f \longmapsto d_{\vec{v}}(f):=\frac{d f(m+t \vec{v})}{d t}(0) \in \mathbb{R} . \tag{4.2}
\end{equation*}
$$

(The difficulty is that we can't make sense of $m+t v$, but let's just go with it.) This only depends on the germ $\bar{f}$ of $f$ at $m$. Furthermore, by the linearity of derivatives and the product rule, this satisfies

$$
\begin{gathered}
d_{\vec{v}}(\bar{f}+\bar{g})=d_{\vec{v}}(\bar{f})+d_{\vec{v}}(\bar{g}), \quad d_{\vec{v}}(\lambda \bar{f})=\lambda d_{\vec{v}}(\bar{f}), \\
\text { and } \quad d_{\vec{v}}(\bar{f} \bar{g})=d_{\vec{v}}(\bar{f}) \bar{g}(m)+\bar{f}(m) d_{\vec{v}}(\bar{g}) .
\end{gathered}
$$

Let us abstract this definition:
Definition 4.2.5. A derivation $X: \mathcal{E}(M, m) \rightarrow \mathbb{R}$ is a function which satisfies

- $X(\bar{f}+\bar{g})=X(\bar{f})+X(\bar{g})$,
- $X(\lambda \bar{f})=\lambda X(\bar{f})$, and

$$
\cdot X(\bar{f} \bar{g})=X(\bar{f}) \bar{g}(m)+\bar{f}(m) X(\bar{g})
$$

Example 4.2.6. The value of $X$ on the constant function 1 is given by

$$
X(1)=X(1 \cdot 1)=X(1) \cdot 1+1 \cdot X(1)=2 \cdot X(1)
$$

so $X(1)=0$. As a consequence of linearity, $X($ constant function $)=0$.
We can add and scale such derivations, making them into a $\mathbb{R}$-vector space: $(X+$ $Y)(f)=X(f)+Y(f)$ and $(\lambda X)(f)=\lambda X(f)$.

Definition 4.2.7. The tangent space $T_{m} M$ is the vector space $\operatorname{Der}(\mathcal{E}(M, m))$ of derivations $X: \mathcal{E}(M, m) \rightarrow \mathbb{R}$.

Let us recap: $\mathcal{E}(M, m)$ us the $\mathbb{R}$-algebra of germs at $m$ of smooth functions $M \rightarrow \mathbb{R}$. We take derivations of this algebra, a notion inspired by directional derivatives; these form a vector space as in desideratum (I).

It remains to show that the vector space $T_{m} M$ is $k$-dimensional if $M$ is $k$-dimensional, as in desideratum (II). To do so, we use that the isomorphism $\left(\varphi_{\alpha}\right)^{*}: \mathcal{E}(M, m) \xrightarrow{\sim} \mathcal{E}_{k}=$ $\mathcal{E}\left(\mathbb{R}^{k}, 0\right)$ induced by a chart induces a linear isomorphism

$$
T_{0} \mathbb{R}^{k}=\operatorname{Der}\left(\mathcal{E}_{k}\right) \xrightarrow{\sim} \operatorname{Der}(\mathcal{E}(M, m))=T_{m} M .
$$

Thus it suffices to prove that $T_{0} \mathbb{R}^{k}$ is $k$-dimensional. Unlike on $M$, on $\mathbb{R}^{k}$ we can make sense of addition, and hence the directional derivatives of (4.2) with respect to each of the $k$ coordinate directions give derivations

$$
\begin{aligned}
\frac{\partial}{\partial x_{i}}: \mathcal{E}_{k} & \longrightarrow \mathbb{R} \\
\bar{f} & \longmapsto \frac{\partial f}{\partial x_{i}}(0) .
\end{aligned}
$$

These are linearly independent, by applying them to the coordinate functions $x=$ $x_{j}:\left(x_{1}, \ldots, x_{k}\right) \mapsto x_{j}$. Every other derivation is a linear combination of these:
Proposition 4.2.8. The derivations $\frac{\partial}{\partial x_{i}}$ form a basis of $T_{0} \mathbb{R}^{k}$, and in particular the latter is $k$-dimensional.

We will use the following lemma:
Lemma 4.2.9. Let $U \subset \mathbb{R}^{k}$ be an open neighborhood and $f: U \rightarrow \mathbb{R}$ a smooth function. Then there exist smooth functions $f_{1}, \ldots, f_{k}: U \rightarrow \mathbb{R}$ such that

$$
f(x)=f(0)+\sum_{i=1}^{k} x_{i} f_{i}(x)
$$

Proof. The fundamental theorem of analysis implies

$$
f(x)-f(0)=\int_{0}^{1} \frac{d}{d t} f\left(t x_{1}, \ldots, t x_{k}\right) d t=\sum_{i=1}^{n} x_{i} \int_{0}^{1} d_{i} f\left(t x_{1}, \ldots, t x_{n}\right) d t
$$

with $d_{i} f$ the partial derivative in the $i$ th coordinate direction. So we have that

$$
f_{i}(x)=\int_{0}^{1} d_{i} f\left(t x_{1}, \ldots, t x_{n}\right) d t
$$

This implies that for germs we have $\bar{f}=f(0)+\sum_{i} \bar{x}_{i} \bar{f}_{i}$.
Proof of Proposition 4.2.8. We prove that $X=\sum_{i=1}^{k} X\left(x_{i}\right) \frac{\partial}{\partial x_{i}}$ by proving that

$$
Y:=X-\sum_{i=1}^{k} X\left(x_{i}\right) \frac{\partial}{\partial x_{i}}
$$

vanishes on all germs. By construction, it vanishes on the coordinate function. Then we have that

$$
\begin{aligned}
Y(\bar{f}) & =Y\left(f(0)+\bar{x}_{i} \overline{f_{i}}\right) \\
& =Y(f(0))+\sum_{i} Y\left(\bar{x}_{i} \overline{f_{i}}\right) \\
& =\sum_{i} Y\left(\bar{x}_{i}\right) \overline{f_{i}}(0) \\
& =0 .
\end{aligned}
$$

Here we use that $\bar{x}_{i}$ evaluates to 0 at the origin, and that $Y\left(\bar{x}_{i}\right)$ vanishes by construction.

## The algebraicists' definition of derivatives

A smooth map $f: M \rightarrow N$ sending $m$ to $n$ induces a map of germs $f^{*}: \mathcal{E}(N, n) \rightarrow$ $\mathcal{E}(M, m)$, which in turn induces a map of tangent spaces

$$
\begin{aligned}
d_{m} f: T_{m} M & \longrightarrow T_{n} N \\
X & \longmapsto X \circ f^{*}
\end{aligned}
$$

This is the derivative of $f$ at $m$. From the properties of $f^{*}$, we easily deduce the basic properties of the derivative, desiderata (I') and (II'):

## Lemma 4.2.10.

(i) $d_{m} f$ is a linear map
(ii) $d_{m} \mathrm{id}=\mathrm{id}$, and
(iii) $d_{m}(g \circ f)=d_{f(m)} g \circ d_{m} f$.

You may recognize (iii) as an incarnation of the chain rule. We will compare it to the chain rule in multivariable calculus later in this section.

Example 4.2.11. If $f: M \rightarrow N$ is a diffeomorphism, then it follows from (ii) and (iii) that $d_{m} f$ is invertible with inverse $d_{f(m)} f^{-1}$.
Example 4.2.12. To compute the derivative, you can often exploit the chain rule. Recall from Problem 4.4.2 that $\pi_{1}: X \times Y \rightarrow X$ has derivative

$$
d_{(x, y)} \pi_{1}: T_{x} X \oplus T_{y} Y=T_{(x, y)}(X \times Y) \longrightarrow T_{x} X
$$

is given by projection onto the first summand. The analogous statement is true for $\pi_{2}: X \times Y \rightarrow Y$.

We will deduce from this that the diagonal map

$$
\begin{aligned}
\Delta: M & \longrightarrow M \times M \\
m & \longmapsto(m, m)
\end{aligned}
$$

has derivative $T_{m} \Delta: T_{m} M \rightarrow T_{m \times m}(M \times M)=T_{m} M \oplus T_{m} M$ given by $v \mapsto(v, v)$. To see this, observe that $\pi_{1} \circ \Delta$ and $\pi_{2} \circ \Delta$ have derivatives given by the first and second components of $T_{m} \Delta$. We apply the chain rule to $\pi_{1} \circ \Delta=\operatorname{id}_{M}=\pi_{2} \circ \Delta$. For example, for the first equality: the first component of $T_{m} \Delta(v)$ is given by

$$
T_{(m, m)} \pi_{1} \circ T_{m} \Delta(v)=T_{m}\left(\pi_{1} \circ \Delta\right)(v)=T_{m}\left(\mathrm{id}_{M}\right)(v)=v .
$$

For example, this implies that the diagonal map has injective derivative everywhere.
Let us finally describe explicitly $T_{m} f$ in terms of charts, and verify desideratum (III'). Fix a pair of charts ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) such that $\phi_{\alpha}(0)=m$ and $\left(U_{\alpha^{\prime}}^{\prime}, V_{\alpha^{\prime}}^{\prime}, \phi_{\alpha^{\prime}}^{\prime}\right)$ such that $\phi_{\alpha^{\prime}}^{\prime}(0)=f(m)$. Let us denote $f(m)$ by $n$. What is the dashed linear map which makes the following diagram commute?

$$
\begin{align*}
T_{m} M \xrightarrow{d_{m} f} & T_{n} N  \tag{4.3}\\
\cong \Uparrow d_{0} \phi_{\alpha} & \cong d_{0} \phi_{\alpha^{\prime}}^{\prime} \\
\mathbb{R}^{k} \cong T_{0} \mathbb{R}^{k} \ldots-\cdots \mathbb{R}^{k^{\prime}} & \cong T_{0} \mathbb{R}^{k^{\prime}} .
\end{align*}
$$

Lemma 4.2.13. It is the total derivative $D_{0}\left(\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)$.
Proof. As $\left(d_{0} \phi_{\alpha^{\prime}}^{\prime}\right)^{-1}=d_{n}\left(\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1}\right)$ by Example 4.2.11, and

$$
d_{n}\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ d_{m} f \circ d_{0} \phi_{\alpha}=d_{0}\left(\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)
$$

by (iii), it suffices to compute explicitly the derivative of $g: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k^{\prime}}$ at the origin; we will substitute $g=\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}$. We write $g_{j}: \mathbb{R}^{k} \rightarrow \mathbb{R}$ for the $j$ th component of $g$, $1 \leq j \leq k^{\prime}$.

Given $\bar{h} \in \mathcal{E}_{k^{\prime}}$, we can use the chain rule to compute that

$$
d_{0} g\left(\frac{\partial}{\partial x_{i}}\right)(\bar{h})=\frac{\partial(\bar{h} \circ g)}{\partial x_{i}}=\sum_{j=1}^{k^{\prime}} \frac{\partial \bar{h}}{\partial y_{j}}(0) \frac{\partial g_{j}}{\partial x_{i}}(0)=\sum_{j=1}^{k^{\prime}} \frac{\partial g_{j}}{\partial x_{i}}(0) \frac{\partial}{\partial y_{j}}(\bar{h}) .
$$

As this is true for all $\bar{h}$, the $\frac{\partial}{\partial y_{j}}$-component is $\frac{\partial g_{j}}{\partial x_{i}}(0)$. These are exactly the entries of the total derivative matrix.

We can use this to justify calling $d_{m}(g \circ f)=d_{f(m)} g \circ d_{m} f$ a chain rule, by proving that under charts it reduces to the chain rule you already know. Fixing a third chart, we have a triple of commutative diagrams (three instances of (4.3))

$$
\begin{aligned}
& T_{m} M d_{m} f \\
& \cong T_{n} N \\
& \cong T_{0} \phi_{\alpha} \cong T_{0} \phi_{\alpha^{\prime}}^{\prime} \\
& \mathbb{R}^{k} \cong T_{0} \mathbb{R}^{k} \xrightarrow{D_{0}\left(\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)} \longrightarrow \mathbb{R}^{k^{\prime}} \cong T_{0} \mathbb{R}^{k^{\prime}},
\end{aligned}
$$

$$
\begin{aligned}
& T_{f(m)} N \longrightarrow T_{g \circ f(m)} P \\
& \cong \uparrow T_{0} \phi_{\alpha^{\prime}}^{\prime} \quad \cong \uparrow T_{0} \phi_{\alpha^{\prime \prime}}^{\prime \prime} \\
& \mathbb{R}^{k^{\prime}} \cong T_{0} \mathbb{R}^{k^{\prime}} \xrightarrow{D_{0}\left(\left(\phi_{\alpha^{\prime \prime}}^{\prime \prime}\right)^{-1} \circ g \circ \phi_{\alpha^{\prime}}^{\prime}\right)} \mathbb{R}^{k^{\prime \prime}} \cong T_{0} \mathbb{R}^{k^{\prime \prime}}, \\
& T_{m} N \longrightarrow T_{g \circ f(m)} P \\
& \begin{array}{rlrl} 
& \cong T_{0} \phi_{\alpha} & & \xlongequal{ } T_{0} \phi_{\alpha^{\prime \prime}}^{\prime \prime} \\
& \cong T_{0} \mathbb{R}^{k} \xrightarrow{D_{0}\left(\left(\phi_{\alpha^{\prime \prime}}^{\prime \prime}\right)^{-1} \circ g \circ f \circ \phi_{\alpha}\right)} \mathbb{R}^{k^{\prime \prime}} \cong T_{0} \mathbb{R}^{k^{\prime \prime}} .
\end{array}
\end{aligned}
$$

Identifying the term $d_{f(m)} g \circ d_{m} f$ in charts using the vertical arrows in the three commutative diagrams pictured above, we get a composition of total derivatives

$$
D_{0}\left(\left(\phi_{\alpha^{\prime \prime}}^{\prime \prime}\right)^{-1} \circ g \circ \phi_{\alpha^{\prime}}^{\prime}\right) \circ D_{0}\left(\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)
$$

By the ordinary chain rule this is the total derivative

$$
D_{0}\left(\left(\phi_{\alpha^{\prime \prime}}^{\prime \prime}\right)^{-1} \circ g \circ \phi_{\alpha^{\prime}}^{\prime} \circ\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)=D_{0}\left(\left(\phi_{\alpha^{\prime \prime}}^{\prime \prime}\right)^{-1} \circ g \circ f \circ \phi_{\alpha}\right),
$$

which is indeed $d_{m}(g \circ f)$ under the above identification.
Thus, we can combine the three squares into a larger commutative diagram combining the general chain rule and the chain rule in local coordinates:


### 4.3 Alternative definitions of tangent spaces and derivatives

Recall that we are giving five definitions of the tangent space $T_{m} M$, and have just given the first. In this section we give three other definitions, leaving a final one to the Problem 4.4.6.

### 4.3.1 The definition for submanifolds of Euclidean space

You probably have an intuition for the tangent space at $m$ to some $k$-dimensional smooth submanifold $M \subset \mathbb{R}^{n}$. Informally, it is the $k$-dimensional affine plane in $\mathbb{R}^{n}$ through $m \in M$, which is the best linear approximation to $M$. Before making this precise, we give an example:

Example 4.3.1. By definition, a point $x \in S^{k} \subset \mathbb{R}^{k+1}$ is given by a unit length vector in $\mathbb{R}^{k+1}$. Then the tangent space $T_{x} S^{k}$ is the $k$-dimensional affine plane given by

$$
T_{x} S^{n}=\{x+v \mid v \perp x\}
$$

Note that, upon translating $m$ back to the origin, this affine plane yields a linear subspace of $\mathbb{R}^{n}$. This gives $T_{m} M$ the structure of an $m$-dimensional real vector space.

To define $T_{m} M$ rigorously, we fix a charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of $M$ such that $m \in V_{\alpha}$, and let $x=\phi_{\alpha}^{-1}(m)$. Then from the inclusion $i: M \rightarrow \mathbb{R}^{n}$, we can construct a smooth map between open subsets of Euclidean space

$$
i \circ \phi_{\alpha}: \mathbb{R}^{k} \supset U_{\alpha} \longrightarrow \mathbb{R}^{n}
$$

The best linear approximation to this smooth map at $x$ is given in terms of the total derivative $D_{x}\left(i \circ \phi_{\alpha}\right)$ as

$$
\mathbb{R}^{k} \supset U_{\alpha} \ni y \longmapsto\left(i \circ \phi_{\alpha}\right)(x)+D_{x}\left(i \circ \phi_{\alpha}\right)(y-x) \in \mathbb{R}^{n}
$$

It is a consequence of the definition of a submanifold that $D\left(i \circ \phi_{\alpha}\right)_{x}$ is an injective linear map; indeed, in terms of some other chart of $\mathbb{R}^{n}$ it is a restriction of the inclusion $\mathbb{R}^{k} \hookrightarrow \mathbb{R}^{n}$. This tells us that:

Definition 4.3.2. One definition of the tangent space $T_{m} M$ is as

$$
T_{m}^{\mathrm{submfd}} M:=m+D_{x}\left(i \circ \phi_{\alpha}\right)\left(\mathbb{R}^{k}\right),
$$

considered as a $k$-dimensional real vector space by its identification with $D_{x}\left(i \circ \phi_{\alpha}\right)\left(\mathbb{R}^{k}\right)$.
We need to verify that this is independent of the choice of chart. This is the case because if we use another chart $\left(U_{\beta}, V_{\beta}, \phi_{\beta}\right)$, we have

$$
i \circ \phi_{\beta}=\left(i \circ \phi_{\alpha}\right) \circ\left(\phi_{\alpha}^{-1} \circ \phi_{\beta}\right)=\left(i \circ \phi_{\alpha}\right) \circ \psi_{\beta \alpha}
$$

so its total derivative is given by $D_{x}\left(i \circ \phi_{\alpha}\right) \circ D_{x^{\prime}}\left(\psi_{\beta \alpha}\right)$. Since $\psi_{\beta \alpha}$ is a diffeomorphism, $D_{x^{\prime}}\left(\psi_{\beta \alpha}\right)$ is a linear isomorphism and hence

$$
D_{x^{\prime}}\left(i \circ \phi_{\beta}\right)\left(\mathbb{R}^{k}\right)=D_{x}\left(i \circ \phi_{\alpha}\right)\left(\mathbb{R}^{k}\right)
$$

## Relation to algebraicists' definition

We have previously identified $T_{0} \mathbb{R}^{n}$ with the $n$-dimensional real vector space spanned by the derivations $\partial / \partial x_{i}$. You can think of this as applying the formalism above to $M=\mathbb{R}^{n}$, using the standard chart $\left(\mathbb{R}^{n}, \mathbb{R}^{n}, \mathrm{id}\right)$.

Given an inclusion $i: M \rightarrow \mathbb{R}^{n}$, where without loss of generality we may assume by translation that $i(m)=0$, we can compute the derivative of $i$ at $m$ with respect to the standard chart of $\mathbb{R}^{n}$ and some chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of $M$ with $\phi_{\alpha}(0)=m$. By the chain rule there is a commutative diagram of linear maps

$$
\begin{array}{rlr}
T_{m} M \xrightarrow{d_{m} i} & T_{0} \mathbb{R}^{n} \\
& \cong T_{0} \phi_{\alpha} & \\
& \cong \mathrm{id} \\
\mathbb{R}^{k} \cong T_{0} \mathbb{R}^{k} \xrightarrow{D_{0}\left(i \circ \phi_{\alpha}\right)} \mathbb{R}^{n} \cong T_{0} \mathbb{R}^{n} .
\end{array}
$$

Because $M$ is a submanifold $d_{m} i$ is injective, as in terms of appropriate charts it is the derivative of the inclusion $\mathbb{R}^{k} \rightarrow \mathbb{R}^{n}$. This tells us that we could have defined $T_{m} M$ as the image of the linear map $d_{m} i$. By the commutative diagram, this linear subspace is the same as the image of the total derivative $D_{0}\left(i \circ \phi_{\alpha}\right)$. Undoing the translation of $i(m)$ to the origin, we recover the $T_{m}^{\mathrm{submf}} M$. We conclude that there is a linear isomorphism

$$
T_{m} M \cong T_{m}^{\text {submfd }} M
$$

### 4.3.2 The physicists' definition

For physicists, a tangent vector is described in terms of a chart (thought of as a local coordinate system), which transforms in a certain way when passing to other local coordinates. That is, an element of $T_{m} M$ is an equivalence class of a chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ such that $\phi_{\alpha}(0)=M$ and a vector $v \in \mathbb{R}^{k}$. The equivalence relation tells us that $v$ transforms as expected: by applying the total derivative of the transition function $\psi_{\beta \alpha}$.

Definition 4.3.3. The physicists' definition of the tangent space of $M$ at $m$ is

$$
T_{m}^{\text {phys }} M=\left(\begin{array}{cc} 
& \mathbb{R}^{k} \\
\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)
\end{array}\right) / \simeq
$$

where the disjoint union is over all charts with $m=\phi_{\alpha}(0)$ and the equivalence relation $\simeq$ is given by

$$
(\alpha, \vec{v}) \simeq(\beta, \vec{w}) \quad \text { if and only if } \quad \vec{w}=D_{0} \psi_{\beta \alpha}(\vec{v})
$$

Remark 4.3.4. This definition reflects the experimental roots of physical theories: the transformation rule under change of local coordinates for physical quantities is determined experimentally, and a mathematical framework is built on top of these results.

Since each $D_{0} \psi_{\beta \alpha}$ is a linear map, addition and scalar multiplication in each copy of $\mathbb{R}^{k}$ induce a vector space structure on $T_{m}^{\text {phys }} M$. Since each copy of $\mathbb{R}^{k}$ is identified with every other copy, this is a $k$-dimensional vector space. Finally, the maps

$$
\begin{aligned}
\mathbb{R}^{k} & \longrightarrow T_{m} M \\
(\alpha, \vec{v}) & \longmapsto\left(D_{0} \phi_{\alpha}\right)\left(\sum_{i} v_{i} \partial / \partial x_{i}\right)
\end{aligned}
$$

are compatible with the equivalence relation, and thus induce a linear map

$$
T_{m}^{\mathrm{phys}}(M) \longrightarrow T_{m} M
$$

On representatives $(\alpha, \vec{v})$, its composition with the linear isomorphism $\left(D_{0} \phi_{\alpha}\right)^{-1}: T_{m} M \rightarrow$ $\mathbb{R}^{k}$ is given by $(\alpha, \vec{v}) \mapsto \vec{v}$, so this is an isomorphism. We conclude that

$$
T_{m} M \cong T_{m}^{\mathrm{phys}}(M)
$$

This identification is compatible with the construction of derivatives. Any smooth $\operatorname{map} f: M \rightarrow N$ induces a map

$$
\begin{aligned}
d_{f}^{\text {phys }}: T_{m}^{\text {phys }} M & \longrightarrow T_{f(m)}^{\text {phys }} N \\
{[\alpha, \vec{v}] } & \longmapsto\left[\alpha^{\prime}, D_{0}\left(\left(\phi_{\alpha^{\prime}}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)(\vec{v})\right] .
\end{aligned}
$$

Under the identification of the left and right sides with the tangent spaces $T_{m} M$ and $T_{f(m)} M$ defined using derivations, this is the ordinary derivative:

Lemma 4.3.5. The following diagram of linear maps commutes


### 4.3.3 The geometers' definition

For geometers, a tangent vector is the derivative of a curve. As such, it is an equivalence class of germs of smooth maps

$$
\bar{\gamma}:(\mathbb{R}, 0) \longrightarrow(M, m) .
$$

Because we want to avoid a circular definition, we can not refer to the derivative of this map. However, we can take a function germ $\bar{g}:(M, m) \rightarrow \mathbb{R}$ and compute

$$
\frac{d}{d t} g \circ \gamma(0),
$$

a derivative of a real-valued function on a neighborhood of the origin in $\mathbb{R}$.
Then the equivalence relation $\approx$ on curves through $m$ is given by

$$
\bar{\gamma} \approx \bar{\eta} \quad \text { if and only if } \quad \frac{d}{d t} g \circ \gamma(0)=\frac{d}{d t} g \circ \eta(0) \text { for all } g:(M, m) \rightarrow \mathbb{R} .
$$

That is, if $\gamma$ and $\eta$ define the same directional derivative.
Definition 4.3.6. The geometers' definition of the tangent space of $M$ at $m$ is

$$
T_{m}^{\text {geom }} M:=\{\operatorname{germs}(\mathbb{R}, 0) \rightarrow(M, m)\} / \approx
$$

There is a map

$$
\begin{aligned}
T_{m}^{\text {geom }} M=\{\operatorname{germs}(\mathbb{R}, 0) \rightarrow(M, m)\} / & \approx \longrightarrow T_{m} M=\operatorname{Der}(\mathcal{E}(M, m)) \\
{[\bar{\gamma}] } & \longmapsto\left(\bar{h} \mapsto \frac{d(h \circ \gamma)}{d t}\right)
\end{aligned}
$$

By evaluation on coordinate functions in a chart, this is seen to be injective. By construction of curves in the same chart, this is seen to be surjective. Hence it is a
bijection. In particular, we can use this to make $T_{m}^{\text {geom }} M$ into a vector space, getting tautologically a linear isomorphism.

$$
T_{m} M \cong T_{m}^{\text {geom }} M
$$

Again, this is compatible with the construction of derivatives. Any smooth map $f: M \rightarrow N$ induces a map

$$
\begin{aligned}
d_{f}^{\text {geom }}: T_{m}^{\text {geom }} M & \longrightarrow T_{f(m)}^{\text {geom }} N \\
{[\bar{\gamma}] } & \longmapsto[\overline{f \circ \gamma}] .
\end{aligned}
$$

Under the identifications with tangent spaces defined using derivations, this is the ordinary derivative:

Lemma 4.3.7. The following diagram of linear maps commutes


### 4.4 Problems

Problem 4.4.1 (Smooth invariance of domain). Prove that $\mathbb{R}^{k}$ and $\mathbb{R}^{l}$ are not diffeomorphic if $k \neq l$. (Hint: look at the derivative of a hypothetical such diffeomorphism.)

Problem 4.4.2 (Maps in or out of products). Let $X, Y$ be smooth manifolds.
(a) Prove that the projection maps $\pi_{1}: X \times Y \rightarrow X$ given by $\pi_{1}(x, y)=x$ and $\pi_{2}: X \times Y \rightarrow Y$ given by $\pi_{2}(x, y)=y$ are both smooth.
(b) Show that

$$
\begin{align*}
T_{(x, y)}(X \times Y) & \longrightarrow T_{x} X \oplus T_{y} Y  \tag{4.4}\\
v & \longmapsto\left(d_{(x, y)} \pi_{1}(v), d_{(x, y)} \pi_{2}(v)\right)
\end{align*}
$$

is an isomorphism of $\mathbb{R}$-vector spaces.
(c) Fixing a point $y \in Y$, there is an injection map

$$
\begin{aligned}
i_{y}: X & \longrightarrow X \times Y \\
x & \longmapsto(x, y)
\end{aligned}
$$

which you may assume is smooth. Prove that its derivative $d_{x} i_{y}: T_{x} X \rightarrow$ $T_{(x, y)}(X \times Y) \cong T_{x} X \oplus T_{y} Y$ is given by $w \mapsto(w, 0)$.
(d) Let $f: X \rightarrow X^{\prime}$ and $g: Y \rightarrow Y^{\prime}$ be smooth maps. Prove that

$$
\begin{aligned}
f \times g: X \times Y & \longrightarrow X^{\prime} \times Y^{\prime} \\
(x, y) & \longmapsto(f(x), g(y))
\end{aligned}
$$

is smooth. Prove that its derivative $d_{(x, y)}(f \times g): T_{(x, y)}(X \times Y) \rightarrow T_{(f(x), g(y))}\left(X^{\prime} \times\right.$ $\left.Y^{\prime}\right)$ is given by $(v, w) \mapsto\left(d_{x} f(v), d_{y} g(w)\right)$ under the isomorphism (4.4).

Problem 4.4.3 (Complex general linear groups). Show that $\mathrm{GL}_{n}(\mathbb{C})$ is a $(2 n)^{2}$-dimensional Lie group.
Problem 4.4.4 (Orthogonal groups). Show that $\mathrm{O}(n) \subset \mathrm{GL}_{n}(\mathbb{R})$ is an $\frac{n(n-1)}{2}$-dimensional Lie group.

Problem 4.4.5 (Derivatives of addition and inverse). Let $G$ be a Lie group as in Definition 4.1.18, and $e \in G$ denote its identity element.
(a) What are the restrictions of $\mu$ to $G \times\{e\}$ and $\{e\} \times G$ ?
(b) Use the chain rule to prove that the derivative $d_{e, e} \mu: T_{e} G \oplus T_{e} G \rightarrow T_{e} G$ is given by addition $(v, w) \mapsto v+w$.
(c) Let $\Delta: G \rightarrow G \times G$ denote the diagonal map $g \mapsto(g, g)$. What is the composition $\mu \circ(\mathrm{id} \times \iota) \circ \Delta$ ?
(d) Use the chain rule to prove that the derivative $d_{e} \iota: T_{e} G \rightarrow T_{e} G$ is given by negation $v \mapsto-v$.

Problem 4.4.6 (The algebraic geometers' definition). In this problem you will give the algebraic geometers' definition of a tangent space.
(a) Prove that there is a unique maximal ideal of $\mathcal{E}(M, m)$, given by $\mathfrak{m}_{m}=\{f \mid$ $f(m)=0\}$.
(b) Prove that for $(M, m)=\left(\mathbb{R}^{k}, 0\right)$, the maximal ideal $\mathfrak{m}_{0}$ is spanned by the coordinate functions $x_{1}, \ldots, x_{k}$.
(c) Prove that $\mathcal{E}(M, m) / \mathfrak{m}_{m}$ is a 1 -dimensional $\mathbb{R}$-vector space, and $\mathfrak{m}_{m} / \mathfrak{m}_{m}^{2}$ is $k$ dimensional if $M$ is $k$-dimensional.
The algebraic geometers' definition of the tangent space of $M$ at $m$ is

$$
T_{m}^{\mathrm{ag}} M:=\left(\mathfrak{m}_{m} / \mathfrak{m}_{m}^{2}\right)^{*} .
$$

(d) Construct a linear map $d_{m}^{\text {ag }} f: T_{m}^{\text {ag }} M \rightarrow T_{f(m)}^{\text {ag }} N$ for each smooth map $f: M \rightarrow N$. Prove it satisfies $d_{m}^{\mathrm{ag}} \mathrm{id}=\mathrm{id}$ and $d_{m}^{\mathrm{ag}}(g \circ f)=d_{f(m)}^{\mathrm{ag}} g \circ d_{m}^{\mathrm{ag}} f$.

## Chapter 5

## Tangent bundles

We now assemble the tangent spaces to tangent bundles, and the derivatives of a smooth map to a map of tangent bundles. This appears in Chapter 2 of [BJ82]. See also [Tu11, Chapters 6, 8].

### 5.1 Vector bundles

The tangent bundle is the prototypical example of a mathematical object which is worth studying by itself.

### 5.1.1 The definition of a vector bundle

Recall that in its most concrete form, the total derivative of a smooth map $f: \mathbb{R}^{m} \rightarrow$ $\mathbb{R}^{n}$ at a point $a \in \mathbb{R}^{m}$ is the $(n \times m)$-matrix of partial derivatives of its components at $x$, a linear map $\mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$. Thus the total derivative of $f$ is a collection of linear maps $\mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ parametrized by $x \in \mathbb{R}^{n}$. On smooth manifolds the domains $\mathbb{R}^{m}$ and targets $\mathbb{R}^{n}$ will depend on $x$ and $f(x)$ respectively; they will be collections of vector spaces parametrized by a manifold. Such an object is called a vector bundle. It is one of the other geometric objects studied by differential topology in addition to smooth manifolds.

Definition 5.1.1. A $k$-dimensional vector bundle over a topological space $X$ is a topology on the disjoint union $E=\bigsqcup_{x \in X} E_{x}$ of a collection of real vector spaces, such that for each $x \in X$ there exists an open subset $V \subset X$ containing $x$ and a homeomorphism

$$
\zeta: \bigsqcup_{x \in V} E_{x} \xrightarrow{\cong} V \times \mathbb{R}^{k}
$$

that restricts to an invertible linear map $E_{x} \rightarrow\{x\} \times \mathbb{R}^{k}$ for each $x \in V$.
Let $p: E \rightarrow X$ denote the function that sends $E_{x}$ the point $\{x\}$. It is a consequence of the definition that $p$ is continuous. The map $p$ is called the projection, $E$ the total base, $X$ the base, and each $E_{x}$ is a fiber. Finally, the pair $(V, \zeta)$ is called a bundle chart. Example 5.1.2. The cartesian product $X \times \mathbb{R}^{k}$ has an evident structure of a $k$-dimensional vector bundle. We call this the trivial $k$-dimensional vector bundle over $X$. The property
in Definition 5.1.1 is often referred to as a local triviality condition, as it is says that $E$ locally looks like such a trivial bundle.
Example 5.1.3. The real projective space $\mathbb{R} P^{n}$ is the space of lines in $\mathbb{R}^{n+1}$. There is a 1-dimensional vector bundle over it with fiber of $L$ given by those $v \in \mathbb{R}^{n+1}$ which lie in $L$. This is the canonical bundle. More precisely, writing $\mathbb{R} P^{n}=S^{n} /\{ \pm 1\}$, we have

$$
E_{[x]}=\left\{v \in \mathbb{R}^{n+1} \mid v=\lambda x \text { for some } \lambda \in \mathbb{R}\right\}
$$

We topologize $\bigsqcup_{[x]} E_{[x]}$ as a subspace of $\mathbb{R} P^{n} \times \mathbb{R}^{n+1}$. The local triviality condition is verified using charts.

### 5.1.2 Maps between vector bundles

Definition 5.1.4. Let $p: E \rightarrow X$ and $p^{\prime}: E^{\prime} \rightarrow X^{\prime}$ be vector bundles (possibly of different dimension). For a continuous map $F: E \rightarrow E^{\prime}$ to be a map of vector bundles, the first requirement is that there is a continuous map $f: X \rightarrow X^{\prime}$ the following diagram commute


Then $F$ restricts to a map of fibers $F_{x}: E_{x} \rightarrow E_{f(x)}^{\prime}$, and the second requirement is that this is a linear map.

Note that $f$ is uniquely determined by $F$, and we say that $F$ covers $f$ or $F$ is over $f$. It is clear that the identity is a map of vector bundles, and that maps of vector bundles are closed under composition.

Definition 5.1.5. An isomorphism of vector bundles is a map of vector bundles which is bijective and whose inverse is also a map of vector bundles.

Example 5.1.6. Over $S^{1}$ we have exactly two 1-dimensional vector bundles up to isomorphism: the trivial one and the "Mobius strip" bundle. More precisely, the latter is the canonical bundle over $\mathbb{R} P^{1} \cong S^{1}$. The latter is given by taking $[0,1] \times \mathbb{R}$ and identifying the endpoints by $(0, v) \sim(1,-v)$.
Example 5.1.7. Let $X \times \mathbb{R}^{m}$ be a trivial bundle. The $(m \times m)$-matrices $\operatorname{Mat}_{m}(\mathbb{R})$ are topologized by identifying them with $\mathbb{R}^{m^{2}}$ through their entries. Then any continuous $\operatorname{map} A: X \rightarrow \operatorname{Mat}_{m}(\mathbb{R})$ gives rise to a map of vector bundles

$$
\begin{aligned}
X \times \mathbb{R}^{m} & \longrightarrow X \times \mathbb{R}^{m} \\
(x, v) & \longmapsto(x, A(x)(v))
\end{aligned}
$$

This is an isomorphism of vector bundles if and only if $A$ takes values in $\mathrm{GL}_{m}(\mathbb{R}) \subset$ $\operatorname{Mat}_{m}(\mathbb{R})$, the subset of invertible matrices.

### 5.1.3 Smooth vector bundles

As for topological manifolds, we can package the data of $k$-dimensional vector bundle over a topological space into an atlas: the collection of bundle charts $(V, \zeta)$ for $(p, E, X)$ with $V$ covering $X$ is called a bundle atlas. As in the case of smooth atlases, we can define maximal bundle atlases and prove that every bundle atlas is contained in a unique maximal bundle atlas.

Such a bundle atlas has transition functions: taking $\left(V_{\alpha}, \zeta_{\alpha}\right)$ and $\left(V_{\beta}, \zeta_{\beta}\right)$, the composition

$$
\left(V_{\alpha} \cap V_{\beta}\right) \times \mathbb{R}^{k} \xrightarrow{\zeta_{\alpha}^{-1}} p^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \xrightarrow{\zeta_{\beta}}\left(V_{\alpha} \cap V_{\beta}\right) \times \mathbb{R}^{k}
$$

is necessary of the form $(x, v) \mapsto\left(x, \xi_{\alpha \beta}(x)(v)\right)$ for a linear map $\xi_{\alpha \beta}(x): \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ depending continuously on $x$.

If the base were a smooth manifold, so are the $V_{\alpha} \cap V_{\beta}$. Recall that $\mathrm{GL}_{k}(\mathbb{R})$ is an open subset of $\mathbb{R}^{k^{2}}$ and hence inherits a smooth structure, we can make sense of whether these transition functions are smooth.

Definition 5.1.8. Suppose $M$ is a smooth manifold. Then a vector bundle $(p, E, M)$ is smooth if all transition functions $\xi_{\beta \alpha}: U_{\alpha} \cap U_{\beta} \rightarrow \mathrm{GL}_{k}(\mathbb{R})$ are smooth.

The proof of the following is left as a problem:
Lemma 5.1.9. If $(p, E, M)$ is a smooth vector bundle then $E$ has a unique smooth structure such that all bundle charts $\zeta_{\alpha}: p^{-1}\left(V_{\alpha}\right) \rightarrow V_{\alpha} \times \mathbb{R}^{k}$ are diffeomorphisms. Then $p: E \rightarrow M$ is a smooth map.

When we have a pair of vector bundles $(p, E, M)$ and $\left(p^{\prime}, E^{\prime}, M^{\prime}\right)$ and a map $F: E \rightarrow$ $E^{\prime}$ of vector bundles over $f: M \rightarrow M^{\prime}$, then we can use the bundle charts to write

$$
\left(V_{\alpha} \cap f^{-1}\left(V_{\alpha^{\prime}}^{\prime}\right)\right) \times \mathbb{R}^{k} \xrightarrow{\zeta_{\alpha}^{-1}} p^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\alpha^{\prime}}^{\prime}\right)\right) \xrightarrow{F}\left(p^{\prime}\right)^{-1}\left(V_{\alpha^{\prime}}^{\prime}\right) \xrightarrow{\zeta_{\alpha^{\prime}}^{\prime}} V_{\alpha^{\prime}}^{\prime} \times \mathbb{R}^{k^{\prime}}
$$

As before, this preserves the first coordinate and hence is encoded by a continuous map $\left(U_{\alpha} \cap f^{-1}\left(V_{\alpha^{\prime}}^{\prime}\right)\right) \rightarrow \operatorname{Lin}\left(\mathbb{R}^{k}, \mathbb{R}^{k^{\prime}}\right)$.

We can ask this to be smooth, and if the vector bundles are smooth this is independent of the choice of bundle charts. If all these maps are smooth, we say that the map $F:(p, E, M) \rightarrow\left(p^{\prime}, E^{\prime}, M^{\prime}\right)$ of vector bundles is smooth. This is in particular always a smooth map between the manifolds $M$ and $M^{\prime}$.

### 5.2 The tangent bundle and the derivative

In the previous chapter, we described how to assign a vector space $T_{m} M$ to each $m \in M$, as well as maps

$$
d_{m} f: T_{m} M \longrightarrow T_{f(m)} N,
$$

which satisfy the desiderata
(I') $d_{m} f$ is a linear map.
$(\mathrm{II}) \quad d_{m} \mathrm{id}=\mathrm{id}$ and $d_{m}(g \circ f)=d_{f(m)} g \circ d_{m} f$.
(III') In local coordinates $T_{m} M$ is $\mathbb{R}^{k}$ and $d_{m} f$ is the total derivative.
We next explain how to patch together the vector spaces $T_{m} M$ to a "vector bundle $T M$ over $M$ " and the linear maps $d_{m} f$ to a map $d f: T M \rightarrow T N$ for each smooth map $f: M \rightarrow N$. These should satisfy analogous desiderata
(I") $d f$ is a map of vector bundles.
$(\mathrm{II} ") d(\mathrm{id})=\mathrm{id}$ and $d(g \circ f)=d g \circ d f$,
(III") In local coordinates $T M$ is given by $\mathbb{R}^{k}$ 's and $d f$ by the total derivatives.

### 5.2.1 Constructing the tangent bundle

To construct the tangent bundle $T M$ of a manifold, we shall employ a general construction. This is the analogue for vector bundles of Example 2.2.14.

Definition 5.2.1. A $k$-dimensional pre-vector bundle over a space $X$ is a disjoint union $E=\bigsqcup_{x \in X} E_{x}$ of a collection of real vector space $E_{x}$, together with a collection $\mathcal{B}=\left\{\left(V_{\alpha}, \zeta_{\alpha}\right)\right\}$ of open subsets covering $X$ and bijections

$$
\zeta_{\alpha}: \bigsqcup_{x \in V_{\alpha}} E_{x} \xrightarrow{\cong} V_{\alpha} \times \mathbb{R}^{k}
$$

that restrict to invertible linear maps $E_{x} \rightarrow\{x\} \times \mathbb{R}^{k}$. Furthermore, we require that all transition functions $\xi_{\alpha \beta}: V_{\alpha} \cap V_{\beta} \rightarrow \mathrm{GL}_{k}(\mathbb{R})$ are continuous.

That is, a pre-vector bundle is essentially a vector bundle that is of yet without a topology on its total space. However, we have:

Lemma 5.2.2. There is exactly one topology on $E$ so that $\mathcal{B}=\left\{\left(V_{\alpha}, \zeta_{\alpha}\right)\right\}$ is a bundle atlas for $(p, E, B)$.

Proof sketch. Give $E$ the smallest topology such that all $\zeta_{\alpha}$ are continuous.
If we replace $X$ by a manifold $M$, we can similarly define $k$-dimensional smooth vector bundles, by demanding that all $\xi_{\alpha \beta}$ are smooth. Using the above construction then makes $(p, E, M)$ into a smooth vector bundle. In particular, we can define the tangent bundle $T M$ by prescribing a smooth pre-vector bundle on $M$ :

- $T M=\bigsqcup_{m \in M} T_{m} M$,
- $\mathcal{B}=\left\{V_{\alpha}, \zeta_{\alpha}\right\}$ where $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ is ranges over the charts of the maximal atlas of $M$, and
- $\zeta_{\alpha}: \bigsqcup_{m \in M} T_{m} M \rightarrow V_{\alpha} \times \mathbb{R}^{k}$ is given by

$$
(m, v) \longmapsto\left(m,\left(d_{\phi_{\alpha}^{-1}(m)} \phi_{\alpha}\right)^{-1}(v)\right) .
$$

Implicitly, we are using here the identifications of $D_{\phi_{\alpha}^{-1}(m)} \mathbb{R}^{k}$ with $\mathbb{R}^{k}$ using the basis $\frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{k}}$.

Definition 5.2.3. The smooth vector bundle $T M$ over $M$ constructed from this prevector bundle is the tangent bundle to $M$.

Example 5.2.4. If $U \subset \mathbb{R}^{k}$ is open, then $T U=U \times \mathbb{R}^{k}$.
Observe that the tangent bundle is itself a smooth manifold. Indeed, there is a unique $2 k$-dimensional smooth structure on $T M$ such that each of the local trivializations $\left.T M\right|_{U} \cong U \times \mathbb{R}^{k}$ induced by a chart of $M$ is a diffeomorphism. As a consequence, the projection map $T M \rightarrow M$ is a smooth map, and the zero section $s_{0}: M \rightarrow T M$ is an embedding whose image is a codimension $k$ submanifold called the 0 -section.

### 5.2.2 The derivative and its properties

It is now easy to define the derivative $d f: T M \rightarrow T N$ of a smooth map $f: M \rightarrow N$. This will be a map of vector bundles which covers $f$, and hence it suffices to give linear maps $d_{m} f: T_{m} M \rightarrow T_{f(n)} N$ and verify that these are continuous and in fact smooth. Of course, we will take these linear maps the derivatives as we defined before.

Lemma 5.2.5. The derivatives $d_{m} f: T_{m} M \rightarrow T_{f(n)} N$ assemble to a smooth bundle map $d f: T M \rightarrow T N$.

Proof. Since being smooth is a local property, it suffices to check this with respect to the bundle charts defining $T M$ and $T N$, i.e. those arising from charts. That is, we need to prove that

$$
\begin{aligned}
\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \times \mathbb{R}^{k} & \longrightarrow V_{\beta}^{\prime} \times \mathbb{R}^{k^{\prime}} \\
(m, v) & \longmapsto\left(f(m),\left[\left(d_{\left(\phi_{\beta}^{\prime}\right)^{-1}(f(m))} \phi_{\beta}^{\prime}\right)^{-1} \circ d_{m} f \circ d_{\left(\phi_{\beta}^{\prime}\right)^{-1}(m)} \phi_{\beta}^{\prime}(m)\right](v)\right)
\end{aligned}
$$

is smooth. To do so, we precompose it with the diffeomorphism

$$
\phi_{\alpha} \times \operatorname{id}: \phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \times \mathbb{R}^{k} \xrightarrow{\cong}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \times \mathbb{R}^{k}
$$

and postcompose it with the inverse of

$$
\phi_{\beta} \times \mathrm{id}: U_{\beta}^{\prime} \times \mathbb{R}^{k} \xrightarrow{\cong} V_{\beta}^{\prime} \times \mathbb{R}^{k}
$$

The result is the map $\phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \times \mathbb{R}^{k} \longrightarrow U_{\beta}^{\prime} \times \mathbb{R}^{k^{\prime}}$ between trivial vector bundles over open subsets of Euclidean space given by

$$
(x, v) \longmapsto\left(\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}(x),\left[\left(d_{\left(\phi_{\beta}^{\prime}\right)^{-1}\left(f\left(\phi_{\alpha}(x)\right)\right)} \phi_{\beta}^{\prime}\right)^{-1} \circ d_{f(x)} f \circ d_{x} \phi_{\alpha}\right](v)\right)
$$

Using the chain rule, we identify the right term as $d_{x}\left(\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)$, which equals the total derivative $D_{x}\left(\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)$. That is, we are dealing with the map

$$
(x, v) \longmapsto\left(\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}(x), D_{x}\left(\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}\right)(v)\right) .
$$

This is evidently linear on each fiber and smooth.

Example 5.2.6. If $U \subset \mathbb{R}^{k}$ and $V \subset \mathbb{R}^{k^{\prime}}$ are open and $f: U \rightarrow V$ is a smooth map, then $d f: T U \rightarrow T V$ is the map

$$
\begin{aligned}
T U=U \times \mathbb{R}^{k} & \longrightarrow T V=V \times \mathbb{R}^{k} \\
(x, v) & \longmapsto\left(f(x), D_{x} f(v)\right)
\end{aligned}
$$

obtained by applying pointwise the total derivative of $f$.
Applying $d_{m}(\mathrm{id})=$ id and $d_{m}(g \circ f)=d_{f(m)} g \circ d_{m} f$ in each fiber, we see that:
Lemma 5.2.7. The derivative satisfies $d(\mathrm{id})=\mathrm{id}$ and $d(g \circ f)=d g \circ d f$.

### 5.3 Linear algebra of vector bundles

We want to generalize our usual definitions and constructions for vector spaces to vector bundles.

### 5.3.1 Subbundles

The generalization of a subspace of a vector space is a subbundle.
Definition 5.3.1. Let $p: E \rightarrow X$ be a $k$-dimensional vector bundle. A subspace $E^{\prime} \subset E$ is a $k^{\prime}$-dimensional subbundle if each $E_{x}^{\prime}:=p^{-1}(x) \cap E^{\prime}$ is a $k^{\prime}$-dimensional linear subspace of $E_{x}=p^{-1}(E)$ and there are local trivializations $\phi: \bigsqcup_{x \in U} E_{x} \cong U \times \mathbb{R}^{k}$ sending $\bigsqcup_{x \in U} E_{x}^{\prime}$ to $U \times \mathbb{R}^{k^{\prime}}$.

If $(p, E, M)$ is a smooth vector bundle, we can make sense of a smooth subbundle, by requiring that the local trivializations are smooth.

### 5.3.2 Kernels

Using this we can make sense of the kernel and image of certain maps of vector bundles. This requires the following technical lemma, whose proof you do not need to know. Let $\operatorname{Lin}\left(\mathbb{R}^{p}, \mathbb{R}^{p^{\prime}}\right)$ denote the space of linear map $\mathbb{R}^{p^{\prime}} \rightarrow \mathbb{R}^{p}$, topologized by identifying it with $\mathbb{R}^{p p^{\prime}}$.

Lemma 5.3.2. If $\Gamma: \mathbb{R}^{n} \rightarrow \operatorname{Lin}\left(\mathbb{R}^{p}, \mathbb{R}^{p^{\prime}}\right)$ is a continuous map with image in the linear maps of rank $r$, then there exists an open neighborhood $W \subset \mathbb{R}^{n}$ of 0 and continuous maps $B: W \rightarrow \mathrm{GL}_{p^{\prime}}(\mathbb{R})$ and $C: W \rightarrow \mathrm{GL}_{p}(\mathbb{R})$ so that $C(w) \Gamma(w) B(w)=\Gamma(0)$ for all $w \in W$. If $\Gamma$ is smooth, then $B$ and $C$ can also be taken to be smooth.

Proof. We may as well change bases to something more convenient: pick a basis of $\mathbb{R}^{p}$ and $\mathbb{R}^{p^{\prime}}$ such that in this basis $\Gamma(0)$ is given by the $\left(p \times p^{\prime}\right)$-matrix (the 0 's are rectangular matrices filled with 0's of the correct size)

$$
\pi_{r}=\left[\begin{array}{cc}
\mathrm{id}_{r} & 0 \\
0 & 0
\end{array}\right]
$$

With respect to these bases, for $w$ in an open neighborhood $W$ of 0 the matrix of $\Gamma(w)$ is given by

$$
\pi_{r}+A=\left[\begin{array}{cc}
\operatorname{id}_{r}+A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right]
$$

with $\|A\|^{2}<1 / 2$ (with $\|A\|^{2}$ the sum of the squared entries). In fact, because the first $r$ rows contains a unique entry $>1 / 2, A_{21}$ and $A_{22}$ have to be 0 for this to have rank $r$.

We will use $C(w)$ to get rid of $A_{11}$ :

$$
C(w)=\left[\begin{array}{cc}
\left(\operatorname{id}_{r}+A_{11}\right)^{-1} & 0 \\
0 & \operatorname{id}_{p^{\prime}-r}
\end{array}\right]
$$

with the inverse in the top-right square existing because each row contains a unique entry $>1 / 2$. We compute that

$$
C(w) \Gamma(w)=\pi_{r}+A=\left[\begin{array}{cc}
\mathrm{id}_{r} & \left(\mathrm{id}_{r}+A_{11}\right)^{-1} A_{12} \\
0 & 0
\end{array}\right]
$$

We will then use $B(w)$ to get rid of the $(r \times p-r)$-matrix $\left(\mathrm{id}_{r}+A_{11}\right)^{-1} A_{12}$ : it will be the $(p \times p)$-matrix given by

$$
B(w)=\left[\begin{array}{cc}
\mathrm{id}_{r} & -\left(\mathrm{id}_{r}+A_{11}\right)^{-1} A_{12} \\
0 & \mathrm{id}_{p-r}
\end{array}\right]
$$

and it is a simple computation that $C(w) \Gamma(w) B(w)=\Gamma(0)$.
Since the construction of $C(w)$ and $B(w)$ depends continuously on the entries of $\Gamma(w)$ these maps are continuous.

Lemma 5.3.3. Suppose $p: E \rightarrow X$ and $p^{\prime}: E^{\prime} \rightarrow X^{\prime}$ are vector bundles, and $G: E \rightarrow E^{\prime}$ is a map of vector bundles so that $G_{x}: E_{x} \rightarrow E_{g(x)}^{\prime}$ has the same rank for all $x \in X$. Then

$$
\operatorname{ker}(G):=\bigsqcup_{x \in X} \operatorname{ker}\left(G_{x}\right)
$$

is a subbundle of $E$. If the vector bundles and the map between them are smooth, then $\operatorname{ker}(G)$ is a smooth subbundle.

Proof. Passing to local trivializations of $p$ and $p^{\prime}$, we may assume that $G$ is a continuous map $U \times \mathbb{R}^{p} \rightarrow V \times \mathbb{R}^{p^{\prime}}$ over a continuous map $g: U \rightarrow V$ so that $G(u,-): \mathbb{R}^{p} \rightarrow \mathbb{R}^{p^{\prime}}$ is linear of fixed rank $r$. In other words, $G$ is described by a $g$ and a continuous map $\Gamma: U \rightarrow \operatorname{Lin}\left(\mathbb{R}^{p^{\prime}}, \mathbb{R}^{p}\right)$ landing in the subspace of linear spaces that have rank $r$. By the previous lemma, on a neighborhood of each point $u_{0} \in U$ we adjust the local trivializations is that $\Gamma$ is constant with value $\pi_{r}$.

### 5.3.3 Images

The image of a vector bundle map is not defined in general. On the one hand, if the underlying map on base spaces is not injective, it will try to assign two fibers to the same point in the target. On the other hand, if the underlying map on base spaces is not surjective, it will not know what fibers to assign to some points in the target. These issues are resolved by restricting our attention to inclusions of base spaces only, and constructing the image of the vector bundle map only over the image of this inclusion.

Definition 5.3.4. Suppose that $p: E \rightarrow X$ is a vector bundle and $Y \subset X$ a subspace, then $\left.E\right|_{Y}:=\bigcup_{y \in Y} E_{y}$ with the subspace topology is a vector bundle over $Y$.

This definition makes sense, because the local trivializations of $E$ restrict to local trivializations of $\left.E\right|_{Y}$.
Example 5.3.5. The local triviality condition in the definition of a $k$-dimensional vector bundle $p: E \rightarrow X$ can rephrased as saying that for all $x \in X$ there exists an open subset $U \subset X$ such that $\left.E\right|_{U}$ is isomorphic to the trivial bundle $U \times \mathbb{R}^{k}$.

A similar argument as for kernels now tells us that:
Lemma 5.3.6. Suppose $p: E \rightarrow X$ and $p^{\prime}: E^{\prime} \rightarrow X^{\prime}$ are vector bundles and $X \subset X^{\prime}$, and $G: E \rightarrow E^{\prime}$ over the inclusion so that $G_{x}: E_{x} \rightarrow E_{x}^{\prime}$ has the same rank for all $x \in X$. Then

$$
\operatorname{im}(G):=\bigsqcup_{x \in X} \operatorname{im}\left(G_{x}\right)
$$

is a subbundle of $\left.E^{\prime}\right|_{X}$. If the vector bundles and the map between them are smooth, then $\operatorname{im}(G)$ is a smooth vector bundle.

### 5.3.4 Quotients

Given a subspace of a vector space, we can take the quotient. Similarly, we can take the fibewise quotient of a vector bundle by a subbundle.

Lemma 5.3.7. Let $E \rightarrow X$ be a vector bundle and $E^{\prime} \subset E$ a subbundle. Then the quotients of the vector space $E_{x}$ by the subspace $E_{x}^{\prime}$ assemble to a vector bundle

$$
E / E^{\prime}:=\bigsqcup_{x \in X} E_{x} / E_{x}^{\prime}
$$

over $X$ using the quotient topology, which we call the quotient bundle. If $E$ was a smooth vector bundle and $E^{\prime}$ a smooth subbundle, then $E / E^{\prime}$ is also a smooth vector bundle.

### 5.4 Problems

Problem 5.4.1 (Construction of smooth vector bundles). Prove Lemma 5.1.9.
Problem 5.4.2 (The tangent bundle of a Lie group). Let $G$ be a Lie group as in Definition 4.1.18, with identity element $e$.
(a) Let $\mu_{g}: G \rightarrow G$ be the map $h \mapsto \mu(g, h)$. Prove that it is smooth and that $d_{e} \mu_{g}: T_{e} G \rightarrow T_{g} G$ is an isomorphism.
(b) Prove that the tangent bundle $T G$ is isomorphic to a trivial vector bundle.

Problem 5.4.3 (Tangent bundles to submanifolds). Let $M \subset \mathbb{R}^{n}$ be a $k$-dimensional smooth submanifold.
(a) Prove that

$$
T^{\text {submfd }} M=\left\{(m, v) \in M \times \mathbb{R}^{n} \mid v+m \in T_{m}^{\text {submfd }} M\right\}
$$

is a $k$-dimensional smooth vector bundle.
(b) Prove that $T M$ and $T^{\text {submfd }} M$ are isomorphic as smooth vector bundles.

## Chapter 6

## Immersions and submersions

In this chapter we continue with implementation of one of the slogans of differential topology: state globally, prove locally. We do so by importing the inverse function theorem and its corollaries into the language of smooth manifolds. The main difficulty is figuring out the correct statements, as most proofs will start by passing to charts and then work on open subsets of Euclidean space.

This covers $1 . \S 3$ of [GP10], as well as a version of pages 51-52.

### 6.1 Globalizing the inverse function theorem

The easiest example of the above slogan is a characterization of diffeomorphisms where you do not need to go through the effort of finding the inverse and proving it is smooth. We start by recalling the statement of the inverse function theorem [DK04a, Theorem 3.2.4]:

Theorem 6.1.1 (Inverse function theorem). Let $U_{0} \subset \mathbb{R}^{n}$ be open and $a \in U_{0}$. Suppose $g: U_{0} \rightarrow \mathbb{R}^{n}$ is a smooth map whose total derivative $D g_{a}$ at a is an invertible linear map. Then there exists an open neighborhood $U \subset U_{0}$ of a such that $g(U)$ is open and

$$
\left.g\right|_{U}: U \longrightarrow g(U)
$$

is a diffeomorphism onto this open subset.
To translate this into the language of smooth manifolds we recall the notions we introduced in the previous lecture. We constructed for each $k$-dimensional smooth manifold $M$ a tangent bundle $T M$, which is a $k$-dimensional smooth vector bundle over $M$. Each smooth map $f: M \rightarrow N$ with $M k$-dimensional and $N k^{\prime}$-dimensional induces a map of vector bundles $d f: T M \rightarrow T N$ called the derivative.

Both of the tangent bundle and the derivative are easy to understand when viewed through the lens of a chart. A chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ of $M$ with $p \in U_{\alpha}$ gives an identification the restriction of $T M$ to $V_{\alpha}$ with $U_{\alpha} \times \mathbb{R}^{k}$. A chart $\left(U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}\right)$ with $f(p) \in V_{\beta}^{\prime}$ gives a similar identification of the restriction of $T N$ to $V_{\beta}^{\prime}$ with $U_{\beta}^{\prime} \times \mathbb{R}^{k^{\prime}}$. Under these identifications, the derivative $d_{p} f: T_{p} M \rightarrow T_{f(p)} N$ is the total derivative of $\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}$
at $\phi_{\alpha}^{-1}(p)$. That is, the following diagram of vector spaces and linear maps commutes:


We shall translate the hypothesis on $d_{p} f$ into one about the bottom linear map, and then apply the inverse function theorem to get:

Lemma 6.1.2. Let $f: M \rightarrow N$ be a smooth map with $M$-dimensional and $N k^{\prime}$ dimensional, and suppose that $d_{p} f: T_{p} M \rightarrow T_{f(p)} N$ is an isomorphism. Then $k=k^{\prime}$ and $f$ is a local diffeomorphism at $p$, i.e. there is an open neighborhood $V$ of $p$ in $M$ such that $\left.f\right|_{V}: V \rightarrow f(V)$ is a diffeomorphism.

Proof. Using (6.1), the hypothesis translates into the statement that the total derivative of the map

$$
\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}: U_{\alpha} \supset \phi_{\alpha}^{-1}\left(V_{\alpha} \cap f^{-1}\left(V_{\beta}^{\prime}\right)\right) \longrightarrow \phi_{\alpha}^{-1}\left(f\left(V_{\alpha}\right) \cap V_{\beta}^{\prime}\right) \subset U_{\beta}^{\prime}
$$

at $\phi_{\alpha}^{-1}(p)$ is an isomorphism. This is only possible if the total derivative is a square matrix, so $k=k^{\prime}$.

When we call this function $g$ and apply the inverse function theorem to it at $a=\phi_{\alpha}^{-1}(p)$, we get an open subset $U \subset \phi_{\alpha}^{-1}\left(f\left(V_{\alpha}\right) \cap V_{\beta}^{\prime}\right)$ such that $g(U)$ is open and $\left.g\right|_{U}: U \rightarrow g(U)$ is a diffeomorphism. Translating this back into $M$ and setting $V:=\phi_{\alpha}(U)$ through the commutative diagram

this is saying that $f(V)=\phi_{\beta}(g(U))$ is open in $N$ and $\left.f\right|_{V}: V \rightarrow f(V)$ is a diffeomorphism.

Theorem 6.1.3. A bijective smooth map $f: M \rightarrow N$ which has a bijective differential at all $p \in M$ is a diffeomorphism.

Proof. Since $f: M \rightarrow N$ is a bijection, it has an inverse $f^{-1}: N \rightarrow M$. To see that this is smooth at $f(p) \in M$, apply the previous lemma and observe that on $f(V), f^{-1}$ of course coincides with $\left(\left.f\right|_{V}\right)^{-1}$. The latter is smooth as the inverse of the diffeomorphism $\left.f\right|_{V}$.

Example 6.1.4. The quotient map $\mathbb{R}^{2} \rightarrow \mathbb{R}^{2} / \mathbb{Z}^{2}$ is a surjective smooth map which has bijective differential at all $p \in \mathbb{R}^{2}$, but it is not a diffeomorphism as it is not even a homeomorphism.

We can avoid having to check that $f$ is surjective by demanding $M$ is compact and $N$ is connected.

Corollary 6.1.5. If $M$ is non-empty compact and $N$ is connected, an injective smooth map $f: M \rightarrow N$ which has a bijective differential at all $p \in M$ is a diffeomorphism.

Proof. In light of the previous theorem it remains to prove that $f$ is surjective. By Lemma 6.1.2 the image of $f$ is open. The image of every compact space under a continuous map is compact and in a Hausdorff space every compact set is closed, so the image of $f$ is both open and closed. This means it is a union of connected components of $N$ and by assumption $N$ has a single such component, hence $f$ must be surjective.

### 6.2 Globalizing the immersion theorem

We next globalize the immersion theorem [DK04a, Section 4.3]. This immersion theorem said:

Theorem 6.2.1 (Immersion theorem). Let $U_{0} \subset \mathbb{R}^{k}$ be an open subset and $a \in U_{0}$. Suppose we have a smooth map $h: U_{0} \rightarrow \mathbb{R}^{k^{\prime}}$ such that the total derivative $D h_{a}$ of $h$ at a is injective. Then $k \leq k^{\prime}$ and there exist open neighborhoods $U \subset U_{0}$ of a and $V \subset \mathbb{R}^{n}$ of $h(a)$, and diffeomorphisms $\phi: \mathbb{R}^{k} \rightarrow U$ and $\phi^{\prime}: \mathbb{R}^{k^{\prime}} \rightarrow V$ such that
(i) $\phi(0)=a$,
(ii) $\phi^{\prime}(0)=h(a)$, and
(iii) the following diagram commutes

with $\iota_{k}$ the inclusion $\left(x_{1}, \ldots, x_{k}\right) \mapsto\left(x_{1}, \ldots, x_{k}, 0, \ldots, 0\right)$.
Let us give a name the condition that the differential is injective at some point in domain:

## Definition 6.2.2.

- A smooth map $f: M \rightarrow N$ is an immersion at $p$ if $d_{p} f: T_{p} M \rightarrow T_{f(p)} N$ is an injective linear map.
- A smooth map $f: M \rightarrow N$ is an immersion if it is an immersion at all $p \in M$.

Applying the immersion theorem to $\left(\phi_{\beta}^{\prime}\right)^{-1} \circ f \circ \phi_{\alpha}$ for charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ and ( $U_{\beta}^{\prime}, V_{\beta}^{\prime}, \phi_{\beta}^{\prime}$ ) around $p$ and $f(p)$ respectively, we deduce:

Lemma 6.2.3. Let $f: M \rightarrow N$ be a smooth map which is an immersion at $p$, with $M$ $k$-dimensional and $N k^{\prime}$-dimensional. Then $k \leq k^{\prime}$ and there exists a chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$
of $M$ around $p$ and a chart $\left(U_{\alpha}^{\prime}, V_{\alpha}^{\prime}, \phi_{\alpha}^{\prime}\right)$ of $N$ around $f(p)$ so that the following diagram commutes

with $\iota_{k}$ the inclusion onto first $k^{\prime}$ coordinates.
Remark 6.2.4. Note that a linear map being injective is an open condition, which is reflected in the above lemma by the observation that if $f$ looks like the standard inclusion in some coordinates at $p$, then it does so near $p$, namely on all of $\phi_{\alpha}\left(U_{\alpha}\right)$.

Unlike being a diffeomorphism, being an immersion is a purely local condition. This means that its image may be pathological. Of course since an immersion need not be injective it may intersect itself, see the first example of Figure 6.1. However, even an injective immersion need not be a homeomorphism onto its image, see the second example of Figure 6.1.
Example 6.2.5. One of the worst examples is the immersion

$$
\begin{aligned}
h: \mathbb{R} & \longrightarrow \mathbb{T}^{2}=\mathbb{R}^{2} / \mathbb{Z}^{2} \\
x & \longmapsto[x, \theta x]
\end{aligned}
$$

with $\theta \in(0,1)$ irrational. This immersion has dense image in $\mathbb{T}^{2}$. To see it is an immersion define $\tilde{h}(x): \mathbb{R} \rightarrow \mathbb{R}^{2}$ by $x \mapsto(x, \theta x)$ and consider the commutative diagram of vector spaces


The linear map $d_{\tilde{h}(x)} q$ is an isomorphism because the map $\tilde{h}$ is a local diffeomorphism, and since $d_{x} \tilde{h}=D_{x} \tilde{h}$ is easily seen to be injective, $d_{x} h$ must also be injective.


Figure 6.1 The image of two different immersions of $\mathbb{R}$ into $\mathbb{R}^{2}$.

The latter is the real problem: we would like $f(M)$ not to intersect $\phi_{\beta}^{\prime}\left(U_{\beta}^{\prime}\right)$ again. If $f$ were a homeomorphism onto its image, then $f\left(V_{\alpha}\right)$ would be open in $f(M)$ and this means that there is an open neighborhood $V^{\prime}$ in $N$ such that $V^{\prime} \cap f(M)=f\left(V_{\alpha}\right)$ so by shrinking $\phi_{\beta}^{\prime}\left(U_{\beta}^{\prime}\right)$ we could arrange that $\phi_{\beta}^{\prime}\left(U_{\beta}^{\prime}\right) \cap f(M)=f\left(V_{\alpha}\right)$. That such a open subset $V^{\prime}$ exists is proven by contradiction: if it did not exist then there would be a sequence of points $y_{i} \in f(M) \backslash f\left(V_{\alpha}\right)$ converging to $y \in f(M)$, which contradicts the fact that $f\left(V_{\alpha}\right)$ is open. In this case the charts from the immersion theorem give the image of $f$ the structure of an $r$-dimensional submanifold of $N$. We will make this precise in a moment.
Remark 6.2.6. The advantage of the condition on an immersion being purely local is that we can classify them up to regular homotopy using an $h$-principle, as discussed in the first lecture.

### 6.2.1 Embeddings

Definition 6.2.7. An embedding is an injective immersion which is a homeomorphism onto its image.

Example 6.2.8. If $m, n$ are integers such that $\operatorname{gcd}(m, n)=1$, then the map

$$
\tilde{e}: \mathbb{R} \ni t \mapsto(m t, n t) \in \mathbb{R}^{2}
$$

is easily seen to be an embedding. Taking the quotient by the action of $\mathbb{Z}^{2}$ on $\mathbb{R}^{2}$ induces an injective smooth map $e: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{R}^{2} / \mathbb{Z}^{2}$ which is automatically proper. To see this its differential is injective everywhere, we use the commutative diagram of smooth maps

and fixing $p \in \mathbb{R}$ we get a commutative diagram of linear maps


The vertical maps are isomorphisms by a previous example, and the top map is injective. Hence the bottom map is also injective.

This gives an example of many embeddings of circles into $\mathbb{T}^{2}$, one in each homotopy class $(m, n) \in \mathbb{Z}^{2}=\pi_{1}\left(\mathbb{T}^{2}\right)$ which $\operatorname{gcd}(m, n)=1$. These are the only elements of the fundamental group which can be represented by embeddings (if we use the convention $\operatorname{gcd}(0,0)=1)$ [Rol90, Theorem 2.C.2].

Proposition 6.2.9. A subset $X \subset M$ is a submanifold if and only if it is the image of an embedding.

Proof. For $\Leftarrow$, observe that we can use the local charts provided by Lemma 6.2.3 to make $e(X)$ a submanifold. For $\Rightarrow$, it suffices to prove that the inclusion $\iota: X \hookrightarrow M$ is an embedding. It is visibly a homeomorphism onto its image, and by computing locally in the charts provided by the fact $\iota$ is an immersion, we see that its differential $d \iota$ is injective everywhere.

In the proof of Theorem 6.2.9, the charts used to make $e(X)$ into a submanifold exhibit $e: X \rightarrow e(X)$ as a bijective smooth map which has bijective differential at all $x \in X$. By Theorem 6.1.3, $e$ is not just a homeomorphism onto its image but a diffeomorphism. Let us record this:

Corollary 6.2.10. If $e: X \hookrightarrow M$ is an embedding then it is a diffeomorphism onto its image.

Let us discuss furter the condition that an embedding is homeomorphism onto its image. If the domain of an injective immersion $X \hookrightarrow M$ is compact, it restricts to a continuous bijection $X \rightarrow \operatorname{im}(X)$ of compact Hausdorff spaces and hence is a homeomorphism onto its image. If the domain is not compact, we can instead add the following condition:

Definition 6.2.11. A continuous $f: X \rightarrow Y$ is proper if $f^{-1}(K) \subset X$ is compact for all compact $K \subset Y$.

Intuitively, a proper map is one that "maps infinity to infinity." In particular, none of the examples with domain $\mathbb{A}^{2}$ or $\mathbb{A}^{3}$ in Chapter 1 is proper. One way to see this is to recall that proper maps between locally compact Hausdorff spaces are closed. This shows that embeddings need not be proper.

Theorem 6.2.12. A proper injective immersion is an embedding.
Proof. It suffices to prove that if $e: X \rightarrow M$ is an proper injective immersion then it is a homeomorphism onto its image.

Since $e$ is presumed continuous and injective, we will use properness to deduce that $e$ is open. Thus we need to show that if $W$ is open in $X$ then $e(W)$ open in $e(X)$. We will do so by contradiction, and hence suppose there is a sequence $y_{1}, y_{2}, \ldots$ in $e(X)$ but not in $e(W)$, and converging to $y \in e(W)$. As $\left\{y, y_{1}, y_{2}, \ldots\right\}$ is compact in $M$, so is its inverse image in $X$ because $e$ is proper. Thus it has an accumulation point, and by passing to a subsequence we may assume that the $e^{-1}\left(y_{i}\right)$ converge to some $z \in X$. Then $e\left(e^{-1}\left(y_{i}\right)\right)$ converges both to $y \in e(W)$ and $e(z) \in e(X)$ so $y=e(z)$ and by injectivity of $e$ thus $e^{-1}(y)=z$. But since $W$ is open in $X$ this means that $e^{-1}\left(y_{i}\right) \in W$ for $i$ large enough, contradicting $y_{i} \notin e(W)$.

Corollary 6.2.13. An injective immersion with compact domain is an embedding.
Proposition 6.2.14. A closed subset $X$ is a submanifold if and only if the image of a proper embedding.

Proof. For $\Leftarrow$, use that proper maps are closed. For $\Rightarrow$, suppose that $K \subset M$ is compact and $\left\{U_{i}\right\}$ is an open cover of $\iota^{-1}(K)$. Then there exists an open cover $\left\{\tilde{U}_{i}\right\}$ of $X \cap K \subset M$ and since $X \cap K$ is closed inside a compact it is compact, and there is a finite subcover $\tilde{U}_{1}, \ldots, \tilde{U}_{n}$. The corresponding open subsets $U_{1}, \ldots, U_{n}$ are finite subcover of $\iota^{-1}(K)$ in $X$.

### 6.3 Globalizing the submersion theorem

We can similarly globalize the submersion theorem [DK04a, Section 4.5].
Theorem 6.3.1 (Submersion theorem). Let $U_{0} \subset \mathbb{R}^{k}$ be open and $a \in U_{0}$. Suppose we have a smooth map $g: U_{0} \rightarrow \mathbb{R}^{k^{\prime}}$ such that the total derivative $D g_{a}$ of $g$ at a is a surjective linear map. Then $k^{\prime} \leq k$ and there exist open neighborhoods $U \subset U_{0}$ of a and $V \subset \mathbb{R}^{k^{\prime}}$ of $g(a)$ and diffeomorphisms $\phi: \mathbb{R}^{k} \rightarrow U$ and $\phi^{\prime}: \mathbb{R}^{k^{\prime}} \rightarrow V$ such that
(i) $\psi(0)=a$,
(ii) $\varphi(0)=g(a)$, and
(iii) the following diagram commutes

with $\pi_{k^{\prime}}$ the projection $\left(x_{1}, \ldots, x_{k}\right) \mapsto\left(x_{1}, \ldots, x_{k^{\prime}}\right)$.

## Definition 6.3.2.

- A smooth map $f: M \rightarrow N$ is a submersion at $p$ if $d_{p} f: T_{p} M \rightarrow T_{f(p)} N$ is a surjective linear map.
- A smooth map $f: M \rightarrow N$ is a submersion if it is a submersion at all $p \in M$.

Lemma 6.3.3. Let $f: M \rightarrow N$ be a smooth map which is a submersion at $p$, with $M$ $k$-dimensional and $N k^{\prime}$-dimensional. Then $k^{\prime} \leq k$ and there exists a chart ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) of $M$ around $p$ and a chart $\left(U_{\alpha}^{\prime}, V_{\alpha}^{\prime}, \phi_{\alpha}^{\prime}\right)$ of $N$ around $f(p)$ so that the following diagram commutes

with $\pi_{k^{\prime}}$ the projection onto first $k^{\prime}$ coordinates.
Remark 6.3.4. Note that a linear map being a submersion is open condition, which is reflected in the above lemma by the observation that if $f$ looks like the standard projection in some coordinates at $p$, then it does so near $p$, namely on all of $\phi_{\alpha}\left(U_{\alpha}\right)$.

However, its main use is that if we denote $c:=f(p)$ it says that $f^{-1}(c)$ is a $\left(k-k^{\prime}\right)$ dimensional submanifold near $p$; in the charts it is just $U_{\alpha} \cap\left\{\left(0, \ldots, 0, x_{k^{\prime}+1}, \ldots, x_{k}\right)\right\}$. Furthermore, as in these chart the tangent spaces to this subset are given by the kernel of the derivative of $\pi_{k^{\prime}}$, the tangent space to $f^{-1}(c)$ at $p$ is given by the kernel of $d_{p} f$ when we identify it with a subspace of $T_{p} M$ using the derivative of the inclusion map $f^{-1}(c) \rightarrow M$.

This leads to the following definition and theorem:

Definition 6.3.5. Let $f: M \rightarrow N$ be a smooth map. Then a point $c \in N$ is called a regular value of $f$ if $f$ is a submersion at all $x \in f^{-1}(c)$.

Theorem 6.3.6 (Preimage theorem). If $f: M \rightarrow N$ is a smooth map and $c \in N a$ regular value, then $f^{-1}(c)$ is a $\left(k-k^{\prime}\right)$-dimensional submanifold of $M$ and $T_{p} f^{-1}(c)=$ $\operatorname{ker}\left(d_{p} f: T_{p} M \rightarrow T_{f(p)} M\right)$ for all $p \in f^{-1}(c)$.

It is often more convenient to remember not the dimension of $f^{-1}(c)$, but how much this is smaller than the dimension of $M$; this is the codimension and in the previous theorem $f^{-1}(c)$ has codimension $k^{\prime}$.

Example 6.3.7. If $f: M \rightarrow N$ is a submersion, then all points in $N$ are regular values.
It may also be helpful to name those points in $N$ that arenot regular values.

Definition 6.3.8. Let $f: M \rightarrow N$ be a smooth map. Then a point $c \in N$ is called a critical value of $f$ if it is not a regular value of $f$.

Example 6.3.9. The map $\mathbb{R}^{k} \rightarrow \mathbb{R}$ given by

$$
\left(x_{1}, \ldots, x_{k}\right) \longmapsto x_{1}^{2}+\ldots+x_{i}^{2}-x_{i+1}^{2}-\ldots-x_{k}^{2}
$$

has 0 has its only critical value; all other $t \in \mathbb{R}$ are regular values.

### 6.4 Problems

Problem 6.4.1 (Images of immersions). Are following subsets of $\mathbb{R}^{2}$ the image of an immersion and/or an embedding $\mathbb{R} \rightarrow \mathbb{R}^{2}$ (you should imagine them continuing indefinitely)? You need to explain your reasoning for each example, but do not need to give proofs.

(i)

(ii)

(iv)

Problem 6.4.2 (Submersions with compact domain).
(a) Suppose $f: M \rightarrow N$ is a submersion with $M$ a compact smooth manifold and $N$ a connected smooth manifold. Show that $f$ is surjective. (Hint: show that its image is both open and closed.)
(b) Show that there exists no submersion from a compact smooth manifold to a Euclidean space of positive dimension.

Problem 6.4.3 (Submersions, immersions, and smooth maps).
(a) Suppose that $f: M \rightarrow N$ is an immersion and $h: P \rightarrow M$ is a continuous map. Prove that $h$ is smooth if and only if $f \circ h$ is.
(b) Suppose that $f: M \rightarrow N$ is a surjective submersion and $g: N \rightarrow P$ is a continuous map. Prove that $g$ is smooth if and only if $g \circ f$ is.

Problem 6.4.4 (A family of surfaces). Prove that the subspace

$$
X=\left\{(x, y, z) \mid\left(x^{4}-x^{2}+y^{2}\right)^{2}+z^{2}=\epsilon\right\} \subset \mathbb{R}^{3}
$$

is a 2-dimensional smooth submanifold for $\epsilon>0$ sufficiently small. Sketch it. What happens when we increase $\epsilon$ ?

Problem 6.4.5 (Special orthogonal groups). Let $O(n) \subset \mathrm{GL}_{n}(\mathbb{R})$ be the subgroup of orthogonal matrices, i.e. $A$ such that $A^{t}=A^{-1}$. This is known as the orthogonal group.
(a) Using the submersion theorem to prove that $O(n)$ is a $\frac{1}{2} n(n-1)$-dimensional manifold.
(b) Prove that $O(n)$ is a Lie group.
(c) Show that $O(n)$ has two path components.

The path component $S O(n) \subset O(n)$ containing the identity is a Lie group known as the special orthogonal group.

Problem 6.4.6 (Some orthogonal Stiefel manifolds). Let $V_{2}\left(\mathbb{R}^{n}\right)$ be the subset of $\left(\mathbb{R}^{n}\right)^{2}$ of pairs $\left(v_{1}, v_{2}\right)$ of vectors such that $\left\|v_{1}\right\|^{2}=1=\left\|v_{2}\right\|^{2}$ and $v_{1} \cdot v_{2}=0$.
(a) Prove that $V_{2}\left(\mathbb{R}^{n}\right)$ is a smooth manifold.
(b) Prove that $V_{2}\left(\mathbb{R}^{3}\right)$ is diffeomorphic to the special orthogonal group $S O(3)$.
(c) Let $W_{n}$ be the subset of $\mathbb{C}^{n}$ of $n$-tuples $\left(z_{1}, \ldots, z_{n}\right)$ satisfying $z_{1}^{2}+\cdots+z_{n}^{2}=0$ and $\left|z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2}=2$. Prove that $W_{n}$ is a smooth manifold which is diffeomorphic to $V_{2}\left(\mathbb{R}^{n}\right)$.

Problem 6.4.7 (Configuration spaces in robotics). Fix an integer $n \geq 1$ and real numbers $r_{i}>0,1 \leq r \leq n$. We consider the space $C$ of configurations of a robot arm with $n$ segments of lengths $r_{1}, \ldots, r_{n}$. We take the attaching point of the arm as the origin, and for simplicity assume that the segments are constrained to move in the plane $\mathbb{R}^{2}$. That is, $C$ is the subspace of $\mathbb{C}^{n} \cong \mathbb{R}^{2 n}$ of points $\left(z_{1}, \ldots, z_{n}\right)$ such that $\left|z_{i}-z_{i-1}\right|=r_{i}$ for $1 \leq i \leq n$ (with the convention that $z_{0}=0$ ).
(a) Use the submersion theorem to show that $C$ is a submanifold of $\mathbb{C}^{n}$. What is its dimension?
(b) Show that $C$ is diffeomorphic to $\left(S^{1}\right)^{n}$.
(c) Is it still a submanifold when we add the requirement that the segments of the arm do not intersect outside the joints? That is, we take the subspace $D \subset C$ of those $\left(z_{1}, \ldots, z_{n}\right)$ such that for all $1 \leq i, j \leq n$ satisfying $i \neq j, j-1$ we have $z_{i} \notin\left\{t z_{j-1}+(1-t) z_{j} \mid t \in[0,1]\right\}$ (again with the convention that $z_{0}=0$ ). You have to explain your answer or give a counterexample, but do not need to give a full proof.


Figure 6.2 A point $\left(z_{1}, z_{2}\right)$ in $C$ for $n=2$, visualized as an arm with two segments.

Problem 6.4.8 (Embeddings between projective spaces). Prove that the following are smooth embeddings:
(a) The standard inclusion $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+2}$ induces a continuous map

$$
\begin{aligned}
i: \mathbb{R} P^{n} & \longrightarrow \mathbb{R} P^{n+1} \\
{\left[x_{0}: \cdots: x_{n}\right] } & \longmapsto\left[x_{0}: \cdots: x_{n}: 0\right] .
\end{aligned}
$$

(b) The Segre embedding is the continuous map

$$
\begin{aligned}
S: \mathbb{C} P^{1} \times \mathbb{C} P^{1} & \longrightarrow \mathbb{C} P^{3} \\
\left(\left[x_{0}: x_{1}\right],\left[y_{0}, y_{1}\right]\right) & \longmapsto\left(\left[x_{0} y_{0}: x_{1} y_{0}: x_{0} y_{1}: x_{1} y_{1}\right]\right)
\end{aligned}
$$

Generalize this to an embedding $\mathbb{C} P^{i} \times \mathbb{C} P^{j} \rightarrow \mathbb{C} P^{(i+1)(j+1)-1}$.
(c) Complexification $\mathbb{R}^{n} \rightarrow \mathbb{C}^{n}$ induces a continuous map

$$
\begin{gathered}
j: \mathbb{R} P^{n} \longrightarrow \mathbb{C} P^{n} \\
{\left[x_{0}: \ldots: x_{n}\right] \longmapsto\left[x_{0}: \ldots: x_{n}\right]}
\end{gathered}
$$

where the left hand side is an equivalence class of $(n+1)$ real numbers, which is considered as an equivalence of $(n+1)$ complex numbers on the right hand side.

## Chapter 7

## Quotients and coverings

In this chapter we discuss smooth manifolds which are evenly covered by another smooth manifold. Such covering maps often arise as quotients by discrete groups, and we end with a discussion of quotients by Lie groups.

### 7.1 Covering spaces

In point-set topology, there is a notion of a covering of one topological space by another. One should imagine many sheets of fabric covering a surface.

Definition 7.1.1. A continuous map $p: E \rightarrow B$ is a covering map if each point $b \in B$ has an open neighborhood $U$ such that $p^{-1}(U)$ can be written as a union $\bigsqcup_{i} V_{i}$ of disjoint open subsets of $E$, such that $\left.p\right|_{V_{i}}: V_{i} \rightarrow U$ is a homeomorphism for each $i$.

Example 7.1.2. Prototypical examples are

$$
\begin{aligned}
\mathbb{R} & \longrightarrow S^{1}=\{z \in \mathbb{C}| | z \mid=1\} \\
t & \longmapsto e^{2 \pi i t},
\end{aligned}
$$

where each $z \in S^{1}$ has infinite pre-image, and

$$
\begin{aligned}
S^{1} & \longrightarrow S^{1} \\
z & \longmapsto z^{n}
\end{aligned}
$$

where each $z \in S^{1}$ has exactly $n$ pre-images.
Is a cover of a smooth manifold again a smooth manifold? If $p: E \rightarrow B$ is a covering map and $B$ is Hausdorff or locally Euclidean then so is $E$, and $E$ is second-countable when $B$ is and $p$ has countable fibers. Thus $E$ satisfies all the point-set topological properties necessary for being a smooth manifold. It remains to lift the smooth structure on $B$ to one on $E$ :

Theorem 7.1.3. If $p: E \rightarrow B$ is a covering map such that $p^{-1}(b)$ is countable for all $b \in B$ and $B$ is a $k$-dimensional smooth manifold, then there is a unique $k$-dimensional smooth structure on $E$ such that $p: E \rightarrow B$ is a local diffeomorphism.


Figure 7.1 A three-fold covering of $S^{1} \sqcup S^{1}$ by $S^{1}$.

Proof. Let us first take care of point-set topological requirements. We start by proving that $E$ is Hausdorff if $B$ is: $e \neq e^{\prime} \in E$ with $p(e) \neq p\left(e^{\prime}\right)$ can be separated by $p^{-1}(U)$ and $p^{-1}\left(U^{\prime}\right)$ where $U, U^{\prime} \subset B$ are disjoint open subsets such that $p(e) \in U, p\left(e^{\prime}\right) \in U^{\prime}$. If $e \neq e^{\prime} \in E$ but $p(e)=p\left(e^{\prime}\right)$, then they must lie in different $V_{i}$ 's and these open subsets separate them.

To see that $E$ is second countable, we first observe that the condition on $p^{-1}(b)$ implies that each disjoint union $\bigsqcup_{i} V_{i}$ as in Definition 7.1.1 is a countable one. Take $\left\{U_{j}\right\}$ a countable basis for the topology of $B$. By possible discarding some of the larger subsets, we may without loss of generality assume that $p^{-1}\left(U_{j}\right)$ is a countable union of open subsets $V_{j, i}$ of $E$ homeomorphic to $U_{i}$. The countable collection $\left\{V_{j, i}\right\}$ is a basis for the topology of $E$.

We shall give a chart around each $e \in E$ : pick $U$ around $b=p(e)$ as in the definition of a covering map, and a chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ around $b$ in $B$ such that $V_{\alpha} \subset U$. If $V_{i}$ is such that $e \in V_{i}$, then we produce a chart around $e$ by taking $U_{\alpha, i}^{\prime}=U_{\alpha}$, taking $V_{\alpha, i}^{\prime}=\left(\left.p\right|_{V_{i}}\right)^{-1}\left(V_{\alpha}\right)$ and setting $\phi_{\alpha, i}^{\prime}$ to be

$$
\phi_{\alpha, i}^{\prime}: \mathbb{R}^{k} \supset U_{\alpha} \xrightarrow{\phi_{\alpha}} V_{\alpha} \xrightarrow{\left(p| |_{V_{i}}\right)^{-1}} V_{\beta}^{\prime} \subset E .
$$

The transition function between $\left(U_{\alpha, i}^{\prime}, V_{\alpha, i}^{\prime}, \phi_{\alpha, i}^{\prime}\right)$ and $\left(U_{\beta, j}^{\prime}, V_{\beta, j}^{\prime}, \phi_{\beta, j}^{\prime}\right)$ is only non-trivial if $V_{\alpha, i}^{\prime} \cap V_{\beta, j}^{\prime} \neq \varnothing$ and then it lies in $V_{i}$. Thus we can write $\phi_{\alpha, i}^{\prime}=\left.p\right|_{V_{i}} ^{-1} \circ \phi_{\alpha}$ and $\phi_{\beta, j}^{\prime}=\left.p\right|_{V_{i}} ^{-1} \circ \phi_{\beta}$, and the transition function is a restriction of $\left(\left.p\right|_{V_{i}} ^{-1} \circ \phi_{\beta}\right)^{-1} \circ\left(\left.p\right|_{V_{i}} ^{-1} \circ \phi_{\alpha}\right)=\phi_{\beta}^{-1} \circ \phi_{\alpha}$ and hence smooth. This completes the construction of the smooth structure on $E$.

To see that $p$ is a local diffeomorphism with respect to this smooth structure, we use that with respect to coordinates given by the charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ and $\left(U_{\alpha, i}^{\prime}, V_{\alpha, i}^{\prime}, \phi_{\alpha, i}^{\prime}\right)$ it is the identity map between the equal open subsets $U_{\alpha, i}^{\prime}$ and $U_{\alpha}$ of $\mathbb{R}^{k}$.

To see that this smooth structure is uniquely determined by this property, we must prove that the identity map of $E$ is smooth with respect to any two smooth structures $\mathcal{A}_{1}, \mathcal{A}_{2}$ on $E$ such that $p: E \rightarrow B$ is a local diffeomorphism. It suffices to verify this
locally in $E$. The diagram

$$
\begin{aligned}
\left(V_{i},\left.\mathcal{A}_{1}\right|_{V_{i}}\right) \xrightarrow{\text { id }} & \left(V_{i},\left.\mathcal{A}_{2}\right|_{V_{i}}\right) \\
& \cong \uparrow\left(\left.p\right|_{V_{i}}\right)^{-1} \\
& \cong|p|_{V_{i}} \\
U_{i} \longrightarrow & \text { id } \\
& U_{i}
\end{aligned}
$$

evidently commutes, and we can think of the left map as a diffeomorphism with respect to $\left.\mathcal{A}_{1}\right|_{V_{i}}$, the right map as a diffeomorphism with respect to $\left.\mathcal{A}_{2}\right|_{V_{i}}$. Since the bottom map is smooth, the top map must also be smooth (e.g. as an application of Problem 6.4.3).

In fact, many local diffeomorphisms arise this way:
Proposition 7.1.4. Suppose $E$ and $B$ are smooth manifolds, and $p: E \rightarrow B$ is a smooth map whose derivative is bijective at all points in $E$. Then $p$ is a covering map if $E$ is compact.

Proof. The conditions imply that $E$ is a local diffeomorphism whose image is a collection of components of $B$ so we may as well assuming $p$ is surjective by discarding some components. For each $b \in B, p^{-1}(b)$ is a finite set and for each $e \in p^{-1}(b)$ the fact that $p$ is a local diffeomorphism gives us an open subset $V_{e}$ of $E$ containing $e$ such that $\left.p\right|_{V_{e}}: V_{e} \rightarrow p\left(V_{e}\right)$ is a diffeomorphism. Using the fact that $E$ is Hausdorff we may assume that the $V_{e}$ are pairwise disjoint. Then let $U=\bigcap_{e} p\left(V_{e}\right)$, which is an open neighborhood of $b$ because it is a finite intersection of open subsets containing $b$.

We claim that $p^{-1}(U)$ is a union of the disjoint open subsets $p^{-1}(U) \cap V_{e}$ of $E$, at least after shrinking $U$. If so, $\left.p\right|_{V_{e}}$ provides not just a homeomorphism $p^{-1}(U) \cap V_{e} \cong U$ but in fact a diffeomorphism and we would be done.

We will give a proof of the claim by contradiction: suppose that no matter how much we shrink $U$ it is always the case that $p^{-1}(U) \backslash \bigcup_{e} V_{e} \neq \varnothing$. Then there exists a sequence of $x_{i} \in E \backslash \bigcup_{e} V_{e}$ such that the $x_{i}$ converges to some $x \in E$ (since $E$ is compact) and the $p\left(x_{i}\right)$ converges to $b$. This means that $x \in p^{-1}(b)$, and hence $x_{i}$ lies in some $V_{e}$ for $i$ large enough. This gives a contradiction.

Example 7.1.5. The Lie group $\mathrm{SO}(n)$ has a path-connected double cover $\operatorname{Spin}(n)$ for $n \geq 3$. Proposition 7.1 .3 shows that $\operatorname{Spin}(n)$ has a unique smooth structure making $\operatorname{Spin}(n) \rightarrow \mathrm{SO}(n)$ a local diffeomorphism.

### 7.2 Quotients by discrete groups

Let us discuss a major source of examples of covering maps; as quotients of sufficiently nice group actions. Recall an action of a discrete group $G$ has to be continuous in the sense that the map

$$
\begin{aligned}
G \times X & \longrightarrow X \\
(g, x) & \longmapsto g x
\end{aligned}
$$

is continuous. This is equivalent to each map $g: X \rightarrow X$ being a homeomorphism.

Definition 7.2.1. Suppose a (discrete) group $G$ acts on a topological space $X$. It acts freely if $g x=x$ for some $x \in X$ implies $g=e$.

We would like a condition on a free action that guarantees the quotient map $q: X \rightarrow$ $X / G$ is a quotient map. The following strengthening of a free action will suffice:

Definition 7.2.2. Suppose a (discrete) group $G$ acts on a topological space $X$. We say this is a covering action if each $x \in X$ has an open neighborhood $U$ such that $g(U) \cap U \neq \varnothing$ if and only if $g=e$.

Lemma 7.2.3. If the action of $G$ on $X$ is a covering action, then the quotient map $q: X \rightarrow X / G$ is a covering map.

Proof. For $q(x) \in X / G$, take the image $q(U)$ in $X / G$ of $U$ in Definition 7.2.2. Then $q^{-1}(q(U))=\bigcup_{g} g U$ and this is a disjoint union because

$$
g U \cap h U \neq \varnothing \quad \Longleftrightarrow \quad h^{-1} g U \cap U \neq \varnothing
$$

and this implies $h^{-1} g=e$ so $g=h$. Furthermore, each $g U$ is open as $U$ is open and $g: X \rightarrow X$ is a homeomorphism. In particular we conclude that $q^{-1}(q(U))$ is open so $q(U)$ is open by definition of the quotient topology.

To see that the restriction of $q$ to a map $g U \rightarrow q(U)$ is a homeomorphism, we first observe that there is a commutative diagram

with horizontal map a homeomorphism. Hence it suffices to prove that this only for $g=e$. As $\left.q\right|_{U}: U \rightarrow q(U)$ is clearly a continuous bijection, it remains to see it is open. But for $W \subset U$ open, $\left.q\right|_{U}(W) \subset q(U)$ is open if and only if $q^{-1}\left(\left.q\right|_{U}(W)\right)$ is. Since $q^{-1}\left(\left.q\right|_{U}(W)\right)=\bigcup_{g} g W$ this is true.

It is clear from the definition of the quotient topology that $X / G$ is second-countable if $X$ is second-countable. However, it is not obvious that $X / G$ is again Hausdorff; this requires a stronger definition:

Definition 7.2.4. Suppose a (discrete) group $G$ acts on a topological space $X$. It acts properly if the map

$$
\begin{aligned}
G \times X & \longrightarrow X \times X \\
(g, x) & \longmapsto(x, g x)
\end{aligned}
$$

is proper, as in Definition 6.2.11.
Concretely the action being proper means that $g(K) \cap K \neq \varnothing$ for only finitely many $g \in G$ whenever $K \subset X$ is compact. For locally compact Hausdorff $X$, this is equivalent to the following:
(i) each $x \in X$ has an open neighborhood $U$ such that $g(U) \cap U \neq \varnothing$ for only finitely many $g \in G$, and
(ii) any two $x, x^{\prime} \in X$ in distinct orbits have open neighborhoods $U, U^{\prime}$ such that $g(U) \cap U^{\prime}=\varnothing$ for all $g \in G$.
Note that smooth manifolds are always locally compact and Hausdorff.
Example 7.2.5. $\mathbb{Z}^{n}$ acts freely and properly on $\mathbb{R}^{n}$ by translation.
Example 7.2.6. If $X$ is locally compact Hausdorff and $G$ is finite then $G$ acts freely and properly if and only if it acts freely. To see this, observe that latter implies that for each $x$ all elements $g x$ for $g \in G$ are distinct. Using the Hausdorff property we can find for each $g \in G$ an open subset $U_{g}$ around $g x$ with the property that $U_{g} \cap U_{h} \neq \varnothing$ if and only if $g=h$. Then $U:=\bigcap_{g} g^{-1}\left(U_{g}\right)$ is an open subset around $x$ which satisfies $g(U) \cap U \neq \varnothing$ if and only if $g=e$. A similar argument shows that any two $x, x^{\prime} \in X$ in distinct orbits have open neighborhoods $U, U^{\prime}$ such that $g(U) \cap U^{\prime}=\varnothing$ for all $g \in G$.

If the action is free in addition to being proper and $X$ is still locally compact Hausdorff, we can shrink $U$ as in (i) and get that each $x \in X$ has an open neighborhood $U$ such that $g(U) \cap U \neq \varnothing$ if and only if $g=e$. Thus it will be a covering action as in Definition 7.2.2.

Lemma 7.2.7. If $G$ acts properly and freely on a locally compact Hausdorff space $X$, then $q: X \rightarrow X / G$ is a covering map and $X / G$ is Hausdorff.

Proof. Lemma 7.2 .3 says that the quotient map $q: X \rightarrow X / G$ is a covering map. It thus suffice to prove the quotient is Hausdorff. For two distinct orbits $[x],\left[x^{\prime}\right] \in X / G$ we can take two representatives and $U, U^{\prime}$ as in (ii). Then $q^{-1}(q(U))=\bigcup_{g \in G} g U$ and $q^{-1}\left(q\left(U^{\prime}\right)\right)=\bigcup_{g \in G} g U^{\prime}$ are disjoint and open, so $q(U)$ and $q\left(U^{\prime}\right)$ are open sets separating $[x]$ and $\left[x^{\prime}\right]$.

If $X / G$ happens to be a smooth manifold, this gives a smooth structure on $X$. We now want to go the other direction, taking $X$ to be a smooth manifold $M$ and assuming that the action is compatible with the smooth structure in the following sense:

Definition 7.2.8. We say that a group $G$ acts smoothly on a smooth manifold $M$ if the action map

$$
G \times M \longrightarrow M
$$

is smooth.

As $G$ is discrete, this is equivalent to each $g: M \rightarrow M$ being a diffeomorphism. It is also equivalent to the map

$$
\begin{aligned}
G \times M & \longrightarrow M \times M \\
(g, m) & \longmapsto(m, g m)
\end{aligned}
$$

being smooth, by Remark 4.1.13.

Theorem 7.2.9. If a discrete group $G$ acts freely, properly, and smoothly on a $k$ dimensional smooth manifold $M$, then there is a unique $k$-dimensional smooth structure on $M / G$ such that $q: M \rightarrow M / G$ is a local diffeomorphism.
Proof. We know from Lemma 7.2.7 that $q$ is a covering map, and that $M / G$ is Hausdorff and second countable. We next produce a smooth atlas on $M / G$. Let us take for each orbit $[p] \in M / G$ an open neighborhood $U$ as in Definition 7.1.1, so that $q^{-1}(U)=\bigsqcup_{i} V_{i}$. Let us also take charts $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ such that $V_{\alpha} \subset V_{i}$ for some $i$ and $[p] \in q\left(V_{\alpha}\right)$. The charts in our atlas for $M / G$ are then given by the ( $\left.U_{\alpha}, q\left(V_{\alpha}\right),\left.q\right|_{V_{i}} \circ \phi_{\alpha}\right)$.

The transition function between ( $U_{\alpha}, q\left(V_{\alpha}\right),\left.q\right|_{V_{i}} \circ \phi_{\alpha}$ ) an ( $U_{\beta}, q\left(V_{\beta}\right), q_{V_{j}} \circ \phi_{\beta}$ ) has non-empty domain and target if and only if $q\left(V_{\alpha}\right) \cap q\left(V_{\beta}\right) \neq \varnothing$, which happens only if $V_{\alpha} \cap q^{-1}\left(q\left(V_{\alpha}\right) \cap q\left(V_{\beta}\right)\right) \subset V_{i}$ and $V_{\beta} \cap q^{-1}\left(q\left(V_{\alpha}\right) \cap q\left(V_{\beta}\right)\right) \subset g\left(V_{i}\right)$ for some $g \in G$. Hence for the sake of computing transition functions we may replace $\left.q\right|_{V_{j}}$ by $\left.q\right|_{g\left(V_{i}\right)}$. Then the transition function is given by

$$
\left(\left.q\right|_{g\left(V_{i}\right)} \circ \phi_{\beta}\right)^{-1} \circ\left(\left.q\right|_{V_{i}} \circ \phi_{\alpha}\right)=\phi_{\beta}^{-1} \circ g \circ \phi_{\alpha},
$$

which is smooth by the assumption that $g: M \rightarrow M$ is a diffeomorphism. This completes the construction of the smooth structure on $M / G$.

To see $q$ is a local diffeomorphism with respect to this smooth structure, we use that in to the coordinates given by the chart ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) and ( $U_{\alpha}, q\left(V_{\alpha}\right), q \circ \phi_{\alpha}$ ) it is the identity map of $U_{\alpha} \subset \mathbb{R}^{k}$.

To see that this smooth structure is uniquely determined by this property, we must prove that the identity map of $M / G$ is smooth with respect to any two smooth structures $\mathcal{A}_{1}, \mathcal{A}_{2}$ on $M / G$ such that $q: M \rightarrow M / G$ is a local diffeomorphism. The diagram

evidently commutes, and we can think of the left map as a local diffeomorphism with respect to $\mathcal{A}_{1}$, of the right map as a local diffeomorphism with respect to $\mathcal{A}_{2}$. Since the top map is smooth, the top-right composite is. Since $q$ is a submersion, the bottom map must also be smooth as a application of Problem 6.4.3.

Example 7.2.10. Since $\mathbb{Z}^{n}$ acts freely, properly, and smoothly on $\mathbb{R}^{n}$ by translation, Theorem 7.2.9 gives another way to construct the smooth structure on the $n$-torus $\mathbb{T}^{n}=\mathbb{R}^{n} / \mathbb{Z}^{n}$.
Example 7.2.11. Fix two coprime integers $p$ and $q$. Let $\mathbb{Z} / p$ act on $S^{3}=\left\{\left(z_{1}, z_{2}\right) \mid\right.$ $\left.\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}=1\right\} \subset \mathbb{C}^{2}$ by

$$
k \cdot\left(z_{1}, z_{2}\right)=\left(e^{2 \pi i k / p} z_{1}, e^{2 \pi i q k / p} z_{2}\right)
$$

This is a free smooth action of the finite group $\mathbb{Z} / p$ on the 3 -dimensional smooth manifold $S^{3}$, so by Theorem 7.2.9, $L(p, q):=S^{3} /(\mathbb{Z} / p)$ is again a 3 -dimensional smooth manifold. These are lens spaces. As an example, let us take $L(2,1)$. This is the quotient of $S^{3}$ by the equivalence relation generated by $\left(z_{1}, z_{2}\right) \sim\left(-z_{1},-z_{2}\right)$, so is diffeomorphic to $\mathbb{R} P^{3}$.

Example 7.2.12. Define the configuration space of $n$ ordered particles in a manifold $M$ as

$$
\operatorname{Conf}_{n}(M):=\left\{\left(m_{1}, \ldots, m_{n}\right) \in M^{n} \mid m_{i} \neq m_{j} \text { if } i \neq j\right\} .
$$

As an open subset of a finite product of manifolds, this has a canonical smooth structure. The permutation action on $M^{n}$ by the symmetric group $\mathfrak{S}_{n}$ is proper and smooth, but not free. The subset $\operatorname{Conf}_{n}(M)$ exactly consists of all free orbits, so the restriction of this action to $\operatorname{Conf}_{n}(M)$ is smooth, proper, and free. Thus the configuration space of $n$ unordered particles

$$
C_{n}(M):=\operatorname{Conf}_{n}(M) / \mathfrak{S}_{n}
$$

again has a canonical smooth structure.

### 7.3 Quotients by Lie groups

Above we gave conditions on an action of a discrete group $G$ on a smooth manifold $M$, so that the quotient $M / G$ is again a smooth manifold. What can we say if we instead we take $G$ to be a Lie group? The definitions, when phrased correctly, go through without modification: as before, we say that $G$ acts smoothly on $M$ if the map

$$
\begin{aligned}
G \times M & \longrightarrow M \times M \\
(g, m) & \longmapsto(m, g m)
\end{aligned}
$$

is smooth, it acts properly if this map is proper, and acts freely if the action is free. A generalization of Theorem 7.2.9 to Lie groups is the following, which we shall not prove [Lee13, Theorem 21.10]:

Theorem 7.3.1. If a Lie group $G$ of dimension $r$ acts freely, properly, and smoothly on a $k$-dimensional smooth manifold $M$, then there is a unique $(k-r)$-dimensional smooth structure on $M / G$ such that $q: M \rightarrow M / G$ is a submersion.

Example 7.3.2 (Complex projective space as quotients). The Lie group $\mathbb{C}^{\times}$of non-zero complex numbers under multiplication acts freely, properly, and smoothly on $\mathbb{C}^{n} \backslash\{0\}$. Thus

$$
\mathbb{C} P^{n}=\left(\mathbb{C}^{n} \backslash\{0\}\right) / \mathbb{C}^{\times}
$$

is a smooth manifold, giving a construction of the complex projective plane without having to give charts by hand.

In many application we fix a Lie group $G$, as well as a Lie subgroup $H \subset G$, which is a subgroup which is also a smooth submanifold. It is evident that the action of $H$ on $G$ by multiplication is smooth and free. Furthermore, as $H$ must be closed [Lee13, Corollary 15.30] it follows that the action is proper. The above theorem says that $G / H$ is a smooth manifold and the quotient map

$$
G \longrightarrow G / H
$$

is a submersion.

Example 7.3.3 (Orthogonal Stiefel manifolds). We identify $O(n-2)$ be the subgroup of $O(n)$ as

$$
O(n-2) \ni A \longmapsto\left[\begin{array}{cc}
A & 0 \\
0 & \mathrm{id}_{2}
\end{array}\right]
$$

which is also the subgroup which fixes the vectors $e_{n-1}, e_{n}$. Then

$$
V_{2}\left(\mathbb{R}^{n}\right):=O(n) / O(n-2)
$$

is a smooth manifold, whose points are given by a pair of orthogonal vectors of length 1 . We saw these manifolds before in Problem 6.4.6. Replacing $n-2$ by $n-r$, we obtain the orthogonal Stiefel manifold

$$
V_{r}\left(\mathbb{R}^{n}\right):=O(n) / O(n-r)
$$

of orthogonal frames of $r$ vectors in $\mathbb{R}^{n}$. Replacing orthogonal groups by general linear groups we similarly obtain ordinary Stiefel manifolds.

### 7.4 Problems

Problem 7.4.1 (Higher-dimensional lens spaces). Fix an integer $p$ and integers $q_{1}, \ldots, q_{n}$ coprime to $p$. The higher-dimensional lens space $L\left(p, q_{1}, \ldots, q_{n}\right)$ is the quotient of $S^{2 n-1}=\left\{\left.\left(z_{1}, \ldots, z_{n}\right)| | z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2}=1\right\} \subset \mathbb{C}^{n}$ by the smooth action

$$
k \cdot\left(z_{1}, \ldots, z_{n}\right)=\left(e^{2 \pi i q_{1} k / p} z_{1}, \ldots, e^{2 \pi i q_{n} k / p} z_{n}\right) .
$$

Prove this admits a unique smooth structure such that the quotient map $q: S^{2 n-1} \rightarrow$ $L\left(p, q_{1}, \ldots, q_{n}\right)$ is a local diffeomorphism.

Problem 7.4.2 (Dold manifolds). Let $\mathbb{Z} / 2$ act on $S^{m} \times \mathbb{C} P^{n}$ by multiplication by -1 on $S^{m}$ and by complex conjugation on $\mathbb{C} P^{n}$. Prove that

$$
D(m, n):=\left(S^{m} \times \mathbb{C} P^{n}\right) / \mathbb{Z} / 2
$$

is a smooth manifold. This is called a Dold manifold.
Problem 7.4.3 (Orthogonal Grassmannians). We can $O(r) \times O(n-r)$ with a subgroup of $O(n)$ by

$$
O(r) \times O(n-r) \ni(A, B) \longmapsto\left[\begin{array}{cc}
A & 0 \\
0 & B
\end{array}\right] \in O(n) .
$$

(a) Show that the quotient

$$
\operatorname{Gr}_{r}\left(\mathbb{R}^{n}\right):=O(n) /(O(r) \times O(n-r))
$$

is a smooth manifold.
(b) Use Gram-Schmidt to explain why we can think of $\operatorname{Gr}_{r}\left(\mathbb{R}^{n}\right)$ as a smooth manifold of $r$-dimensional linear subspaces of $\mathbb{R}^{n}$.

The smooth manifold $\operatorname{Gr}_{r}\left(\mathbb{R}^{n}\right)$ is called the orthogonal Grassmannian of $r$-planes in $\mathbb{R}^{n}$.
Problem 7.4.4 (Complex polynomials without double roots). Let $X_{n} \subset \mathbb{C}^{n}$ be the subset of those $\left(a_{0}, \ldots, a_{n-1}\right)$ such that $p(z)=a_{0}+a_{1} z+\cdots+a_{n-1} z^{n-1}+z^{n}$ has only has roots with multiplicity one.
(a) Prove that $X_{n}$ is a smooth manifold of dimension $2 n$.
(b) Prove that $X_{n}$ is diffeomorphic to the configuration space

$$
C_{n}(\mathbb{C})=\left\{\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n} \mid z_{i} \neq z_{j} \text { if } i \neq j\right\} / \mathfrak{S}_{n}
$$

of $n$ unordered particles, where the symmetric group $\mathfrak{S}_{n}$ acts by $\sigma \cdot\left(z_{1}, \ldots, z_{n}\right)=$ $\left(z_{\sigma(1)}, \ldots, z_{\sigma(n)}\right)$. (Hint: fundamental theorem of algebra.)
(c) Is the subspace of $X_{n}$ of polynomials which have at least one real root always a smooth submanifold? Give a proof or give a counterexample.

## Chapter 8

## Three further examples of manifolds

In this chapter we describe three more manifolds, each interesting and an example of a more general construction.

### 8.1 The Poincaré homology sphere

We start with one of the first manifolds ever described, due to Poincaré. For more constructions, see [KS79].

## The quaternions

Our construction starts with the quaternions $\mathbb{H}$. These are an associative $\mathbb{R}$-algebra, generated as an $\mathbb{R}$-vector space by elements $1, i, j, k$ which satisfy the relations

$$
\begin{gathered}
i^{2}=j^{2}=k^{2}=-1, \quad i j=-j i, \quad i k=-k i, j k=-k j \\
i j=k, \quad j k=i, k i=j
\end{gathered}
$$

This is visibly not commutative, e.g. $i j=k$ but $j i=-k$. The elements which commute with every other element, the center, is given by $\mathbb{R} \cdot 1$. As a $\mathbb{R}$-vector space, it is 4 -dimensional, with a basis given by $1, i, j, k$.

This is a so-called division algebra, which means that algebraically it behaves like a non-commutative four-dimensional version of the complex numbers. Firstly, the quaternions have a conjugation operation

$$
\overline{a+b i+c j+d k}:=a-b i-d j-c k
$$

Lemma 8.1.1. Conjugation is linear and an antihomomorphism, i.e. satisfies $\overline{x y}=\overline{y x}$.
In terms of this, we define $\|x\|^{2}:=x \bar{x}$. Explicitly, this is given by

$$
\|a+b i+c j+d k\|:=\sqrt{a^{2}+b^{2}+c^{2}+d^{2}}
$$

and hence is visibly a norm (in fact the usual Euclidean one).
Every non-zero element of $\mathbb{H}$ has a unique multiplicative inverse, which can be written in terms of the conjugation and norm

$$
x^{-1}=\frac{\bar{x}}{\|x\|^{2}}
$$

## The 3 -sphere as a Lie group

The subset $S^{3} \subset \mathbb{H}$ of quaternions with norm 1 is a smooth manifold; it is just the subspace $\left\{a+b i+c j+d k \mid a^{2}+b^{2}+c^{2}+d^{2}=1\right\} \subset \mathbb{H}$. The multiplication and inversion of $\mathbb{H}$ restrict to $S^{3}$. This uses the following lemma:

Lemma 8.1.2. $\|x y\|=\|x|\|| | y\|$
Proof. Since the conjugation is an anti-homomorphism, we have

$$
\|x y\|^{2}=x y \overline{x y}=x y \bar{y} \bar{x}=\|x\|^{2}\|y\|^{2} .
$$

This exactly says that the product of two elements of norm 1 has norm 1. It also implies that the inverse of an element of norm 1 has norm 1: more generally, if $x \neq 0$ we have

$$
1=\|1\|=\left\|x x^{-1}\right\|=\|x\|\left\|x^{-1}\right\|
$$

so $\left\|x^{-1}\right\|=\|x\|^{-1}$.
To see that both multiplication and inverse are smooth maps on $\mathbb{H} \backslash\{0\}$, observe they are given by polynomials in $a, b, c, d$. In fact, inverse is particularly easy: $g^{-1}=\bar{g}$. Hence their restriction to the submanifold $S^{3}$ is also smooth, and we conclude that $S^{3}$ is a Lie group.

Remark 8.1.3. $S^{1}$ and $S^{3}$ are the only spheres that admit the structure of a Lie group.
Example 8.1.4. In fact, this is isomorphic to the Lie group $S U(2)$ of unitary $(2 \times 2)$ matrices with complex entries and determinant 1 . The correspondence is given by thinking of a quaternion $a+b i+c j+d k \in \mathbb{H}$, on which $S^{3}$ acts, as a pair $(a+b i, c+d i)$ of complex numbers, on which $S U(2)$ acts. Explicitly, the isomorphism of Lie groups is given by

$$
S^{3} \ni a+b i+c j+d k \longmapsto\left[\begin{array}{cc}
a+b i & c+d i \\
-c+d i & -a-b i
\end{array}\right] \in S U(2)
$$

The Poincaré homology sphere via the binary icosahedral group
It follows from Theorem 7.2 .9 that if $G \subset S^{3}$ is a finite subgroup, $S^{3} / G$ admits a 3 -dimensional smooth structure such that the quotient map

$$
S^{3} \longrightarrow S^{3} / G
$$

is a local diffeomorphism.
Example 8.1.5. Taking $G=\{ \pm 1\}$, we obtain $S^{3} /\{ \pm 1\}=\mathbb{R} P^{3}$.
Our next goal is construct a particular rather large finite subgroup of $S^{3}$. The first observation is that for $g \in S^{3}$ the conjugation

$$
S^{3} \ni h \longmapsto g h g^{-1} \in S^{3}
$$

preserves the subset of quaternions of the form $b i+c j+d k$.


Figure 8.1 Haeckel's "Fig. 1: Circogonia icosahedra, n. sp., $\times 80$. The entire shell, with twelve radial tubes and twenty triangular faces. In the centre of one face is the mouth, with six teeth." (from https://en.wikisource.org/wiki/Report_on_the_Radiolaria/Plates12\# /media/File:Radiolaria_(Challenger)_Plate_117.jpg.

We can identify this subset with $\mathbb{R}^{3}$ through $b i+c j+d k \longleftrightarrow(b, c, d)$. Under this identification the norm on $\mathbb{H}$ corresponds to the Euclidean norm, and thus we get an action of $S^{3}$ on $\mathbb{R}^{3}$ which is orthogonal. The resulting homomorphism $S^{3} \rightarrow S O(3)$ has kernel of order 2 . That the kernel has order at least 2 is easy to see: both $x,-x \in \mathbb{H}$ map to the same linear transformation. We leave it as an exercise to the reader that there are no further elements in the kernel.

Let the icosahedral group $I \subset S O(3)$ be the subgroup of symmetries of the icosahedron, and let $I^{*}$ be its inverse image in $S^{3}$. $I^{*}$ has order 120 . The quotient manifold is the Poincaré homology sphere:

$$
P:=S^{3} / I^{*} .
$$

Remark 8.1.6. Why is the Poincaré homology sphere interesting? As you might expect, it was first constructed by Poincaré, though he did not construct it this way. Poincaré produced it as a counterexample to the first version of the Poincaré conjecture: it has the same homology as a 3 -sphere, but it is not homeomorphic to $S^{3}$ because it
has fundamental group isomorphic to $I^{*}$. The correct Poincaré conjecture says that a 3-dimensional differentiable manifold that is homotopy equivalent to $S^{3}$ is diffeomorphic to it. This was eventually proven by Perelman in a series of papers in 2002-2003, for which he received a Fields medal. ${ }^{1}$
Remark 8.1.7. For a while some scientists thought the cosmic microwave background radiation was most consistent with the universe having space-like direction $S^{3} / I^{*}$ instead of $\mathbb{R}^{3}$, though with the acquisition of more data this is no longer the case. ${ }^{2}$

### 8.2 The K3-manifold

Our second example come from algebraic geometry, and is a particular case of a general construction of a hypersurface in complex projective space.

Recall from Problem 2.3.6 the complex projective spaces $\mathbb{C} P^{k}$, defined as

$$
\mathbb{C} P^{k}=\left(\mathbb{C}^{k+1} \backslash\{0\}\right) / \sim
$$

where the equivalence relation $\sim$ is generated by $\left(z_{0}, \ldots, z_{k}\right) \sim\left(\lambda z_{0}, \ldots, \lambda z_{k}\right)$ for $\lambda \in$ $\mathbb{C} \backslash\{0\}$. In other words, we are taking the quotient of the free action of the non-zero invertible complex numbers $\mathbb{C}^{\times}$by scalar multiplication on $\mathbb{C}^{k+1} \backslash\{0\}$. We denote the equivalence class of $\left(z_{0}, \ldots, z_{k}\right)$ by $\left[z_{0}: \cdots: z_{k}\right]$. It is a $2 k$-dimensional smooth manifold, covered by the $k+1$ charts

$$
\begin{aligned}
\phi_{j}: \mathbb{C}^{k} & \longrightarrow \mathbb{C} P^{k} \\
\left(z_{1}, \ldots, z_{k}\right) & \longmapsto\left[z_{1}: \cdots: z_{j-1}: 1: z_{j}: \cdots: z_{k}\right] .
\end{aligned}
$$

The image $V_{j}$ of $\phi_{j}$ is given by $\left\{\left[z_{0}: \ldots: z_{k}\right] \mid z_{j} \neq 0\right\}$.
Suppose we are interested in subsets of $\mathbb{C} P^{n}$ given by points which satisfy some equation, e.g. $f\left(z_{0}, \ldots, z_{k}\right)=0$. Whether or not a point $\left[z_{0}: \ldots: z_{k}\right]$ satisfies this equation ought to be independent of the choice of representative, and one way to guarantee this is the case is to assume that $f$ is homogeneous:

$$
f\left(\lambda z_{0}, \ldots, \lambda z_{k}\right)=\lambda^{d} f\left(z_{0}, \ldots, z_{k}\right)
$$

for some $d \geq 1$. If so, if $f$ vanishes on all representatives of $\left[z_{0}, \ldots, z_{k}\right]$ when it vanishes on one of them.

We shall now restrict our attention to such $f$ which are polynomial, homogeneous polynomials. These are polynomials in $z_{0}, \ldots, z_{k}$ in which every term has the same total degree $d$.
Example 8.2.1. The polynomial $z_{0}^{2}+z_{1}^{2}$ of $z_{0}, z_{1}$ is homogeneous, but $z_{0}+z_{1}^{2}$ is not.
We now use the submersion theorem to answer the following question: when does the zero set of homogeneous polynomial describe a smooth submanifold of $\mathbb{C} P^{k}$ ?

[^6]Theorem 8.2.2 (Hypersurfaces in complex projective spaces). Let p be a homogeneous polynomial of $z_{0}, \ldots, z_{k}$ such that

$$
\left\{\left(z_{0}, \ldots, z_{k}\right) \mid p\left(z_{0}, \ldots, z_{k}\right)=0\right\} \cap \bigcap_{j=0}^{k}\left\{\left(z_{0}, \ldots, z_{k}\right) \left\lvert\, \frac{\partial}{\partial z_{j}} p\left(z_{0}, \ldots, z_{k}\right)=0\right.\right\}=\{0\}
$$

then the subspace

$$
\left\{\left[z_{0}: \ldots: z_{k}\right] \mid p\left(z_{0}, \ldots, z_{k}\right)=0\right\} \subset \mathbb{C} P^{k}
$$

is a $2 k-2)$-dimensional smooth submanifold.
This statement requires an explanation. We can identify the domain $\mathbb{C}^{k+1}$ with $\mathbb{R}^{2 k}$ by $z_{j} \longleftrightarrow x_{j}+i y_{j}$, and similarly identify the target $\mathbb{C}$ with $\mathbb{R}^{2}$. Then $p$ is not only differentiable as a function $\mathbb{R}^{2 k+2} \rightarrow \mathbb{R}^{2}$, is in fact complex-differentiable as a function $\mathbb{C}^{k+1} \rightarrow \mathbb{C}$. That is, for each $1 \leq i \leq k$ the limit $\frac{p\left(z_{0}, \ldots, z_{j}+h, \ldots, z_{k}\right)}{h}$ with as $\mathbb{C} \ni h \rightarrow 0$ exists, and these limits are the partial derivatives $\frac{\partial p}{\partial z_{j}}\left(z_{0}, \ldots, z_{k}\right)$.

Proof. Let us write $X:=\left\{\left[z_{0}: \ldots: z_{k}\right] \mid p\left(z_{0}, \ldots, z_{k}\right)=0\right\}$. If suffices to prove that $X \cap V_{j}$ is a smooth submanifold for all $0 \leq j \leq k$. To do so, we may pass to the local coordinates provided by the chart $\phi_{j}$, i.e. prove that $\phi_{j}^{-1}\left(X \cap V_{j}\right) \subset \mathbb{C}^{k}$ is a smooth submanifold. This is given by the vanishing set of the polynomial $q_{j}$ given by $p\left(z_{1}, \ldots, z_{j-1}, 1, z_{j}, \ldots, z_{k}\right)$ of the $k$ variables $z_{1}, \ldots, z_{k}$ (it is not homogeneous).

We now ought to identify the domain $\mathbb{C}^{k}$ with $\mathbb{R}^{2 k}$ and the target $\mathbb{C}$ with $\mathbb{R}^{2}$, and show that when $q_{j}\left(x_{1}+i y_{1}, \ldots, x_{k}+i y_{k}\right)=0$, the $(2 \times 2 k)$-matrix of partial derivatives of the real and imaginary part of $q_{j}$ with respect to $x_{1}, \ldots, x_{k}$ and $y_{1}, \ldots, y_{k}$ is surjective. However, it is more convenient not to leave the world of complex numbers, as $q_{j}$ is complex-differentiable with respect to the $k$ complex variables $z_{1}, \ldots, z_{k}$. In this case, we can form a $(1 \times k)$-matrix of complex numbers

$$
\left[\begin{array}{lll}
\frac{\partial q_{j}}{\partial z_{1}}\left(z_{1}, \ldots, z_{k}\right) & \cdots & \frac{\partial q_{j}}{\partial z_{k}}\left(z_{1}, \ldots, z_{k}\right) .
\end{array}\right]
$$

This is surjective if and only if the $(2 \times 2 k)$-matrix with real entries mentioned before is surjective.

Thus the condition is that when $q_{j}$ vanishes, at least one of the partial derivatives of $q_{j}$ does not vanish. We will get a contradiction with the hypothesis from the assumption that $q_{j}$ and all its partial derivatives vanish simultaneously. We start by relating these vanishing for $q_{j}$ and partial derivatives back to $p$ :

$$
\begin{aligned}
q_{j} \text { vanishes at }\left(z_{1}, \ldots, z_{k}\right) & \Longleftrightarrow p \text { vanishes at }\left(z_{1}, \ldots, z_{j-1}, 1, z_{j}, \ldots, z_{k}\right), \\
\frac{\partial q_{j}}{\partial z_{r}} \text { vanishes at }\left(z_{1}, \ldots, z_{k}\right) & \Longleftrightarrow \frac{\partial p}{\partial z_{r^{\prime}}} \text { vanishes at }\left(z_{1}, \ldots, z_{j-1}, 1, z_{j}, \ldots, z_{k}\right),
\end{aligned}
$$

with $r^{\prime}=r$ if $r<j$ and $r^{\prime}=r+1$ if $r \geq j$. This gives us information about all partial derivatives except $\frac{\partial p}{\partial z_{j}}$.

To understand this remaining partial derivative, we use a fact due to Euler:

$$
\begin{equation*}
\sum_{j=0}^{k} \frac{\partial p}{\partial z_{j}}\left(z_{0}, \ldots, z_{k}\right) \cdot z_{j}=d \cdot p\left(z_{0}, \ldots, z_{k}\right) \tag{8.1}
\end{equation*}
$$

with $d$ the degree of $p$. To prove this, consider the function $p\left(\lambda z_{0}, \ldots, \lambda z_{k}\right)-\lambda^{d} p\left(z_{0}, \ldots, z_{k}\right)$. This vanishes identically because $p$ is homogeneous of degree $d$, hence so its derivative with respect to $\lambda$. Evaluating this derivative at $\lambda=1$ gives (8.1). If we use this at the point $\left(z_{1}, \ldots, z_{j-1}, 1, z_{j}, \ldots, z_{k}\right)$, we know the right hand side vanishes as do all terms on the left hand side expect one. We get that

$$
\frac{\partial p}{\partial z_{j}}\left(z_{1}, \ldots, z_{j-1}, 1, z_{j}, \ldots, z_{k}\right)=0
$$

which contradicts the hypothesis. This completes the proof.
Remark 8.2.3. Implicitly we used the complex version of the submersion theorem, [DK04a, Section 3.7].

A smooth manifold obtained as in Theorem 8.2.2 is called a hypersurface. The example which plays such an important role in algebraic geometry is the K3-manifold, ${ }^{3}$ also known as the Fermat quartic. It is obtained by taking the homogeneous polynomial $p$ given by $z_{0}^{4}+z_{1}^{4}+z_{2}^{4}+z_{3}^{4}$ :

$$
\left.K 3:=\left\{\left[z_{0}: \cdots: z_{3}\right]\right) \mid z_{0}^{4}+z_{1}^{4}+z_{2}^{4}+z_{3}^{4}=0\right\} \subset \mathbb{C} P^{3}
$$

It is easy to verify that the polynomial $p$ satisfies the conditions in Theorem 8.2.2, so this is a 4 -dimensional smooth manifold: if $\frac{\partial}{\partial z_{j}} p\left(z_{0}, z_{1}, z_{2}, z_{3}\right)=0$ then $z_{j}=0$, so all partial derivatives vanish simultaneously only at the origin.
Remark 8.2.4. Why is the K3 manifold interesting? It plays an important role in algebraic geometry and the study of 4-dimensional smooth manifolds.

When one does algebraic geometry over $\mathbb{C}$, out of a smooth $k$-dimensional variety one can extract a smooth $2 k$-dimensional manifold ("taking the analytic topology"). In particular, the K3 manifold can be obtained this way from not one but many algebraic surfaces. There are roughly three types of algebraic surfaces: Fano surfaces (which are "easy"), surfaces of general type (which are "hard"), and Calabi-Yau surfaces (which are "intermediate"). The latter class contains only complex 2-dimensional tori and the K3 surfaces, and all K3 surfaces have the same underlying 4-dimensional smooth manifold: the K3 manifold that we constructed above.

Because it has an algebraic origin, the gauge-theoretic invariants used to study exotic smooth structures on smooth 4-manifolds can be computed for $K 3$ using more algebraic approaches. This gives a starting point for constructing exotic smooth 4-manifolds: start with $K 3$, make a modification to it, and study how this changes the gauge-theoretic invariants.

### 8.3 The Whitehead manifold

Our final example is quite peculiar. It is an example of a 3-dimensional smooth manifold which from the perspective of algebraic topology looks like $\mathbb{R}^{3}$, but is not in

[^7]fact diffeomorphic to it. It is an example of infinite phenomena leading to pathological objects in differential topology.

We start with the following injective immersion $S^{1} \times \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$ of an open torus. Let us denote its complement in $\mathbb{R}^{3}$ by $W_{1}$. This contains another, curiously linked, open torus; its complement in $\mathbb{R}^{3}$ is denoted by $W_{2}$. We can keep iterating this procedure, finding a linked copy of $S^{1} \times \mathbb{R}^{2}$ in the previous copy of $S^{1} \times \mathbb{R}^{2}$, and denoting its complement by $W_{n}$.

The Whitehead manifold is then defined to be increasing union

$$
W:=\bigcup_{n} W_{n}
$$

This is an open subset of $\mathbb{R}^{3}$ and hence a smooth 3 -dimensional manifold. It is the complement of the intersection of all the linked open tori, which is known as the Whitehead continuum.
Remark 8.3.1. Why is the Whitehead manifold interesting? The Whitehead manifold is a contractible 3-dimensional smooth manifold which is not diffeomorphic or even homeomorphic to $\mathbb{R}^{3}$. (Surprisingly, it is homeomorpic to a union of two copies of $\mathbb{R}^{3}$ intersecting in another copy of $\mathbb{R}^{3}$ [Gab11].)

The reason is that being contractible does not take into account the "topology at infinity," i.e. how $W \backslash K_{n}$ behaves as for a sequence $K_{n}$ of compact codimension 0 submanifolds exhausting $W$. This is a general phenomenon: if you want to use algebraic topology to study non-compact manifolds you need to take into account the topology at infinity.

### 8.4 Problems

Problem 8.4.1 (Klein quartic). Prove that the subspace

$$
X=\left\{[x: y: z] \in \mathbb{C} P^{2} \mid x^{3} y+y^{3} z+z^{3} x=0\right\} \subset \mathbb{C} P^{2}
$$

is a 2-dimensional compact submanifold. It is called the Klein quartic. What is its genus?
Problem 8.4.2 (Milnor manifolds). Let $m \leq n$. Prove that the subspaces

$$
H(m, n):=\left\{\left(\left[z_{0}, \ldots, z_{m}\right],\left[w_{0}, \ldots, w_{n}\right]\right) \mid \sum_{j=0}^{m} z_{j} w_{j}=0\right\} \subset \mathbb{C} P^{m} \times \mathbb{C} P^{n}
$$

are $2(m+n-1)$-dimensional smooth submanifolds. These are called Milnor manifolds.

## Chapter 9

## Partitions of unity and the weak Whitney embedding theorem

In this chapter we prove that every compact manifold can be embedded into a Euclidean space, using partitions of unity.

### 9.1 The weak Whitney embedding theorem

We now prove that every smooth manifold $M$ arises a smooth submanifold of some $\mathbb{R}^{N}$, by constructing an embedding $M \hookrightarrow \mathbb{R}^{N}$ when $M$ is compact. The result is true even for non-compact smooth manifolds, but proving that requires more care and is done in Section 12.1. Thus we could have set up the theory by demanding every smooth manifold is of this form, as [GP10] does.

The new tool in our argument is the existence of partitions of unity, which is one of the main reasons that we demanded $M$ was second-countable and Hausdorff. Recall that the support $\operatorname{supp}(\eta) \subset M$ of a continuous function $\eta: M \rightarrow[0,1]$ is the closure of the open subset $\eta^{-1}((0,1])$.

Definition 9.1.1. Let $\mathcal{W}=\left\{W_{i}\right\}_{i \in I}$ be an open cover of $M$. Then a partition of unity subordinate to $\mathcal{W}$ is a collection of smooth function $\eta_{i}: M \rightarrow[0,1]$ with the following properties:
(i) $\operatorname{supp}\left(\eta_{i}\right) \subset W_{i}$,
(ii) each $p \in M$ has an open neighborhood on which only finitely many $\eta_{i}$ are non-zero,
(iii) for all $p \in M, \sum_{i} \eta_{i}(p)=1$.

Theorem 9.1.2. Every open cover $\mathcal{W}=\left\{W_{i}\right\}_{i \in I}$ of $M$ admits a subordinate partition of unity.

The main use of partitions of unity is to construct a function (or something similar) on $W_{i}$, usually the codomain of a chart, multiply it with $\eta_{i}$ and extend the result by 0 elsewhere. The result is then defined on all of $M$.

Theorem 9.1.3 (Whitney). Every compact $k$-dimensional smooth manifold $M$ has an embedding into some Euclidean space $\mathbb{R}^{N}$.

Proof. Since $M$ is compact it is covered by the codomains $V_{i}$ of finitely many charts $\left(U_{i}, V_{i}, \phi_{i}\right)$ for $1 \leq i \leq r$. Let $\eta_{i}: M \rightarrow \mathbb{R}$ be a subordinate partition of unity subordinate to this cover. We then define

$$
\overline{\eta_{r}(p) \phi_{r}^{-1}}(p):= \begin{cases}\eta_{r}(p) \phi_{r}^{-1}(p) & \text { if } p \in V_{i} \\ 0 & \text { otherwise }\end{cases}
$$

(Thus $\overline{\eta_{r}(p) \phi_{r}^{-1}}$ ought to be interpreted as a compound symbol.) This is smooth as the support of $\eta_{i}$ is contained in $V_{i}$ and $\eta_{i}$ is smooth.

Then we define the following map

$$
\begin{aligned}
\rho: M & \longrightarrow \mathbb{R}^{r(k+1)} \\
p & \longmapsto\left(\eta_{1}(p), \overline{\eta_{1}(p) \phi_{1}^{-1}}(p), \ldots, \eta_{r}(p), \overline{\eta_{r}(p) \phi_{r}^{-1}}(p)\right)
\end{aligned}
$$

Since each of the components of $\rho$ is smooth, so is $\rho$.
We must now verify $\rho$ is injective and has injective differential for all $p \in M$ (it is automatically proper because $M$ is compact). We start with injectivity and suppose that $\rho(p)=\rho\left(p^{\prime}\right)$. Since the $\eta_{i}$ are a partition of unity we can pick an $\eta_{i}$ such that $\eta_{i}(p)=\eta_{i}\left(p^{\prime}\right) \neq 0$. From this we deduce that both $p$ and $p^{\prime}$ are in $V_{i}$. We can then divide the equation $\eta_{i}(p) \phi_{i}^{-1}(p)=\eta_{i}\left(p^{\prime}\right) \phi_{i}^{-1}\left(p^{\prime}\right)$ by $\eta_{i}(p) \neq 0$ to get $\phi_{i}^{-1}(p)=\phi_{i}^{-1}\left(p^{\prime}\right)$ and apply the injective map $\phi_{i}$ to deduce $p=p^{\prime}$.

Next we verify $\rho$ has injective differential everywhere. Let $p \in M$ such that $\eta_{i}(p) \neq 0$ and set $q=\phi_{i}^{-1}(p)$. Since projections are smooth and on $\eta_{i}^{-1}((0,1])$ division by $\eta_{i}$ is a smooth map, the following is a smooth map $\eta_{i}^{-1}((0,1]) \rightarrow \mathbb{R}^{k}$ :

$$
q \longmapsto \rho(q) \stackrel{\text { proj }}{\longmapsto} \eta_{i}(q) \phi_{i}^{-1}(q) \stackrel{\text { divide }}{\longmapsto} \phi_{i}^{-1}(q) .
$$

It is visibly equal to $\phi_{i}$, so it has bijective differential $d_{p} \phi_{i}$ at $p$. By the chain rule we can write

$$
d_{p} \phi_{i}=d_{\rho(p)}(\text { divide } \circ \operatorname{proj}) \circ d_{p} \rho
$$

and since the left hand side is bijective the term $d_{p} \rho$ on the right hand side must be injective.

Example 9.1.4. The embeddings produced by this result are unnecessary wasteful. For example, at best it produces an embedding of $S^{n}$ into $\mathbb{R}^{2 n+2}$, even though we know it can be embedded into $S^{n+1}$. We shall later prove that every compact $k$-dimensional manifold embeds into $\mathbb{R}^{2 k+1}$.

### 9.1.1 Tangent bundles of submanifolds

Suppose $M$ is a $k$-dimensional manifold and $Z \subset M$ is a submanifold of codimension $r$. Then both $M$ and $Z$ have tangent bundles $T M$ and $T Z$. The inclusion $i: Z \hookrightarrow M$ is an injective map whose derivative is injective at all $z \in Z$. Thus the map $d i: T Z \rightarrow T M$
is injective; it maps at most one fiber to each $T_{p} M$ and on that fiber it is injective. We claim that this allows us to think of $T Z$ as a subbundle of $\left.T M\right|_{Z}$.

Indeed, taking $E=T Z, X=Z, E^{\prime}=T N, X^{\prime}=M$ and $G=d i$ in Lemma 5.3.6, we see that $\operatorname{im}(d i)$ is a subbundle of $\left.T M\right|_{Z}$. Of course it is also true that $\operatorname{ker}(d i)$ is subbundle of $T Z$, but it is 0-dimensional. This makes precise the statement that " $T Z$ is a subbundle of $\left.T M\right|_{Z}$."
Example 9.1.5. By the Whitney embedding theorem, $T M$ is a subbundle of $\left.T \mathbb{R}^{N}\right|_{M}$, which is the trivial bundle of dimension $N$ over $M$. We conclude that the tangent bundle to a compact manifold is always a subbundle of a trivial vector bundle.

### 9.2 Existence of partitions of unity

We now prove the existence of partitions of unity. But before doing so, we prove a few results about the point-set topology of $M$. These results are the main reason we demanded that $M$ was second-countable and Hausdorff.

Lemma 9.2.1. $M$ is a union of countable many open subsets with compact closure.
Proof. Let $\left\{W_{i}\right\}_{i \in I}$ denote the countable basis for the topology of $M$ and let $\mathcal{A}=$ $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ be the atlas of $M$. If there is an $V_{\alpha}$ containing $W_{i}$, pick one and call it $V_{i}$. This gives a collection of open $\left\{V_{i}\right\}_{i \in I^{\prime}}$ indexed by a subset $I^{\prime} \subset I$. We have $\bigcup_{i \in I^{\prime}} V_{i}=M$, because the $V_{\alpha}$ cover $M$ by definition of an atlas and $V_{\alpha}$ is a union of elements of the basis $\left\{W_{i}\right\}_{i \in I}$ by definiton of a basis for a topology.

Given a chart $\left(U_{i}, V_{i}, \phi_{i}\right)$ for $i \in I^{\prime}$, take all open balls $B_{\epsilon_{j}}\left(x_{j}\right) \subset U_{i}$ in its domain such that $\epsilon_{j}>0$ is rational, $x_{j} \in U_{i}$ has rational coordinates and $\bar{B}_{\epsilon_{j}}\left(x_{j}\right) \subset U_{i}$. We denote these

$$
W_{i}^{j}:=\phi_{i}\left(B_{\epsilon_{j}}\left(x_{j}\right)\right)
$$

indexed by some countable set $J_{i}$. The collection of all of these is a countable union of countable sets, so is countable. We will prove that $\left\{W_{i}^{j}\right\}_{i \in I^{\prime}, j \in J_{i}}$ is the sought-after collection of open subsets.

To see that the $W_{i}^{j}$ cover $M$, we remark that for fixed $i$ we have $\bigcup_{j \in J_{i}} W_{i}^{j}=V_{i}$ and then varying $i$ we have

$$
\bigcup_{i \in I^{\prime}} \bigcup_{j \in J_{i}} W_{i}^{j}=\bigcup_{i \in I^{\prime}} V_{i}=M
$$

The image of the compact set $\bar{B}_{\epsilon_{j}}\left(x_{j}\right)$ under $\phi_{i}$ is compact. Because $M$ is Hausdorff each compact subset is closed and thus the closure of $\phi_{i}\left(B_{\epsilon_{j}}\left(x_{j}\right)\right)$ is contained in $\phi_{i}\left(\bar{B}_{\epsilon_{j}}\left(x_{j}\right)\right)$. Hence it is a closed subset of a compact set and hence compact.

Lemma 9.2.2. There are compact subsets $K_{i} \subset M$, indexed by integers $i \geq 0$, and open subsets $V_{i+1 / 2} \subset M$ such that $K_{0} \subset V_{1 / 2} \subset K_{1} \subset V_{1+1 / 2} \subset \cdots$ and $\bigcup_{i \geq 0} K_{i}=M$.

Proof. Let $M=\bigcup_{i \in \mathbb{N}} W_{i}$ with $\overline{W_{i}}$ compact. We define the $K_{i}$ inductively, starting with $K_{0}=\overline{W_{0}}$. Suppose we have defined $K_{n-1}$, then let $N$ be the smallest integer $\geq n$ such that $K_{n-1} \subset W_{1} \cup \cdots \cup W_{N}$. Set $V_{n-1 / 2}:=W_{1} \cup \cdots \cup W_{N}$ and $K_{n}:=\overline{W_{1}} \cup \cdots \cup \overline{W_{N}}$.

If $\mathcal{U}$ is an open cover of $X$, a second open cover $\mathcal{V}$ is a refinement if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. One can deduce easily from the previous lemma that $M$ is paracompact, i.e. every open cover has a refinement to a locally finite subcover and it is then a standard fact in point-set topology that partitions of unity by continuous functions exist. We instead want partitions of unity by smooth functions, so there is no getting around using the fact that $M$ is a smooth manifold. We first prove a slightly weaker version of Theorem 9.1.2 and along the way we will prove that $M$ is paracompact.

Proposition 9.2.3. Every open cover $\mathcal{W}=\left\{W_{i}\right\}_{i \in I}$ of $M$ has a refinement which admits a subordinate partition of unity.

Proof. Let $K_{0} \subset V_{1 / 2} \subset K_{1} \subset V_{1+1 / 2} \subset \cdots$ be as above $M$ and $\mathcal{W}=\left\{W_{i}\right\}_{i \in I}$ be the open cover. Any $p \in M$ lies in a unique $K_{n} \backslash K_{n-1}$, which has $V_{n+1 / 2} \backslash K_{n-1}$ as an open neighborhood. We can then pick a chart $\left(U_{\beta}, V_{\beta}, \phi_{\beta}\right)$ of $M$, a point $z \in U_{\beta}$, and $\delta>0$, such that $B_{\delta}(z) \subset U_{\beta}, p=\phi_{\beta}(z)$ and $\phi_{\beta}\left(B_{\delta}(z)\right) \subset W_{i} \cap V_{n+1 / 2} \backslash K_{n-1}$ for some $i$.

Ranging over all $p \in M$ (and thus implicitly all $n \geq 0$ ), the open sets $\phi_{\beta}\left(B_{\delta / 3}(z)\right)$ in particular cover the compact set $K_{m+1} \backslash V_{m-1 / 2}$, hence there is a finite subcover $\phi_{\beta_{i}^{m}}\left(B_{\delta_{i}^{m} / 3}\left(z_{i}^{m}\right)\right), 1 \leq i \leq j_{m}$ of $K_{m+1} \backslash V_{m-1 / 2}$. Taking the $\left\{\phi_{\beta_{i}^{m}}\left(B_{\delta_{i}^{m} / 3}\left(z_{i}^{m}\right)\right)\right\}_{1 \leq i \leq j_{m}}$ for all $m$, these give a cover of $M$, as

$$
\bigcup_{m \geq 0} K_{m+1} \backslash V_{m-1 / 2} \supset \bigcup_{m \geq 0} K_{m+1} \backslash K_{m}=M
$$

By construction $\phi_{\beta_{i}^{m}}\left(B_{\delta_{i}^{m} / 3}\left(z_{i}^{m}\right)\right)$ is contained in $W_{i}$, so this is a refinement of $\mathcal{W}$. It is locally finite since the open subsets $\phi_{\beta_{i}^{m}}\left(B_{\delta_{i}^{m}}\left(z_{i}^{m}\right)\right)$ can only intersect the open subset $V_{n+1 / 2} \backslash K_{n-1}$ for $n=m-1, m$. At this point we have proven that $M$ is paracompact.

In Problem 9.3.1 you will show that there exists a smooth function $\tilde{\rho}_{i}^{m}: U_{\beta_{i}^{m}} \rightarrow[0,1]$ which vanishes outside $B_{\delta_{i}^{m} / 2}\left(z_{i}^{m}\right)$ and is equal to 1 on $B_{\delta_{i}^{m} / 3}\left(z_{i}^{m}\right)$. We can then define a smooth map $\tilde{\eta}_{i}^{m}: M \rightarrow[0,1]$ by

$$
\tilde{\eta}_{i}^{m}(p)= \begin{cases}\tilde{\rho}_{i}^{m}\left(\phi_{\beta_{i}^{m}}^{-1}(p)\right) & \text { if } p \in V_{\beta} \\ 0 & \text { otherwise }\end{cases}
$$

Since the collection of open subsets $\phi_{\beta_{i}^{m}}\left(B_{\delta_{i}^{m} / 3}\left(z_{i}^{m}\right)\right)$ covers $M$ and the collection of open subsets $\phi_{\beta_{i}^{m}}\left(B_{\delta_{i}^{m}}\left(z_{i}^{m}\right)\right)$ is locally finite, we have that

$$
p \longmapsto \sum \tilde{\eta}_{i}^{m}(p)
$$

is locally equal to a finite sum of non-zero terms, so is a smooth map $M \rightarrow \mathbb{R}_{>0}$. We then define $\eta_{i}^{m}: M \rightarrow[0,1]$ by

$$
\eta_{i}^{m}:=\frac{\tilde{\eta}_{i}^{m}}{\sum \tilde{\eta}_{i}^{m}}
$$

This is the desired partition of unity subordinate to the refinement of $\mathcal{W}$ given by the $\phi_{\beta_{i}^{m}}\left(B_{\delta_{i}^{m} / 3}\left(z_{i}^{m}\right)\right)$.
Remark 9.2.4. If $M$ is compact, the proof of Theorem 9.1.2 greatly simplifies as you can forget about the $K_{i}$ and $V_{i+1 / 2}$ 's.

The above construction has multiple functions with support in $W_{i}$. Instead, it is often more convenient to have one function for each $W_{i}$ in $\mathcal{W}$.

Proof of Theorem 9.1.2. By the previous proposition we can find a refinement $\mathcal{W}^{\prime}=$ $\left\{W_{j}\right\}_{j \in J}$ of $\mathcal{W}=\left\{W_{i}\right\}_{i \in I}$ and a partition of unity $\left\{\eta_{j}^{\prime}: M \rightarrow[0,1]\right\}$ subordinate to it.

For $j \in J$, fix a $W_{i}$ such that $W_{j}^{\prime} \subset W_{i}$. This gives a function $\lambda: J \rightarrow I$. We claim that

$$
\eta_{i}:=\sum_{j \in J^{-1}(i)} \eta_{j}^{\prime}
$$

gives the desired partition of unity. By property (ii), this is a locally finite sum and hence a smooth function. By property (i), the sum of the $\eta_{i}$ is 1 everywhere. From property (i), we know that $\operatorname{supp}\left(\eta_{j}^{\prime}\right) \subset W_{j}^{\prime}$ and hence is also contained in $W_{i}$. Now observe that

$$
\operatorname{supp}\left(\eta_{i}\right)=\overline{\left.\eta_{i}^{-1}((0,1])\right)}=\overline{\bigcup_{j \in J^{-1}(i)}\left(\eta_{j}^{\prime}\right)^{-1}((0,1])} .
$$

By property (ii), the latter is a closure of a locally finite union of open subsets. This is equal to the union of the closures, by an elementary argument in point-set topology. So we conclude that

$$
\operatorname{supp}\left(\eta_{i}\right)=\bigcup_{j \in J^{-1}(i)} \overline{\left(\eta_{j}^{\prime}\right)^{-1}((0,1])}=\bigcup_{j \in J^{-1}(i)} \operatorname{supp}\left(\eta_{j}^{\prime}\right) \subset W_{i}
$$

This finishes the proof.

### 9.3 Problems

Problem 9.3.1 (A bump function).
(a) Prove that

$$
\begin{aligned}
f: \mathbb{R} & \longrightarrow \mathbb{R} \\
x & \longmapsto \begin{cases}e^{-1 / x^{2}} & \text { if } x>0 \\
0 & \text { if } x \leq 0\end{cases}
\end{aligned}
$$

is smooth.
(b) Observe that $g(x)=f(x) f(1-x)$ is smooth, positive on $(0,1)$, and 0 outside of this interval. Prove that

$$
h(x)=\frac{\int_{-\infty}^{x} g(y) d y}{\int_{-\infty}^{\infty} g(y) d y}
$$

is smooth, equal to 0 when $x \leq 0$ and equal to 1 when $x \geq 1$.
(c) Construct a smooth function on $\mathbb{R}^{k}$ which is 1 on an open neighborhood of the origin and is supported in the unit ball.

Problem 9.3.2 (Smooth Urysohn theorem). Use partitions of unity to prove that if $A, B \subset M$ are disjoint closed subsets of a smooth manifold $M$, then there is a smooth function $\lambda: M \rightarrow[0,1]$ such that $\left.\lambda\right|_{A}=0$ and $\left.\lambda\right|_{B}=1$.

Problem 9.3.3 (Charts from coordinate axes). Suppose that $M$ is a $k$-dimensional smooth manifold and $e: M \rightarrow \mathbb{R}^{N}$ is a smooth embedding. Prove that for each $p \in M$ there is an open subset $U \subset M$ containing $p$ and integers $i_{1}, \ldots, i_{k}$ in $\{1, \ldots, N\}$ such that

$$
\begin{aligned}
M \supset U & \longrightarrow \mathbb{R}^{k} \\
p & \longmapsto\left(\pi_{i_{1}} \circ e(p), \ldots, \pi_{i_{k}} \circ e(p)\right)
\end{aligned}
$$

is a diffeomorphism onto an open subset. Here $\pi_{i_{j}}: \mathbb{R}^{N} \rightarrow \mathbb{R}$ is the projection on the $i_{j}$ th coordinate.

## Chapter 10

## Transversality and the improved preimage theorem

In this chapter we improve the pre-image theorem to give a sufficient condition under which pre-images of submanifolds are submanifolds. This will have many applications, among them a generalization of the Whitney embedding theorem to non-compact manifolds.

### 10.1 The preimage theorem restated

Recall that given a submanifold $Z \subset M$, with $i: Z \rightarrow M$ denoting the inclusion, we have that by considering the image of $d i$ we can consider $T Z$ as a subbundle of $\left.T M\right|_{Z}$. This makes precise the statement that " $T Z$ is a subbundle of $\left.T M\right|_{Z}$."

Many submanifolds arise through the pre-image theorem: we have a smooth map $f: M \rightarrow N$ with regular value $c$ and $Z=f^{-1}(c)$. The pre-image theorem said that $Z$ is then $\left(k-k^{\prime}\right)$-dimensional submanifold of $M$ and $T_{p} f^{-1}(c)=\operatorname{ker}\left(d_{p} f: T_{p} M \rightarrow T_{f(p)} M\right)$ for all $p \in f^{-1}(c)$. The latter part about the tangent spaces to $Z$, can be improved to a statement about tangent bundles. The proof is identical, but it is only now that we can phrase it:

Theorem 10.1.1 (Preimage theorem). If $f: M \rightarrow N$ is a smooth map and $c \in N$ a regular value, then $Z:=f^{-1}(c)$ is a $\left(k-k^{\prime}\right)$-dimensional submanifold of $M$ and $T Z=\left.\operatorname{ker}\left(d f:\left.T M\right|_{Z} \rightarrow T N\right) \subset T M\right|_{Z}$.

Example 10.1.2. Recall that $S^{n-1}$ can be written as $g^{-1}(1)$ with $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ given by $\left(x_{1}, \ldots, x_{n}\right) \mapsto x_{1}^{2}+\ldots+x_{n}^{2}$. The map $g$ is smooth and has total derivative $\left[2 x_{1}, \ldots, 2 x_{n}\right]$, so all non-zero real numbers are regular values of $g$. In particular, $S^{n-1}$ is an $(n-1)$ dimensional differentiable manifold and $T S^{n-1}$ is the kernel of the total derivative maps; for $x=\left(x_{1}, \ldots, x_{n}\right) \in S^{n-1}$ the kernel of $\left[2 x_{1}, \ldots, 2 x_{n}\right]$ is just the ( $n-1$ )-dimensional plane $x^{\perp}$ of vectors orthogonal to $x$.

### 10.2 Transversality

The most important geometric notion in differential topology is transversality. This condition tells you in terms of tangent spaces when submanifolds (or the image of a map and a submanifold) intersect nicely.

### 10.2.1 Submanifolds locally

We start by recalling the definition of a submanifold, and describe how in suitable local coordinates all submanifolds are the inverse images of projection maps.

Suppose that we have a $k^{\prime}$-dimensional differentiable manifold $N$ with a submanifold $Z \subset N$ of codimension $r$ (that is, $Z$ is $\left(k^{\prime}-r\right)$-dimensional). Then for each $z \in Z$ we have a local parametrization, that is, open subsets $U \subset \mathbb{R}^{k^{\prime}}$ and $V \subset N$ as well as a diffeomorphism $\phi: U \rightarrow V$ so that $\phi^{-1}(Z \cap V)=U \cap\left(\{0\} \times \mathbb{R}^{k^{\prime}-r}\right)$. That is, on $U$ we can define $\pi_{r}: U \rightarrow \mathbb{R}^{r}$ projecting onto the first $r$ coordinates and $\phi^{-1}(Z \cap V)=\pi_{r}^{-1}(0)$. Thus we see that

$$
Z \cap V \subset V=\phi\left(\pi_{r}^{-1}(0)\right) \subset V
$$

If we want to explicit understand $T_{z} Z \subset T_{z} N$, then we may as well identify it in $U$ by applying the linear isomorphism $d_{z} \phi^{-1}$. Here it is the tangent space to $U \cap\left(\{0\} \times \mathbb{R}^{k^{\prime}-r}\right)$ at $\phi^{-1}(z)$, which is just $\{0\} \times \mathbb{R}^{k^{\prime}-r}$. Applying the inverse $d_{\phi^{-1}(z)} \phi$ of $d_{z} \phi^{-1}$, we see that $T_{z} Z$ is the following $\left(k^{\prime}-r\right)$-dimensional linear subspace of $T_{z} N$ :

$$
\begin{equation*}
T_{z} Z=d_{\phi^{-1}(z)} \phi\left(\{0\} \times \mathbb{R}^{k^{\prime}-r}\right) \subset T_{z} N \tag{10.1}
\end{equation*}
$$

### 10.2.2 Improving the pre-image theorem

Now suppose we have a smooth map $f: M \rightarrow N$. We will give a criterion that tells us when $f^{-1}(Z)$ is a differentiable submanifold of $M$.

To find a local parametrization of $f^{-1}(Z) \subset M$ near $p \in f^{-1}(Z)$, we might as well find one of $f^{-1}(Z \cap V) \subset f^{-1}(V) \subset M$. The advantage of passing to this open subset is that on $f^{-1}(V)$ we can use projection to define the smooth map

$$
g:=f^{-1}(V) \subset M \xrightarrow{f} V \subset N \xrightarrow{\phi^{-1}} U \subset \mathbb{R}^{k^{\prime}} \xrightarrow{\pi_{r}} \mathbb{R}^{r} .
$$

This has the property that

$$
g^{-1}(0)=f^{-1}\left(\phi\left(\pi_{r}^{-1}(0)\right)\right)=f^{-1}\left(\phi\left(\phi^{-1}(Z \cap V)\right)\right)=f^{-1}(Z \cap V)
$$

The pre-image theorem then tells us that $f^{-1}(Z \cap V)$ is a submanifold of $f^{-1}(V) \subset M$ of codimension $r$ whenever 0 is regular value of $g$. That is, $g$ should be a submersion at all $p \in f^{-1}(Z \cap V)$.

So we need to understand when $d_{p} g: T_{p} M \rightarrow T_{0} \mathbb{R}^{r}$ is surjective. Writing

$$
d_{p} g=d_{\phi^{-1} f(p)} \pi_{r} \circ d_{f(p)} \phi^{-1} \circ d_{p} f
$$

we first observe that for $d_{p} g$ to be surjective, $\operatorname{im}\left(d_{f(p)} \phi^{-1} \circ d_{p} f\right)$ should be a linear subspace of $T_{\phi^{-1} f(p)} \mathbb{R}^{k^{\prime}}=\mathbb{R}^{k^{\prime}}$ which surjects onto $T_{0} \mathbb{R}^{r}=\mathbb{R}^{r}$ under the linear map
$d_{\phi^{-1} f(p)} \pi_{r}: \mathbb{R}^{k^{\prime}} \rightarrow \mathbb{R}^{r}$. This is the case exactly when $\operatorname{im}\left(d_{f(p)} \phi^{-1} \circ d_{p} f\right)+\operatorname{ker}\left(d_{\phi^{-1} f(p)} \pi_{r}\right)=$ $\mathbb{R}^{k^{\prime}}$. Using the fact that $\operatorname{ker}\left(d_{\phi^{-1} f(p)} \pi_{r}\right)=\{0\} \times \mathbb{R}^{k^{\prime}-r}$ we obtain the requirement

$$
\operatorname{im}\left(d_{f(p)} \phi^{-1} \circ d_{p} f\right)+\{0\} \times \mathbb{R}^{k^{k^{\prime}-r}}=\mathbb{R}^{k^{\prime}}
$$

Let us apply the linear isomorphism $d_{\phi^{-1} f(p)} \phi$ to translate this back to a statement about linear subspaces of the original tangent space $T_{f(p)} N$. By the chain rule $d_{\phi^{-1} f(p)} \phi$ sends $\operatorname{im}\left(d_{f(p)} \phi^{-1} \circ d_{p} f\right)$ to $\operatorname{im}\left(d_{p} f\right)$, and by (10.1) it sends $\{0\} \times \mathbb{R}^{k^{\prime}-r}$ to $T_{f(p)} Z$. Since a linear isomorphism preserves sums, we see that $d_{p} g$ is surjective if and only if

$$
\operatorname{im}\left(d_{p} f\right)+T_{f(p)} Z=T_{f(p)} N .
$$

Let us give this condition a name:
Definition 10.2.1. Let $Z \subset N$ be a submanifold. We say that $f: M \rightarrow N$ is transverse to $Z$ at $p \in f^{-1}(Z)$, denoted $f \pitchfork_{p} Z$, when $\operatorname{im}\left(d_{p} f\right)+T_{f(p)} Z=T_{f(p)} N$.

Definition 10.2.2. Let $Z \subset N$ be a submanifold. We say that $f: M \rightarrow N$ is transverse to $Z$, denoted $f \pitchfork Z$, when $f$ is transverse to $Z$ at all $p \in f^{-1}(Z)$.

Example 10.2 .3. A smooth map $f: \mathbb{R} \rightarrow \mathbb{R}^{2}$ is transverse to $\mathbb{R} \times\{0\}$ if and only if the derivative $\partial f_{2} / \partial t$ is non-zero whenever $f(t)$ crosses the $x$-axis.


Figure 10.1 Examples of smooth functions $\mathbb{R} \rightarrow \mathbb{R}^{2}$.

Then the above discussion tells us that $f: M \rightarrow N$ being transverse to $Z$ at all $p \in f^{-1}(Z \cap V)$ implies that $f^{-1}(Z \cap V)=g^{-1}(0)$ is a submanifold. Varying the local parametrizations, we see that $f$ being transverse to $Z$ implies $f^{-1}(Z)$ is a submanifold. We can say a bit more; by the pre-image theorem the tangent space to $f^{-1}(Z \cap V)=g^{-1}(0)$ at $p$ is given by the kernel of $d_{p} g$, i.e. $\left(d_{p} g\right)^{-1}(0)$, which is equal to $\left(d_{p} f\right)^{-1}\left(T_{f(p)} Z\right)$.

Theorem 10.2.4 (Improved preimage theorem). Let $Z \subset N$ a submanifold of codimension $r$ and suppose that $f: M \rightarrow N$ that is transverse to $Z$. Then $f^{-1}(Z) \subset M$ is also a submanifold of codimension $r$ and $T f^{-1}(Z)=\left.(d f)^{-1}(T Z) \subset T M\right|_{f^{-1}(Z)}$.

Remark 10.2.5. This is an improvement of the preimage theorem, because we can recover the preimage theorem by take $Z$ to be a point $c$. Since the tangent space to the (rather boring) 0 -dimensional manifolds $c$ is 0 -dimensional, $f$ is transverse to $c$ at $p \in f^{-1}(c)$ if and only if $d_{p} f$ is surjective.
Example 10.2.6. Suppose $Z \subset N$ is a collection of points and $M$ is of smaller dimension than $N$. Then $f: M \rightarrow N$ is transverse to $Z$ if and only $\operatorname{im}(f) \cap Z=\varnothing$, as it is not possible for the sum of a 0 -dimensional and $<k^{\prime}$-dimensional subspace to equal a $k^{\prime}$-dimensional vector space.
Example 10.2.7. Though $f \pitchfork Z$ implies that $f^{-1}(Z)$ is a submanifold, the converse is not true: the inclusion $i: Z \rightarrow N$ is very much not transverse to $Z$, but $i^{-1}(Z)=Z$.
Remark 10.2.8. You may want to try to come up with the definition of two smooth maps $f: M_{1} \rightarrow N$ and $g: M_{2} \rightarrow N$ being transverse, and then prove that $\left\{\left(m_{1}, m_{2}\right) \mid f\left(m_{1}\right)=\right.$ $\left.g\left(m_{2}\right)\right\} \subset M_{1} \times M_{2}$ is a submanifold.

### 10.2.3 Transversality for submanifolds

The case that is of most geometric interest is when $f$ is the inclusion $j: Y \rightarrow N$ of another submanifold. In that case, it is more convenient to forget about the maps $i: Z \rightarrow$ $N$ and $j: Y \rightarrow N$ and state the transversality condition in terms of the submanifolds:
Definition 10.2.9. Let $Y, Z \subset N$ be submanifolds. Then $Y$ and $Z$ are transverse at $p \in Y \cap Z$, denoted $Y \pitchfork_{p} Z$, if $T_{p} Y+T_{p} Z=T_{p} N$.

Definition 10.2.10. Let $Y, Z \subset N$ be submanifolds. Then $Y$ and $Z$ are transverse, denoted $Y \pitchfork Z$, if $Y$ and $Z$ are transverse at all $p \in Y \cap Z$.

Example 10.2.11. If $Y \cap Z=\varnothing, Y \pitchfork Z$ because there are no points in $p \in Y \cap Z$ at which any conditions are imposed.

The improved pre-image theorem says that if $Y \pitchfork Z$ then $Y \cap Z$ is a submanifold of $Y$, and hence a submanifold of $N$. (If this sounds surprising, you should go through the definitions again and verify that a submanifold of a submanifold is a submanifold). At each $p \in Y \cap Z, T_{p}(Y \cap Z)=T_{p} Y \cap T_{p} Z$. This in particular implies that

$$
\operatorname{codim}(Y \cap Z)=\operatorname{codim}(Y)+\operatorname{codim}(Y)
$$

You should think of $Y \pitchfork Z$ as saying that $Y$ and $Z$ intersect nicely. Let us make this more precise:
Example 10.2.12. Two linear subspaces $U$ and $V$ in $\mathbb{R}^{n}$ of codimension $r$ and $s$ respectively intersect transversally if and only if $U \cap V$ is a linear subspace of codimension $r+s$.

The direction $\Rightarrow$ is a consequence of the general formula for the codimension of a transverse intersection. For the direction $\Leftarrow$, we note that at each $p \in U \cap V$ we can identity $T_{p} U$ and $T_{p} V$ with $U$ and $V$ again. To compute their sum $U+V$ we use the inclusion-exclusion formula for the dimension of a sum of two linear subspaces:
$\operatorname{dim}(U+V)=\operatorname{dim}(U)+\operatorname{dim}(V)-\operatorname{dim}(U+V)=(n-r)+(n-s)-(n-r-s)=n$.
Hence $U+V=\mathbb{R}^{n}$ and $U$ and $V$ intersect transversally at $p \in U \cap V$.


Figure 10.2 Examples of 1-dimensional submanifolds of $\mathbb{R}^{2}$.

Any transverse intersection locally is of the form in Example 10.2.12 in the right coordinates:

Lemma 10.2.13. $Y \pitchfork_{p} Z$ if and only if there is a chart $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ such that $\phi_{\alpha}^{-1}(Y)$ and $\phi_{\alpha}^{-1}(Z)$ are given by the intersection with $U_{\alpha}$ of two linear subspaces intersecting transversally.

Proof. $\Leftarrow$ follows from transversality being preserved by diffeomorphisms, so we focus on $\Rightarrow$. Since the intersection is non-empty the codimensions $r$ and $s$ of $Y$ and $Z$ satisfiy $r+s \leq k^{\prime}$.

The proof of the improved pre-image theorem provides a chart $\left(U_{1}, V_{1}, \phi_{1}\right)$ in which $\phi^{-1}(Z)=U_{1} \cap\left(\{0\} \times \mathbb{R}^{k^{\prime}-r}\right)$. We may assume that $\phi_{1}(0)=p$ by translating. Translated to this chart, $Y \pitchfork_{p} Z$ says that $T_{0} \phi_{1}^{-1}(Y)+\{0\} \times \mathbb{R}^{k^{\prime}-r}=\mathbb{R}^{k^{\prime}}$. Thus by applying a linear isomorphism of $\mathbb{R}^{k^{\prime}}$ preserving $\{0\} \times \mathbb{R}^{k^{\prime}-r}$ we may assume that $T_{0} \phi_{1}^{-1}(Y)=\mathbb{R}^{k^{\prime}-s} \times\{0\}$.

So it remains to fix $\phi_{1}^{-1}(Y)$. Consider the map $\pi: \phi_{1}^{-1}(Y) \rightarrow \mathbb{R}^{k^{\prime}-s} \times\{0\}$ given by restricting the projection map $\mathbb{R}^{k^{\prime}} \rightarrow \mathbb{R}^{k^{\prime}-s} \times\{0\}$. The derivative of $\pi$ at 0 is the identity and hence bijective. Inverse function theorem then tells us that $\pi$ is a local diffeomorphism. Thus near the origin,

$$
\phi_{1}^{-1}(Y)=\left\{(w, \rho(w)) \in \mathbb{R}^{k^{\prime}-s} \times \mathbb{R}^{s}\right\}
$$

for a smooth map $\rho: \mathbb{R}^{k^{\prime}-s} \rightarrow \mathbb{R}^{s}$ with $\rho(0)=0$. Thus there exists an open subset $U_{2}$ of the origin in $\mathbb{R}^{k^{\prime}}$ so that the diffeomorphism $\bar{\rho}: \mathbb{R}^{k^{\prime}} \rightarrow \mathbb{R}^{k^{\prime}}$ given by $\bar{\rho}(w, v)=(w, v+\rho(w))$ maps $U_{2} \cap\left(\mathbb{R}^{k^{\prime}-s} \times\{0\}\right)$ onto a neighborhood of the origin in $\phi_{1}^{-1}(Y)$. Note that $\bar{\rho}$ preserves $\{0\} \times \mathbb{R}^{k-r}$, we only translate in the last $s$ coordinates and $s \leq k^{\prime}-r$ as $k^{\prime} \geq r+s$. Thus the desired chart is

$$
\left(U_{2}, V_{2}, \phi_{2}\right):=\left(U_{2}, \phi_{1} \circ \bar{\rho}\left(V_{2}\right), \phi_{1} \circ \bar{\rho}\right)
$$

### 10.3 Another construction of the Poincaré homology sphere

As an extended example, we will now give an alternative and at first sight completely unrelated construction of the Poincaré homology sphere $P=S^{3} / I^{*}$, which we first saw in Section 8.1.

To do so, we consider the map

$$
\begin{aligned}
f: \mathbb{C}^{3} & \longrightarrow \mathbb{C} \\
\left(z_{1}, z_{2}, z_{3}\right) & \longmapsto z_{1}^{2}+z_{2}^{3}+z_{3}^{5}
\end{aligned}
$$

We claim that

$$
X=f^{-1}(0) \cap\left(\mathbb{C}^{3} \backslash 0\right)
$$

is a codimension 2 submanifold of $\mathbb{C}^{3} \backslash\{0\}$. We of course would like to use the submersion theorem, and we could do by identifying the domain $\mathbb{C}^{3}$ with $\mathbb{R}^{6}$ by $z_{j} \longleftrightarrow x_{j}+i y_{j}$, and similarly identify the target $\mathbb{C}$ with $\mathbb{R}^{2}$. We would then need to verify that the total derivative, a $(2 \times 6)$-matrix, is surjective.

However, it is much convenient to keep working with complex numbers: as a polynomial, $p$ is not only differentiable as a function $\mathbb{R}^{6} \rightarrow \mathbb{R}^{2}$, is in fact complex-differentiable as a function $\mathbb{C}^{3} \rightarrow \mathbb{C}$. We can compile these into a $(1 \times 3)$-matrix of complex numbers

$$
\left[\frac{\partial f}{\partial z_{1}}\left(z_{1}, z_{2}, z_{3}\right) \quad \frac{\partial f}{\partial z_{2}}\left(z_{1}, z_{2}, z_{3}\right) \quad \frac{\partial f}{\partial z_{3}}\left(z_{1}, z_{2}, z_{3}\right)\right] .
$$

This complex total derivative is surjective if and only if the total derivative is surjective.
In our case, the complex total derivative is given by

$$
\left[\begin{array}{lll}
2 z_{1} & 3 z_{2}^{2} & 5 z_{3}^{4} \tag{10.2}
\end{array}\right]
$$

and hence surjective for all $\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3} \backslash\{0\}$. We conclude that $X \subset \mathbb{C}^{3} \backslash\{0\}$ is a 4-dimensional smooth manifold, or equivalently codimension 2.

To reduce the dimension by one, we will intersect with the sphere $S^{5}:=\left\{\left(z_{1}, z_{2}, z_{3}\right) \mid\right.$ $\left.\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}+\left|z_{3}\right|^{2}=1\right\}$, of codimension 1 . We claim this is transverse to $X$. To see this is the case, we use that the tangent bundle to $X$ at $x=\left(z_{1}, z_{2}, z_{3}\right) \in X$ is given by the kernel of the matrix (10.2); this has fibers isomorphic to $\mathbb{C}^{2} \cong \mathbb{R}^{4}$ so is 4 -dimensional. A particular vector in this kernel is

$$
w=\left(z_{1} / 2, z_{2} / 3, z_{3} / 5\right)
$$

The tangent bundle to $S^{5}$ at $x \in S^{5}$ is given by those vectors orthogonal to $x$; this is 5 -dimensional. It is convenient to work with complex numbers, and observe that $w=\left(w_{1}, w_{2}, w_{3}\right) \in T_{x} \mathbb{C}^{3}$ being orthogonal to $x$ is equivalent to

$$
\operatorname{Re}(\bar{x} \cdot w)=0
$$

Let us evaluate this on the above vector in $T_{x} X$ : we get

$$
\operatorname{Re}(\bar{x} \cdot w)=\left|z_{1}\right|^{2} / 2+\left|z_{2}\right|^{2} / 3+\left|z_{3}\right|^{2} / 5
$$

and since $\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}+\left|z_{3}\right|^{2}=1$, we see this is at least $1 / 5$ so non-zero. Thus $T_{x} X \not \subset T_{x} S^{5}$ and by a dimension count we conclude that $T_{x} X+T_{x} S^{5}=T_{x} \mathbb{C}^{3}$. Thus $f^{-1}(0) \cap S^{5}$ is a submanifold of $S^{5}$. Its codimension is $2+1$, so it is 3 -dimensional. It is in fact diffeomorphic to the Poincaré homology sphere [KS79, p. 128-132].

Remark 10.3.1. A smooth manifold which arises as the transverse intersection of a zero set of a complex polynomial (here $z_{1}^{2}+z_{2}^{3}+z_{3}^{5}$ ) with a small sphere around a singularity (here we took the sphere of radius one around the origin), is called a link of a singularity.

These have been studied in detail, see e.g. [Mil68]. Particularly interesting are the Brieskorn spheres $\Sigma\left(k_{1}, \ldots, k_{n}\right)$, constructed as the links of the singularity at the origin of polynomials

$$
z_{1}^{k_{1}}+\cdots+z_{n}^{k_{n}}
$$

These give examples of smooth manifolds which are homeomorphic of spheres but not diffeomorphic to them [Bri66]. For examples, the cases

$$
\Sigma(2,2,2,3,6 k-1)
$$

give all examples of exotic 7 -spheres up to diffeomorphism.

### 10.4 Problems

Problem 10.4.1 (Brieskorn manifolds). Verify that all Brieskorn spheres are $(2 n-3)$ dimensional manifolds.

Problem 10.4.2 ( $\mathbb{R} P^{3}$ as a link of a singularity). Recall the smooth manifold $W_{n}$ from Problem 6.4.6. Use the map

$$
\begin{aligned}
f: \mathbb{C}^{2} & \longrightarrow \mathbb{C}^{3} \\
\left(w_{1}, w_{2}\right) & \longmapsto\left(w_{1}^{2}+w_{2}^{2}, i\left(w_{1}^{2}-w_{2}^{2}\right), 2 i w_{1} w_{2}\right)
\end{aligned}
$$

to produce a diffeomorphism $\mathbb{R} P^{3} \rightarrow W_{3}$. Conclude that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R} P^{3}$.
Problem 10.4.3 (Transversality and eigenvalues). Prove that if $A: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ is a linear map, then $\operatorname{graph}(A)=\left\{(v, A v) \mid v \in \mathbb{R}^{k}\right\} \subset \mathbb{R}^{k} \times \mathbb{R}^{k}$ is transverse to the diagonal $\operatorname{graph}(\mathrm{id}) \subset \mathbb{R}^{k} \times \mathbb{R}^{k}$ if and only if 1 is not an eigenvalue of $A$.

Problem 10.4.4 (Whitney's double point immersion). Consider the following smooth map:

$$
\begin{aligned}
\alpha: \mathbb{R} & \longrightarrow \mathbb{R}^{2} \\
x & \longmapsto\left(\frac{1}{1+x^{2}}, x-\frac{2 x}{1+x^{2}}\right)
\end{aligned}
$$

(a) Prove it is an immersion.
(b) Prove that it fails to be injective at a single pair of points in the domain $\mathbb{R}$. That is, prove its image has a single self-intersection.
(c) Prove that this self-intersection is transverse.

Problem 10.4.5 (Compact exhaustion by submanifolds). Suppose that $M$ is a smooth manifold. Prove that there exists a sequence $K_{0} \subset K_{1} \subset \cdots \subset M$ of compact codimension 0 submanifolds (possibly with non-empty boundary) such that $\cup_{i} K_{i}=M$.

## Chapter 11

## Stable and generic classes of smooth maps

It is a standard strategy to study the effect of small deformations on mathematical objects. On the one hand such deformations can make the object more generic, and hence easier to understand, and on the other hand small enough deformations often preserve salient properties.

To start applying this strategy to certain types of smooth maps, we will need to do the following:
(i) make precise what we mean by a "deformation,"
(ii) understand which types of smooth maps are "stable", i.e. preserved by small deformations, and
(iii) understand what a "generic smooth map" looks like.

### 11.1 Homotopies of smooth maps

A reasonable definition of deforming of a smooth map $f_{0}$ is to situate it in a family of smooth maps $f_{s}$ which depends smoothly on the parameter $s$. Restricting the parameter $s$ to lie in the closed interval $[0,1]$, we get the following definition:

Definition 11.1.1. A homotopy is a smooth map $H: M \times[0,1] \rightarrow N$.
Example 11.1.2. Out of a smooth map $f: M \rightarrow N$, we can construct a constant homotopy $H: M \times[0,1] \rightarrow N$ by $H(p, t):=f(p)$. This homotopy does not deform $f$ at all!

We have not officially said what it means to have a smooth map with domain $M \times[0,1]$; we will later define manifolds with boundary, but for now it suffices to say that it should extend to a smooth map whose domain is an open neighborhood of $M \times[0,1]$ in $M \times \mathbb{R}$.

Since the restrictions of smooth maps are smooth, each $\left.f\right|_{M \times\{t\}}: M \rightarrow N$ is a smooth map. In particular this is the case for $f_{0}:=\left.f\right|_{M \times\{0\}}$ and $f_{1}:=\left.f\right|_{M \times\{1\}}$ and we say that $H$ is a homotopy from $f_{0}$ to $f_{1}$.

Definition 11.1.3. Two smooth maps $f_{0}, f_{1}: M \rightarrow N$ are homotopic, denoted $f_{0} \sim f_{1}$, if there is a homotopy from $f_{0}$ to $f_{1}$.

Lemma 11.1.4. Homotopy is an equivalence relation of smooth maps $M \rightarrow N$.

Proof. The constant homotopy shows it is reflexive. To see it is symmetric, note that if $H: M \times[0,1] \rightarrow N$ is a homotopy from $f_{0}$ to $f_{1}$ then $\bar{H}(p, t):=H(p, 1-t)$ is a homotopy from $f_{1}$ to $f_{0}$. In Problem 11.5.1 you will show it is transitive.

### 11.2 Stable classes of maps

A class of smooth maps is stable if it is preserved by small perturbations, in the following sense:

Definition 11.2.1. A subset $U$ of the set of all smooth maps $M \rightarrow N$ is stable if for each $f_{0} \in U$ and smooth map $H: M \times \mathbb{R}^{r} \rightarrow N$ starting at $f_{0}$ there exists an $\epsilon>0$ such that $\left.H\right|_{M \times\{x\}} \in U$ for all $\|x\|<\epsilon$.

This definition has a consequence for homotopies:
Lemma 11.2.2. If $U$ is stable then for each $f_{0} \in U$ and homotopy $H: M \times[0,1] \rightarrow N$ starting at $f_{0}$ there exists an $\epsilon>0$ such that $\left.H\right|_{M \times\{t\}} \in U$ for all $t<\epsilon$.

Proof. There exists a smooth map $\eta: \mathbb{R} \rightarrow[0,1]$ such that $\eta(t)=0$ for $t \leq 0$ and $\eta^{\prime}(t)>0$ for $t>0$. Now apply the condition in the definition of stable classes of maps to $H \circ(\mathrm{id} \times \eta): M \times \mathbb{R} \rightarrow N$.

Remark 11.2.3. If we were to go to the trouble of defining a suitable topology on the set $C^{\infty}(M, N)$ of smooth maps $M \rightarrow N$, open subsets of $C^{\infty}(M, N)$ would be stable.

This remark makes us suspect that subsets which are defined by "open conditions" should be stable. Let us look at an example: in the space $\operatorname{Lin}\left(\mathbb{R}^{p}, \mathbb{R}^{p}\right)$ of all linear maps $\mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ the invertible linear maps are open (as they are defined by the condition that the determinant is non-zero). This means that if an invertible $A \in \operatorname{Lin}\left(\mathbb{R}^{p}, \mathbb{R}^{p}\right)$ is perturbed slightly, it remains invertible. Since a map $f: M \rightarrow N$ is a local diffeomorphism if and only if all derivatives $d_{p} f$ are invertible, one might expect that this condition should be preserved by a small perturbation of $f$, as it gives rise to a small perturbation of each $d_{p} f$. Thus, if we could somehow "bound the determinant of the $d_{p} f$ " away from 0 , any small perturbation of $f$ will remain a local diffeomorphism.

The problem with this vague argument is of course that one can't make sense of the determinant of a linear map between two different vector spaces. The idea is to use the determinant in finitely many charts, and to guarantee $M$ is covered by finitely many charts we assume it is compact.

Let us now make it precise:
Theorem 11.2.4. If $M$ is compact, then the following classes of smooth maps $f: M \rightarrow N$ are stable:
(i) local diffeomorphisms,
(ii) immersions,
(iii) submersions,
(iv) maps transverse to a submanifold $Z \subset N$,
(v) embeddings,
(vi) diffeomorphisms.

Proof. The case (i) is a special case of both (ii) and (iii). Case (ii) is very similar to case (iii) and proven in Guillemin \& Pollack, so we will only prove the latter. So suppose $f_{0}: M \rightarrow N$ is a submersion and $H: M \times[0,1] \rightarrow N$ a homotopy starting at $f_{0}$. We can find a finite collection of charts $\left\{\left(U_{i}, V_{i}, \phi_{i}\right)\right\}, 1 \leq i \leq r$, such that $\bigcup_{i} V_{i}=M$ and $f\left(V_{i}\right) \subset V_{j(i)}^{\prime}$ for some chart $\left(U_{j(i)}^{\prime}, V_{j(i)}^{\prime}, \phi_{j(i)}^{\prime}\right)$ of $N$. Taking a partition of unity $\eta_{i}: M \rightarrow[0,1]$, we find compact subsets $\operatorname{supp}\left(\eta_{i}\right) \subset V_{i}$ which also cover $M$. Each compact subset $f_{0}\left(\operatorname{supp}\left(\eta_{i}\right)\right)$ is contained in an open subset $V_{j(i)}^{\prime}$. Hence there exists a $\delta_{i}>0$ such that $H\left(\operatorname{supp}\left(\eta_{i}\right) \times\left[0, \delta_{i}\right]\right) \subset V_{j(i)}^{\prime}$. For suppose no such $\delta_{i}>0$ exists, then there is a sequence $\left(p_{k}, t_{k}\right)$ with $p_{k} \in \operatorname{supp}\left(\eta_{i}\right), t_{k} \rightarrow 0$ and $H\left(p_{k}, t_{k}\right) \in N \backslash V_{j(i)}^{\prime}$. Since $M$ is compact, without loss of generality $p_{k}$ converges to $p$. Since $N \backslash V_{j(i)}^{\prime}$ is closed, we get $N \backslash V_{j(i)}^{\prime} \ni \lim _{k} H\left(p_{k}, t_{k}\right)=H(p, 0)=f_{0}(p)$ and thus a contradiction as $f_{0}\left(\operatorname{supp}\left(\eta_{i}\right)\right) \subset V_{j(i)}^{\prime}$. So if we take $\delta=\min \left(\delta_{i} \mid 1 \leq i \leq r\right)>0$ we have that $H\left(\operatorname{supp}\left(\eta_{i}\right) \times[0, \delta]\right) \subset V_{j(i)}^{\prime}$ for all $1 \leq i \leq r$.

All this setup has the following goal: whether there is an $\epsilon \in(0, \delta)$ such that $\left.H\right|_{M \times\{t\}}$ for all $t<\epsilon$ is a submersion is equivalent to whether each of the finitely many functions

$$
f_{t}^{i}:=\left.\left(\phi_{j(i)}^{\prime}\right)^{-1} \circ H\right|_{\operatorname{supp}\left(\eta_{i}\right) \times\{t\}} \circ \phi_{i}
$$

has surjective total differential all points in its domain for all $t<\epsilon$.
Each $f_{t}^{i}$ is a smooth map from the compact subset $\phi_{i}^{-1}\left(\operatorname{supp}\left(\eta_{i}\right)\right) \subset \mathbb{R}^{k}$ to the open subset $V_{j(i)}^{\prime} \subset \mathbb{R}^{k^{\prime}}$. Consider now the continuous function

$$
\phi_{i}^{-1}\left(\operatorname{supp}\left(\eta_{i}\right)\right) \times[0, \delta] \ni(p, t) \longmapsto \begin{gathered}
\text { maximum of absolute value of determinants } \\
\text { of }\left(k^{\prime} \times k^{\prime}\right) \text {-submatrices of } d_{p} f_{t}^{i}
\end{gathered}
$$

The right hand side is $>0$ if and only if there is a square submatrix of full rank, which happens if and only if it is surjective. Hence we know that for $t=0$, the total derivatives at all $x \in \phi_{i}^{-1}\left(\operatorname{supp}\left(\eta_{i}\right)\right)$ are surjective and hence the above function is strictly positive. Since $\phi_{i}^{-1}\left(\operatorname{supp}\left(\eta_{i}\right)\right)$ is compact, it is bounded away from 0 for $t=0$, and by continuity thus for all $t$ in some small interval $\left[0, \epsilon_{i}\right] \subset[0, \delta]$ with $\epsilon_{i}>0$. The argument this is similar to the above argument that $H\left(\operatorname{supp}\left(\eta_{i}\right) \times\left[0, \delta_{i}\right]\right) \subset V_{j(i)}^{\prime}$ for some $\delta_{i}>0$, and I recommend you work it out yourself. Taking $\epsilon=\min \left(\epsilon_{i} \mid 1 \leq i \leq r\right)$ gives the desired $\epsilon>0$.

We may reduce the case (iv) to the case (iii) by picking finitely many local parametrizations covering the intersection of $Z$ with an open neighborhood of $f_{0}(M)$. In the coordinates coming from each of these local parametrizations, $Z$ is given by $\{0\} \times \mathbb{R}^{r}$ and by composing with the projection $\pi_{k^{\prime}-r}$ onto the first $k^{\prime}-r$ coordinates we can rephrase $f \pitchfork Z$ in terms of $\pi_{k^{\prime}-r} \circ f$ being a submersion.

For (vi) we may reduce to the case that $M$ and $N$ are connected by considering each connected component separately. But an embedding $f: M \rightarrow N$ between compact
connected manifolds of the same dimension is the same as diffeomorphism. Hence (vi) reduces to (v).

Furthermore, (v) reduces to (ii) as soon as we prove that there must exist an $\epsilon>0$ such that each $\left.H\right|_{M \times\{t\}}$ is injective for $t<\epsilon$. Suppose this is not the case, then we will derive a contradiction. Then if we define $\tilde{H}: M \times[0,1] \rightarrow N \times[0,1]$ by $\tilde{H}(p, t)=(H(p, t), t)$ we can find a collection of pairs $\left(p_{i}, t_{i}\right),\left(p_{i}^{\prime}, t_{i}\right) \in M \times[0,1]$ with $t_{i} \rightarrow 0, p_{i} \neq p_{i}^{\prime}$ and $\tilde{H}\left(p_{i}, t_{i}\right)=\tilde{H}\left(p_{i}^{\prime}, t_{i}\right)$. Using the fact that $M$ is compact, by passing to a subsequence we can assume that both sequences $p_{i}$ and $p_{i}^{\prime}$ converge to $p$ and $p^{\prime}$ in $M$. Then $f_{0}(p)=\lim H\left(p_{i}, t_{i}\right)=\lim H\left(p_{i}^{\prime}, t_{i}\right)=f_{0}\left(p^{\prime}\right)$ and since $f_{0}$ is injective $p=p^{\prime}$. We may compute that

$$
d_{(p, 0)} \tilde{H}=\left[\begin{array}{cc}
d_{p} f_{0} & * \\
0 & 1
\end{array}\right]: T_{p} M \oplus \mathbb{R} \rightarrow T_{f_{0}(p)} N \oplus \mathbb{R},
$$

which is injective. Hence $\tilde{H}$ is an embedding near ( $p, 0$ ), so in particular injective and hence $\left(p_{i}, t_{i}\right)=\left(p_{i}^{\prime}, t_{i}\right)$ for $i$ large enough, contradicting the construction of the sequences $p_{i}$ and $p_{i}^{\prime}$.

Example 11.2.5. If $Z$ is a compact submanifold of $M$, then any sufficiently small perturbation of the inclusion map $i: Z \hookrightarrow M$ is still an embedding. Concretely, when you pick any smooth function $g: S^{1} \rightarrow \mathbb{R}^{2}$, there exists some $\epsilon>0$ such that

$$
\begin{aligned}
i_{t}: S^{1} & \longrightarrow \mathbb{R}^{2} \\
p & \longmapsto p+\operatorname{tg}(p)
\end{aligned}
$$

is an embedding for $t<\epsilon$.

### 11.3 Generic classes of smooth maps

A class of smooth maps is generic if we can deform any smooth map to such a map by an arbitrarily small perturbation. It will be technically convenient to allow these perturbations to be indexed by $\mathbb{R}^{r}$ instead of $\mathbb{R}$.

Definition 11.3.1. A subset $D$ of the set of all smooth maps $M \rightarrow N$ is generic if for all $f_{0}: M \rightarrow N$ there exists an $r \geq 0$ and a smooth map $H: M \times \mathbb{R}^{r} \rightarrow N$ such that $\left.M\right|_{M \times\{0\}}=f_{0}$ and for all $\epsilon>0$ there exists an $x \in \mathbb{R}^{r}$ with $\|x\|<\epsilon$ such that $\left.H\right|_{M \times\{x\}} \in D$.

Remark 11.3.2. If we were to define a suitable topology on the set $C^{\infty}(M, N)$ of smooth maps $M \rightarrow N$, dense open subsets of $C^{\infty}(M, N)$ would be generic.

Example 11.3.3. We will later prove that if the set of all smooth maps $M \rightarrow N$ transverse to $Z$ is generic. Thus every smooth map $f: M \rightarrow N$ can be approximated by maps transverse to $Z$.

The main tool to find generic classes of smooth maps is Sard's theorem, often applied to homotopies or families of maps but incredibly useful in general:

Theorem 11.3.4 (Sard). If $f: M \rightarrow N$ is a smooth map, then the critical values of $f$ have measure zero.

A subset $C$ of $\mathbb{R}^{p}$ has measure zero if there is a countable collection of rectangles $R_{i} \subset \mathbb{R}^{p}$ such that $C \subset \bigcup_{i=1}^{\infty} R_{i}$ and $\sum_{i=1}^{\infty} \operatorname{vol}\left(R_{i}\right)<\epsilon$. A subset $C$ of $M$ has measure zero if for each chart $\left\{\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)\right\}$ of $M$ the subset $\phi_{\alpha}^{-1}(C)$ has measure zero.

Corollary 11.3.5. If $f_{i}: M \rightarrow N$ is a countable number of smooth maps, then the set of $c \in N$ which are regular values for all $f_{i}$ is dense.

Proof. The countable union $\bigcup_{i} \operatorname{crit}\left(f_{i}\right) \subset N$ of measure zero subsets has measure zero, so it suffices to observe that the complement of a measure zero subset $C$ is dense. If it were not dense, $C$ would have non-empty interior and in some chart contain a small ball of some definite volume $>0$.

Let us give some first applications of Sard's theorem:
Example 11.3.6. There are space-filling curves, continuous maps $f:[0,1] \rightarrow[0,1]^{2}$ which are surjective. However, no smooth space-filling curve can exist: a regular value of such a smooth map is a point in $[0,1]^{2}$ which is not in the image of $f$, and the regular values need to be dense in $[0,1]^{2}$ by Sard's lemma.

The following is an elaboration of that idea:
Definition 11.3.7. A path-connected differentiable manifold $M$ is said be to $m$-connected if every smooth map $f: S^{i} \rightarrow M$ is homotopic to a constant map for $i \leq m$.

Remark 11.3.8. To connect this definition to a more familiar one in algebraic topology involving continuous maps instead of smooth maps, one uses the fact that every continuous map is homotopic to a smooth one.

Corollary 11.3.9. The sphere $S^{k}$ is $(k-1)$-connected.
Proof. As before, the regular values of smooth map $f: S^{i} \rightarrow S^{k}$ for $i \leq k-1$ are those that are not in the image of $f$. Since these must be dense $f$ must miss some point $x_{0} \in S^{k}$. We can then identify $S^{k} \backslash\left\{x_{0}\right\}$ with $\mathbb{R}^{k}$ and consider $f$ as a smooth map $f: S^{i} \rightarrow \mathbb{R}^{k}$. This is homotopic to a constant map by the homotopy $H: S^{i} \times[0,1] \rightarrow \mathbb{R}^{k}$ given by $H(p, t)=t f(p)$.

Next chapter we will use Sard's lemma to improve the Whitney embedding theorem.

### 11.4 The proof of Sard's theorem

The following is the proof of Sard's theorem, Theorem 11.3.4, which is essentially a result in multivariable calculus and as such not part of the course proper. Its proof is the standard one, and is included for completeness. It needs one fact regarding sets of measure 0, a special case of Fubini's theorem. This falls within the realm of measure theory, so we will assert it without proof (but see Appendix 1 of [GP10] in the case $C$ is closed).

Lemma 11.4.1. Suppose that we are given an open subset $U \subset \mathbb{R}^{k+1}$ and a subset $C \subset U$ such that $C \cap\left(\{t\} \times \mathbb{R}^{k}\right)$ has measure 0 for all $t \in \mathbb{R}$. Then $C$ has measure 0 .

Theorem 11.4.2. The set of critical values of any smooth map $f: M \rightarrow N$ has measure 0.

Proof. When we proved that partitions of unity exist, we prove that there exists a countable collection of charts $\left\{\left(U_{i}, V_{i}, \phi_{i}\right)\right\}$ covering $M$ and a countable collection of charts $\left\{\left(U_{i(j)}^{\prime}, V_{i(j)}^{\prime}, \phi_{i(j)}^{\prime}\right)\right\}$ covering $N$ such that $f\left(V_{i}\right) \subset V_{i(j)}^{\prime}$. The set $\operatorname{Crit}(f)$ of critical values of $f$ is equal to

$$
\operatorname{Crit}(f)=\bigcup_{i} \phi_{j(i)}^{\prime}\left(\operatorname{Crit}\left(\left(\phi_{j(i)}^{\prime}\right)^{-1} \circ f \circ \phi_{i}\right)\right) .
$$

We observed in the proof of Corollary 11.3.5 that subsets of measure 0 are closed under taking countable unions, so it suffices to prove Sard's theorem for each of the functions on the right hand side. That is, it suffices to prove Sard's theorem for smooth maps $f: U \rightarrow \mathbb{R}^{k^{\prime}}$ with $U \subset \mathbb{R}^{k}$ open. We will prove this by induction over $k$.

In the case $k=0$, there are either no critical values (when $k^{\prime}=0$ ) or a single one (when $k^{\prime}>0$ ), so this initial case is true. For the induction step from $k-1$ to $k$, we let $C \subset U$ denote the set of critical points of $f$ and filter it by

$$
C \supset C_{1} \supset C_{2} \supset \cdots,
$$

letting $C_{i}$ be the subset where all partial derivatives of order $1 \leq r \leq i$ vanish. Now we will write $C$ as $\left(C \backslash C_{1}\right) \cup \bigcup_{i \geq 1} C_{i}$. We have to prove $f(C)$ has measure 0 . As $f(C)=f\left(C \backslash C_{1}\right) \cup \bigcup_{i \geq 1} f\left(C_{i}\right)$ it suffices to prove that $f\left(C \backslash C_{1}\right)$ and $f\left(C_{i}\right)$ for $i \geq 1$ have measure 0 .

This is done in three steps:
The case $f\left(C \backslash C_{1}\right)$. If $k^{\prime}=1$ then $C=C_{1}$ and there is nothing to prove, so assume $k^{\prime} \geq 2$. At $c \in C \backslash C_{1}, \frac{\partial f_{i}}{\partial x_{j}}(c) \neq 0$ for some $i$ and $j$. Without loss of generality (reordering the coordinate directions) we may assume $i=1$ and $j=1$. Define a smooth map

$$
\begin{aligned}
h: U & \longrightarrow \mathbb{R}^{k} \\
\left(x_{1}, \ldots, x_{k}\right) & \longmapsto\left(f_{1}(x), x_{2}, \ldots, x_{k}\right),
\end{aligned}
$$

which is easily seen to have bijective total derivative at $c$. Applying the inverse function theorem, we see it is a local diffeomorphism, i.e. there is an open neighborhood $V$ around $c$ such that $h$ restricts to a diffeomorphism $U \supset V \rightarrow h(V) \subset \mathbb{R}^{k^{\prime}}$.
Now consider the composition of its inverse with $f$

$$
\begin{aligned}
f \circ h^{-1}: h(V) & \longrightarrow \mathbb{R}^{k^{\prime}} \\
\left(x_{1}, \ldots, x_{k}\right) & \longmapsto\left(x_{1}, f_{2}(h(x)), \ldots, f_{n}(h(x))\right) .
\end{aligned}
$$

This sends the manifold $h(V) \cap\left(\{t\} \times \mathbb{R}^{k-1}\right)$ to $\{t\} \times \mathbb{R}^{k^{\prime}-1}$, and a point $\left(t, c^{\prime}\right)$ is a critical point of $f$ if and only if $c^{\prime}$ is a critical point of

$$
\bar{f}_{t}:=\left(f_{2}(t,-), \ldots, f_{n}(t,-)\right): h(V) \cap\left(\{t\} \times \mathbb{R}^{k-1}\right) \longrightarrow\{t\} \times \mathbb{R}^{k^{\prime}-1} .
$$

Applying the inductive hypothesis to each of these, we see that the set of critical values of $\bar{f}_{t}$ has measure zero.
Letting $C\left(\bar{f}_{t}\right)$ denote the critical points of $\bar{f}_{t}$, the application of Fubini's theorem discussed above then tells us that

$$
\bigcup_{t}\{t\} \times \bar{f}_{t}\left(C\left(\bar{f}_{t}\right)\right)
$$

also has measure 0. But that union is exactly the subset of the critical values of $g \circ h^{-1}$ where not all first order partial derivatives vanish. Since $h^{-1}$ is a diffeomorphism, these are also the subset of such critical values of $\left.g\right|_{V}$. Thus $f\left(\left(C \backslash C_{1}\right) \cap V\right)$ has measure 0 . Since a countable collection of $V$ 's cover $C \backslash C_{1}$ (using second countability of $M$ ), we conclude that $f\left(C \backslash C_{1}\right)$ has measure 0 .

The case $f\left(C_{i} \backslash C_{i+1}\right)$. Starting as in the previous case, at $c \in C_{i} \backslash C_{i+1}$ we know that $\frac{\partial^{i+1} f_{j}}{\partial x_{k_{1}} \cdots \partial x_{k_{i+1}}} \neq 0$ for some $j$ and $k_{1}$, and without loss of generality we can assume both are equal to 1 . Then we define

$$
\begin{aligned}
h: U & \longrightarrow \mathbb{R}^{k} \\
\left(x_{1}, \ldots, x_{k}\right) & \longmapsto\left(\frac{\partial^{i} f_{1}}{\partial x_{k_{2}} \cdots \partial x_{k_{i+1}}}, x_{2}, \ldots, x_{k}\right) .
\end{aligned}
$$

As before, $h$ is a diffeomorphism onto its image when restricted to an open neighborhood $V$ of $c$. It also maps $C_{i}$ into $\{0\} \times \mathbb{R}^{k-1}$, because the first entry involves an $i$ th partial derivative. Thus $f \circ h^{-1}$ only has critical points of type $C_{i}$ in $\{0\} \times \mathbb{R}^{k-1}$, and we can apply the inductive hypothesis to $\left.\left(f \circ h^{-1}\right)\right|_{\{0\} \times \mathbb{R}^{k-1}}$ to see its critical values have measure 0. An argument as in the first step finishes the argument.
The case $C_{i}$. Finally, one proves that $C_{N}$ has measure 0 for $N>k / k^{\prime}-1$. Then $C_{i}=\left(C_{i} \backslash C_{i+1}\right) \cup \cdots \cup\left(C_{N-1} \backslash C_{N}\right) \cup C_{N}$, all of which have measure 0 . To see this final case, it is convenient to assume $U=(0,1)^{k}$, with $f$ extending to an open neighborhood of $[0,1]^{k}$. We may make this assumption because countably many rescaled versions of closed cubes with these properties cover $U$. If $c \in C_{N}$, the Taylor approximation to order $\leq N$ of $f$ at $c$ vanishes, in the sense that $\|f(c+h)-f(c)\| \leq D\|h\|^{N+1}$ for some constant $D>0$ and $\|h\|<\epsilon_{0}$, cf. [DK04a, Theorem 2.8.3].
Since $C_{N}$ is closed in $[0,1]^{k}$ it is compact, and the constants $D$ and $\epsilon_{0}$ depend continuously on $c \in C_{N}$ we may find constants $D>0$ and $\epsilon_{0}>0$ that work for all $c \in C_{N}$. Then subdivide $[0,1]^{k}$ into cubes with sides $1 / L$ where $1 / L<\epsilon_{0} / 2$. Then $f$ must map of each the cubes that intersects $C_{N}$ into a disk of radius $\leq D(\sqrt{k} / L)^{N+1}$. Hence $C_{N}$ is contained in a set of volume $\leq L^{k} D^{\prime}(\sqrt{k} / L)^{k^{\prime}(N+1)}$. If $N>k / k^{\prime}-1$ the exponent $L$ is $<k-k^{\prime} k / k^{\prime}=0$, so goes this volume goes to 0 as $L \rightarrow \infty$.

### 11.5 Problems

Problem 11.5.1 (Concatenation of homotopies).
(a) Suppose that $H: M \times[0,1] \rightarrow N$ is a homotopy from $f_{0}$ to $f_{1}$. Construct a different homotopy $\tilde{H}: M \times[0,1] \rightarrow N$ from $f_{0}$ to $f_{1}$ such that $\tilde{H}(-, t)=f_{0}$ for $t<1 / 4$ and $\tilde{H}(-, t)=f_{1}$ for $t>3 / 4$. (Hint: use Problem 9.3.1.)
(b) Use part (a) to show that the relation of homotopy is transitive, i.e. $f_{0} \sim f_{1}$ and $f_{1} \sim f_{2}$ implies $f_{0} \sim f_{2}$.

Problem 11.5.2 (The fundamental group). For a smooth manifold $M$ with chosen basepoint $m_{0} \in M$, we consider the set of smooth maps $\gamma: S^{1} \rightarrow M$ sending $1 \in S^{1}$ to $m_{0} \in M$. We say that two such smooth maps $\gamma_{0}, \gamma_{1}: S^{1} \rightarrow M$ are homotopic rel endpoints if there is a homotopy $H: M \times[0,1] \rightarrow N$ from $\gamma_{0}$ to $\gamma_{1}$ such that $H(1, t)=m_{0}$ for all $t \in[0,1]$.
(a) Prove that being homotopic rel endpoints is an equivalence relation.

We denote the set of homotopy classes rel endpoints by $\pi_{1}\left(M, m_{0}\right)$, the fundamental group $M$ at $m_{0}$. As the name suggests it has a group structure, which you will construct below:
(b) Use the ideas of Problem 11.5.1 to prove that concatenation of loops gives a well-defined map

$$
\pi_{1}\left(M, m_{0}\right) \times \pi_{1}\left(M, m_{0}\right) \longrightarrow \pi_{1}\left(M, m_{0}\right)
$$

(c) Show that concatenation makes $\pi_{1}\left(M, m_{0}\right)$ into a group. (Hint: the inverse is given by reversing loops.)

Problem 11.5.3 (Spaces of planar polygons). Let us fix positive real numbers $\ell_{0}, \ldots, \ell_{n}$. Then the space of polygons with edge lengths $\ell_{0}, \ldots, \ell_{n}$ is the space of ordered $n$-tuples of points $p_{1}, \ldots, p_{n}$ in the plane $\mathbb{R}^{2}$ such that for each $1 \leq i \leq n$ the distance from $p_{i}$ to $p_{i+1}$ (with the convention that $p_{n+1}=p_{1}$ ) is given by $\ell_{i}$, up to rotation and translation.
(a) Explain why we may replace working up to rotation and translation by the conditions that $p_{1}=0$ and that $p_{n}$ lies in the half-line $\mathbb{R}_{>0}:=\left\{(x, 0) \in \mathbb{R}^{2} \mid x>\right.$ $0\} \subset \mathbb{R}^{2}$. That is, why we instead study the subspace $\mathcal{M}\left(\ell_{1}, \ldots, \ell_{n}\right)$ defined as

$$
\left\{\left(p_{1}, \ldots, p_{n}\right) \mid\left\|p_{i+1}-p_{i}\right\|=\ell_{i} \text { for } 1 \leq i \leq n, p_{1}=0, \text { and } p_{n} \in \mathbb{R}_{>0}\right\} \subset\left(\mathbb{R}^{2}\right)^{n}
$$

We will investigate for which edge lengths this is a manifold. To do so, it is helpful to introduce the topological space $\mathcal{A}\left(\ell_{1}, \ldots, \ell_{n-1}\right)$ defined as:

$$
\left\{\left(p_{1}, \ldots, p_{n}\right) \mid\left\|p_{i+1}-p_{i}\right\|=\ell_{i} \text { for } 1 \leq i \leq n-1, p_{1}=0, \text { and } p_{n} \in \mathbb{R}_{>0}\right\} \subset\left(\mathbb{R}^{2}\right)^{n}
$$

(b) Prove that $\mathcal{A}\left(\ell_{1}, \ldots, \ell_{n-1}\right)$ is a ( $n-2$ )-dimensional smooth manifold.

There is a smooth map

$$
\begin{aligned}
\pi: \mathcal{A}\left(\ell_{1}, \ldots, \ell_{n-1}\right) & \longrightarrow \mathbb{R}_{>0} \\
\left(p_{1}, \ldots, p_{n}\right) & \longmapsto \text { first coordinate of } p_{n}
\end{aligned}
$$

(c) Prove that for a dense set of $\left(\ell_{1}, \ldots, \ell_{n}\right) \subset\left(\mathbb{R}_{>0}\right)^{n}, \mathcal{M}\left(\ell_{1}, \ldots, \ell_{n}\right)$ is a $(n-3)$ dimensional smooth manifold. (Hint: Sard's theorem.)

## Chapter 12

## Two applications of Sard's theorem

In the previous chapter we proved Sard's theorem, Theorem 11.3.4: the set of critical values of a smooth map $f: M \rightarrow N$ has measure 0 . Today we give two applications: (i) the strong Whitney embedding theorem, (ii) the Brouwer fixed point theorem. This is in Sections $1 . \S 8,2 . \S 1$ and $2 . \S 2$ of [GP10], and uses results from Appendices 1 and 2 of [GP10].

### 12.1 The strong Whitney embedding theorem

Let's recall the weak Whitney embedding theorem, Theorem, Theorem 9.1.3: any compact manifold $M$ can be embedded into some Euclidean space. Today we prove the stronger statement that any compact $k$-dimensional manifold $M$ can be embedded into $\mathbb{R}^{2 k+1}$, and deduce from it that a non-compact $k$-dimensional manifold $M$ can be embedded into $\mathbb{R}^{2 k+2}$.

### 12.1.1 The compact case

Theorem 12.1.1 (Strong Whitney embedding theorem). If $M$ is a compact $k$-dimensional smooth manifold, then there exists an embedding of $M$ into $\mathbb{R}^{2 k+1}$.

This is a direct consequence of the following proposition using the weak Whitney embedding theorem (Theorem 9.1.3) and the fact that all injective immersions with compact domain are embeddings, since every continuous map with compact domain is proper (Corollary 6.2.13).

Proposition 12.1.2. If $M$ is a $k$-dimensional smooth manifold with an injective immersion of $M$ into $\mathbb{R}^{N}$ for some $N$, then there exists an injective immersion of $M$ into $\mathbb{R}^{2 k+1}$.

Proof. If $N \leq 2 k+1$ there is nothing to prove. If this is not the case, we will show that we can reduce $N$ to $2 k+1$, one dimension at a time. That is, we suppose that $N \geq 2 k+2$ and show that $M$ also has an injective immersion into $\mathbb{R}^{N-1}$. Let $i: M \rightarrow \mathbb{R}^{N}$ denote the injective immersion.

Consider the following two smooth maps

$$
\begin{aligned}
f^{\mathrm{inj}}: M \times M \backslash\{(m, m) \mid m \in M\} & \longrightarrow S^{N-1} \\
\left(p, p^{\prime}\right) & \longmapsto \frac{i(p)-i\left(p^{\prime}\right)}{\left\|i(p)-i\left(p^{\prime}\right)\right\|}, \\
f^{\mathrm{tang}}: T M \backslash 0 \text {-section } & \longrightarrow S^{N-1} \\
v & \longmapsto \frac{d i(v)}{\|d i(v)\|} .
\end{aligned}
$$

These maps were chosen because of the meaning we can ascribe to their regular values. For each $x \in S^{N-1}$ there is a linear projection $\pi_{x}: \mathbb{R}^{N} \rightarrow x^{\perp}$. If $x \notin \operatorname{im}\left(f^{\text {inj }}\right)$ then $\pi_{x} \circ i$ is injective, and if $x \notin \operatorname{im}\left(f^{\text {tang }}\right)$ then the derivative of $\pi_{x} \circ i$ is injective. In particular, if $x \notin \operatorname{im}\left(f^{\text {inj }}\right) \cup \operatorname{im}\left(f^{\text {tang }}\right)$, then $\pi_{x} \circ i: M \rightarrow x^{\perp} \cong \mathbb{R}^{N-1}$ is an injective immersion of $M$ into a Euclidean space of lower dimension.

Both $M \times M \backslash\{(m, m) \mid m \in M\}$ and $T M \backslash M$ are $2 k$-dimensional. As $N-1>2 k$, this means $x$ is disjoint from the images of $f^{\text {inj }}$ and $f^{\text {tang }}$ if and only if $x$ is a regular value of $f^{\text {inj }}$ and $f^{\text {tang }}$. By Sard's theorem, Theorem 11.3.4, such joint regular values are dense.

In fact, since the derivative is linear, to see that $\pi_{x} \circ i$ has injective differential, we only need to avoid the image of

$$
\begin{aligned}
\bar{f}^{\operatorname{tang}}:\{v \in T M \mid\|d i(v)\|=1\} & \longrightarrow S^{N-1} \\
v & \longmapsto d i(v) .
\end{aligned}
$$

Its domain is $(2 k-1)$-dimensional, so we can go one dimension further if only care about guaranteeing that the derivative remains injective. We can do a better by still picking $x$ to be a regular value of $f^{\text {inj }}: M \times M \backslash\{(m, m) \mid m \in M\} \rightarrow S^{2 k}$. In that case the intersection points of the immersion will be transverse. If $M$ is compact, then there must be a finite number of them since transverse intersection points are isolated.

Corollary 12.1.3. If $M$ is a compact $k$-dimensional smooth manifold, then there exists an immersion of $M$ into $\mathbb{R}^{2 k}$ with finitely many transverse intersections.

Example 12.1.4 (Whitney double point). We can always add more self-intersections, by inserting in a local chart one of the following maps, due to Whitney [Whi44, Section 1.2]. These are immersions with a single transverse double point that are approximately linear outside a compact set:

$$
\begin{aligned}
\alpha_{k}: \mathbb{R}^{k} & \longrightarrow \mathbb{R}^{2 k} \\
\left(x_{1}, \ldots, x_{k}\right) & \longmapsto\left(\frac{1}{u}, x_{1}-2 \frac{x_{1}}{u}, \frac{x_{1} x_{2}}{u}, x_{2}, \frac{x_{1} x_{3}}{u}, x_{3}, \cdots, \frac{x_{1} x_{k}}{u}, x_{k}\right)
\end{aligned}
$$

with $u=\left(1+x_{1}^{2}\right) \cdots\left(1+x_{k}^{2}\right)$.
Their existence is used in the proof that every compact $k$-dimensional smooth manifold embeds into $\mathbb{R}^{2 k}$ [Whi44, Theorem 5]. This is the best possible bound: $\mathbb{R} P^{2^{n}}$ does not embed in $\mathbb{R}^{2^{n+1}-1}$.

### 12.1.2 Non-compact case

We continue with a discussion of the non-compact case. It is based on a double application of Proposition 12.1.2 and the following lemma:

Lemma 12.1.5. Every smooth manifold $M$ admits a proper smooth function $\lambda: M \rightarrow$ $[0, \infty)$.

Proof. Using Lemma 9.2.2, pick compact subsets $K_{i}$ and open subsets $V_{i+1 / 2}$ of $M$ such that $K_{0} \subset V_{1 / 2} \subset K_{1} \subset V_{1+1 / 2} \subset \cdots$ and $\bigcup_{i} K_{i}=M$. Applying Theorem 9.1.2, let $\eta_{i}: M \rightarrow[0,1]$ be a partition of unity subordinate to the open cover by $V_{i+1 / 2} \backslash K_{i}-1$. Then we define

$$
\begin{aligned}
\lambda: M & \longrightarrow[0, \infty) \\
p & \longmapsto \sum_{i} i \eta_{i}(p) .
\end{aligned}
$$

This sum is locally finite so smooth, and if $\lambda(p) \leq i$ then at least one of the $\eta_{j}$ for $j \leq i$ has to be non-zero, so $p \in K_{i+1}$. Thus $\lambda^{-1}([0, i])$ is a closed subset of the compact set $K_{i+1}$ and hence $\lambda$ is proper.

Theorem 12.1.6. If $M$ is a $k$-dimensional smooth manifold, then there exists an embedding of $M$ into some Euclidean space $\mathbb{R}^{N}$.

Proof. Using Lemma 9.2.2, pick compact subsets $K_{i}$ and open subsets $V_{i+1 / 2}$ of $M$ such that $K_{0} \subset V_{1 / 2} \subset K_{1} \subset V_{1+1 / 2} \subset \cdots$ and $\bigcup_{i} K_{i}=M$. Then $K_{i+1} \backslash V_{i-1 / 2}$ is compact, and hence can be covered by finitely many charts. The proof of the weak Whitney embedding theorem then provides an injective immersion of an open neighborhood $W_{i}$ of $K_{i+1} \backslash V_{i-1 / 2}$ in $V_{i+3 / 2} \backslash K_{i-1}$ into some Euclidean space. By Proposition 12.1.2 we may assume this Euclidean space is in fact $\mathbb{R}^{2 k+1}$.

Thus we have an open cover by $W_{i} \subset M$ so that $W_{i} \cap W_{j} \neq \varnothing$ is only possible if $|i-j| \leq 2$, which come with injective immersion $\rho_{i}: W_{i} \rightarrow \mathbb{R}^{2 k+1}$. Now pick a partition of unity $\eta_{i}: M \rightarrow[0,1]$ subordinate to the $W_{i}$ 's and define smooth maps

$$
p \longmapsto \overline{\eta_{i}(p) \rho_{i}(p)}:= \begin{cases}\eta_{i}(p) \rho_{i}(p) & \text { if } p \in W_{i} \\ 0 & \text { otherwise }\end{cases}
$$

We can then define for each $i$ a new smooth map

$$
\begin{aligned}
\tilde{\rho}_{i}: M & \longrightarrow \mathbb{R}^{9(2 k+2)} \\
p & \longmapsto\left(\eta_{i}(p), \overline{\eta_{i}(p) \rho_{i}(p)}\right)
\end{aligned} \begin{aligned}
& \text { put in the } j \text { th copy of } \mathbb{R}^{2 k+2} \\
& 1 \leq j \leq 9, \text { if } i \equiv j(\bmod 9)
\end{aligned}
$$

and zeroes in all other entries, and take

$$
\begin{aligned}
\rho: M & \longrightarrow \mathbb{R}^{1+9(2 k+2)} \\
p & \longmapsto\left(\sum_{i} i \eta_{i}(p), \sum_{i} \tilde{\rho}_{i}(p)\right) .
\end{aligned}
$$

This is smooth since each sum is locally finite.
This is proper because $\sum_{i} i \eta_{i}(p)$ is proper, as in Lemma 12.1.5. For each $p$ there is an open neighborhood on which only five terms in each sum are possibly non-zero; if $p \in W_{i}$ then only the terms $i-2, i-1, i, i+1, i+2$ can be non-zero. In the second entry all of these open subsets map to a different copy of $\mathbb{R}^{2 k+2}$, so the differential is injective by the same argument as used in the weak Whitney embedding theorem.

For injectivity, we further observe that if $p \in W_{i}$ then $i-2 \leq \sum_{i} i \eta_{i}(p) \leq i+2$. That is, if $l:=\sum_{i} i \eta_{i}(p)$, then $p \in \bigcup_{j=-2}^{2} W_{[l]+j}$. From this we conclude that if $\rho(p)=\rho\left(p^{\prime}\right)$, then both $p$ and $p^{\prime}$ are in $\bigcup_{j=-2}^{2} W_{[l]+j}$. On this open subset only nine terms in the second sum are possibly non-zero, all of which map to a different copy of $\mathbb{R}^{2 k+2}$. Again we can apply the proof of the weak Whitney embedding theorem to deduce injectivity.

In fact, we can now reduce the dimension again:
Corollary 12.1.7. If $M$ is a $k$-dimensional smooth manifold, then there exists an embedding of $M$ into $\mathbb{R}^{2 k+2}$.

Proof. We start with an embedding as in the previous lemma. Proposition 12.1.2 gives us an injective immersion of $M$ into $\mathbb{R}^{2 k+1}$. If we pick a proper smooth function $\lambda: M \rightarrow[0, \infty)$ as in Lemma 12.1.5, we get an embedding $i:=(\lambda, e): M \rightarrow \mathbb{R}^{2 k+2}$.

Remark 12.1.8. In fact, by the argument on pp. 53-54 of [GP10] you can decrease the dimension once more to get an embedding $M \hookrightarrow \mathbb{R}^{2 k+1}$ by a projecting along a suitable $x \in S^{2 k+1}$.

### 12.2 Manifolds with boundary

A $k$-dimensional smooth manifold $M$ is a second countable Hausdorff space with a $k$-dimensional smooth atlas. The atlas provides a local identification of $M$ with an open subset of $\mathbb{R}^{k}$, such that transition functions are smooth.

Unfortunately, using these definitions such reasonable spaces as $D^{n}$ and $M \times[0,1]$ are not smooth manifolds, because a point in $\partial D^{n}$ resp. $M \times\{0,1\}$ does not admit an open neighborhood homeomorphic to an open subset of $\mathbb{R}^{k}$. To allow these examples, we need to broaden our scope and allow manifolds to have boundary. These are locally modeled on $[0, \infty) \times \mathbb{R}^{k-1}$ instead of $\mathbb{R}^{k}$.

### 12.2.1 Definitions

Definition 12.2.1. A $k$-dimensional smooth atlas with boundary for topological space $M$ is a collection of triples $\left(U_{\alpha}, V_{\alpha}, \phi_{\alpha}\right)$ consisting of open subsets $U_{\alpha} \subset[0, \infty) \times \mathbb{R}^{k-1}$, $V_{\alpha} \subset M$ and homeomorphisms $\phi_{\alpha}: U_{\alpha} \rightarrow V_{\alpha}$, so that $\bigcup V_{\alpha}=X$ and all maps

$$
\phi_{\beta}^{-1} \circ \phi_{\alpha}: \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \longrightarrow \phi_{\beta}^{-1}\left(V_{\alpha} \cap V_{\beta}\right)
$$

are smooth maps between open subsets of $[0, \infty) \times \mathbb{R}^{k-1}$ (they are then automatically diffeomorphisms since they have smooth inverses). The triples ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) are called charts and the maps $\phi_{\beta}^{-1} \circ \phi_{\alpha}$ are called transition functions.

Here we use that we already know what a smooth map between open subsets of $[0, \infty) \times \mathbb{R}^{k-1}$ is; it is a function which locally extends to a smooth function on an open subset of $\mathbb{R}^{k}$. All of the previously discussed machinery goes through, starting with the definitions:

Lemma 12.2.2. Every $k$-dimensional smooth atlas with boundary is contained in a unique maximal $k$-dimensional smooth atlas with boundary.

Definition 12.2.3. A $k$-dimensional smooth manifold with boundary is a Hausdorff second countable topological space $X$ with a maximal $k$-dimensional smooth atlas with boundary.

Example 12.2.4. If $M$ is a $k$-dimensional smooth manifold in the ordinary, it is also a $k$-dimensional smooth manifold with boundary. Its boundary just happens to be empty.
Example 12.2.5. If $M$ is a ( $k-1$ )-dimensional smooth manifold, then $M \times[0,1]$ is a $k$-dimensional smooth manifold with boundary.

Suppose that a diffeomorphism between open subsets of $[0, \infty) \times \mathbb{R}^{k-1}$ sends a point in $(0, \infty) \times \mathbb{R}^{k-1}$ to a point in $\{0\} \times \mathbb{R}^{k-1}$. Its derivative is bijective, so the inverse function says it is local diffeomorphism. This means that it must also hit some points in $(-\infty, 0) \times \mathbb{R}^{k-1}$, which is not allowed. Hence a diffeomorphism must send points in $\{0\} \times \mathbb{R}^{k-1}$ to points in $\{0\} \times \mathbb{R}^{k-1}$. Hence the following is a reasonable definition:

Definition 12.2.6. The boundary $\partial M$ of a $k$-dimensional smooth manifold $M$ with boundary is the subset of those points that are in the image of $\{0\} \times \mathbb{R}^{k-1}$ under a chart.

The charts of $M$ restrict to charts for $\partial M$, and we get a smooth ( $k-1$ )-dimensional atlas for $\partial M$.

### 12.2.2 Theorems

Let us now explain the modifications that need to be made to the theory when including manifolds with boundary. We will only state the results here, you should read their proofs in $2 . \S 1$ of [GP10].

We can give the definitions of a smooth map between manifolds with boundary, tangent bundles, and derivatives, as before. These behave with respect to the boundary as follows: a smooth map $f: M \rightarrow N$ between manifolds with boundary restricts to a smooth map $\partial f: \partial M \rightarrow N$. At $p \in \partial M$, the tangent space $T_{p} \partial M$ is a $(k-1)$-dimensional linear subspace of $T_{p} M$, and $d_{p} \partial f=\left.\left(d_{p} f\right)\right|_{T_{p} \partial M}$.

The pre-image theorem and Sard's lemma generalize in the following manner:
Theorem 12.2.7 (Pre-image theorem for manifolds with boundary). Let $f: M \rightarrow N$ be a smooth map with $M$ a manifold with boundary, $N$ a manifold without boundary, and $Z \subset N$ a submanifold without boundary. If $f \pitchfork Z$ and $\partial f \pitchfork Z$, then $f^{-1}(Z) \subset M$ is a manifold with boundary $\partial\left(f^{-1}(Z)\right)=(\partial f)^{-1}(Z)$.

Moreover, the codimension of $f^{-1}(Z)$ is equal to the codimension of $Z$ and $T f^{-1}(Z)=$ $d f^{-1}(T Z)$.

Theorem 12.2.8 (Sard's theorem for manifolds with boundary). For any smooth map $f: M \rightarrow N$ with $M$ a manifold with boundary and $N$ a manifold without boundary, the subset of points in $N$ which are critical values of either $f$ or $\partial f$ has measure 0 .

### 12.3 The Brouwer fixed point theorem

The Brouwer fixed point theorem says that every continuous map $F: D^{n} \rightarrow D^{n}$ has a fixed point. This is deduced from the theorem that there are no continuous maps $f: D^{n} \rightarrow \partial D^{n}$ which are the identity on $\partial D^{n}$.

We will prove a version of this result, which is stronger because it concerns all manifolds with boundary, but weaker because it concerns only smooth maps. The latter is however easily remedied by the use of certain smooth approximation results.

To prove our generalization we use another fact, which is proven in Appendix 2 of [GP10] or the Appendix of [Mi197].

Theorem 12.3.1 (Classification of 1-dimensional manifolds). Every compact connected 1 -dimensional manifold is diffeomorphic to either $S^{1}$ or $[0,1]$.

Corollary 12.3.2. The boundary of every compact 1 -dimensional manifold is an even number of points.

Using this trivial observation, we prove Hirsch's generalization of the Brouwer fixed point theorem:

Theorem 12.3.3 (Hirsch). Let $M$ be a compact manifold with boundary. Then there is no smooth map $M \rightarrow \partial M$ which is the identity on $\partial M$.

Proof. Suppose for the sake of contradiction that such an $f: M \rightarrow \partial M$ does exist. By Sard's theorem we can pick an $p \in \partial M$ which is a regular value of both $f$ and $\partial f$. This means that $f^{-1}(p) \subset M$ is a 1 -dimensional manifold with boundary. It is closed in a compact space hence compact, and thus by Theorem 12.3.1 has an even number of boundary points. But $\partial f^{-1}(p)=(\partial f)^{-1}(p)=\{p\}$ since $\partial f=\operatorname{id}_{\partial M}$. This is a contradiction.

Remark 12.3.4. One easily generalizes this proof to say that there is no smooth map $M \rightarrow \partial M$ which is injective on $\partial M$.

Let us deduce from this the Brouwer fixed point theorem for smooth maps:
Corollary 12.3.5 (Smooth Brouwer fixed point theorem). If $F: D^{n} \rightarrow D^{n}$ is a smooth map, it has a fixed point.

Proof. For a proof by contradiction, we suppose that $F$ has no fixed points. Then

$$
\begin{aligned}
f: D^{n} & \longrightarrow \partial D^{n} \\
x & \longmapsto \text { intersection with } \partial D^{n} \text { of half-line starting at } F(x) \text { through } x
\end{aligned}
$$

is a well-defined smooth function $f: D^{n} \rightarrow \partial D^{n}$ that is the identity on $\partial D^{n}$.


Figure 12.1 The map $f$ in the proof of the Brouwer fixed point theorem.

Example 12.3.6. There is an anecdotal application of the Brouwer fixed point theorem to physics. Trying to balance a pencil on a table, it seems intuitive that there is an equilibrium point. You can of course prove this in an idealized setting, but it seems hard if we use some realistic model of the forces acting upon and within the pencil.

Suppose there is no equilibrium point, then the pencil would always fall with eraser facing some direction. This gives a map from the upper hemisphere $S_{+}^{2}$ to $S^{1}$, which is clearly the identity on the boundary. The claim is that the Brouwer fixed point theorem rules this out, so an equilibrium point must exist. However, it is far from obvious that the described map is continuous (see the section "Courant-Robbins Train" of [Ste11]).

### 12.4 Problems

Problem 12.4.1 (Classification of 1-dimensional manifolds). Read Appendix 2 of [GP10] and the Appendix of [Mil56b]. Which proof of the classification of 1-dimensional smooth manifolds do you prefer, and why?

## Chapter 13

## Transverse maps are generic

Today we prove a result announced in Chapter 11: the set of smooth maps $f: M \rightarrow N$ transverse to $Z \subset M$ is generic. As an application we deduce the tubular neighborhood theorem. This is $2 . \S 3$ of [GP10].

### 13.1 Transverse maps are generic

Recall the following definition:
Definition 13.1.1. A subset $D$ of the set of all smooth maps $M \rightarrow N$ is generic if for all $f_{0}: M \rightarrow N$ we can find a perturbation $H: M \times \mathbb{R}^{r} \rightarrow N$ with $\left.H\right|_{M \times\{0\}}=f_{0}$ so that for all $\epsilon>0$ there exists an $x \in \mathbb{R}^{r}$ with $\|x\|<\epsilon$ such that $\left.H\right|_{M \times\{x\}} \in D$.

The pertubation $H: M \times \mathbb{R}^{r} \rightarrow N$ is a particular example of a family of smooth maps as below, where $S=\mathbb{R}^{r}$. This is just a change of perspective; we think of a $F$ not as a single map $M \times S \rightarrow N$ but a collection of maps $M \rightarrow N$ parametrized by $S$.

Definition 13.1.2. Let $S$ be a smooth manifold, then a family of smooth maps $M \rightarrow N$ indexed by $S$ is a smooth map $F: M \times S \rightarrow N$.

Since the restriction of a smooth map to a submanifold is smooth, $f_{s}:=\left.F\right|_{M \times\{s\}}$ is a smooth map for each $s \in S$.

Theorem 13.1.3. Suppose that $F: M \times S \rightarrow N$ is a family of smooth maps $M \rightarrow N$, where $M$ may have boundary but $S$ and $N$ do not. Let $Z \subset N$ be a submanifold without boundary. If $F \pitchfork Z$ and $\partial F \pitchfork Z$, then there is a dense set of $s \in S$ such that $f_{s} \pitchfork Z$ and $\partial f_{s} \pitchfork Z$.

As usual when applying Sard's theorem, we will actually prove that the complement of those $s \in S$ such that $f_{s} \pitchfork Z$ and $\partial f_{s} \pitchfork Z$ has measure zero.

Proof. Let $W=f^{-1}(Z) \subset M \times S$, a submanifold with boundary $\partial W=W \cap(\partial M \times S)$ by the improved preimage theorem. Thus we can ask for regular values of the restriction $\left.\pi\right|_{W}: W \rightarrow S$ of the projection $M \times S \rightarrow S$, as well as its restriction $\left.\pi\right|_{\partial W}: \partial W \rightarrow S$ to the boundary. Such common regular values are dense by Sard's theorem.

We claim that $f_{s} \pitchfork Z$ if and only if $s$ is a regular value of $\left.\pi\right|_{W}$, and similarly $\partial f_{s} \pitchfork Z$ if and only if $s$ is a regular value of $\left.\pi\right|_{\partial W}$. Let us only prove the first equivalence, the second one being similar.

Let us first use the hypothesis that $F \pitchfork Z$, at $(p, s) \in W \subset M \times S$ mapping to $z \in Z$ under $F$. Then the projections induce a linear isomorphism $T_{p, s}(M \times S) \cong T_{p} M \oplus T_{s} S$, and transversality exactly means that

$$
d_{(p, s)} F\left(T_{p} M \oplus T_{s} S\right)+T_{z} Z=T_{z} N .
$$

By the preimage theorem, we may describe $T_{(p, s)} W$ as $\left(d_{(p, s)} F\right)^{-1}\left(T_{z} Z\right) \subset T_{p} M \oplus T_{s} S$. The derivative $\left.d_{(p, s)} \pi\right|_{W}: T_{(p, s)} W \rightarrow T_{s} S$ is the restriction of projection $T_{p} M \oplus T_{s} S \rightarrow T_{s} S$ to this subspace.

We next want to show that $\left.d_{(p, s)} \pi\right|_{W}: T_{(p, s)} W \rightarrow T_{s} S$ is surjective if and only if $d_{(p, s)} F\left(T_{p} M\right)+T_{z} Z=T_{z} N$. This is the linear-algebraic lemma following this proof, applied with $U=T_{p} M, U^{\prime}=T_{s} S, V=T_{z} N, W=T_{z} Z, T=d_{(p, s)} F$.

Finally, observe that because $d_{(p, s)} F\left(T_{p} M\right)=d_{p} f_{s}\left(T_{p} M\right)$, the statement $d_{(p, s)} F\left(T_{p} M\right)+$ $T_{z} Z=T_{z} N$ is true if and only if $f_{s} \pitchfork Z$ at $z$.

Lemma 13.1.4. If $T: U \oplus U^{\prime} \rightarrow V$ is a linear map of finite-dimensional vector spaces, $W \subset V$ such that $T\left(U \oplus U^{\prime}\right)+W=V$. Then $\pi_{2}: T^{-1}(W) \rightarrow U^{\prime}$ is surjective if and only if $T(U)+W=V$.

Proof. For $\Rightarrow$; if $\pi_{2}: T^{-1}(W) \rightarrow U^{\prime}$ is surjective, it admits a section $s: U^{\prime} \rightarrow T^{-1}(W)$. Then we have

$$
T\left(u+u^{\prime}\right)+W=T\left(u+u^{\prime}\right)+T\left(-s\left(u^{\prime}\right)\right)+W=T\left(u+u^{\prime}-s\left(u^{\prime}\right)\right)
$$

and since $\pi_{2}\left(u+u^{\prime}-s\left(u^{\prime}\right)\right)=0, u+u^{\prime}-s\left(u^{\prime}\right) \in U$.
For $\Leftarrow$, take $u^{\prime} \in U^{\prime}$ and note that because $T(U)+W=V$ we can find $u \in U$ and $w \in W$ such that $T\left(u^{\prime}\right)=T(u)+w$. Then $u^{\prime}-u \in T^{-1}(W)$ and $\pi_{2}\left(u^{\prime}-u\right)=u^{\prime}$.

We will now prove the maps transverse to $Z$ are generic by showing that for every $f_{0}: M \rightarrow N$ there exists a smooth map $F: M \times \mathbb{R}^{r} \rightarrow N$ such that $\left.F\right|_{M \times\{0\}}=f_{0}$ and which satisfies $F \pitchfork Z$ and $\partial F \pitchfork Z$.

To construct $F$ we shall embed $N$ into an Euclidean space $\mathbb{R}^{r}$ using the Whitney embedding theorem, Theorem 12.1.6, and consider the rather uninteresting family

$$
\begin{aligned}
\tilde{F}: M \times \mathbb{R}^{r} & \longrightarrow \mathbb{R}^{r} \\
(p, s) & \longmapsto f_{0}(p)+s
\end{aligned}
$$

This is obviously a submersion so transverse to the submanifold $Z \subset N \subset \mathbb{R}^{r}$, and by the previous theorem there is a dense set of $s \in \mathbb{R}^{r}$ such that $\tilde{f}_{s}:=\left.\tilde{F}\right|_{M \times\{s\}}$ is transverse to $Z$.

The obvious problem is now that $\tilde{f}_{s}$ doesn't map $M$ into $N$ any more. To fix this, we shall use the following theorem to "project back into $N$ ":

Theorem 13.1.5 (Regular neighborhood theorem). For every submanifold $N \hookrightarrow \mathbb{R}^{r}$ without boundary, there exists an open neighborhood $U \subset \mathbb{R}^{r}$ of $N$ with a submersion $\pi_{N}: U \rightarrow N$ that is the identity on $N$. At $n \in N \subset U$, the linear map $d_{n} \pi: T_{n} \mathbb{R}^{r}=$ $T_{n} N \oplus T_{n} N^{\perp} \rightarrow T_{n} N$ is given by orthogonal projection onto $T_{n} N$.

Remark 13.1.6. In fact, if $M$ is compact $U$ can be obtained by picking a small enough $\epsilon>0$, letting $U$ be the set of points of distance $<\epsilon$ to $N$ and $\pi_{N}$ be the map sending $x \in U$ to the unique closest point in $N$ (so implicitly we are saying you can find an $\epsilon>0$ such that this exists and is unique). This follows from the proof of Theorem 13.3.4. For non-compact $M, \epsilon$ is replaced by a smooth positive-valued function.

We shall prove the regular neighborhood theorem in Section 13.3, and first finish the proof of genericity.

Theorem 13.1.7. Suppose $M$ is a manifold possibly with boundary, $N$ is a manifold without boundary and $Z \subset N$ is a submanifold without boundary. If $f_{0}: M \rightarrow N$ is a smooth map, then there exists an $r \geq 0$ and a smooth map $H: M \times \mathbb{R}^{r} \rightarrow N$ starting at $f_{0}$ so that for all $\epsilon>0$ there exists an $x \in \mathbb{R}^{r}$ with $\|x\|<\epsilon$ such that $\left.H\right|_{M \times\{x\}}$ is transverse to $Z$.

Proof. Embed $N$ into a Euclidean space $\mathbb{R}^{r}$ and identify $N$ with its image in $\mathbb{R}^{r}$. Take $U \subset \mathbb{R}^{r}$ and $\pi_{N}: U \rightarrow N$ as in the regular neighborhood theorem. Since $U \subset N$ is an open neighborhood, we can find a smooth function $\epsilon: N \rightarrow(0, \infty)$ such that for each $p^{\prime} \in N$ and $x \in \mathbb{R}^{r}$ satisfying $\|x\|<\epsilon\left(p^{\prime}\right), p^{\prime}+x \in U$. Then we define the smooth map

$$
\begin{aligned}
F: M \times \mathbb{R}^{r} & \longrightarrow N \\
(p, s) & \longmapsto \pi_{N}\left(f_{0}(p)+\epsilon\left(f_{0}(p)\right) \frac{s}{1+\|s\|^{2}}\right)
\end{aligned}
$$

By construction, $\left.F\right|_{M \times\{0\}}=\pi_{N} \circ f_{0}=f_{0}$ because $\pi_{N}$ is the identity on $N$.
Since $\pi_{N}$ is a submersion, $F$ is a submersion if and only if the map $M \times \mathbb{R}^{r} \rightarrow U$ given by $(p, s) \mapsto f_{0}(p)+\epsilon\left(f_{0}(p)\right) \frac{s}{1+\|s\|^{2}}$ is. But when we fix $p \in M$ this is a diffeomorphism of $\mathbb{R}^{r}$ onto a little ball, so has surjective differential at each point in $M \times \mathbb{R}^{r}$. The same argument shows that $\partial F: \partial M \times \mathbb{R}^{r} \rightarrow N$ is a submersion.

Now that we have established that $F$ and $\partial F$ are submersions, they are clearly transverse to $Z$ and Theorem 13.1.3 gives the desired conclusion.

Picking a point $s \in \mathbb{R}^{r}$ such that $\left.F\right|_{M \times\{s\}} \pitchfork Z$ and $\left.\partial F\right|_{\partial M \times\{s\}} \pitchfork Z$, the homotopy $H: M \times[0,1] \rightarrow N$ given by $(p, t) \mapsto F(p, t s)$ proves:

Corollary 13.1.8. Suppose $M$ is a manifold possibly with boundary, $N$ is a manifold without boundary and $Z \subset N$ is a submanifold without boundary. Then any smooth map $f_{0}: M \rightarrow N$ is homotopic to $f_{1}: M \rightarrow N$ satisfying $f_{1} \pitchfork Z$ and $\partial f_{1} \pitchfork Z$.

### 13.2 Isotoping submanifolds

In the Chapter 1, we discussed how to deform embeddings. This is the notion of isotopy, intuitively a one-parameter family of embeddings. Let us recall the definition:

Definition 13.2.1. A homotopy $H: M \times[0,1] \rightarrow N$ is an isotopy if $M \times[0,1] \ni(x, t) \mapsto$ $(H(m, t), t) \in N \times[0,1]$ is an embedding.

This is implied by $H$ being a smooth proper map such that $\left.H\right|_{M \times\{t\}}$ is an embedding for all $t \in[0,1]$, as then the map $M \times[0,1] \rightarrow N \times[0,1]$ is a proper injective immersion. Note that if $M$ is compact, we may drop the hypothesis that this map is proper.

Suppose we are given two submanifolds $Y, Z \subset M$ without boundary, with $Y$ compact. We can then consider the inclusion $i: Y \hookrightarrow M$ as a smooth map. By Theorem 13.1.7 we can find a map $F: Y \times \mathbb{R}^{r} \rightarrow M$ such that the set of $s \in \mathbb{R}^{r}$ such that $\left.F\right|_{Y \times\{s\}}: Y \rightarrow M$ is transverse to $Z$, is dense.

Since the class of maps transverse to $Z$ are stable when the domain is compact, Theorem 11.2.4 (iv), we can find $\epsilon>0$ such that $\left.F\right|_{Y \times\{s\}}: Y \rightarrow M$ is still an embedding if $\|s\|<\epsilon$. Take such an $s$ with $\left.H\right|_{Y \times\{s\}} \pitchfork Z$. Then the homotopy

$$
\begin{aligned}
H: Y \times[0,1] & \longrightarrow M \\
(y, t) & \longmapsto F(y, t s)
\end{aligned}
$$

is an isotopy of embeddings of $Y$ into $M$ from $i$ to an embedding transverse to $Z$. In other words, the maps $\left.H\right|_{Y \times\{t\}}$ tells us how to move submanifold $Y$ to a new position at which it is transverse to $Z$.

If $Y$ is $r$-dimensional and $Z$ is $s$-dimensional, satisfying $r+s<k$, then $Y$ is transverse to $Z$ if and only if $Y \cap Z=\varnothing$. Thus we have shown that in these conditions any two submanifolds can be made disjoint by moving one of them.
Example 13.2.2. Suppose we take $S^{1}=\left\{(x, y, 0) \mid x^{2}+y^{2}\right\} \subset \mathbb{R}^{3}$ and any other embedding $i: S^{1} \rightarrow \mathbb{R}^{3}$. This gives us two submanifolds of $\mathbb{R}^{3}$ which are diffeomorphic to $S^{1}$. They may very well be linked in a complicated way in $\mathbb{R}^{3}$. However, if we increase the dimension by 1 they become unlinked.

That is, we claim that we can isotope $i\left(S^{1}\right) \subset \mathbb{R}^{4}$ in the complement of $S^{1} \subset \mathbb{R}^{4}$ so that it becomes disjoint from the disk $D^{2} \subset \mathbb{R}^{4}$. This follows by applying the above observations with $Y=i\left(S^{1}\right), Z=D^{2} \backslash S^{1}$ and $N=\mathbb{R}^{4} \backslash S^{1}$, as the dimensions of $Y$ and $Z$ add up to $3<4$.

### 13.3 The regular neighborhood theorem

It remains to prove Theorem 13.1.5. This uses a new vector bundle associated to a submanifold $Z \subset M$, the normal bundle. Over $Z$ we have two vector bundles, the trivial bundle $\left.T M\right|_{Z}$ and its subbundle $T Z$.

Definition 13.3.1. The normal bundle $N Z$ is the vector bundle over $Z$ given by $\left.T M\right|_{Z} / T Z$.

When $M=\mathbb{R}^{r}$ and $Z=N$, this admits a more concrete definition. In that case $\left.T \mathbb{R}^{r}\right|_{N}=N \times \mathbb{R}^{r}$ and comes with a preferred inner product on each fiber (the restriction of usual Euclidean inner product). The orthogonal complements $\left(T_{p} N\right)^{\perp}$ assemble to a vector bundle $T N^{\perp}$ over $N$, explicitly given by

$$
\left\{(p, v) \in N \times \mathbb{R}^{r} \mid v \perp T_{p} N\right\}
$$

Orthogonal projection gives a map $\left.T \mathbb{R}^{r}\right|_{N} \rightarrow T N^{\perp}$ whose kernel is exactly $T N$. Thus there is an induced isomorphism

$$
\left.T \mathbb{R}^{r}\right|_{N} / T N \xrightarrow{\cong} T N^{\perp}
$$

of vector bundles over $N$.
Example 13.3.2. Let us verify $T N^{\perp}$ is a vector bundle. Suppose we have a local trivialization $\phi: N \cap V \cong U \cap \mathbb{R}^{k^{\prime}} \subset \mathbb{R}^{r}$. For $x \in \mathbb{R}^{k^{\prime}}$, the bilinear map $\left(v, v^{\prime}\right) \mapsto\left\langle d_{x} \phi(v), d_{x} \phi\left(v^{\prime}\right)\right\rangle$ is an inner product on $T_{x} \mathbb{R}^{r}$. We can think of this as a symmetric matrix $A_{x}$ whose entries vary smoothly with $x ; v \cdot A_{x} v^{\prime}=\left\langle d_{x} \phi(v), d_{x} \phi\left(v^{\prime}\right)\right\rangle$. Every positive semidefinite symmetric matrix $A$ has a unique decomposition $A=B^{t} B$ with $B$ again positive semidefinite, and the entries of $B$ depend smoothly on those of $A$. Thus we can identify $T N^{\perp}$ with the subbundle

$$
\left\{\left(x, B^{-1} v\right) \mid x \in \mathbb{R}^{k^{\prime}}, v \in\{0\} \times \mathbb{R}^{k^{\prime}-r}\right\} \subset\left(\mathbb{R}^{k^{\prime}} \times \mathbb{R}^{r}\right)
$$

visibly admitting a local trivialization.
Furthermore, it is clear from this description that the transitions between local trivializations are smooth, so $T N^{\perp}$ is a smooth vector bundle. In particular, $T N^{\perp}$ is a manifold and the projection map $\pi: T N^{\perp} \rightarrow N$ is a submersion.

We now prove the regular neighborhood theorem, which said that given a $N \hookrightarrow \mathbb{R}^{r}$ without boundary, there exists an open neighborhood $U \subset \mathbb{R}^{r}$ of $N$ and a submersion $\pi_{N}: U \rightarrow N$ that is the identity on $N$. Furthermore, the linear map $d_{p} \pi: T_{p} \mathbb{R}^{r}=$ $T_{p} N \oplus T_{p} N^{\perp} \rightarrow T_{p} N$ is given by orthogonal projection onto $T_{p} N$.

Proof of Theorem 13.1.5. Define the smooth map

$$
\begin{aligned}
h: T N^{\perp} & \longrightarrow \mathbb{R}^{r} \\
(p, v) & \longmapsto p+v .
\end{aligned}
$$

Because $T N^{\perp}$ is $r$-dimensional, so is the tangent space $T_{(p, 0)} N^{\perp}$. As the manifold $T N^{\perp}$ contains the submanifolds $N \times\{0\}$ and $\{p\} \times T_{p} N^{\perp}$, which intersect only at $(p, 0)$, $T_{(p, 0)} N^{\perp}$ contains their tangent spaces at $(p, 0)$, given by $T_{p} N$ and $T_{p} N^{\perp}$ respectively. This gives a linear map

$$
T_{p} N \oplus T_{p} N^{\perp} \longrightarrow T_{(p, 0)} T N^{\perp}
$$

which we claim is an isomorphism. Since both sides have the same dimension, and this map is an inclusion on each summand, it suffices to prove that $T_{p} N$ and $T_{p} N^{\perp}$ intersect only in $\{0\}$. This follows from the fact that the map $d_{(p, 0)} \pi: T_{(p, 0)} N^{\perp} \rightarrow T_{p} N$ is the identity on $T_{p} N$ and 0 on $T_{p} N^{\perp}$

With respect to this direct sum decomposition, the linear map

$$
d_{(p, 0)} h: T_{(p, 0)} T N^{\perp} \longrightarrow \mathbb{R}^{r}
$$

is given by sending the summand $T_{p} N$ onto $T_{p} N \subset T_{p} \mathbb{R}^{r}$ and the summand $T_{p} N^{\perp}$ to $T_{p} N^{\perp} \subset T_{p} \mathbb{R}^{r}$. In particular, it is bijective.

By the inverse function theorem, Theorem 6.1.1, it is a local diffeomorphism near $N$. As it is an embedding on $N$, it is injective on an open neighborhood $V$ of $N$ by Lemma
13.3.3 (take $A=N, M=T N^{\perp}, N=\mathbb{R}^{r}$ ). Let $U:=h(V)$, an open subset of $N$ in $\mathbb{R}^{r}$, and set

$$
\pi_{N}:=\pi \circ h^{-1}: U \longrightarrow V \longrightarrow N
$$

Since $\pi_{N}$ is a composition of a diffeomorphism and a submersion, it is a submersion. Since $\pi$ and $h$ are the identity on $N$, so is $\pi_{N}$. To prove the addendum, it remains to observe that $d_{(p, 0)} \pi: T_{(p, 0)} T N^{\perp} \cong T_{p} N \oplus T_{p} N^{\perp} \rightarrow T_{p} N$ is projection onto the first summand.

Lemma 13.3.3. If $A \subset M$ is closed and $f: M \rightarrow N$ is a smooth map which is a local diffeomorphism near $A$ and injective on $A$, then $f$ is injective near $A$.

Proof. We first prove the case that $A$ is compact. For contradiction, suppose there is pair of sequence of points $p_{i} \in M, p_{i}^{\prime} \in M$ so that $p_{i} \neq p_{i}^{\prime}, f\left(p_{i}\right)=f\left(p_{i}^{\prime}\right)$, which get arbitrarily close to $A$. By compactness of $A$, we may assume they converge: $p_{i} \rightarrow p \in A$ and $p_{i}^{\prime} \rightarrow p^{\prime} \in A$. Then by continuity $f(p)=f\left(p^{\prime}\right)$, so $p=p^{\prime}$ since $f$ is injective on $p$. But since $f$ is a local diffeomorphism near $p$ it is injective near $p$ and hence $p_{i}=p_{i}^{\prime}$ for $i$ large enough.

In general, take the subset $D=\left\{\left(p, p^{\prime}\right) \in M \times M \mid p \neq p^{\prime}, f(p)=f\left(p^{\prime}\right)\right\}$. By assumption on $A$, it is disjoint from $A \times A$. Its closure is contained in the union of $D$ with the diagonal, but the local diffeomorphism condition implies that every point in the diagonal has an open neighborhood disjoint from $D$. Thus $D$ is closed and its complement is open. By exhausting $M$ with compact subsets, e.g. using Lemma 9.2.2, and applying the above argument, this open subset contains a product neighborhood $W_{p, q} \times W_{p, q}^{\prime} \subset M \times M$ of each point $(p, q) \in A \times A$; by replacing $W_{p, q}$ with $W_{p, q} \cap W_{q, p}^{\prime}$ we may assume that $W_{p, q}=W_{q, p}$ for all $(p, q) \in A \times A$. Then $\bigcup_{p, q} W_{p, q} \subset M$ is the desired open neighborhood.

### 13.3.1 The tubular neighborhood theorem

We will now slightly generalize Theorem 13.1 .5 , replacing $\mathbb{R}^{r}$ with an arbitrary manifold and choosing a smaller but nicer neighborhood:

Theorem 13.3.4 (Tubular neighborhood theorem). For every submanifold $Z \hookrightarrow N$, there is an open neighborhood $W$ of $Z$ in $N$ and a diffeomorphism $\phi: N Z \longrightarrow W$ that is the identity on $Z$.

Proof. Given an embedding $N \hookrightarrow \mathbb{R}^{r}$, let $U, V$ and $\pi_{N}$ be as in the proof of the regular neighborhood theorem. We can then identify $N Z$ with the orthogonal complement $T Z^{\perp}$ of $T Z$ in $T N$. The only thing we will use of this observation is that the orthogonal projection map $\left.T N\right|_{Z} \rightarrow N Z$ has a section.

Define the smooth map

$$
\begin{aligned}
\tilde{h}: N Z & \longrightarrow \mathbb{R}^{r} \\
(p, v) & \longmapsto p+v .
\end{aligned}
$$

and take $W^{\prime}=\tilde{h}^{-1}(V)$.

The map $\pi_{N} \circ \tilde{h}$ has bijective differential at the 0 -section, because by the chain rule it is the composition of $d_{p} \tilde{h}: T_{p} Z \oplus N Z \cong T_{p} N \rightarrow T_{p} N$, which is the identity, and $T_{p} \pi_{N}: T_{p} \mathbb{R}^{r} \cong T_{p} N \oplus T_{p} N^{\perp} \rightarrow T_{p} N$ the projection onto the first summand. It also is the identity on $N$.

By the same argument as before, we find an open neighborhood $W^{\prime \prime}$ of the 0 -section in $N Z$ on which is an embedding. We can find a smooth function $\epsilon: Z \rightarrow(0, \infty)$ such that

$$
W=\{(p, v) \in N Z \mid\|v\|<\epsilon(p)\} \subset W^{\prime \prime},
$$

where $\|-\|$ is the norm from the inner product on $T Z^{\perp} \subset \mathbb{R}^{r}$. The diffeomorphism is given by

$$
\begin{aligned}
\phi: \nu_{Z} & \longrightarrow W \\
(p, v) & \longmapsto\left(p, \epsilon(p) \frac{v}{1+\|v\|^{2}}\right) .
\end{aligned}
$$

This completes the proof.

### 13.3.2 Collars

In a manifold $M$ with boundary $\partial M$, the boundary admits particularly nice open neighborhoods:

Definition 13.3.5. A collar of $\partial M$ is a open neighborhood $V \subset M$ of $\partial M$ with a diffeomorphism $\phi: V \rightarrow \partial M \times[0,1)$ that is the identity on $\partial M$.

Theorem 13.3.6. Every manifold with boundary admits a collar.
We will construct the two components $V:[0,1)$ and $V \rightarrow \partial M$ independently.
Lemma 13.3.7. There exists a smooth map $\chi: M \rightarrow[0, \infty)$ such that (i) $\chi^{-1}(0)=\partial(M)$ and (ii) for each $p \in \partial M$ there exists a $v \in T_{p} M \backslash T_{p} \partial M$ with $d \chi(v) \neq 0$.
Proof. Pick charts $\phi_{\alpha}: \mathbb{R}^{k-1} \times[0, \infty) \supset U_{\alpha} \rightarrow V_{\alpha} \subset M$ whose codomains cover $M$. The local coordinates gives a smooth function

$$
\begin{aligned}
f_{\alpha}: V_{\alpha} & \longrightarrow[0, \infty) \\
p & \longmapsto \pi_{2} \circ \phi_{\alpha}^{-1}(p),
\end{aligned}
$$

with $\pi_{2}: \mathbb{R}^{k-1} \times[0, \infty) \rightarrow[0, \infty)$ the projection onto the second coordinate.
Let us now pick an partition of unity subordinate to the open cover $\left\{V_{\alpha}\right\}$, given by smooth functions $\lambda_{\alpha}: V_{\alpha} \rightarrow[0,1]$. The function $\lambda_{\alpha} f_{\alpha}$ extends by zero to a smooth function $\overline{\lambda_{\alpha} f_{\alpha}}: M \rightarrow[0, \infty)$. Then the function

$$
\begin{aligned}
\chi: M & \longrightarrow[0, \infty) \\
p & \longmapsto \sum_{\alpha} \overline{\lambda_{\alpha} f_{\alpha}}(p)
\end{aligned}
$$

has the desired properties. We will leave the verification of this to the reader.

Lemma 13.3.8. There exists an open neighborhood $U \subset M$ of $\partial M$ with a smooth map $r: V \rightarrow \partial M$ that is the identity on $\partial M$.

Proof. The weak Whitney embedding theorem, Theorem 9.1.3, also holds for manifolds with boundary, so we may pick an embedding $e: M \hookrightarrow \mathbb{R}^{N}$ and consider $M$ as a submanifold of Euclidean space. We may then apply the regular neighborhood theorem, Theorem 13.1.5, to $\partial M$, resulting in an open neighborhood $U \subset \mathbb{R}^{N}$ of $\partial M$ with a smooth map $\pi_{\partial M}: U \rightarrow \partial M$ that is the identity on $\partial M$. We then have $V:=U \cap M$ and $r=\left.\pi_{\partial M}\right|_{V}$.

Proof of Theorem 13.3.6. We may combine $\left.\chi\right|_{V}$ and $r$ to a smooth map

$$
f:=r \times\left.\chi\right|_{V}: V \longrightarrow M \times[0, \infty)
$$

By construction, this is the identity and has bijective derivative on $\partial M$. By the inverse function theorem, it is thus a local diffeomorphism near $\partial M$. As a consequence of Lemma 13.3.3, it is injective onto some smaller open neighborhood $V^{\prime}$ of $\partial M$. Picking a smooth function $\epsilon: \partial M \rightarrow(0, \infty)$ such that $\{(q, t) \in \partial M \times[0, \infty) \mid t \in[0, \epsilon(q))\} \subset f\left(V^{\prime}\right)$. Setting

$$
\begin{aligned}
U & :=f^{-1}(\{(q, t) \in \partial M \times[0, \infty) \mid t \in[0, \epsilon(q))\}) \\
\phi & :=\left.f\right|_{U}
\end{aligned}
$$

is the desired diffeomorphism.
Collars are unique up to isotopy. They have great use in reducing questions about manifolds with boundary to separate questions about the boundary and the interior.

### 13.4 Problems

Problem 13.4.1 (Transversality and normal bundles). Let $Y, Z \subset N$ be submanifolds. Prove that $Y \pitchfork Z$ if and only if for all $p \in Y \cap Z, N_{p} Y \cap N_{p} Z=\{0\}$.

Problem 13.4.2 (Collared embeddings). Use collars to prove that there exists an embedding $e: M \rightarrow \mathbb{R}^{N} \times[0, \infty)$ such that $e^{-1}\left(\mathbb{R}^{N} \times\{0\}\right)=\partial M$.

Problem 13.4.3 (Smooth maps and submanifolds). Suppose that $X \subset M$ is a submanifold. Prove that a continuous map $f: X \rightarrow N$ is smooth if and only if it extends to a smooth amp $\tilde{f}: M \rightarrow N$.

Problem 13.4.4 (Smooth approximation). It is a consequence of the Stone-Weierstrass approximation theorem that for all open subsets $U \subset \mathbb{R}^{k}$, compact subsets $K \subset U$, $\epsilon>0$, and continuous maps $f: U \rightarrow \mathbb{R}$, there exists a smooth map $g: U \rightarrow \mathbb{R}$ such that $|g(x)-f(x)|<\epsilon$ for all $x \in K$.
(a) Prove that for each compact $k$-dimensional smooth manifold $M, \epsilon>0$, and continuous map $f: M \rightarrow \mathbb{R}$, there exists a smooth map $g: M \rightarrow \mathbb{R}$ such that $|g(x)-f(x)|<\epsilon$ for all $x \in M$.
(b) Is this result still true when we drop the assumption that $M$ is compact?

Problem 13.4.5 (Gluing manifolds with boundary). Suppose that $M_{0}$ and $M_{1}$ are $d$ dimensional smooth manifolds, and that we are given a diffeomorphism $\varphi: \partial M_{0} \rightarrow \partial M_{1}$. Use the existence of collars to produce a smooth structure on the topological space $M_{0} \cup_{\varphi} M_{1}$ such that the inclusions $M_{0} \rightarrow M_{0} \cup_{\varphi} M_{1}$ and $M_{1} \rightarrow M_{0} \cup_{\varphi} M_{1}$ are smooth embeddings.

Problem 13.4.6 (Configuration spaces are path-connected). Recall that the configuration space of $r$ ordered points in $M$ is given by

$$
\operatorname{Conf}_{r}(M):=\left\{\left(m_{1}, \ldots, m_{r}\right) \mid m_{i} \neq m_{j} \text { if } i \neq j\right\} \subset M^{r} .
$$

In other words, it is the complement in $M^{r}$ of the thick diagonal $\Delta=\left\{\left(m_{1}, \ldots, m_{r}\right) \mid\right.$ $m_{i}=m_{j}$ for some $\left.i \neq j\right\}$.
(a) Write $\Delta$ as a union of $\binom{r}{2}$ submanifolds $\Delta_{i j}$. rove that if $X$ is compact, then maps $X \rightarrow M^{r}$ which are transverse to all $\Delta_{i j}$ are generic.
(b) Use (a) to prove that if $M$ is path-connected of dimension $k \geq 2$ then $\operatorname{Conf}_{r}(M)$ is path-connected.

## Chapter 14

## Mod 2 intersection theory

In this chapter we use a slight technical strengthening of the theorem that transverse maps are generic to develop mod 2 intersection theory; this constructs invariants by counting transverse intersection points.

### 14.1 A strongly relative transversality theorem

In the previous chapter we proved that if $M$ and $N$ are manifolds without boundary, and $Z \subset M$ is a submanifold without boundary, then any smooth map $f_{0}: M \rightarrow N$ can be homotoped to a map $f_{1}: M \rightarrow N$ such that $f_{1} \pitchfork Z$.

Sometimes you already know that $f_{0}$ is transverse to $Z$ on an open neighborhood $U$ of closed subset $C \subset M$, and you don't want to modify $f_{0}$ near $C$. In fact, you might want to control more precisely where you modify $f_{0}$ and fix a closed subset $D \subset M$ (where we definitely want to modify $f_{0}$ ) and an open subset $V \subset M$ containing $D \backslash U$ (outside of which we definitely do not want to modify $f_{0}$ ). Many results in differential topology admit such refined forms, which are referred to as strong relative results.

Theorem 14.1.1 (Strongly relative transversality theorem). Suppose that $M$ is a compact manifold with boundary, $N$ is a manifold without boundary, and $Z$ is a submanifold without boundary. Fix the following data:

- a smooth map $f_{0}: M \rightarrow N$,
- a closed subset $C \subset M$ such that $f_{0} \pitchfork Z$ and $\partial f_{0} \pitchfork Z$ on an open neighborhood $U$ of $C$,
- a closed subset $D \subset M$ and open neighborhood $V \subset M$ containing $D \backslash U$.

Then there is an open neighborhood $U^{\prime} \subset M$ of $C \cup D$, as well as an $r \geq 0$ and a smooth map $F: M \times \mathbb{R}^{r} \rightarrow N$ with $\left.F\right|_{M \times\{0\}}=f_{0}$ such that
(i) $\left.F\right|_{M \times\{s\}}=f_{0}$ on $M \backslash V$,
(ii) for each $\epsilon>0$ there exists an $s \in \mathbb{R}^{r}$ such that $\left.F\right|_{M \times\{s\}}$ and $\left.\partial F\right|_{M \times\{s\}}$ are transverse to $Z$ on $U^{\prime}$.

As preparation, we construct a particular smooth function which controls where we manipulate $f_{0}$ :


Figure 14.1 The data in Theorem 14.1.1. Eventually $U^{\prime}$ will be an open neighborhood of $C \cup D$ inside $U \cup V$.

Lemma 14.1.2. Suppose we are given closed subsets $C, D \subset M$, an open neighborhood $U \subset M$ of $C$ and an open neighborhood $V \subset M$ of $D \backslash U$. Then there exists a smooth function $\gamma: M \rightarrow[0,1]$ with the following properties:

- it has support in $V$,
- is 0 on an open neighborhood of $C$, and
- is 1 on an open neighborhood of $D \backslash U$.

Proof. Take a partition of unity subordinate to $V \backslash C, U$, and $M \backslash(C \cup D)$; we call them $\eta_{V \backslash C}, \eta_{U}$ and $\eta_{M \backslash(C \cup D)}$. The function $\eta_{V \backslash C}: M \rightarrow[0,1]$ is the desired $\gamma$.

By construction, it has support in $V \backslash C \subset V$. Both $\operatorname{supp}\left(\eta_{U}\right)$ and $\operatorname{supp}\left(\eta_{M \backslash(C \cup D)}\right)$ are closed subsets not containing $D \backslash U$, so the complement of their union contains an open neighborhood of $D \backslash U$; necessarily $\eta_{V \backslash C}=1$ there. Similarly, only $U$ contains $C$ so $\eta_{U}=1$ on $C$, and hence $\eta_{V \backslash C}=0$ on $C$.

The proof is now a small variation on the proof that maps transverse to $Z$ are generic, using $\gamma$ to control the size of deformations.

Proof of Theorem 14.1.1. Embed $N$ into $\mathbb{R}^{r}$ and take a regular neighborhood $\pi_{N}: U \rightarrow$ $N$. We can find a smooth function $\epsilon: N \rightarrow(0, \infty)$ such that for each $p^{\prime} \in N$ and $x \in \mathbb{R}^{r}$ satisfying $\|x\|<\epsilon\left(p^{\prime}\right), p^{\prime}+x \in U$. Then we define the smooth map

$$
\begin{aligned}
F: M \times \mathbb{R}^{r} & \longrightarrow N \\
(p, s) & \longmapsto \pi_{N}\left(f_{0}(p)+\gamma(p) \epsilon\left(f_{0}(p)\right) \frac{s}{1+\|s\|^{2}}\right) .
\end{aligned}
$$

By construction $\left.F\right|_{M \times\{0\}}=\pi_{N} \circ f_{0}=f_{0}$, because $\pi_{N}$ is the identity on $N$. Furthermore $f_{s}=f_{0}$ on the complement of $V^{\prime}:=\gamma^{-1}((0,1]) \subset V$.

When we fix $p \in V^{\prime}$ we get a submersion, and the argument of Theorem 13.1.7 tells us that for a dense set of $s \in \mathbb{R}^{r}$, we have that $f_{s}$ and $\partial f_{s}$ are transverse to $Z$ at $p \in \gamma^{-1}((0,1])$. Furthermore, $f_{0}$ and $\partial f_{0}$ were already transverse to $Z$ at $p \in U$, and since an open neighborhood $W \subset U$ of $C$ is contained in $M \backslash V^{\prime}$, the same is true for $f_{s}$ and $\partial f_{s}$ at $p \in W$. We conclude that $f_{s}$ and $\partial f_{s}$ are transverse to $Z$ at $p \in V^{\prime} \cup W$. This is an open neighborhood $(D \backslash U) \cup C$. Finally, if we take $s$ small enough, the stability of transverse maps will guarantee $f_{s}$ is transverse to $Z$ an open neighborhood $W^{\prime}$ of a closed subset $D^{\prime}$ of $D$ contained in $U$ and satisfying $C \cup D \subset V^{\prime} \cup W \cup W^{\prime}$.

To apply this result, it is helpful to know that $f_{0}$ and $\partial f_{0}$ are transverse to $Z$ on an open neighborhood $U$ of $C$ if and only if they are transverse to $Z$ on $C$. One direction is obvious, the other holds if $Z$ is closed:

Lemma 14.1.3. If $Z$ is closed and $f_{0}$ and $\partial f_{0}$ are transverse to $Z$ on a closed subset $C \subset M$, then there exists an open neighborhood $U$ of $C$ such that $f_{0}$ and $\partial f_{0}$ are transverse to $Z$.

The idea is essentially the same as the stability of maps transverse to $C$.
Proof. We prove that such an open neighborhood exists for each $p \in C$. If $p \notin f_{0}^{-1}(Z)$ then $M \backslash f^{-1}(Z)$ works because $Z$ is closed.

If $p \in f_{0}^{-1}(Z)$, pick a local parametrization $\phi: \mathbb{R}^{k^{\prime}} \supset U^{\prime} \rightarrow V^{\prime} \subset N$ of $Z$ near $f_{0}(x)$. If $Z$ is codimension $r, Z \cap V^{\prime}=\phi\left(\{0\} \times \mathbb{R}^{k^{\prime}-r}\right)$. Then $f_{0}$ is transverse to $Z \cap V^{\prime}$ at $p^{\prime}$ if and only if the derivative at $p^{\prime}$ of $\pi_{r} \circ \phi^{-1} \circ f_{0}$ is surjective. Because surjective linear maps are open, if this is true at $p$ it is true for all $p^{\prime}$ in an open neighborhood of $p$.

Corollary 14.1.4. Suppose $M, N, Z$ are all without boundary, $M$ compact. If $f_{0}, f_{1}: M \rightarrow$ $N$ are homotopic and both transverse to $Z$, then there exists a homotopy $H: M \times[0,1] \rightarrow$ $N$ from $f_{0}$ to $f_{1}$ which is transverse to $Z$.

Proof. Apply Theorem 14.1 .1 with $f_{0}$ a given homotopy $\tilde{H}: M \times[0,1] \rightarrow N, C=$ $M \times\{0,1\}$ and $D=M \times[0,1]$. The open neighborhood $U$ is provided by Lemma 14.1.3 and the open neighborhood $V$ is an open subset of $M \times(0,1)$ containing $M \times[0,1] \backslash U$. Pick an $s \in R^{r}$ such that $\left.F\right|_{M \times[0,1] \times\{s\}}$ and $\left.\partial F\right|_{M \times[0,1] \times\{s\}}$ are transverse to $Z$. Then $\left.F\right|_{M \times[0,1] \times\{s\}}$ is the desired homotopy $H$.

### 14.2 Mod 2 intersection theory

Suppose that $Y, Z \subset M$ are compact submanifolds and that $\operatorname{dim}(Y)+\operatorname{dim}(Z)=$ $\operatorname{dim}(M)$, the dimension of $M$. If $Y \pitchfork Z$, then $Y \cap Z$ is a compact 0-dimensional submanifold and hence a finite number of points. If $Y$ is not transverse to $Z$, we know we make it so by an isotopy. However, the number of points in the intersection make depend on the way we make $Y$ transverse to $Z$, see Figure 14.2. However, a bit of experimentation suggests that whenever we change the number of intersection points,


Figure 14.2 Two transverse perturbations with a different number of intersection points.
we either add or remove two points; the number of intersection points mod 2 might be independent of the transverse perturbation!

Let us prove this in a bit more generality:
Definition 14.2.1. Let $Y$ be a compact manifold, $M$ be a manifold, and $Z \subset M$ be a submanifold, all without boundary and satisfying $\operatorname{dim}(Y)+\operatorname{dim}(Z)=\operatorname{dim}(M)$.

Let $f_{0}: Y \rightarrow M$ be a smooth map, then the mod 2 intersection number $I_{2}\left(f_{0}, Z\right)$ of $f_{0}$ with $Z$ is defined as follows: take $f_{1}$ homotopic to $f_{0}$ with $f_{1} \pitchfork Z$, and set

$$
I_{2}\left(f_{0}, Z\right):=\# f_{1}^{-1}(Z) \quad(\bmod 2)
$$

If $f_{0}$ is the inclusion of $Y$ as a submanifold, we shall use the notation $I_{2}(Y, Z):=$ $I_{2}\left(f_{0}, Z\right)$.

Lemma 14.2.2. The number $I_{2}\left(f_{0}, Z\right) \in \mathbb{Z} / 2$ is well-defined.
Proof. Suppose that $f_{1}$ and $f_{1}^{\prime}$ are two different smooth maps homotopic to $f_{0}$ and transverse to $Z$. Since homotopy is an equivalence relation, $f_{1}$ is homotopic to $f_{1}^{\prime}$. Then Corollary 14.1.4 provides a homotopy $H: Y \times[0,1] \rightarrow M$ from $f_{1}$ to $f_{1}^{\prime}$ which is transverse to $Z$. This means that $H^{-1}(Z)$ is a 1-dimensional submanifold of $Y \times[0,1]$ with boundary

$$
\partial H^{-1}(Z)=(\partial H)^{-1}(Z)=\left(f_{1}^{-1}(Z) \times\{0\}\right) \cup\left(\left(f_{1}^{\prime}\right)^{-1}(Z) \times\{0\}\right)
$$

It is compact because $Y \times[0,1]$ is compact.
Since $\partial H^{-1}(Z)$ is even by the classification of compact 1-dimensional manifolds, Theorem 12.3.1, we see that

$$
\# f_{1}^{-1}(Z)+\#\left(f_{1}^{\prime}\right)^{-1}(Z)=\# \partial H^{-1}(Z) \equiv 0 \quad(\bmod 2)
$$

Example 14.2.3. If $M=\mathbb{R}^{n}$, then $I_{2}(f, Z)$ vanishes when $\operatorname{dim}(Y)>0$. To see this, observe that $M \backslash Z$ is non-empty and open, and hence contains a ball. By composing $f$ with translation and scaling we can homotope $f$ so that its image lies in this little ball and hence disjoint from $Z$.

Example 14.2.4. Let $M$ be the Moebius strip, $Y=Z$ the central circle. Then $I_{2}(Y, Z)=1$ because it is easy to find a small perturbation of $Y$ which makes it intersect $Z$ transversally in a single point.

Here are some basic properties of this invariant of smooth maps $Y \rightarrow M$.
Proposition 14.2.5. The mod 2 intersection number has the following properties:
(i) If $f, g: Y \rightarrow M$ are homotopic then $I_{2}(f, Z)=I_{2}(g, Z)$.
(ii) If $f: Y \rightarrow M$ is homotopic to a constant map and $\operatorname{dim}(Y)>0$, then $I_{2}(f, Z)=$ 0.
(iii) If $Y=\partial W$ for a compact manifold $W$ and $f: \partial W \rightarrow M$ extends to a smooth map $W \rightarrow M$ then $I_{2}(f, Z)=0$.
(iv) If we have a pair of smooth maps $f: X \rightarrow Y, g: Y \rightarrow M$ with $X$ compact, $\operatorname{dim}(X)+\operatorname{dim}(Z)=\operatorname{dim}(M)$, and $g$ transverse to $Z$, then $I_{2}\left(f, g^{-1}(Z)\right)=$ $I_{2}(g \circ f, Z)$.

Proof. Part (i) follows from the definitions and the fact that homotopy is an equivalence relation. Part (ii) follows because such an $f$ is homotopic to a map disjoint from $Z$. Part (iii) follows from the fact that we may assume $f$ transverse to $Z$ and then the extension can be also chosen transverse to $Z$. In this case $f^{-1}(Z)$ is the boundary of a compact 1 -dimensional manifold and must be an even number of points. Part (iv) follows by noting that we may assume $f$ transverse to $g^{-1}(Z)$ and then both intersection numbers count the same set.

### 14.3 Applications of mod 2 intersection theory

### 14.3.1 Contractible compact manifolds

Let's start with an easy consequence:
Proposition 14.3.1. The point is the only contractible compact manifold.
Proof. Suppose $Y$ is contractible but not a point. Then Proposition 14.2.5 (ii) applied to id: $Y \rightarrow Y$ implies $1=I_{2}(\mathrm{id},\{p\})=0$ for any $p \in Y$, an obvious contradiction.

Remark 14.3.2. As the Whitehead manifold in Section 8.3 shows, this is false without the compactness assumption.

### 14.3.2 The mod 2 degree of maps

In the extreme case that $\operatorname{dim}(Y)=\operatorname{dim}(M)$ and $M$ is connected, we can define:
Definition 14.3.3. The $\bmod 2$ degree $\operatorname{deg}_{2}(f)$ of a smooth map $f: Y \rightarrow M$ is given by $I_{2}(f,\{p\})$ for some $p \in M$.

Lemma 14.3.4. This is well-defined.
Proof. We claim that $p \mapsto I_{2}(f,\{p\})$ is locally constant. Indeed, we may assume that $f$ is transverse to $\{p\}$. Then by the inverse function theorem and the fact that $Y$ is compact, there exists an open neighborhood $U$ of $p$ such that $f^{-1}(U)$ is a finite disjoint union $\bigsqcup_{i=1}^{k} V_{i}$ with $\left.p\right|_{V_{i}}: V_{i} \rightarrow U$ a diffeomorphism. This means that the number of points in the pre-image of $f$ is locally constant, hence so is this number modulo 2 .

Example 14.3.5. The identity map id: $M \rightarrow M$ has mod 2 degree equal to 1 .
Example 14.3.6. If $\phi: Y \rightarrow M$ is a diffeomorphism, then it is transverse to all points in

Example 14.3.7. If $q: E \rightarrow B$ is a covering map of degree $d$ with $B$ connected and $E$ compact, then $\operatorname{deg}_{2}(f) \equiv d(\bmod 2)$.

We can translate the properties of Proposition 14.2.5 into properties for $\operatorname{deg}_{2}$ :
Proposition 14.3.8. Suppose $Y$ is compact, $\operatorname{dim}(Y)=\operatorname{dim}(M)$, and $M$ is connected, The mod 2 degree has the following properties:
(i) If $f, g: Y \rightarrow M$ are homotopic then $\operatorname{deg}_{2}(f)=\operatorname{deg}_{2}(g)$.
(ii) If $f: Y \rightarrow M$ is homotopic to a constant map and $\operatorname{dim}(Y)>0$ then $\operatorname{deg}_{2}(f)=0$.
(iii) If $Y=\partial W$ for a compact manifold $W$ and $f: \partial W \rightarrow M$ extends to a smooth map $W \rightarrow M$ then $\operatorname{deg}_{2}(f)=0$.
(iv) If we have a pair of smooth maps $f: X \rightarrow Y, g: Y \rightarrow M$ with $X$ and $Y$ compact, $Y$ and $M$ connected and $\operatorname{dim}(X)=\operatorname{dim}(Y)=\operatorname{dim}(M)$, then $\operatorname{deg}_{2}(g \circ f)=$ $\operatorname{deg}_{2}(g) \cdot \operatorname{deg}_{2}(f)$.

Proof. Only (iv) is not obvious. By homotoping $g$ we can make it transverse to $p \in M$, and by homotoping $f$ we can make it transverse to $g^{-1}(p) \subset Y$. Then $g \circ f$ is transverse to $p$ and

$$
\begin{aligned}
\operatorname{deg}_{2}(g \circ f) & =\#(g \circ f)^{-1}(p) \\
& =\# f^{-1}\left(g^{-1}(p)\right) \\
& =\sum_{q \in g^{-1}(p)} \# f^{-1}(q) \\
& \equiv \# g^{-1}(p) \cdot \operatorname{deg}_{2}(f) \\
& =\operatorname{deg}_{2}(g) \cdot \operatorname{deg}_{2}(f),
\end{aligned}
$$

where we have used that all values $\# f^{-1}(q)(\bmod 2)$ are equal to $\operatorname{deg}_{2}(f)$ by the argument used to prove that $\operatorname{deg}_{2}$ is well-defined (this uses that $Y$ is connected).

### 14.3.3 Winding numbers

If $M$ is compact manifold of dimension $k$ and $f: M \rightarrow \mathbb{R}^{k+1}$ is a smooth map, then for $x \notin \operatorname{im}(f)$ we can define a smooth map

$$
\begin{aligned}
w_{f, z}: M & \longrightarrow S^{k} \\
x & \longmapsto \frac{f(x)-z}{\|f(x)-z\|}
\end{aligned}
$$

and then let define the mod 2 winding number $W_{2}(f, z)$ of $f$ around $z$ to be $\operatorname{deg}_{2}\left(w_{f, z}\right)$. It only depends on the connected component of $\mathbb{R}^{k+1} \backslash \operatorname{im}(f)$ containing $z$.

If $M=\partial W$ with $W$ compact and $f$ extends to a smooth map $F: W \rightarrow \mathbb{R}^{k+1}$, we can often compute $W_{2}(f, z)$ in terms of $F$ :

Proposition 14.3.9. If $z$ is a regular value of $F$, then $W_{2}(f, z) \equiv \# F^{-1}(z)(\bmod 2)$ (that is, $I_{2}(F, z)$ ).


Proof. Because $z$ is a regular value, we can find a small open disk $U$ around $z$ avoiding $f(\partial W)$, such that $f^{-1}(U)$ is a finite disjoint union $\bigsqcup_{i=1}^{r} V_{i}$ with $\left.p\right|_{V_{i}}: V_{i} \rightarrow U$ a diffeomorphism, with $r=\# F^{-1}(z)$. Then $\bar{W}:=W \backslash \bigsqcup_{i=1}^{r} V_{i}$ is another compact manifold with boundary and $F$ restricts to a smooth map $\bar{F}: \bar{W} \rightarrow \mathbb{R}^{k+1}$.

Since this avoids $z$, there is a smooth map

$$
\begin{aligned}
\bar{F}: \bar{W} & \longrightarrow S^{k} \\
x & \longmapsto \frac{F(x)-z}{\|F(x)-z\|}
\end{aligned}
$$

and by Sard's theorem we can find a $p \in S^{k}$ such that $\bar{F}$ and $\partial \bar{F}$ are transverse to $p$.
Hence $\bar{F}^{-1}(p)$ is one-dimensional compact submanifold on $\bar{W}$, so its boundary is an even number of points. This implies that

$$
\begin{aligned}
0 & \equiv \# \partial \bar{F}^{-1}(p) \quad(\bmod 2) \\
& =\# w_{f, z}^{-1}(p)+\sum_{i=1}^{r} \# w_{\left.F\right|_{\partial V_{i}, z}}^{-1}(p) \\
& =W_{2}(f, z)+\sum_{i=1}^{r} W_{2}\left(\left.F\right|_{\partial V_{i}}, z\right)
\end{aligned}
$$

Thus we may as well compute each $W_{2}\left(\left.F\right|_{\partial V_{i}}, z\right)$.
Since $\left.F\right|_{\partial V_{i}}$ is a diffeomorphism, is given by the composition of the inclusion $i: \partial U \hookrightarrow$ $\mathbb{R}^{k+1}$ with a diffeomorphism, each of these is equal to $W_{2}(i, z)$. Since $w_{i, z}: \partial U \rightarrow S^{k}$ is given by a composition of translation and scaling, it is a diffeomorphism; by Example 14.3.6 $W_{2}(i, z)=1$. We conclude that $W_{2}(f, z) \equiv k(\bmod 2)$, as desired.

### 14.4 Problems

Problem 14.4.1 (Spheres are not products). Let $M$ and $N$ be compact connected smooth manifolds of dimension $k$ and $n-k$ respectively, and suppose that $k>0$ and $n-k>0$. Fixing $q_{0} \in N$ there is an inclusion $i_{q_{0}}: M \rightarrow M \times N$ given by $p \mapsto\left(p, q_{0}\right)$.
(a) Prove that if $S^{n}$ is diffeomorphic to $M \times N$ then $i_{q_{0}}$ is homotopic to a constant map.
(b) Prove that $S^{n}$ is not diffeomorphic to $M \times N$ using intersection theory.

## Chapter 15

## Two applications of mod 2 intersection theory

We continue of Chapter 14 with our discussion of mod 2 intersection theory and its applications. This includes some applications from [Mat03] and Section $2 . \S 5$ of [GP10].

### 15.1 The Borsuk-Ulam theorem

Recall that if $M$ is compact smooth manifold of dimension $k$ and $f: M \rightarrow \mathbb{R}^{k+1}$ is a smooth map, then for $x \notin \operatorname{im}(f)$ we can define a smooth map

$$
\begin{aligned}
w_{f, z}: M & \longrightarrow S^{k} \\
x & \longmapsto \frac{f(x)-z}{\|f(x)-z\|} .
\end{aligned}
$$

The mod 2 winding number $W_{2}(f, z)$ of $f$ around $z$ is then $\operatorname{deg}_{2}\left(w_{f, z}\right)$. As an application of mod 2 winding numbers we will prove the Borsuk-Ulam theorem. Before doing so, let us start with an easier example of how conditions on a smooth map constrain its winding number:

Proposition 15.1.1. If a smooth map $f: S^{k} \rightarrow \mathbb{R}^{k+1} \backslash\{0\}$ satisfies $f(-x)=f(x)$, then $W_{2}(f, 0)=0$.

Proof. We start with the observation that $f$ is homotopic as a smooth map $S^{k} \rightarrow$ $\mathbb{R}^{k+1} \backslash\{0\}$ to $f /\|f\|$ by $(p, t) \mapsto f /(1-t+t\|f\|)$, and that this satisfies the same symmetry condition. Hence, without loss of generality we are dealing with a smooth map $f: S^{k} \rightarrow S^{k}$. Then $w_{f, 0}=f$, and we are equivalently proving a result about the degree of $f$. The symmetry condition implies that $f$ factors as


Since $q$ is a double cover, $\operatorname{deg}_{2}(q) \equiv 0(\bmod 2)$ by Example 14.3.7, and Proposition 14.3.8 (iv) hence $\operatorname{deg}_{2}(f)=\operatorname{deg}_{2}(q) \operatorname{deg}_{2}(\bar{f})=0$ as well.

Theorem 15.1.2 (Borsuk-Ulam). If a smooth map $f: S^{k} \rightarrow \mathbb{R}^{k+1} \backslash\{0\}$ satisfies $f(-x)=$ $-f(x)$, then $W_{2}(f, 0)=1$.

Proof. As above, without loss of generality we may assume we are dealing with a smooth $\operatorname{map} f: S^{k} \rightarrow S^{k}$, and we may use $W_{2}(f, 0)$ and $\operatorname{deg}_{2}(f)$ interchangeably. The proof is by induction over $k$, with Proposition 14.3.9 playing a major role in the induction step.

We start with the initial case $k=0$. Then $f: S^{0} \rightarrow S^{0}$ is either the identity id : $S^{0} \rightarrow S^{0}$ or -id. Both id and -id are diffeomorphisms and hence have degree 1 by Example 14.3.6.

For the induction step, we assume the result is true for $k-1$ and prove it for $k$. We are given a smooth map $f: S^{k} \rightarrow S^{k}$ satisfying $f(-x)=-f(x)$, and define $g:=\left.f\right|_{S^{k-1}}$ which also satisfies $g(-x)=-g(x)$. By Sard's theorem there exists an $a \in \operatorname{int}\left(S_{+}^{k}\right)$, with $S_{+}^{k}=S^{k} \cap[0, \infty) \times \mathbb{R}^{k}$ the upper hemisphere, which is a regular value of both $f$ and $g$. By symmetry $-a$ is also a regular value of both $f$ and $g$. We can manipulate $\operatorname{deg}_{2}(f)$ a bit:

$$
\operatorname{deg}_{2}(f) \equiv \# f^{-1}(a)=\frac{1}{2}\left(\# f^{-1}(a)+\# f^{-1}(-a)\right)
$$

To apply Proposition 14.3 .9 we want to go from $S^{k}$ to something diffeomorphic to $\mathbb{R}^{k}$. Let $\pi: \mathbb{R}^{k+1} \rightarrow a^{\perp} \cong \mathbb{R}^{k}$ by the orthogonal projection. That $g \pitchfork\{a,-a\}$ means that the image of $g$ is disjoint from $a$ and $-a$ and hence $\pi \circ g$ avoids 0 . Since furthermore $f \pitchfork\{a,-a\},\left.\pi \circ f\right|_{S_{+}^{k}}$ is transverse to 0 and we have

$$
\#\left(\left.\pi \circ f\right|_{S_{+}^{k}}\right)^{-1}(0)=\#\left(\left.f\right|_{S_{+}^{k}}\right)^{-1}(a)+\#\left(\left.f\right|_{S_{+}^{k}}\right)^{-1}(-a)=\frac{1}{2}\left(\# f^{-1}(a)+\# f^{-1}(-a)\right)
$$

This means that $\operatorname{deg}_{2}(f) \equiv \#\left(\left.\pi \circ f\right|_{S_{+}^{k}}\right)^{-1}(0)$.
Now Proposition 14.3.9 applies with $W=S_{+}^{k}, F=\left.f\right|_{S_{+}^{k}}$ and $z=0$. It says that

$$
\#\left(\left.\pi \circ f\right|_{S_{+}^{k}}\right)^{-1}(0) \equiv W_{2}(\pi \circ g, 0) \quad(\bmod 2)
$$

But $W_{2}(\pi \circ g, 0)=\operatorname{deg}_{2}(\pi \circ g)$ and since $\pi$ is linear, $\pi \circ g(-x)=-\pi \circ g(x)$ so that the inductive hypothesis applies and thus $\operatorname{deg}_{2}(\pi \circ g)=1$.

### 15.1.1 Applications of the Borsuk-Ulam theorem

In this section deduces several famous consequences of Theorem 15.1.2.
Corollary 15.1.3. If a smooth map $f: S^{k} \rightarrow \mathbb{R}^{k+1} \backslash\{0\}$ satisfies $f(-x)=-f(x)$, then $f$ intersects every line through the origin at least once.

Proof. If the image of $f$ does not intersect $\ell$, we compute that $W_{2}(f, 0)=0$ using an element $p \in S^{k} \cap \ell$, contradicting Theorem 15.1.2.

This corollary can be restated in a number of equivalent forms. We purposefully are a bit whether the maps are smooth or not; by an application of the Weierstrass approximation theorem the results for smooth maps imply those for continuous maps.

Theorem 15.1.4. The following are equivalent:
(i) If $f: S^{k} \rightarrow \mathbb{R}^{k+1} \backslash\{0\}$ satisfies $f(-x)=-f(x)$, then $f$ intersects every line through the origin at least once.
(ii) If $g: S^{k} \rightarrow \mathbb{R}^{k}$ satisfies $g(-x)=-g(x)$, then $g$ has a zero.
(iii) Every $h: S^{k} \rightarrow \mathbb{R}^{k}$ has an $x$ such that $h(x)=h(-x)$.
(iv) There is no $F: S^{k} \rightarrow S^{k-1}$ satisfying $F(-x)=-F(x)$.
(v) There is no $G: D^{k} \rightarrow S^{k-1}$ satisfying $G(-x)=-G(x)$ for $x \in \partial D^{k}$.

Proof.

- We start with (i) $\Rightarrow$ (ii). If $g$ has no zero then

$$
\begin{aligned}
f: S^{k} & \longrightarrow \mathbb{R}^{k+1} \backslash\{0\} \\
x & \longmapsto(g(x), 0)
\end{aligned}
$$

avoids the $x_{k+1}$-axis, contradicting (i).

- For $(\mathrm{ii}) \Rightarrow(\mathrm{i})$, if $f$ avoids $\ell$ and $\pi: \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k}$, then taking $g(x)=\pi \circ f(x)$ would contradict (ii).
- For (ii) $\Rightarrow$ (iii), take $g(x)=h(x)-h(-x)$.
- For $(\mathrm{iii}) \Rightarrow(\mathrm{ii})$, there is an $x$ such that $-g(x)=g(-x)=g(x)$ so $g(x)=0$.
- For (ii) $\Leftrightarrow$ (iv), we just normalize.
- For $(\mathrm{iv}) \Rightarrow(\mathrm{v})$ use that from such an $G$ we could produce an $F$ by picking a diffeomorphism $\phi: S_{+}^{k} \rightarrow D^{k}$ that is the identity on the boundary and setting $F(x)=G(\phi(x))$ for $x \in S_{+}^{k}$ and $F(x)=-G(\phi(-x))$ for $x \in \operatorname{int}\left(S_{-}^{k}\right)$.
- For $(\mathrm{v}) \Rightarrow$ (iv) use that from such an $F$ we could produce a $G$ by taking $\left.F\right|_{S_{+}^{k}} \circ$ $\phi^{-1}: D^{k} \rightarrow S^{k-1}$.

Example 15.1.5. Theorem 15.1.4 (v) gives another proof that there is no continuous map $D^{k} \rightarrow \partial D^{k}$ which is the identity on $\partial D^{k}$, a special case of Theorem 12.3.3.

Part (iii) of Theorem 15.1.4 has several famous geometric applications; see [Mat03] for even more:

Corollary 15.1.6 (Lusternik-Schnirelmann). If $U_{0}, \ldots, U_{k}$ is an open cover of $S^{k}$ then there is an $i \in\{0, \ldots, k\}$ such that $U_{i} \cap\left(-U_{i}\right) \neq \varnothing$.

Here $\left(-U_{i}\right)$ is of course the set $\left\{z \in S^{k} \mid-z \in U_{i}\right\}$.
Proof. We first prove that if $C_{0}, \ldots, C_{k}$ is a cover of $S^{k}$ by closed sets then there is an $i$ such that $C_{i} \cap\left(-C_{i}\right) \neq \varnothing$. Consider the continuous function

$$
\begin{aligned}
g: S^{k} & \longrightarrow \mathbb{R}^{k} \\
x & \longmapsto\left(\left(d\left(x, C_{1}\right), \ldots, d\left(x, C_{n}\right)\right)\right.
\end{aligned}
$$

with $d(-,-)$ the ordinary Euclidean metric on $\mathbb{R}^{k+1}$. By Theorem 15.1 .4 (iii) there must be an $x$ such that $g(x)=g(-x)$. If the $i$ th entry of $g(x)$ is 0 , then $x,-x \in C_{i}$. If none of the entries of $g(x)$ are 0 , then $x,-x \notin \bigcup_{i=1}^{n} C_{i}$ and hence $x,-x \in C_{n+1}$.

The version for open covers follows using the fact that a partition of unity subordinate to an open cover $U_{0}, \ldots, U_{k}$ of $S^{k}$ such that all $U_{i} \cap\left(-U_{i}\right)=\varnothing$ for all $i$, which exists by Theorem 9.1.2, provides a closed cover $\operatorname{supp}\left(\eta_{i}\right)$ of $S^{k}$ with the same property.


Figure 15.1 A cover of $S^{1}$ by two open subsets. The open subset $U_{1}$ contains two antipodal points.

Corollary 15.1.7 (Ham-Sandwich). Let $M_{1}, \ldots, M_{n}$ be bounded measurable subsets of $\mathbb{R}^{n}$ of positive measure. Then there exists an affine hyperplane $h \subset \mathbb{R}^{n}$ such that each of both of the half-spaces $h^{ \pm}$bounded by $h$ we have $\mu\left(M_{i} \cap h^{+}\right)=\mu\left(M_{i} \cap h^{-}\right)$for all $1 \leq i \leq n$.

Proof. Without loss of generality $M_{1}, \ldots, M_{n} \subset B_{1}(0)$. For each $x \in S^{k}$ we can define a subspace $h_{x}^{+}$when $x_{k+1} \neq \pm 1, h_{x}^{+}:=\left\{\left(v_{1}, \ldots, v_{k}\right) \in \mathbb{R}^{k} \mid \sum_{i=1}^{k} x_{i} v_{i} \geq x_{k+1}\right\}$. When $x=e_{k+1}$ we take $h_{x}^{+}=\varnothing$, then $x=-e_{k+1}$ we take $h_{x}^{+}=\mathbb{R}^{k}$.

We define a function

$$
\begin{aligned}
g: S^{k} & \longrightarrow \mathbb{R}^{k} \\
x & \longmapsto\left(\mu\left(M_{1} \cap h_{x}^{+}\right), \ldots, \mu\left(M_{k} \cap h_{x}^{+}\right)\right) .
\end{aligned}
$$

We will leave it to Theorem 3.1.1 in [Mat03] the proof that this is continuous. By Theorem 15.1.4 (iii) there must be an $x$ such that $g(x)=g(-x)$. Since $h_{-x}^{+}$is $h_{x}^{-}$, this means that $\left(\mu\left(M_{1} \cap h_{x}^{+}\right), \ldots, \mu\left(M_{k} \cap h_{x}^{+}\right)\right)=\left(\mu\left(M_{1} \cap h_{x}^{-}\right), \ldots, \mu\left(M_{k} \cap h_{x}^{-}\right)\right)$.

In other words, you can slice even an irregular sandwich with a slice of ham and a slice of cheese, such that the bread, ham and cheese are all divided in half.


Figure 15.2 There exists a half-plane which contains half of both the red and the blue figure (this is probably not it).

### 15.2 The Jordan-Brouwer separation theorem

### 15.2.1 Its proof

One can also use the ideas behind mod 2 intersection theory to deduce the famous Jordan-Brouwer separation theorem. Section $2 . \S 5$ of [GP10] deduces it from winding numbers, but I think this direct proof is clearer.

Theorem 15.2.1. If $Z \subset S^{n}$ is a compact connected non-empty submanifold of dimension $n-1$, then $S^{n} \backslash Z$ is a disjoint union of two connected open subsets, each of which has closure a compact submanifold with boundary $Z$.

By removing a point from $S^{n} \backslash Z$, we reduce to the case $\mathbb{R}^{n} \backslash Z$; in this case we only get the second claim for one of the both components but since we could have removed any point the same is true for the other component.
Example 15.2.2. In dimension 2 we are saying that a curve in the plane divides it into two pieces. See https://www.maths.ed.ac.uk/~v1ranick/papers/jordan-revised for some examples of complicated curves if you think this is obviously true.

Proof of Theorem 15.2.1. Pick an $x_{0} \in \mathbb{R}^{n} \backslash Z$. To simplify very end of the proof, we will assume that $x_{0}$ lies outside some closed disk around the origin containing the compact subset $Z$.

We claim that there is locally constant assignment $d: \mathbb{R}^{n} \backslash Z \rightarrow \mathbb{Z} / 2$, given at $x \in \mathbb{R}^{n} \backslash Z$ by picking a smooth path $\gamma$ from $x$ to $x_{0}$ which is transverse to $Z$ and taking $d(x)$ to be $\# \gamma^{-1}(Z)(\bmod 2)$. Let us prove that this is well-defined.

To show that such a $\gamma$ exists, observe that for each $x \in \mathbb{R}^{n} \backslash Z$ there is an open ball $B_{\epsilon}(x) \subset \mathbb{R}^{n} \backslash Z$ around $x$. We define a smooth map

$$
\begin{aligned}
F:[0,1] \times B_{\epsilon}(x) & \longrightarrow \mathbb{R}^{n} \\
(t, y) & \longmapsto t y+(1-t) x_{0},
\end{aligned}
$$

which is visibly a submersion when restricted to fixed $t \in[0,1]$, so $F \pitchfork Z, \partial F \pitchfork Z$ (in fact, $\partial F$ avoids $Z$ all-together). By Theorem 13.1.3 there exists a dense set of $y \in B_{\epsilon}(x)$ such that $\left.F\right|_{[0,1] \times\{y\}} \pitchfork Z$. Now we let $\gamma$ be the concatenation of the linear path from $x$ to $y$ and $\left.F\right|_{[0,1] \times\{y\}}$.

We claim that $\# \gamma^{-1}(Z)$ is independent of the choice of $\gamma$. Given two choices $\gamma, \gamma^{\prime}$, consider the map

$$
\begin{aligned}
G:(0,1) \times[0,1] & \longrightarrow \mathbb{R}^{n} \\
(t, s) & \longmapsto s \gamma(t)+(1-s) \gamma^{\prime}(t),
\end{aligned}
$$

This is transverse to $Z$ on an open neighborhood of the closed subset

$$
C=(0, \epsilon] \times[0,1] \cup[1-\epsilon, 1) \times[0,1] \cup(0,1) \times\{0,1
$$

so by the strongly relative transversality theorem, Theorem 14.1.1, there is a homotopic map which coincides with $G$ near $C$ and is tranverse to $Z$. Then $G^{-1}(Z)$ is a 1-dimensional submanifold, which is contained in some compact subset of $(0,1) \times[0,1]$, since $G$ avoids $Z$ on $\left(0, \epsilon^{\prime}\right) \times[0,1] \cup\left(1-\epsilon^{\prime}, 1\right) \times[0,1]$ for some $\epsilon^{\prime}>0$. Hence it is a compact 1 -dimensional submanifold, and hence its boundary contains an even number of points by Theorem 12.3.1. This implies that the difference between $\# \gamma^{-1}(Z)$ and $\#\left(\gamma^{\prime}\right)^{-1}(Z)$ is even.


Figure 15.3 Proving that $d: \mathbb{R}^{n} \backslash Z \rightarrow \mathbb{Z} / 2$ takes both values.
By construction, this function $d$ is constant on connected components. To see it takes both values, look at a chart exhibiting $Z$ as a submanifold, i.e. a diffeomorphism $\phi: \mathbb{R}^{n} \subset U \rightarrow V \subset \mathbb{R}^{n}$ such that $\phi^{-1}(Z \cap V)=\left(\{0\} \times \mathbb{R}^{n-1}\right) \cap U$. Suppose that $d$ takes value 0 on say $\phi\left(\left((-\infty, 0) \times \mathbb{R}^{n-1}\right) \cap U\right)$. Then by concatenating $\gamma$ with the image under $\phi$ of a straight line segment connecting a point $x$ in $(-\infty, 0) \times \mathbb{R}^{n-1}$ with a point $x^{\prime}$ in $(0, \infty) \times \mathbb{R}^{n-1}$ we see that $d$ takes value 1 on $\phi\left(\left((0, \infty) \times \mathbb{R}^{n-1}\right) \cap U\right)$. That is, crossing $Z$ changes $d$ by 1 . We conclude that $\mathbb{R}^{n} \backslash Z$ has at least 2 connected components.

To show it has exactly two connected components we need to use that $Z$ is connected. For any fixed $x \in \mathbb{R}^{n} \backslash Z$, let $V \subset Z$ be the subset of points $z \in Z$ such that any open
neighborhood $U$ of $z$ in $\mathbb{R}^{n}$ contains a point which has a path to $x$ avoiding $Z$. This is closed and open by looking at charts exhibiting $Z$ as a submanifold, and is non-empty by looking at a point in $Z$ closest to $x$. Thus, $V$ is union of connected components of $Z$ and hence all of $Z$.

Now let us look at opposite sides of $Z$ in a fixed chart; by the above argument, each $x \in \mathbb{R}^{n} \backslash Z$ can be connected to a point within this chart by a path avoiding $Z$. This includes $x_{0}$ and so can be used to divide the points of $\mathbb{R}^{n} \backslash Z$ into two path-components (possibly empty); those that connect to $x_{0}$ and those that do not. Hence $\mathbb{R}^{n} \backslash Z$ has at most two connected components and hence exactly two, given by $d^{-1}(0)$ and $d^{-1}(1)$ respectively.

To see that the closure of $d^{-1}(0)$ is a manifold with boundary we need to find charts near boundary points. Note that for each local trivialization of $Z$, exactly one of $\phi\left(\left((-\infty, 0) \times \mathbb{R}^{n-1}\right) \cap U\right)$ and $\phi\left(\left((0, \infty) \times \mathbb{R}^{n-1}\right) \cap U\right)$ lies in $d^{-1}(0)$, say the latter, and then $\left.\phi\right|_{\left.\left([0, \infty) \times \mathbb{R}^{n-1}\right) \cap U\right)}$ is the desired chart near the boundary. The same argument applies to $d^{-1}(1)$.

Finally, any points $x$ with $\|x\| \geq\left\|x_{0}\right\|$ can be connected to $x_{0}$ by a path avoiding $Z$, so the closure $d^{-1}(1)$ is bounded and hence compact.

Let us reflect on the proof. What did we really use about $\mathbb{R}^{n}$ ? Only that it is connected and simply-connected. That is, for the definition of $d$ we only need to be able connect $x$ to $x_{0}$ by some path $\gamma$. To show it is well-defined, we need that any two choices $\gamma$ and $\gamma^{\prime}$ are homotopic relative to their endpoints. Thus, the same proof gives the following generalization of the Jordan-Brouwer separation theorem:

Theorem 15.2.3. Suppose $M$ is a simply-connected connected compact manifold of dimension $n$ and $Z \subset M$ is a compact connected non-empty submanifold of dimension $n-1$, then $M \backslash Z$ is a disjoint union of two connected open subsets, each of which has closure a compact submanifold with boundary $Z$.

### 15.2.2 The Schoenflies theorem

In particular, if $i: S^{k-1} \hookrightarrow S^{k}$ is a smooth embedding then $i\left(S^{k-1}\right)$ divides $S^{k}$ into two connected components, and the closure of each of these is a compact submanifold with boundary. What are these manifolds with boundary? Of course, taking $i$ to be the standard inclusion we get two disks $D^{k}$. Can other manifolds appear? The answer is "no" in low dimensions:

Theorem 15.2.4 (Schoenflies, Alexander). If $k \leq 3$, for each embedding i: $S^{k-1} \hookrightarrow S^{k}$ the closures of both components of $S^{k} \backslash S^{k-1}$ are diffeomorphic to $D^{k}$.

You can find a proof for $k=3$ in [Hat07, Theorem 3.3], which you should be able to adapt to $k=2$ without much difficulty.

However, in high dimensions there can be. One of the successes of differential topology is the determination of dimensions in which this can happen in terms of other well-studied objects in algebraic topology (the groups of exotic spheres). In particular, in dimension $\leq 140$ we have [BHHM17]:

Theorem 15.2.5. If $5 \leq k \leq 140$, for each embedding $i$ : $S^{k-1} \hookrightarrow S^{k}$ the closures of both components of $S^{k} \backslash S^{k-1}$ are diffeomorphic to $D^{k}$ if and only if $k=5,6,12,56,61$.

There is one dimension remaining for $k \leq 140: k=4$. One of the big remaining open questions of manifold theory asks about this case:

Conjecture 15.2.6 (Smooth Schoenflies conjecture in dimension 4). Given an embedding $i$ : $S^{3} \hookrightarrow S^{4}$, the closures of both components of $S^{4} \backslash S^{3}$ are diffeomorphic to $D^{4}$.

### 15.2.3 Codimension one knots

Just we called (isotopy classes of) embeddings of $S^{1}$ in $S^{3}$ are knots, we refer to (isotopy classes of) embeddings $S^{k-r} \hookrightarrow S^{k}$ as codimension $r$ knots. The most interesting case is, unsurprisingly, codimension 2 . What about codimension 1 ?

If for each embedding $i: S^{k-1} \hookrightarrow S^{k}$ the closure of one of the components of $S^{k} \backslash S^{k-1}$ are diffeomorphic to $D^{k}$, there exists only one embedding $S^{k-1} \rightarrow S^{k}$ up to isotopy:

Theorem 15.2.7. If an embedding $i: S^{k-1} \hookrightarrow S^{k}$ has the property that the closure of one of the components of $S^{k} \backslash S^{k-1}$ is diffeomorphic to $D^{k}$, then $i$ is isotopic to the standard inclusion $S^{k-1} \rightarrow S^{k}$.

It will follow from:
Proposition 15.2.8. Every embedding $S^{k-1} \hookrightarrow \mathbb{R}^{k}$ which extends to an embedding $D^{k} \hookrightarrow \mathbb{R}^{k}$ is isotopic to either the standard inclusion $i$, or $i$ composed with a reflection.

Proof. We prove that every embedding $D^{k} \hookrightarrow \mathbb{R}^{k}$ is isotopic to one given by applying invertible linear map $A \in \mathrm{GL}_{k}(\mathbb{R})$ to $D^{k}$. The result follows from the observation that the two different connected components of $\mathrm{GL}_{k}(\mathbb{R})$ contain the identity and a reflection respectively.

We claim that embeddings $D^{k} \hookrightarrow \mathbb{R}^{k}$ up to isotopy are in bijection with injective immersions $\mathbb{R}^{k} \hookrightarrow \mathbb{R}^{k}$ up to homotopy through injective immersions. This bijection is given in one direction by the composing with the embedding $i: D^{k} \hookrightarrow \mathbb{R}^{k}$, and in the other by composing with the injective immersion $h: \mathbb{R}^{k} \hookrightarrow D^{k}$ given by $z \mapsto \frac{z}{1+\|z\|^{2}}$. It is easy to see that $h \circ i$ is isotopic to $\mathrm{id}_{D^{k}}$, and $i \circ h$ admits an homotopy through injective immersions to $\mathrm{id}_{\mathbb{R}^{k}}$.

Now apply Lemma 15.2 .9 , which classifies injective immersions $\mathbb{R}^{k} \hookrightarrow \mathbb{R}^{k}$ up to homotopy through injective immersions.

Lemma 15.2.9. Every injective immersion $f: \mathbb{R}^{k} \hookrightarrow \mathbb{R}^{k}$ is homotopic through injective immersions to an invertible linear transformation.

Proof sketch. Identify $[0,1]$ with $[1, \infty]$ and take

$$
\begin{aligned}
H: \mathbb{R}^{n} \times[1, \infty] & \longrightarrow \mathbb{R}^{n} \\
\quad(x, t) & \longmapsto \begin{cases}\frac{1}{t} \cdot h(t x) & \text { if } t<\infty, \\
D_{0} h(x) & \text { if } t=\infty .\end{cases}
\end{aligned}
$$

To see that this is smooth at $t=\infty$ apply Taylor's theorem.

We can now complete the argument:
Proof of Theorem 15.2.7. We may assume $e_{k+1} \in S^{k} \backslash S^{k-1}$ is not in the image of the extension, and removing this point, we may as well work in $\mathbb{R}^{k}$. The result follows by observing that the embeddings $S^{k-1} \hookrightarrow S^{k}$ given by $i$ and $i$ composed with a reflection are isotopic, as the action of $\mathrm{GL}_{k}(\mathbb{R})$ on $S^{n}$ extends to an action of $\mathrm{GL}_{k+1}(\mathbb{R})$ and there is an element of $\mathrm{GL}_{k}(\mathbb{R})$ with determinant +1 which acts on $i$ by reflection.

Thus Theorem 15.2.5 tells us the following about the existence of codimension one knots.

Corollary 15.2.10. If $4 \neq k \leq 140$ and $k=0,1,2,3,5,6,12,56,61$, then every embedding $S^{k-1} \hookrightarrow S^{k}$ is isotopic to the standard inclusion.

### 15.3 Problems

Problem 15.3.1. Use the Jordan-Brouwer separation theorem to prove that if $M \subset \mathbb{R}^{k}$ is a compact codimension 1 submanifold, then its normal bundle $N M$ is trivial.

Problem 15.3.2. Adapt the proof of Lemma 15.2 .9 to prove that every diffeomorphism of $\mathbb{R}^{k}$ is isotopic to an invertible linear transformation.

## Chapter 16

## Knot theory

In this chapter, we return to the knots that we first encountered in Chapter 1 and apply some of the techniques we have learned so far. In particular, we will use transversality and intersection theory. You can read more about knot theory in [Ada04, Sos02, Rol90].

### 16.1 Knot diagrams

Recall the following definition from the first lecture:
Definition 16.1.1. A knot is an isotopy class of embeddings $S^{1} \hookrightarrow \mathbb{R}^{3}$.
(a) This is an interesting notion exactly because we are requiring both the map $S^{1} \rightarrow \mathbb{R}^{3}$ and its derivative to be injective. Dropping conditions, all smooth maps $S^{1} \rightarrow \mathbb{R}^{3}$ are homotopic to a constant map by linear interpolation. If we drop injectivity of the map, we are dealing with immersions and Proposition 1.2.4 sketched an arugment that there is only one of these up to regular homotopy. If we drop injectivity of the derivative, then we can remove any knotting by moving it into a ball and shrinking this ball to a point.
(b) Our knots come with a choice of orientation, intuitively a "direction of travel" along the knot. The reverse of a knot is obtained by reversing this direction of travel, or equivalently, composing the embedding with a reflection of the circle. It is not necessarily true that a knot and its reverse are isotopic.
(c) There are other reasonable equivalence relations one can put on embeddings $S^{1} \hookrightarrow \mathbb{R}^{3}$. One could ask for embeddings up to diffeomorphisms of $\mathbb{R}^{3}$, or up to diffeomorphisms of $\mathbb{R}^{3}$ isotopic to the identity ("ambient isotopy"). The latter is the same equivalence relation as isotopy of embeddings, by Chapter 25, and so is the former once we require the diffeomorphisms are orientation-preserving.
(d) Some references use piecewise-linear embeddings instead of smooth embeddings. That is, for them a knot consists of straight line segments. These are of a more combinatorial nature and simplify some technicalities. In dimension $\leq 3$, piecewise linear embeddings up to piecewise-linear isotopy are the same as smooth embeddings up to isotopy [Moi77]. Warning: this type of statement tends to be false in higher dimensions.

We draw knots as follows: we project it to knot to the plane, and we indicate the vertical ordering of intersections at the crossings (see Figure 16.1). Let me be more precise: a knot diagram is an immersion $S^{1} \rightarrow \mathbb{R}^{2}$ with only distinct transverse self-intersections, together with the data at each self-intersection which of the two segments crosses over. From this data we can reconstruct the knot up to isotopy; $x$ - and $y$-coordinates can be recovered from the diagram, and the $z$-coordinate can be inferred up to linear interpolation from the crossing.


Figure 16.1 An example of a knot diagram. We leave out the orientations for the sake of simplicity. (In this case, the knot is isotopic to its reverse.)

The following may seem obvious, but a rigorous proof requires some care. It uses the same techniques used for the strong Whitney embedding theorem, in particular Proposition 12.1.2.

Proposition 16.1.2. Every knot has a knot diagram.
Proof. We will prove that there exists a direction $x \in S^{2}$ such that the composition of $e$ with orthogonal projection $\pi_{x}: \mathbb{R}^{3} \rightarrow x^{\perp} \cong \mathbb{R}^{2}$ gives a knot diagram. This uses the maps

$$
\begin{aligned}
f^{\text {inj }}: S^{1} \times S^{1} \backslash\left\{(p, p) \mid p \in S^{1}\right\} & \longrightarrow S^{2} \\
\left(p, p^{\prime}\right) & \longmapsto \frac{e(p)-e\left(p^{\prime}\right)}{\left\|e(p)-e\left(p^{\prime}\right)\right\|}, \\
\tilde{f}^{\mathrm{tang}}:\left\{v \in T S^{1} \mid\|v\|=1\right\} & \longrightarrow S^{2} \\
v & \longmapsto \frac{d e(v)}{\|\operatorname{de}(v)\|},
\end{aligned}
$$

where we use the identification of $T S^{1}$ with $S^{1} \times \mathbb{R}$ to define $\|v\|$.
By Sard's theorem, Theorem 11.3.4, there is a point $x \in S^{2}$ which is a simultaneous regular value of both these maps. As the domain of $f^{\text {tang }}$ is 1-dimensional (the domain is just two copies of a circle), a regular value is a point which not in its image. That $x$ is not in its image means that $\pi_{x} \circ e$ has injective differential and thus $\pi_{x} \circ e$ is an immersion.

The domain and target of $f^{\mathrm{inj}}$ are both 2 -dimensional, so if a point $x$ is a regular means then if $e(p)-e\left(p^{\prime}\right)$ is a (necessarily non-zero) multiple of $x$, the derivative of $f^{\text {inj }}$ at $\left(p, p^{\prime}\right)$ is surjective. This implies that even though $\pi_{x} \circ e(p)=\pi_{x} \circ e\left(p^{\prime}\right)$, their derivatives are linearly independent. Thus the intersections are transverse.

The only issue is that $\pi_{x} \circ e$ may have multiple self-intersections at the same location in $\mathbb{R}^{2}$. Since these self-intersections are transverse there are only finitely many, and by a small isotopy we can make them distinct.

A much stronger transversality result for families, a parametrized multi-jet transversality theorem, allows one to modify the projection of an isotopy so that it is given by a finite number of local moves:

Theorem 16.1.3 (Reidemeister). Any two knot diagrams differ by a finite sequence of the following moves (see Figure 16.2):

- isotopies of $\mathbb{R}^{2}$,
- I: introducing or removing a twist,
- II: introducing or removing a poke,
- III: sliding over a crossing.


Figure 16.2 The three Reidemeister moves (from Fomenko-Matveev [FM97]).

Remark 16.1.4. The parametrized multi-jet transversality theorem concerns modifying families of smooth functions so that they are transverse to certain "submanifolds" of the space of all smooth functions. The study of the space of smooth functions and its subspaces is the subject of singularity theory, see e.g. [GG73].

The upshot of this result is that we can define knot invariants using knot diagrams: we need to assign something to each diagrams which is invariant under the above moves. For example, tricolorings give the easiest example showing that some knots are not isotopic to the standard inclusion $S^{1} \hookrightarrow \mathbb{R}^{3}$.

Definition 16.1.5. Given a knot diagram and the three colors \{red, blue, green\}, a tricoloring is an assignment of a color to each line segment such that the three strands coming into each crossing either have all three colors, or just a single color.

one color is allowed

three colors are allowed

two colors is not allowed

Figure 16.3 Rules for tricoloring.

Example 16.1.6. The trivial knot diagram (just a circle) has 3 tricolorings. The trefoil below has 9 tricolorings:


Proposition 16.1.7. The number of tricolorings is an invariant of the knot; i.e. it is invariant under isotopies and Reidemeister moves.

Proof sketch. You need to show that under a Reidemeister move, each tricoloring of the right hand side corresponds to a unique tricoloring of the left hand side and vice versa.

Example 16.1.8. The trefoil is not isotopic to the standard inclusion.

### 16.2 Links and linking numbers

One can also consider embeddings of more than one circle into $\mathbb{R}^{3}$. See Figure 16.4 for two examples.

Definition 16.2.1. A link with $k$ components is an isotopy classes of embeddings $e: \bigsqcup_{k} S^{1} \hookrightarrow \mathbb{R}^{3}$.

We did not use anything about the circle in Section 16.1 except that it is a compact 1-dimensional smooth manifold. The same is true for the disjoint union $\bigsqcup_{k} S^{1}$, so it is still the case that we can represent them by diagrams (i.e. a sufficiently nice projection to the plane), and that two such diagrams are related by isotopies and the three Reidemeister moves. In particular, one could use tricolorings or other invariants of knots defined in terms of diagrams to distinguish links.


Figure 16.4 Two links, the trivial link and the Hopf link.

However, there is a new invariant which we can define if $k=2$, that is, we have a link with two components. Let $S_{1}^{1}$ and $S_{2}^{1}$ denote the two circles, then there is a map

$$
\begin{aligned}
g: S_{1}^{1} \times S_{2}^{1} & \longrightarrow S^{2} \\
\left(p, p^{\prime}\right) & \longmapsto \frac{e(p)-e\left(p^{\prime}\right)}{\left\|e(p)-e\left(p^{\prime}\right)\right\|} .
\end{aligned}
$$

As a map between compact manifolds of the same dimension, it has a degree $\operatorname{deg}_{2}(w) \in$ $\mathbb{Z} / 2$. An isotopy of $e$ induces a homotopy of $w$, so this is an invariant of the link; the mod 2 linking number. It is in fact a mod 2 reduction of the Gauss linking number, which is an integer and requires a discussion of oriented manifolds and integral intersection theory to define.
Example 16.2.2. If two circles are unlinked, this linking number is 0 . For the Hopf link, it is 1 . Hence these links are not isotopic.

To prove the first case, we observe that we can pick our preferred representative within an isotopy class. In the case of unlinked circles, we may assume that the circles are contained in balls of radius 1 , the first centered at $(10,0,0)$ and the second at $(-10,0,0)$. In this case, $-e_{1}$ is not in the image of $g$ in $S^{2}$ as we see in Figure 16.5, so using it to compute $\operatorname{deg}(g)$ we see that it is 0 .


Figure 16.5 The linking number of the trivial link is 0 , because $g$ is given by the direction of the arrow from $p^{\prime} \in S_{2}^{1}$ to $p \in S_{1}^{1}$.

### 16.3 A result of Klein

Knot theory goes back to the end of the 19th century, as a hypothetical model for atoms due to Thomson and Maxwell (later Lord Kelvin) [Kel67]. In his 1877 foundational paper on knot theory, Peter Tait wrote [Tai76]:

Klein himself made the very singular discovery that in space of four dimensions there cannot be knots.

Put more dramatically in a poem of Maxwell ${ }^{1}$ :
My soul's an amphicheiral knot
Upon a liquid vortex wrought
By Intellect in the Unseen residing,
While thou dost like a convict sit
With marlinspike untwisting it
Only to find my knottiness abiding;
Since all the tools for my untying
In four-dimensioned space are lying,
Where playful fancy intersperses
Whole avenues of universes;
Where Klein and Clifford fill the void
With one unbounded, finite homaloid,
Whereby the Infinite is hopelessly destroyed.
Klein's "very singular discovery," phrased in modern terms, is that every embedding $S^{1} \hookrightarrow \mathbb{R}^{4}$ extends to an embedding $D^{2} \hookrightarrow \mathbb{R}^{4}$. You can probably come up with an elementary proof yourself ("switch the crossings using fourth dimension"), but let us prove the following weaker statement using transversality:

Proposition 16.3.1. Every embedding $S^{1} \hookrightarrow \mathbb{R}^{5}$ extends to an embedding $D^{2} \hookrightarrow \mathbb{R}^{5}$.
Proof. Given an embedding $i: S^{1} \hookrightarrow \mathbb{R}^{5}$, we can write down an embedding $j: D^{2} \hookrightarrow \mathbb{R}^{9}$ as follows. We write $D^{2}$ as $S^{1} \times[0,1] \cup S_{+}^{2}$, the latter being the upper hemisphere, and take

$$
\begin{aligned}
j: S^{1} \times[0,1] \cup S_{+}^{2} & \longrightarrow \mathbb{R}^{5} \times \mathbb{R} \times \mathbb{R}^{3} \\
p & \longmapsto \begin{cases}(\eta(t) i(x), t,(1-\eta(t)) x) & \text { if } p=(q, t) \in S^{1} \times[0,1], \\
(0,1+z,(x, y, z)) & \text { if } p=((x, y), z) \in S_{+}^{2} \subset \mathbb{R}^{2} \times[0, \infty),\end{cases}
\end{aligned}
$$

with $\eta:[0,1] \rightarrow[0,1]$ smooth, 1 near 0 , and 0 near 1 .
We can now repeat the proof of Theorem 12.1.1 with Sard's theorem for manifolds with boundary, finding a dense set of choices of $x \in S^{k-3}$ such that $\pi_{x} \circ j$ is still an embedding. In particular, we can pick one which is arbitrarily close to the last basis vector at each step. In particular, the map $\left.\pi_{x}\right|_{\mathbb{R}^{5}}: \mathbb{R}^{5} \rightarrow \mathbb{R}^{8}$ will be close to the standard inclusion $\mathbb{R}^{5} \hookrightarrow \mathbb{R}^{8}$, and hence there is an invertible linear map $A: \mathbb{R}^{8} \rightarrow \mathbb{R}^{8}$ such that $A \circ \pi_{x}: \mathbb{R}^{9} \rightarrow \mathbb{R}^{8}$ is the identity on $\mathbb{R}^{5}$. Thus $A \circ \pi_{x} \circ j: D^{2} \rightarrow \mathbb{R}^{8}$ is an embedding extending $i$. We can repeat this argument three more times to get an embedding $j: D^{2} \rightarrow \mathbb{R}^{5}$ extending $i$.

Remark 16.3.2. Klein's observation was noticed in non-mathematical circles and it became part of popular culture. For example, the American magician and medium Henry Slade was performing "magic tricks" claiming that he solves knots in fourth dimension. He was taken seriously by a German astrophysicist J.K.F. Zöllner, who organized a number of seances with Slade in 1877 and 1878 [Car07, Chapter II].

[^8]

Figure 16.6 The knot that Slade's spirits untied in the fourth dimension.

### 16.4 Problems

Problem 16.4.1. Generalize Proposition 16.3 .1 to show that any embedding $S^{r} \hookrightarrow \mathbb{R}^{k}$ extends to an embedding $D^{r+1} \hookrightarrow \mathbb{R}^{k}$ when $2(r+1)<k$.

## Chapter 17

## Orientations and integral intersection theory

The next part will be devoted defining de Rham cohomology, developing computational tools for it, and drawing interesting topological conclusions from it. A prerequisite for some of this material will be the notion of an orientation. We define this today, and give a taste of Chapter 3 of [GP10], which we will not cover in detail in the course.

Convention 17.0.1. All vector spaces are finite-dimensional and over $\mathbb{R}$ unless mentioned otherwise.

### 17.1 What is an orientation on a manifold?

We start with an intuitive description of orientations, before giving rigorous definitions: an orientation of a manifold is "a smooth family of orientations of each of the tangent spaces $T_{p} M$."
An orientation on a vector space such as $T_{p} M$ specifies for each of its ordered bases whether it is "positively oriented" or "negatively oriented," with the following requirement: if one ordered basis can be obtained from another by applying an invertible matrix $A$ to each of its vectors, then they are similarly oriented if and only if $\operatorname{det}(A)>0$. Since $\mathrm{GL}_{n}(\mathbb{R})$ has two path components, this is equivalent to saying homotopic bases are similarly oriented and reflecting a single basis vector changes the orientation of the basis.

That an orientation depends smoothly on $p \in M$ means that if you move a positively oriented basis around $M$, it stays positive (and of course the same is true for negatively oriented bases).
Example 17.1.1. For the circle $S^{1}$, an orientation is a choice of "positive direction" along the circle. There are two such choices: counterclockwise and clockwise.

Example 17.1.2. The real projective plane $\mathbb{R} P^{2}$ admits no orientation. Suppose it did, then starting with a basis $e_{1}, e_{2}$ at some point, say positively oriented, we can move it around $\mathbb{R} P^{2}$ and return to $e_{1},-e_{2}$. This must simultaneously be positively oriented (since moving a basis around shouldn't change how it's oriented) and negatively oriented (since it is obtained from a positively oriented by reflecting a basis vector). This gives a contradiction.

You can find more examples in the following table:


Figure 17.1 Moving a basis around $\mathbb{R} P^{2}$ can return it with opposite orientation.

| orientable | not orientable |
| :---: | :---: |
| spheres $S^{n}$ | real projective spaces $\mathbb{R} P^{2 n}(n \geq 1)$ |
| surfaces of genus $g \geq 1$ | Klein bottle |
| Lie groups |  |
| Lens spaces |  |
| Poincaré homology sphere |  |
| Complex projective spaces |  |
| Quaternionic projective spaces |  |
| $K 3$ surface |  |
| Whitehead manifold |  |

Example 17.1.3. An LCD display is made from a nematic crystal, consisting of long thin filaments. These prefer to be aligned the same way, so locally such a crystal has a order parameter given by a direction in $\mathbb{R}^{3}$. This is an element of $\mathbb{R} P^{2}$, a non-orientable manifold. ${ }^{1}$


Figure 17.2 An nematic crystal (from https://en.wikipedia.org/wiki/Liquid_crystal).

[^9]
### 17.2 A recollection of multilinear algebra

Linear algebra concerns not only the study of vector spaces and linear maps between them, but also of multilinear maps with various properties. This is closely related to the study of tensor products and variations thereof.

### 17.2.1 Tensor products

Definition 17.2.1. A bilinear map is a function $b: V \times V^{\prime} \rightarrow W$ which is linear in each variable.

Definition 17.2.2. The tensor product $V \otimes V^{\prime}$ is the quotient of the free $\mathbb{R}$-vector space on the set $V \times V^{\prime}$, whose basis elements we shall denote $\left(v, v^{\prime}\right)$, by the subspace spanned by the elements

$$
\begin{gathered}
\left(\left(v_{1}+v_{2}\right), v^{\prime}\right)-\left(v_{1}, v^{\prime}\right)-\left(v_{2}, v^{\prime}\right) \\
\left(v,\left(v_{1}^{\prime}+v_{2}^{\prime}\right)\right)-\left(v, v_{1}^{\prime}\right)-\left(v, v_{2}^{\prime}\right) \\
(a v, w)-a(v, w) \\
(v, a w)-a(v, w)
\end{gathered}
$$

We will denote the equivalence class of $(v, w)$ by $v \otimes w$.
Example 17.2.3. The tensor product $\mathbb{R}^{k} \otimes \mathbb{R}^{l}$ has a basis given by $e_{i} \otimes e_{j}^{\prime}$ for $1 \leq i \leq k$, $1 \leq j \leq l$.

The relations are designed to make

$$
\begin{aligned}
b_{0}: V \times V^{\prime} & \longrightarrow V \otimes V^{\prime} \\
\left(v, v^{\prime}\right) & \longmapsto v \otimes v^{\prime}
\end{aligned}
$$

bilinear. It is in fact the initial bilinear map:
Lemma 17.2.4. For every bilinear map $b: V \times V^{\prime} \rightarrow W$ there is a unique linear map $\beta: V \otimes V^{\prime} \rightarrow W$ such that $b=\beta \circ b_{0}$.

Proof. There is a unique linear map $\mathbb{R}\left[V \times V^{\prime}\right] \rightarrow W$ given by $\left(v, v^{\prime}\right) \mapsto b\left(v, v^{\prime}\right)$. Since $b$ is bilinear this factors over $V \otimes V^{\prime}$, determining a linear map $\beta: V \otimes V^{\prime} \rightarrow W$ satisfying $b\left(v, v^{\prime}\right)=\beta\left(v \otimes v^{\prime}\right)=\beta\left(b_{0}\left(v, v^{\prime}\right)\right)$. Since $V \otimes V^{\prime}$ is generated by the elements $b_{0}\left(v, v^{\prime}\right)$, this determines $\beta$ uniquely.

Remark 17.2.5. This universal property satisfied by the tensor product determines it uniquely up to linear isomorphism.

There is a similar correspondence of multilinear maps $V_{1} \times \cdots \times V_{k} \rightarrow W$ with linear $\operatorname{map} V_{1} \otimes \cdots \otimes V_{k} \rightarrow W$.

Example 17.2.6. The universal property tells us what tensor product of a single or no vector spaces is. A multilinear map $V \rightarrow W$ is just a linear map, so a tensor product of a single vector space $V$ is just $V$ again.

The empty product of sets is a point, because such a product receives a unique map from every other set. A multilinear map from an empty product is hence a map from a point to $V$, with no condition imposed, so just an element of $V$. This is the same as a linear map $\mathbb{R} \rightarrow V$. Hence an empty tensor product is $\mathbb{R}$ itself.

### 17.2.2 Alternating multilinear maps

When all vector spaces $V_{i}$ in the domain of a multilinear map are the same $V$, we can require additional symmetry properties. Of specific interest to us are the alternating multilinear maps, though the story for symmetric multilinear maps is similar:

Definition 17.2.7. An alternating multilinear map is a multilinear map $w: V^{k} \rightarrow W$ which satisfies $w\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)=(-1)^{\epsilon(\sigma)} w\left(v_{1}, \ldots, v_{k}\right)$ for all $v_{1}, \ldots, v_{k} \in V$ and permutations $\sigma$ of $\{1, \ldots, k\}$. Here $\epsilon(\sigma) \in \mathbb{Z} / 2$ is the sign of the permutation.
Example 17.2 .8 . The sign of a permutation is uniquely determined by demanding it is a homomorphism and it sends a transposition to the unique non-identity element of $\mathbb{Z} / 2$.

There is also an initial alternating multilinear map.
Definition 17.2.9. The $k$ th exterior power $\Lambda^{k} V$ is the quotient of $V^{\otimes k}$ by the subspace spanned by the elements

$$
v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}-(-1)^{\epsilon(\sigma)} v_{1} \otimes \ldots \otimes v_{k} \text { with } \sigma \in \Sigma_{k}
$$

We will denote the image of $v_{1} \otimes \cdots \otimes v_{k}$ by $v_{1} \wedge \cdots \wedge v_{k}$.
Example 17.2.10. $\Lambda^{2} \mathbb{R}^{n}$ has a basis $e_{i} \wedge e_{j}$ for $1 \leq i<j \leq n$. It is a well-known mistake to think that every element of an exterior product is of the form $v_{1} \wedge v_{2}$. This is not the case, e.g. $e_{1} \wedge e_{2}+e_{3} \wedge e_{4}$ can't be written this way.
Example 17.2.11. $\Lambda^{0} V$ is the quotient of $(V)^{\otimes 0}=\mathbb{R}$ by the trivial subspace, so is equal to $\mathbb{R}$.

The subspace in Definition 17.2.9 is designed to make

$$
\begin{aligned}
w_{0}: V^{k} & \longrightarrow \Lambda^{k} V \\
\left(v_{1}, \ldots, v_{k}\right) & \longmapsto v_{1} \wedge \cdots \wedge v_{k}
\end{aligned}
$$

alternating multilinear. This satisfies:
Lemma 17.2.12. For every alternating multilinear map $w: V^{k} \rightarrow W$ there is a unique linear map $\omega: \Lambda^{k} V \rightarrow W$ such that $w=\omega \circ w_{0}$.

Remark 17.2.13. This universal property tells us that the map $V^{\otimes k} \rightarrow \Lambda^{k} V$ corresponding to a natural assignment of an alternating multilinear map $w(b): V^{k} \rightarrow W$ to each multilinear map $b: V^{k} \rightarrow W$. This is given by anti-symmetrizing:

$$
w(b)\left(v_{1}, \ldots, v_{k}\right)=\frac{1}{n!} \sum_{\sigma \in \Sigma_{k}} b\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)
$$

The construction of $\Lambda^{k} V$ is natural in $V:$ whenever we have a linear map $A: V \rightarrow V^{\prime}$, there is an alternating multilinear map

$$
\begin{aligned}
V^{\times k} & \longrightarrow \Lambda^{k}\left(V^{\prime}\right) \\
\left(v_{1}, \ldots, v_{k}\right) & \longmapsto A\left(v_{1}\right) \wedge \cdots \wedge A\left(v_{k}\right),
\end{aligned}
$$

which induces a unique linear map $\Lambda^{k}(A): \Lambda^{k}(V) \rightarrow \Lambda^{k}\left(V^{\prime}\right)$. This is explicitly given by

$$
\Lambda^{k}(A)\left(v_{1} \wedge \cdots \wedge v_{k}\right)=A\left(v_{1}\right) \wedge \cdots \wedge A\left(v_{k}\right)
$$

From this formula or the universal property one easily deduces the following:

## Lemma 17.2.14.

- $\Lambda^{k}(B A)=\Lambda^{k}(B) \Lambda^{k}(A)$,
- $\Lambda^{k}(\mathrm{id})=\mathrm{id}$.


### 17.2.3 The top exterior power and orientations

Let us take a closer look at the case $V=\mathbb{R}^{k}$. Then $\Lambda^{k} \mathbb{R}^{k}$ has a basis with a single element $e_{1} \wedge \cdots \wedge e_{k}$, i.e. it is one-dimensional.
Example 17.2.15. For $k=2, \mathbb{R}^{2} \otimes \mathbb{R}^{2}$ is spanned by $e_{1} \otimes e_{1}, e_{1} \otimes e_{2}, e_{2} \otimes e_{1}$ and $e_{2} \otimes e_{2}$. In $\Lambda^{2}\left(\mathbb{R}^{2}\right)$ some additional antisymmetry rules are imposed. These for example say $e_{1} \wedge e_{2}=-e_{2} \wedge e_{1}$. But they also say $e_{1} \wedge e_{1}=-e_{1} \wedge e_{1}$ so $e_{1} \wedge e_{1}=0$, and similarly $e_{2} \wedge e_{2}=0$. Thus $\Lambda^{2}\left(\mathbb{R}^{2}\right)$ is indeed 1-dimensional spanned by $e_{1} \wedge e_{2}$.

Thus for each linear map $A: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$, the induced linear map $\Lambda^{k}(A): \Lambda^{k}\left(\mathbb{R}^{k}\right) \rightarrow$ $\Lambda^{k}\left(\mathbb{R}^{k}\right)$ is given by multiplication with a number, which for now we denote $d(A)$.
Example 17.2.16. For a matrix

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

we can compute $d(A)$ by determining which multiple of $e_{1} \wedge e_{2}$ the element $\Lambda^{2}(A)\left(e_{1} \wedge e_{2}\right)$ is equal to. The latter is given by

$$
\begin{aligned}
A\left(e_{1}\right) \wedge A\left(e_{2}\right) & =\left(a e_{1}+c e_{2}\right) \wedge\left(b e_{1}+d e_{2}\right) \\
& =a b e_{1} \wedge e_{1}+a d e_{1} \wedge e_{2}+c b e_{2} \wedge e_{1}+c d e_{2} \wedge e_{2} \\
& =(a d-b c) e_{1} \wedge e_{2}
\end{aligned}
$$

As the previous example shows, you are already familiar with the number $d(A)$.
Lemma 17.2.17. $d(A)=\operatorname{det}(A)$.
Sketch of proof. There are two ways to prove this.
You could use that the determinant is uniquely determined a small number of properties, namely that $\operatorname{det}(B A)=\operatorname{det}(B) \operatorname{det}(A)$ and its value on elementary matrices, upper-diagonal matrices, and permutation matrices. Indeed, using elementary matrices and permutation matrices you can row reduce all matrices to upper-diagonal ones.

You then just need to verify that $d(B A)=d(B) d(A)$, which follows from $\Lambda^{k}(B A)=$ $\Lambda^{k}(B) \Lambda^{k}(A)$, and that $d$ takes the same value as det on elementary matrices, upperdiagonal matrices and permutation matrices.

Alternatively, you could just compute $A\left(e_{1}\right) \wedge \cdots \wedge A\left(e_{k}\right)$ directly. By linearity in each entry and observing that those terms where a basis vector is repeated are 0 , you get

$$
\begin{aligned}
A\left(e_{1}\right) \wedge \cdots \wedge A\left(e_{k}\right) & =\sum_{\sigma}\left(\prod_{i=1}^{k} A_{i \sigma(i)}\right) e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)} \\
& =\sum_{\sigma}\left(\prod_{i=1}^{k}(-1)^{\epsilon(\sigma)} A_{i \sigma(i)}\right) e_{1} \wedge \cdots \wedge e_{k} \\
& =\operatorname{det}(A) e_{1} \wedge \cdots \wedge e_{k} . \square
\end{aligned}
$$

An invertible matrix $\operatorname{det}(A)$ is a composition of rotations and an upper-diagonal matrix with positive entries on the diagonal if and only if its determinant is positive. If the determinant is negative, then it is a composition of such matrices with a reflection in a hyperplane. If we think intuitively of an orientation has a notion of "handedness" (of "chirality" if you want a fancier term), then rotations and upper-diagonal matrices with positive entries on the diagonal should preserve this, but reflection should reverse this. This makes the following definition reasonable:

Definition 17.2.18. An orientation of a finite-dimensional $\mathbb{R}$-vector space $V$ is a choice of a non-zero element of $\Lambda^{\operatorname{dim}(V)}(V)$ up to scaling by a positive real number.

This definition is set up so that an invertible linear map $A$ preserves an orientation if and only if $\operatorname{det}(A)>0$.

### 17.3 Orientations

### 17.3.1 Fiberwise constructions

We have already seen how natural constructions on vector spaces lead to natural construction on vector bundles, by repeating this construction fiberwise:

| vector spaces | vector bundles |
| :---: | :---: |
| direct sum $V \oplus V^{\prime}$ | direct sum $E \oplus E^{\prime}$ |
| quotient $V / V^{\prime}$ | quotient $E / E^{\prime}$ |
| image im $\left(A: V \rightarrow V^{\prime}\right)$ | image $\operatorname{im}\left(G: E \rightarrow E^{\prime}\right)$ (if rank constant) |
| kernel $\operatorname{ker}\left(A: V \rightarrow V^{\prime}\right)$ | $\operatorname{kernel} \operatorname{ker}\left(G: E \rightarrow E^{\prime}\right)$ (if rank constant) |

We proved that these constructions produce vector bundles by going to local trivializations, and then observing that the corresponding constructions on general linear maps are continuous or even smooth in the entries.

Let us repeat this with the top exterior power:

Definition 17.3.1. Let $p: E \rightarrow X$ be a vector bundle of dimension $k$. Then its top exterior power $\Lambda^{k}(p): \Lambda^{k}(E) \rightarrow X$ is the vector bundle of dimension 1 given by $\bigsqcup_{x \in X} \Lambda^{k}\left(E_{x}\right)$. We topologize this as follows: for every local trivializations $\psi: p^{-1}(U)=$ $\bigsqcup_{x \in U} E_{x} \rightarrow U \times \mathbb{R}^{k}$ we define declare that the local trivialization $\left(\Lambda^{k}(p)\right)^{-1}(U)=$ $\bigsqcup_{x \in U} \Lambda^{k}\left(E_{x}\right) \rightarrow U \times \Lambda^{k}\left(\mathbb{R}^{k}\right)$ given by taking $(x, v) \mapsto\left(x, \Lambda^{k}\left(\psi_{x}\right)(v)\right)$ is a homeomorphism.

The transition functions of $\Lambda^{k}(E)$ are given by the determinant of the transition functions of $E$. Thus $\Lambda^{k}(E)$ will be a smooth vector bundle if $E$ is a smooth vector bundle.

Using this observation and similar ones for other exterior power or tensor products we can extend our table as follows:

| vector spaces | vector bundles |
| :---: | :---: |
| top exterior power $\Lambda^{\operatorname{dim}(V)}(V)$ | top exterior power $\Lambda^{\operatorname{dim}(E)}(E)$ |
| tensor product $V \otimes V^{\prime}$ | tensor product $E \otimes E^{\prime}$ |
| exterior power $\Lambda^{r}(V)$ | exterior power $\Lambda^{r}(E)$ |
| symmetric power $\operatorname{Sym}^{r}(V)$ | symmetric power $\operatorname{Sym}^{r}(E)$ |
| dual $V^{*}$ | dual $E^{*}$ |

### 17.3.2 Riemannian metrics

When thinking about smooth vector bundles it is sometimes helpful to have a Riemannian metric around:

Definition 17.3.2. A Riemannian metric is a section $g$ of $(E \otimes E)^{*}$ such that on each fiber $g_{x}: E_{x} \otimes E_{x} \rightarrow \mathbb{R}$ is a positive definite symmetric bilinear form.

Lemma 17.3.3. Every smooth vector bundle $p: E \rightarrow X$ admits a Riemannian metric, and this is unique up to homotopy.

Proof. For each local trivialization $\psi: p^{-1}(U) \rightarrow U \times \mathbb{R}^{k}$ we can define on $U$ the pullback along $\psi^{-1}$ of the standard Riemann metric: for $v, v^{\prime} \in E_{x}$,

$$
\left(\psi^{-1}\right)^{*} g_{\mathrm{std}}\left(v, v^{\prime}\right):=g_{\mathrm{std}}\left(\psi_{x}^{-1}(v), \psi_{x}^{-1}\left(v^{\prime}\right)\right) .
$$

Now take a partition of unity subordinate to an open cover of $X$ by open subsets $U$ of a local trivialization; $\eta_{i}: M \rightarrow[0,1]$ supported in $U_{i}$. Then we define

$$
g:=\sum_{i} \eta_{i} \cdot\left(\psi_{i}^{-1}\right)^{*} g_{\mathrm{std}} .
$$

This is positive define and symmetric since these properties are preserved by convex linear combinations. For uniqueness, observe we can linearly interpolate between any two Riemannian metric.

The main application of this is:

Lemma 17.3.4. If $E^{\prime} \subset E$ is a subbundle, then there is another subbundle $E^{\prime \prime} \subset E$ such that $E^{\prime} \oplus E^{\prime \prime} \cong E$. This subbundle $E^{\prime \prime}$ is isomorphic to $E / E^{\prime}$.

Proof. Equip $E$ with a Riemannian metric. Then we can take $E^{\prime \prime}=\left(E^{\prime}\right)^{\perp}$, given by fibers $\left(E^{\prime}\right)_{x}^{\perp}:=\left(E_{x}^{\prime}\right)^{\perp}$. To get the second part, we observe that the map of vector bundles $E \rightarrow\left(E^{\prime}\right)^{\perp}$ given on fibers by orthogonal projection $E_{x} \rightarrow\left(E^{\prime}\right)_{x}^{\perp}$ with kernel given by $E^{\prime}$ and hence induces an isomorphism $E / E^{\prime} \rightarrow\left(E^{\prime}\right)^{\perp}$.

### 17.3.3 Orientations of vector bundles

Recall that a map which picks a single element of each fiber is called a section:
Definition 17.3.5. A section of a smooth vector bundle $p: E \rightarrow X$ is a smooth map $s: X \rightarrow E$ such that $p \circ s=\operatorname{id}_{X}$.

Example 17.3.6. Every smooth vector bundle has a 0 -section $s_{0}: X \rightarrow E$ picking out the 0 in each fiber.

Example 17.3.7. A smooth section of $T M$ is also known as a smooth vector field.
When we have a section $s: X \rightarrow E$ of a smooth vector bundle and a smooth function $g: X \rightarrow \mathbb{R}$, we can use fiberwise scalar multiplication to produce a new section $g \cdot s$.

Definition 17.3.8. An orientation of a smooth vector bundle $p: E \rightarrow B$ is an everywhere non-zero section $s$ of $\Lambda^{\operatorname{dim}(E)} E$, up to the equivalence relation of scalar multiplication by an everywhere positive smooth function.

Thus, an orientation on $E$ is smooth choice of non-zero elements of each $\Lambda^{\operatorname{dim}(E)} E_{x}$ up to scaling, that is, a smooth choice of orientation of each of vector spaces $E_{x}$.
Example 17.3.9. Trivial vector bundles always admit an orientation.
Example 17.3.10. A much more interesting example is the Moebius strip, i.e. the tautological bundle over $\mathbb{R} P^{1}$. We use the following straightforward observation: every section $s$ of a smooth vector bundle $p: E \rightarrow B$ is homotopic to the 0 -section. Indeed, take $H: B \times[0,1] \rightarrow E$ given by

$$
(p, t) \longmapsto t \cdot s(p)
$$

Using this we prove that the tautological bundle $\gamma$ over $\mathbb{R} P^{1}$ (the one whose total space is the Moebius strip) does not admit an orientation. Let us identify $\mathbb{R} P^{1}$ with the 0 -section. If this bundle did admit an orientation, there would be an everywhere non-zero section $s$ and we would have $I_{2}\left(s, \mathbb{R} P^{1}\right)=0$. But we also know that $I_{2}\left(s, \mathbb{R} P^{1}\right)=I_{2}\left(\mathbb{R} P^{1}, \mathbb{R} P^{1}\right)$, and latter is 1 by exhibiting a particular section transverse to the 0 -section. This gives a contradiction.

A vector bundle $E$ is said to be orientable if it admits an orientation.
Lemma 17.3.11. A vector bundle $E$ is orientable if $\Lambda^{\operatorname{dim}(E)} E$ is isomorphic to a trivial line bundle. Furthermore, an orientation is a trivialization of $\Lambda^{\operatorname{dim}(E)} E$ up to scalar multiplication by a smooth positive function.

Proof. Indeed, a representative $s: X \rightarrow \Lambda^{\operatorname{dim}(E)} E$ of an orientation furnishes an isomorphism

$$
\begin{aligned}
X \times \mathbb{R} & \cong \Lambda^{\operatorname{dim}(E)} E \\
(b, t) & \longmapsto t \cdot s(b) .
\end{aligned}
$$

Conversely, an isomorphism $\phi: \Lambda^{\operatorname{dim}(E)} \cong X \times \mathbb{R}$ gives an everywhere non-vanishing section $s: X \rightarrow \Lambda^{\operatorname{dim}(E)}$ by $x \mapsto \phi^{-1}(x, 1)$.

If $E$ is orientable, how many orientations does it admit? Given an orientation represented by $s$, any other orientation $s^{\prime}$ differs by scalar multiplication of $s$ with an everywhere non-zero smooth function $f$. If we multiply $f$ with an everywhere positive smooth function we get the same $s^{\prime}$, so the orientations are given by the set of everywhere non-zero smooth functions up to multiplication by everywhere positive smooth function. In other words, for each connected component of $X$ we have to pick a choice of sign. We conclude that:

Lemma 17.3.12. Let $\pi_{0}(X)$ denote the set of connected components of $X$, then if $E$ is orientable the set of orientations is (non-canonically) given by the set of functions

$$
\pi_{0}(B) \longrightarrow\{ \pm 1\}
$$

Given orientations for smooth vector bundles $E, E^{\prime}$ over $X$, you can produce a direct sum orientation on $E \oplus E^{\prime}$. The observation you need is that there is a natural isomorphism

$$
\begin{aligned}
& \Lambda^{\operatorname{dim}(E)} E \otimes \Lambda^{\operatorname{dim}\left(E^{\prime}\right)} E^{\prime} \cong \\
&\left(v_{1} \wedge \cdots \wedge v_{\operatorname{dim}(E)}\right) \otimes\left(v_{1}^{\prime} \wedge \cdots \wedge v_{\operatorname{dim}\left(E^{\prime}\right)}^{\prime}\right)\left.\longmapsto v_{1} \wedge \cdots \wedge v_{\operatorname{dim}(E)} \wedge E^{\prime}\right)\left(E \oplus E^{\prime}\right) \\
& \wedge \cdots \wedge v_{\operatorname{dim}\left(E^{\prime}\right)}^{\prime}
\end{aligned}
$$

Thus trivializations of $\Lambda^{\operatorname{dim}(E)} E$ and $\Lambda^{\operatorname{dim}\left(E^{\prime}\right)} E^{\prime}$ give a trivialization of $\Lambda^{\operatorname{dim}(E)} E \otimes$ $\Lambda^{\operatorname{dim}\left(E^{\prime}\right)} E^{\prime}$.

Conversely, if $E=E^{\prime} \oplus E^{\prime \prime}$ with $E$ and $E^{\prime}$ oriented, the trivializations of $E$ and $E^{\prime}$ give isomorphisms

$$
B \times \mathbb{R} \cong \Lambda^{\operatorname{dim}\left(E^{\prime}\right)+\operatorname{dim}\left(E^{\prime \prime}\right)}\left(E^{\prime} \oplus E^{\prime \prime}\right) \cong \Lambda^{\operatorname{dim}\left(E^{\prime}\right)} E^{\prime} \otimes \Lambda^{\operatorname{dim}\left(E^{\prime \prime}\right)} E^{\prime \prime} \cong \Lambda^{\operatorname{dim}\left(E^{\prime \prime}\right)} E^{\prime \prime}
$$

so an orientation of $E^{\prime \prime}$.

### 17.3.4 Orientations of manifolds

If $M$ is a $k$-dimensional manifold, then $T M$ is a $k$-dimensional smooth vector bundle $M$ and hence $\Lambda^{k} T M$ is a 1-dimensional smooth vector bundle $M$, called the orientation line bundle.

Definition 17.3.13. An orientation of $M$ is an orientation of $T M$.
Remark 17.3.14. An orientation of $M$ is equivalent to a choice of "oriented" atlas inside its maximal atlas, where all transition functions are required to have total derivatives with positive determinant.

Let us give two examples of manifolds that are orientable and one which is not:
Example 17.3.15. If $M=S^{1}$, the tangent bundle is isomorphic to a trivial bundle and since $\Lambda^{\operatorname{dim}(E)} E=E$ for any 1-dimensional vector bundle so is its top exterior power. It hence admits exactly two orientations. These correspond to the clockwise and counterclockwise directions of the circle.

Example 17.3.16. If $M=*$, we have that $\Lambda^{0} T M=\mathbb{R}$, so the point admits exactly two orientations. However, the one represented by $1 \in \mathbb{R}$ should obviously be our preferred choice.

Example 17.3.17. We claim that $\mathbb{R} P^{2}$ admits no orientation. If it did then so would $\left.T \mathbb{R} P^{2}\right|_{\mathbb{R} P^{1}}$. This vector bundle is isomorphic to $T \mathbb{R} P^{1} \oplus N \mathbb{R} P^{1} \cong \mathbb{R} \oplus \gamma$, with $\gamma$ the canonical bundle over $\mathbb{R} P^{1}$. This means its orientation line bundle is $\Lambda^{2}(\mathbb{R} \oplus \gamma) \cong \gamma$ and we proved above that $\gamma$ does not admit an everywhere non-vanishing section, i.e. is not trivializable.

There are several constructions which produce new orientations on manifold form old ones:

Example 17.3.18. Given a manifold $M$ with orientation, we can produce another orientation by multiplying a representative section $s: M \rightarrow \Lambda^{k} T M$ with -1 . This is called reversing the orientation and we shall occasionally use the notion $-M$ for this.

Example 17.3.19. If $M$ and $N$ are manifolds with orientations, then we get a direct sum orientation on $M \times N$, as $T_{\left(p, p^{\prime}\right)}(M \times N) \cong T_{p} M \oplus T_{p^{\prime}} N$.

To phrase this in terms of vector bundles, we need a generalization of the restriction of vector bundles: given any map $f: X^{\prime} \rightarrow X$ we can pull back a vector bundle $p: E \rightarrow X$ to $X^{\prime}$ by setting $f^{*} E=\bigsqcup_{x^{\prime} \in X} E_{f\left(x^{\prime}\right)}$. In the language of vector bundles we have $T(M \times N) \cong \pi_{1}^{*} T M \oplus \pi_{2}^{*} T N$.
Example 17.3.20. If $Z \subset N$ is a submanifold and both $N$ and $Z$ are oriented, then the isomorphism $\left.T N\right|_{Z} \cong N Z \oplus T Z$ shows that $N Z$ also comes with an orientation.
Example 17.3.21. Suppose we have a smooth map $f: M \rightarrow N$ with $M$ and $N$ oriented, and $Z \subset M$ an oriented submanifold such that $f \pitchfork Z$. Then $f^{-1}(Z)$ is a submanifold and its tangent bundle satisfies $\left.f^{*} N Z \oplus T f^{-1}(Z) \cong T M\right|_{f^{-1}(Z)}$. Since both $\left.T M\right|_{f^{-1}(Z)}$ and $f^{*} N Z$ comes with orientations, we get an orientation of $T f^{-1}(Z)$.

### 17.3.5 Induced orientation on the boundary

If $M$ is a manifold with boundary $\partial M$, then its boundary $\partial M$ inherits an orientation, canonically so once we fix a single convention. To do so, it is convenient to pick a Riemannian metric on $M$, that is, on $T M$. Then the restriction $\left.T M\right|_{\partial M}$ inherits a Riemannian metric and thus splits as $T \partial M \oplus(T \partial M)^{\perp}$, the latter being a line bundle.

By Lemma 13.3.7, there exist a smooth function $\chi: M \rightarrow[0, \infty)$ such that $\chi^{-1}(0)=$ $\partial M$ and for each $p \in \partial M, d_{p} \chi$ is non-vanishing on some vector $v \in T_{p} M \backslash T_{p} \partial M$. This vector $v$ decomposes as a sum of a vector $v_{\partial} \in T_{p} \partial M$ and a vector $v_{\perp} \in\left(T_{p} \partial M\right)^{\perp}$. Since $\chi$ is constant on $\partial M, v_{\partial}$ is zero so $v_{\perp}$ is non-zero. Hence the restriction $d_{p} \chi:\left(T_{p} \partial M\right)^{\perp} \rightarrow \mathbb{R}$ is non-zero.

We call a vector $v \in\left(T_{p} \partial M\right)^{\perp}$ such that $d_{p} \chi(v)<0$ outward pointing. Such a vector is unique up to scaling by a positive real number. In particular, there is a canonical section $n$ of $\left(\left.T M\right|_{\partial M}\right)^{\perp}$ given at $p \in \partial M$ by the unique element $n_{p}$ of $\left(T_{p} \partial M\right)^{\perp}$ such that $d_{p} \chi\left(n_{p}\right)=1$.

Every vector $v \in V$ provides a linear map $v \wedge-: \Lambda^{k-1}(V) \rightarrow \Lambda^{k}(V)$. This generalizes to a map of vector bundles

$$
\begin{aligned}
\Lambda^{k-1}(T \partial M) & \longrightarrow \Lambda^{k}\left(\left.T M\right|_{\partial M}\right) \\
w & \longmapsto n \wedge w
\end{aligned}
$$

of vector bundles, by thinking of $\Lambda^{k-1}(T \partial M)$ as a linear subspace of $\Lambda^{k-1}\left(\left.T M\right|_{\partial M}\right)$ using the inclusion of $T \partial M$ into $\left.T M\right|_{\partial M}$.
Lemma 17.3.22. If an orientation of $M$ is represented by the section $s$ of $\Lambda^{k} T M$, then there is a unique orientation of $M$ which is represented by a section $\bar{s}$ of $\Lambda^{k-1} T \partial M$ satisfying $n \wedge \bar{s}=s$.

Proof. For each $p \in \partial M$, fix a basis $e_{1}, \ldots, e_{k-1}$ of $T_{p} \partial M$. By adding $n_{p}$ we get a basis of $T_{p} M$. Then $\bar{s}(p)$ is by definition $\bar{c}(p) \cdot e_{1} \wedge \cdots \wedge e_{k-1}$ for some $\bar{c} \in \mathbb{R}$, and $s(p)$ similarly is $c(p) \cdot n_{p} \wedge e_{1} \wedge \cdots \wedge e_{k-1}$ for some $c(p) \in \mathbb{R}$. From the equation

$$
n_{p} \wedge\left(\bar{c}(p) \cdot e_{1} \wedge \cdots \wedge e_{k-1}\right)=c(p) \cdot n_{p} \wedge e_{1} \wedge \cdots \wedge e_{k-1}
$$

we read off $\bar{c}(p)=c(p)$, so $\bar{s}$ is uniquely determined by $n$ and $s$.
Firstly $\bar{s}$, up to multiplication by a positive smooth function, is independent of the choice of representative $s$ : if $s$ changes by multiplying it with positive smooth function, so does $\bar{s}$.

Next, we have to verify the orientation is independent of the choice of Riemannian metric $g$ and smooth function $\chi$. Modifying the latter just changes $n$ by scalar multiplication by a positive smooth function, and hence has the same effect on $\bar{s}$. If we vary $g$, then $n_{p}$ gets replaced by $n_{p}^{\prime}=a n_{p}+\sum_{i=1}^{k-1} a_{i} e_{i}$ with $a>0$ so

$$
n_{p}^{\prime} \wedge\left(\bar{c}(p) \cdot e_{1} \wedge \cdots \wedge e_{k-1}\right)=a \cdot n_{p} \wedge\left(\bar{c}(p) \cdot e_{1} \wedge \cdots \wedge e_{k-1}\right)
$$

and again $\bar{s}$ just changes by scalar multiplication by a positive smooth function.
Definition 17.3.23. If $M$ is oriented, we shall consider $\partial M$ as oriented by the orientation produced in the previous lemma. We refer to this as the induced orientation.
Example 17.3.24. There is a preferred choice of orientation on $[0,1]$, namely using $1 \in \Lambda^{1} T_{p}[0,1] \cong T_{p}[0,1] \cong \mathbb{R}$. Then

$$
\partial[0,1] \cong\{1\}-\{0\}
$$

where, for an oriented manifold $N,-N$ denotes the same manifold with opposite orientation.

More generally, if $M$ is oriented without boundary, then

$$
\partial([0,1] \times M)=M \times\{1\}-M \times\{0\}
$$

However, if we do $\partial(M \times[0,1])$ we get $(-1)^{\operatorname{dim}(M)}(M \times\{1\}-M \times\{0\})$. This is an unfortunate clash of our conventions for orientations and notation for homotopies.

Example 17.3.25. Generalizing Example 17.3 .21 to the case that $M$ has boundary and $f \pitchfork Z, \partial f \pitchfork Z$ we get that $\partial f^{-1}(Z)=(\partial f)^{-1}(Z)$ comes with two orientations: one as the boundary of an oriented manifold and one as the inverse image of an oriented manifold. These are not equal but satisfy

$$
\partial f^{-1}(Z)=(-1)^{\operatorname{codim}(Z)}(\partial f)^{-1}(Z)
$$

### 17.4 Integral intersection theory

Chapter 3 of [GP10] upgrades the mod 2 intersection theory to an integral version. The main input is the observation that

$$
\partial[0,1] \cong\{1\}-\{0\}
$$

and the classification of compact 1-dimensional manifolds lead to the following result:
Proposition 17.4.1. If $M$ is a compact oriented 1-dimensional manifold, then the number of positively-oriented points in $\partial M$ is equal to the number of negatively-oriented points.

So we can define intersection numbers with values in $\mathbb{Z}$ instead of $\mathbb{Z} / 2$ :
Definition 17.4.2. Suppose that $Y$ is a compact oriented manifold without boundary, $M$ is an oriented manifold and $Z \subset M$ is an oriented submanifold such that $\operatorname{dim}(Y)+$ $\operatorname{dim}(Z)=\operatorname{dim}(M)$.

Let $f_{0}: Y \rightarrow M$ be a smooth map. Then the intersection number $I\left(f_{0}, Z\right)$ is defined as follows: take $f_{1}$ homotopic to $f_{0}$ and transverse to $Z$, and set

$$
I\left(f_{0}, Z\right)=\sum_{p \in f_{1}^{-1}(Z)} \text { orientation of } p
$$

One proceeds as before, using Proposition 17.4.1 in place of the fact that the number of points in the boundary of a compact 1-dimensional manifold is even, to prove that $I\left(f_{0}, Z\right)$ is well-defined and establish its basic properties. You can then easily define integral versions of the degree of a map and the winding numbers, and use these to great effect.

Example 17.4.3. With these definitions in hand, the mod 2 linking numbers of Section 16.2 generalize to integer linking numbers.

### 17.5 Problems

Problem 17.5.1 (Codimension 1 submanifolds are orientable). Use the Jordan-Brouwer separation theorem to prove that if $M \subset \mathbb{R}^{k}$ is a compact codimension 1 submanifold, then it is orientable.

Problem 17.5.2. Define a degree $\operatorname{deg}(f) \in \mathbb{Z}$ of a smooth map $f: M \rightarrow N$ between compact oriented smooth manifolds of the same dimension, which reduces to $\operatorname{deg}_{2}(f)$ modulo 2.

Problem 17.5.3. Use partitions of unity to prove that any vector $v \in T_{p} M$ is the value at $x$ of some smooth vector field $X$ on $M$.

Problem 17.5.4 (The degree of multiplication).
(a) Suppose that $f: M \rightarrow N$ is a proper map between connected oriented smooth manifolds of the same dimension. Give the definition of a degree $\operatorname{deg}(f) \in \mathbb{Z}$ which specializes to the usual one when $M$ is compact.
(b) Indicate why $\operatorname{deg}(f)$ of part (a) is well-defined and invariant under homotopies which are also proper maps. You do not need to give full proofs.
Let $\mathcal{P}_{n}$ be the space of monic polynomials of degree $n$ with real coefficients. Through the coefficients it may be identified with $\mathbb{R}^{n}$, making it an $n$-dimensional smooth oriented manifold. There is a multiplication map

$$
\begin{aligned}
\mu: \mathcal{P}_{n} \times \mathcal{P}_{m} & \longrightarrow \mathcal{P}_{n+m} \\
(p, q) & \longmapsto p q .
\end{aligned}
$$

(c) Prove that $\mu$ is smooth and proper.

Thus $\mu$ has a well-defined degree as in part (a) and (b), which we will compute now when $n$ and $m$ are even. Let $\mathcal{Q}_{n}$ be the $(n+1)$-dimensional real vector space of all polynomials of degree $\leq n$ with real coefficients.
(d) For $p \in \mathcal{P}_{n}$, use the map $\mathcal{Q}_{n-1} \rightarrow \mathcal{P}_{n}$ given by $u \mapsto p+u$ to identify the vector space $T_{p} \mathcal{P}_{n}$ with $\mathcal{Q}_{n-1}$.
(e) With respect to these identifications, show that $T_{(p, q)} \mu$ is given by

$$
\begin{aligned}
\mathcal{Q}_{n-1} \times \mathcal{Q}_{m-1} & \longrightarrow \mathcal{Q}_{n+m-1} \\
(u, v) & \longmapsto u q+v p .
\end{aligned}
$$

The determinant of the linear map of part (e) with respect to the standard bases given by mononials is known as the resultant $R(p, q)$. You may use without proof that it can also be computed as $R(p, q)=\prod_{i, j}\left(\alpha_{i}-\beta_{j}\right)$ where $\alpha_{i}$ runs over the complex roots of $p$ (with multiplicity) and $\beta_{j}$ over the complex roots of $q$ (with multiplicity).
(f) Prove that the resultant is positive if the roots of $p$ and $q$ are all distinct and non-real.
(g) Prove that $r \in \mathcal{P}_{n+m}$ is a regular value of $\mu$ if it is a product of pairwise distinct, irreducible, quadratic monic polynomials.
(h) Show that the degree of $\mu$ when $n=2 k$ and $m=2 l$, is given by the binomial coefficient $\binom{k+l}{l}$.
(i) Show that the degree of $\mu$ when $n=2 k+1$ and $m=2 l$, is again given by the binomial coefficient $\binom{k+l}{l}$.
(j) Show that the degree of $\mu$ when $n=2 k+1$ and $m=2 l+1$, is zero. (Hint: is it surjective?)

## Chapter 18

## Integration on manifolds

Today we define differential forms and one of their the raisons-d'etre: integration. This is Section $4 . \S 4$ of [GP10], but you should also take a look at Sections $4 . \S 1-3$ if you haven't done so already.

### 18.1 Differential forms

We start with a discussion of differential forms, with a focus of forms of top degree.

### 18.1.1 The definition of differential forms

Every smooth manifold has a tangent bundle $T M$, which you are already familiar with, and a cotangent bundle $T^{*} M$. The fibers $T_{p}^{*} M$ of the cotangent bundle, called cotangent spaces, are the linear duals $\left(T_{p} M\right)^{*}$ of the tangent spaces. If $M$ has dimension $k$, both are smooth vector bundles of dimension $k$.

Definition 18.1.1. A 1 -form on $M$ is a smooth section of $T^{*} M$.
We can produce a 1-form from a smooth function $f: M \rightarrow \mathbb{R}$. Recall that the fibers $T_{m} M$ of the tangent bundle are derivations on germs $\mathcal{E}(M, m)$ near $m$ of smooth functions $M \rightarrow \mathbb{R}$. In particular, these assign a number to each the germ $\bar{f}$ of $f$. We get an element $(d f)_{m}$ of $\left(T_{m} M\right)^{*}$ by taking

$$
\begin{aligned}
d f: T_{m} M & \longrightarrow \mathbb{R} \\
X & \longmapsto X(\bar{f}) .
\end{aligned}
$$

This produces an element of $T_{m} M$ for each $m$, hence a section. To see it is smooth we use charts:

Example 18.1.2. If $\phi: \mathbb{R}^{k} \supset U \rightarrow V \subset M$ is a chart around $p \in M$, we get an isomorphism of $T_{p} M$ with $T_{\phi^{-1}(p)} \mathbb{R}^{k}$. The latter one thinks of as the $\mathbb{R}$-vector space with basis $\frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{k}}$ (this is just alternative notation for the standard basis vector $e_{1}, \ldots, e_{k}$, but now considered as elements of $\left.T_{\phi^{-1}(p)} \mathbb{R}^{k} \cong \mathbb{R}^{k}\right)$. This in turn gives rise to a dual basis
$d x_{1}, \ldots, d x_{k}$ of $T_{p}^{*} M$. Thus every 1 -form $\alpha$ can be written in local coordinates as

$$
\alpha(x)=\sum_{i=1}^{k} a_{i}(x) d x_{i} .
$$

We saw above that any smooth function $f: M \rightarrow \mathbb{R}$ gives rise to a 1 -form $d f$. In terms of the above coordinates this is given by

$$
T_{p}^{*} M \ni(d f)_{p}:=\sum_{i=1}^{k} \frac{\partial f}{\partial x_{i}}(p) d x_{i} .
$$

To see this, observe that $(d f)_{p}$ by construction evaluates on $\frac{\partial}{\partial x_{i}}$ to $\frac{\partial f}{\partial x_{i}}(p)$.
Example 18.1.3. The 1 -form $-y d x+x d y$ on $\mathbb{R}^{2}$ restricts to a 1 -form on $S^{1} \subset \mathbb{R}^{2}$ which is nowhere-vanishing.

We extended the notion of a 1 -form to a $p$-form as follows:
Definition 18.1.4. A $p$-form is a smooth section of $\Lambda^{p} T^{*} M$.
Example 18.1.5. As $\Lambda^{0} T^{*} M=\mathbb{R}$, a smooth 0 -form is a smooth function. As $\Lambda^{1} T^{*} M=$ $T^{*} M$, this recovers the definition of a smooth 1 -form given above.

Since the value at $p \in M$ of a smooth section of a smooth vector bundle $E$ lie in $\mathbb{R}$-vector spaces $E_{p}$, so we can define addition of smooth sections by pointwise addition. Similarly, we can scale a smooth section with any smooth real-valued function. The result is either operation is again smooth section, making the set $\Gamma(M, E)$ into a $C^{\infty}(M ; \mathbb{R})$ module. Since $C^{\infty}(M ; \mathbb{R})$ contains $\mathbb{R}$ as the constant functions, $\Gamma(M, E)$ is in particular an $\mathbb{R}$-vector space.

Definition 18.1.6. $\Omega^{p}(M)$ is the $\mathbb{R}$-vector space $\Gamma\left(M, \Lambda^{p} T^{*} M\right)$ of $p$-forms.
Definition 18.1.7. $\Omega^{*}(M)$ is the graded $\mathbb{R}$-vector space of differential forms on $M$, given by putting the $p$-forms $\Omega^{p}(M)$ in degree $p$. When the degree plays no role, we refer to a $p$-form as a differential form of degree $p$.

Recalling that $M$ is $k$-dimensional, we see that $\Lambda^{p} T^{*} M=0$ if $p>k$, and hence there are no non-zero differential forms of degree larger than the dimension of $M$. In this lecture our main interest is the case $p=k$. Then $\Lambda^{k} T^{*} M$ is one-dimensional, and we shall refer to the $k$-forms as top forms.
Example 18.1.8. A chart $\phi: \mathbb{R}^{k} \supset U \rightarrow V \subset M$ induces a local trivialization of $T M$. In turn, this gives a local trivializations of $T^{*} M$ and hence of $\Lambda^{p} T^{*} M$. For this we see that each $p$-form $\omega \in \Omega^{p}(V)$ can be written in local coordinates as

$$
\omega(x)=\sum_{I} a_{I}(x) d x_{I}
$$

where for each index set $I=1 \leq i_{1}<\ldots<i_{p} \leq k$, we write

$$
d x_{I}:=d x_{i_{1}} \wedge \cdots \wedge d x_{i_{p}},
$$

and $a_{I}: U \rightarrow \mathbb{R}$ is a some smooth function.
In particular, every top form can be written in local coordinates as

$$
\omega(x)=a(x) d x_{1} \wedge \cdots \wedge d x_{k}
$$

for a smooth function $a: U \rightarrow \mathbb{R}$.
Example 18.1.9. Recall that an orientation was an everywhere non-vanishing section of $\Lambda^{k} T M$, up to scaling by everywhere positive function. Recall that a Riemannian metric is a smooth family of non-degenerate bilinear forms on $T M$, and always exists. Such a Riemannian metric gives an isomorphism of $T M$ and $T^{*} M$ by sending a vector $v \in T_{p} M$ to the linear functional $w \mapsto\langle w, v\rangle$ in $T_{p}^{*} M$. This isomorphism induces an isomorphism between the line bundles $\Lambda^{k} T M$ and $\Lambda^{k} T^{*} M$, and hence an orientation is also the same as an everywhere non-vanishing top form up to scaling.

### 18.1.2 The wedge product

We defined a wedge product

$$
\wedge: \Omega^{p}(M) \otimes \Omega^{q}(M) \rightarrow \Omega^{p+q}(M),
$$

induced by the corresponding wedge product on the exterior powers of the fiber. This has the following property:

Theorem 18.1.10. The wedge product makes $\Omega^{*}(M)$ into a graded-commutative $\mathbb{R}$ algebra. That is, the wedge product has the following properties:
(1) It is unital with unit given by the function that is constant 1.
(2) It is bilinear.
(3) It is associative.
(4) If $\omega$ has degree $p$ and $\rho$ has degree $q$, then $\omega \wedge \rho$ has degree $p+q$ and

$$
\omega \wedge \rho=(-1)^{p q} \rho \wedge \omega
$$

Remark 18.1.11. Observe that $\Omega^{0}(M)=C^{\infty}(M ; \mathbb{R})$, and the wedge product $\Omega^{0}(M) \otimes$ $\Omega^{p}(M) \rightarrow \Omega^{p}(M)$ is equal to the multiplication of the $C^{\infty}(M ; \mathbb{R})$-module structure. Hence we can replace linearity by $C^{\infty}(M ; \mathbb{R})$-linearity.

We can use the wedge products to produce many top forms, e.g. by wedging together $k$ 1-forms as below:
Example 18.1.12. Given $k$ smooth functions $f_{1}, \ldots, f_{k}: \mathbb{R}^{k} \rightarrow \mathbb{R}$, we can produce a top form

$$
d f_{1} \wedge \cdots \wedge d f_{k},
$$

whose value in local coordinates is given by

$$
\left(d f_{1} \wedge \cdots \wedge d f_{k}\right)_{p}=\operatorname{det}\left(\frac{\partial f_{j}}{\partial x_{i}}\right) d x_{1} \wedge \cdots \wedge d x_{k} .
$$

If you don't see how to do this computation, please ask about it in office hours or sections.

### 18.1.3 Pullback of differential forms

One of the advantages of differential forms is that we can pull them back along any smooth map, unlike vector fields, which can only be pushed forward along a diffeomorphism:

Theorem 18.1.13. Each smooth map $f: M \rightarrow N$ induces a map $f^{*}: \Omega^{*}(N) \rightarrow \Omega^{*}(M)$ of graded-commutative $\mathbb{R}$-algebras by applying to $p$-forms the map $\Lambda^{p}\left(d_{p} f\right)^{*}$ in each fiber. Pullback has the following properties:
(1) On functions (that is, 0-forms), $f^{*}$ is given by precomposition, $f^{*} g=g \circ f$.
(2) $(g \circ f)^{*}=f^{*} \circ g^{*}$ and (id)* $=\mathrm{id}$.
(3) It commutes with wedge products:

$$
f^{*}(\omega \wedge \rho)=f^{*}(\omega) \wedge f^{*}(\rho) .
$$

(4) It commutes with taking derivatives of functions:

$$
f^{*} d g=d\left(f^{*} g\right) .
$$

Example 18.1.14. Let's compute some pullback in local coordinates. Suppose $f: \mathbb{R}^{k} \supset$ $U \rightarrow V \subset \mathbb{R}^{k^{\prime}}$ is a smooth map and recall that $d x_{i}^{\prime} \in \Omega^{1}(V)$ is the dual to the vector field $\frac{\partial}{\partial x_{i}^{\prime}}$ that is constant equal to $e_{i}^{\prime}$. Then

$$
f^{*} d x_{i}^{\prime}=d f^{*} x_{i^{\prime}}^{\prime}=d f_{i^{\prime}} \sum_{i=1}^{k} \frac{\partial f_{i^{\prime}}}{\partial x_{i}} d x_{i},
$$

with $f_{i^{\prime}}$ the $i^{\prime}$ th component of $f$.
A similar formula exists for $p$-forms, but we will focus on the case of top forms. Suppose a $p$-form $\omega \in \Omega^{p}(V)$ is given by

$$
\omega\left(x^{\prime}\right)=a\left(x^{\prime}\right) d x_{1}^{\prime} \wedge \cdots \wedge d x_{k^{\prime}}^{\prime} .
$$

Since pullback commutes with wedge product, its pullback $f^{*} \omega \in \Omega^{p}\left(V^{\prime}\right)$ must given by

$$
\begin{aligned}
f^{*} \omega(x) & =a(f(x)) f^{*}\left(d x_{1}^{\prime} \wedge \cdots \wedge d x_{k^{\prime}}^{\prime}\right) \\
& =a(f(x)) f^{*}\left(d x_{1}^{\prime}\right) \wedge \cdots \wedge f^{*}\left(d x_{k^{\prime}}^{\prime}\right)
\end{aligned}
$$

and above we saw how to compute each term $f^{*}\left(d x_{i^{\prime}}^{\prime}\right)$ in terms of the partial derivatives of $f_{i^{\prime}}$.

Given a submanifold $X \subset M$ with inclusion denoted $i: X \hookrightarrow M$, we can restrict a $p$-form $\omega \in \Omega^{p}(M)$ to $X$ :

$$
\Omega^{*}(M) \ni \omega \longmapsto i^{*} \omega \in \Omega^{p}(X) .
$$

If $X$ is $p$-dimensional, this gives a top form on $X$.
Example 18.1.15. The restriction to $S^{1} \subset \mathbb{R}^{2}$ of the 1 -form $\omega=x d x+y d y$ is identically 0 . This is because that $\omega$ is $d g$ with $g: \mathbb{R}^{2} \rightarrow \mathbb{R}$ given by $\frac{1}{2}\left(x^{2}+y^{2}\right)$. Hence $i^{*} \omega=i^{*} d g=$ $d\left(i^{*} g\right)$ and since $i^{*} g$ constant equal to 1 its derivative vanishes.

### 18.2 Integration of differential forms

Our next goal is the integration of $k$-forms over $k$-dimensional manifolds. This theory of integration has a number of features which may differ from what you are used to:
(I) It is only defined for oriented manifolds.
(II) Over a $k$-dimensional oriented manifold, you can only integrate top forms (so only $k$-forms, not functions).
(III) We will only define integration of compactly-supported top forms.

### 18.2.1 Integration on $\mathbb{R}^{k}$

First suppose that

$$
\omega=a(x) d x_{1} \wedge \cdots \wedge d x_{k}
$$

is a top form on an open subset $U \subset \mathbb{R}^{k}$. Then, as the notation suggests, we shall define

$$
\int_{U} \omega=\int_{U} a(x) d x_{1} \wedge \cdots \wedge d x_{k}:=\int_{U} a(x) d x_{1} \cdots d x_{k}
$$

and to guarantee that the integral exist we assume $a$ has compact support in $U$. This is not really necessary as some integrals of functions without compact support do converge, but it is the only case we shall use. For smooth compactly-supported functions both the Riemann and Lebesgue integral exist and are equal, so we don't need to worry about the technical details too much.

Example 18.2.1. The order of the entries of $d x_{1} \wedge \cdots \wedge d x_{k}$ is important: if $U=\operatorname{int}\left(D^{2}\right)$ and $\omega=d y \wedge d x$, then (ignoring the compact support requirement)

$$
\int_{U} \omega=\int_{\operatorname{int}\left(D^{2}\right)} d y \wedge d x=-\int_{\operatorname{int}\left(D^{2}\right)} d x \wedge d y=-\pi .
$$

How does the integral of a top form transform under a change of coordinates? That is, suppose we have a diffeomorphism $\psi: U^{\prime} \rightarrow U$. Then on the one hand, we have by definition of the integral that

$$
\begin{aligned}
\int_{U^{\prime}} \psi^{*} \omega & =\int_{U^{\prime}} \psi^{*} a\left(x^{\prime}\right) \psi^{*} d x_{1} \wedge \cdots \wedge \psi^{*} d x_{k} \\
& =\int_{U^{\prime}}(a \circ \psi)\left(x^{\prime}\right) d \psi_{1} \wedge \cdots \wedge d \psi_{k} \\
& =\int_{U^{\prime}}(a \circ \psi)\left(x^{\prime}\right) \operatorname{det}\left(\frac{\partial \psi_{j}}{\partial x_{i}^{\prime}}\right) d x_{1}^{\prime} \wedge \cdots \wedge d x_{k}^{\prime}
\end{aligned}
$$

and recognizing the matrix that we are taking the determinant of as the total derivative of $\psi$, we get

$$
\begin{equation*}
\int_{U^{\prime}} \psi^{*} \omega=\int_{U^{\prime}}(a \circ \psi)\left(x^{\prime}\right) \operatorname{det}\left(D_{x^{\prime}} \psi\right) d x_{1}^{\prime} \cdots d x_{k}^{\prime} . \tag{18.1}
\end{equation*}
$$

On the other hand, the change-of-variables formula from multivariable calculus [DK04b, Theorem 6.6.1] says:

Theorem 18.2.2. With notation as above,

$$
\begin{equation*}
\int_{U} a(x) d x_{1} \cdots d x_{k}=\int_{U^{\prime}}(a \circ \psi)\left(x^{\prime}\right)\left|\operatorname{det}\left(D_{x^{\prime}} \psi\right)\right| d x_{1}^{\prime} \cdots d x_{k}^{\prime} \tag{18.2}
\end{equation*}
$$

Remark 18.2.3. To see that the absolute values signs belong in this formula, observe that in the integral you use the values of $a$ and the volumes of blocks, without a sign.

That is, (18.1) and (18.2) could differ by a sign (or even worse if $U$ has many components) and to avoid this, we have to understand when the sign of $\operatorname{det}\left(D_{x^{\prime}} \psi\right)$ is positive. This determinant also appears as the multiple of $e_{1} \wedge \cdots \wedge e_{k}$ one obtains when applying

$$
\Lambda^{k}\left(D_{x^{\prime}} \psi\right): \Lambda^{k} T_{x^{\prime}} U^{\prime} \cong \mathbb{R} \cdot\left(e_{1} \wedge \cdots \wedge e_{k}\right) \longrightarrow \Lambda^{k} T_{x} U \cong \mathbb{R} \cdot\left(e_{1} \wedge \cdots \wedge e_{k}\right)
$$

to $e_{1} \wedge \cdots \wedge e_{k}$. We said that $\psi$ preserves orientation if this multiple is positive. That is, we conclude the following:

Corollary 18.2.4. If $\omega \in \Omega^{k}(U)$ is a compactly-supported top form and $\psi: \mathbb{R}^{k} \supset U^{\prime} \rightarrow$ $U \subset \mathbb{R}^{k}$ is an orientation-preserving diffeomorphism, then

$$
\int_{U^{\prime}} \psi^{*} \omega=\int_{U} \omega
$$

### 18.2.2 Integration on manifolds

We shall define the integral of a compactly-supported top form $\omega$ over an oriented manifold $M$ in several steps.

Theorem 18.2.5. There is a unique construction of an integral of top forms on oriented $k$-dimensional manifolds with the following properties:
(1) If the manifold has an orientation-preserving diffeomorphism to an open subset of $\mathbb{R}^{k}$, it is the integral defined above (note that this is independent of the choice of such diffeomorphism by Corollary 18.2.4).
(2) If $\omega$ is supported in $U \subset M$ then $\int_{M} \omega=\int_{U} \omega$.
(3) It is linear.

Proof. Desiderata (1) and (2) imply that if $\omega$ happens to be supported in the image of an orientation-preserving chart $\phi: \mathbb{R}^{k} \supset U \rightarrow V \subset M$ (using the standard orientation on $U_{\alpha}$ inherited from $\mathbb{R}^{k}$ ), we must define

$$
\int_{V} \omega:=\int_{U} \phi^{*} \omega
$$

If $M$ is oriented, we can find an open cover of $M$ by charts $\phi_{\alpha}: \mathbb{R}^{k} \supset U_{\alpha} \rightarrow V_{\alpha} \subset M$ so that all transition functions are orientation-preserving. Now pick a partition of unity $\eta_{\alpha}$ subordinate to the $V_{\alpha}$, and observe that $\omega=\sum_{\alpha} \eta_{\alpha} \omega_{\alpha}$ which is a finite sum because the support of $\omega$ is compact. Thus desideratum (3) forces us to define

$$
\int_{M} \omega:=\sum_{\alpha} \int_{V_{\alpha}} \eta_{\alpha} \omega_{\alpha}
$$

which makes sense because it is a finite sum.
We need to verify that this is independent of the choice of open cover and partition of unity. Take a second collection of charts $\phi_{\beta}^{\prime}: \mathbb{R}^{k} \supset U_{\beta}^{\prime} \rightarrow V_{\beta}^{\prime} \subset M$ and a subordinate partition of unity $\rho_{\beta}^{\prime}$. Using the fact that $\sum_{\beta} \rho_{\beta}^{\prime}=1$ and the sums are finite so may be interchanged, we get

$$
\begin{aligned}
\sum_{\alpha} \int_{V_{\alpha}} \eta_{\alpha} \omega & =\sum_{\alpha} \int_{V_{\alpha}} \eta_{\alpha}\left(\sum_{\beta} \rho_{\beta}^{\prime} \omega\right) \\
& =\sum_{\alpha} \sum_{\beta} \int_{V_{\alpha} \cap V_{\beta}^{\prime}} \eta_{\alpha} \rho_{\beta}^{\prime} \omega
\end{aligned}
$$

and by symmetry this is also $\sum_{\beta} \int_{V_{\beta}^{\prime}} \rho_{\beta}^{\prime} \omega$.
Example 18.2.6. If $-M$ denotes $M$ with opposite orientation, then $\int_{-M} \omega=-\int_{M} \omega$.
Example 18.2.7. If $\omega$ is a $p$-form for $p<k$, we can't integrate it over the $k$-dimensional manifold $M$. However, we can integrate it over an oriented submanifold $X \subset M$ of dimension $p$ :

$$
\int_{X} \omega:=\int_{X} i^{*} \omega
$$

with $i: X \hookrightarrow M$ the inclusion.
Remark 18.2.8. From the construction in Theorem 18.2.5, we see that if you have a preferred collection of $\left\{\left(U_{i}, V_{i}, \phi\right)\right\}$ of $M$ such that $\bigcup_{i} V_{i}=M$, you can use only these charts in your construction of the integral.

Using this definition, Corollary 18.2.4 generalizes to manifolds:
Corollary 18.2.9. If $f: M \rightarrow N$ is an orientation-preserving diffeomorphism and $\omega \in \Omega^{k}(N)$ is a compactly-supported top form, then

$$
\int_{M} f^{*} \omega=\int_{N} \omega .
$$

This definition of the integral is useful for proving theorems, but hard to use in practical computations. In practice one does the following. We start with two observations: the above construction goes through for Riemann-integrable forms, not just smooth ones, and for manifolds with boundary (or even corners).

Now suppose one has a finite collection of orientation-preserving embeddings $\varphi_{i}: \mathbb{R}^{k} \supset$ $N_{i} \rightarrow M$ of submanifolds with boundary (or even corners), which only intersect at their boundary. Then we can decompose a smooth $\omega$ as a finite sum of Riemann-integrable forms $\sum_{i} 1_{\varphi_{i}\left(N_{i}\right)} \omega$ with $1_{\varphi_{i}\left(N_{i}\right)}$ the indicator function of $\varphi_{i}\left(N_{i}\right)$, and evaluate the integral as

$$
\int_{M} \omega=\sum_{i} \int_{\varphi_{i}\left(N_{i}\right)} 1_{\varphi_{i}\left(N_{i}\right)} \omega=\sum_{i} \int_{N_{i}} \varphi_{i}^{*} \omega .
$$

Example 18.2.10. This tells you that to compute the integral of a 2 -form over $S^{2}$, you decompose $S^{2}$ into the two hemisphere, parametrize these by a disk, and you take the sum of the values of the integral of the pullback of the 2 -form to both disks. In other words, it's what you have been doing in multivariable calculus all along.

## Chapter 19

## The exterior derivative and Stokes' theorem

Stokes theorem is a generalization of the formula

$$
\int_{0}^{1} \frac{\partial f}{\partial x} d x=f(1)-f(0)
$$

To state it, we first need to generalize the derivative to a function to differential forms; the exterior derivative. The proof of Stokes' theorem will then follow from an easily proven version in charts. This material can be found in Sections $4 . \S 5$ and $4 . \S 7$ of [GP10].

### 19.1 The exterior derivative

As for the integral, we shall first define the exterior derivative on open subsets of $\mathbb{R}^{k}$ and then extend it to arbitrary smooth manifolds using charts.

Suppose we are given a $p$-form on an open subset $U \subset \mathbb{R}^{k}$,

$$
\omega=\sum_{I} a_{I} d x_{I},
$$

the sum ranging over all $1 \leq i_{1}<\ldots<i_{p} \leq k$ and $d x_{I}:=d x_{i_{1}} \wedge \cdots \wedge d x_{i_{p}}$. Then the exterior derivative is given by taking the $i$ th partial derivative of each of the coefficients and wedging with $d x_{i}$ :

$$
d \omega=\sum_{i} \sum_{I} \frac{\partial a_{I}}{\partial x_{i}} d x_{i} \wedge d x_{I}
$$

Some of the terms in this sum vanish, when $i$ is among the indexing set $I$. More generally, signs appears when shuffling $d x_{i}$ into its standard position.

Example 19.1.1. If $f: \mathbb{R}^{3} \supset U \rightarrow \mathbb{R}$ is a smooth function, i.e. a 0 -form, then its exterior derivative is

$$
d f=\frac{\partial f}{\partial x_{1}} d x_{1}+\frac{\partial f}{\partial x_{2}} d x_{2}+\frac{\partial f}{\partial x_{3}} d x_{3} .
$$

This coincides with the definition we used before. This is related to the gradient of the function $f$.

Example 19.1.2. If we have a 1 -form on $U \subset \mathbb{R}^{3}$,

$$
\alpha=a_{1} d x_{1}+a_{2} d x_{2}+a_{3} d x_{3}
$$

then its exterior derivative is

$$
\begin{aligned}
d \alpha= & \left(\frac{\partial a_{1}}{\partial x_{1}} d x_{1}+\frac{\partial a_{1}}{\partial x_{2}} d x_{2}+\frac{\partial a_{1}}{\partial x_{3}} d x_{3}\right) \wedge d x_{1} \\
& +\left(\frac{\partial a_{2}}{\partial x_{1}} d x_{1}+\frac{\partial a_{2}}{\partial x_{2}} d x_{2}+\frac{\partial a_{2}}{\partial x_{3}} d x_{3}\right) \wedge d x_{2} \\
& +\left(\frac{\partial a_{3}}{\partial x_{1}} d x_{1}+\frac{\partial a_{3}}{\partial x_{2}} d x_{2}+\frac{\partial a_{3}}{\partial x_{3}} d x_{3}\right) \wedge d x_{3} \\
= & \left(\frac{\partial a_{2}}{\partial x_{1}}-\frac{\partial a_{1}}{\partial x_{2}}\right) d x_{1} \wedge d x_{2}+\left(\frac{\partial a_{3}}{\partial x_{1}}-\frac{\partial a_{1}}{\partial x_{3}}\right) d x_{1} \wedge d x_{3}+\left(\frac{\partial a_{3}}{\partial x_{2}}-\frac{\partial a_{2}}{\partial x_{3}}\right) d x_{2} \wedge d x_{3}
\end{aligned}
$$

This is related to the curl of the vector field with components $\left(a_{1}, a_{2}, a_{3}\right)$.
Example 19.1.3. If we have a 2 -form on $U \subset \mathbb{R}^{3}$,

$$
\omega=a_{1} d x_{2} \wedge d x_{3}-a_{2} d x_{1} \wedge d x_{3}+a_{3} d x_{1} \wedge d x_{2}
$$

then its exterior derivative is

$$
d \omega=\left(\frac{\partial a_{1}}{\partial x_{1}}+\frac{\partial a_{2}}{\partial x_{2}}+\frac{\partial a_{3}}{\partial x_{3}}\right) d x_{1} \wedge d x_{2} \wedge d x_{3}
$$

This is related to the divergence of the vector field with components $\left(a_{1}, a_{2}, a_{3}\right)$.
The exterior derivative has the following properties, and the following also serves as a definition:

Theorem 19.1.4. The exterior derivative is the unique operation $\Omega^{*}(U) \rightarrow \Omega^{*+1}(U)$ with the following properties:
(i) For smooth functions $f \in \Omega^{0}(U), d f=\sum_{i=1}^{k} \frac{\partial f}{\partial x_{i}} d x_{i}$.
(ii) It is linear, $d(\omega+\nu)=d \omega+d \nu$.
(iii) It is a derivation for the wedge product, for a p-form $\omega$ and a $q$-form $\nu, d(\omega \wedge \nu)=$ $d(\omega) \wedge \nu+(-1)^{p} \omega \wedge d(\nu)$.
(iv) It is a differential, $d(d \omega)=0$.

Proof. We first verify that the exterior derivative satisfies the above properties. Property (i) is true by definition, and property (ii) follows from the fact that partial derivatives are linear. Properties (iii) and (iv) are slightly harder; the former is essentially the product rule and the latter the fact that partial derivatives commute.

By linearity of $d$ and the fact that $\wedge$ distributes over finite sums, it suffices to prove (iii) in the case that $\omega=a_{I} d x_{I}$ and $\nu=b_{J} d x_{J}$. Then $\omega \wedge \nu=a_{I} b_{J} d x_{I} \wedge d x_{J}$ and we
have that

$$
\begin{aligned}
d(\omega \wedge \nu) & =\sum_{i=1}^{k} \frac{\partial a_{I} b_{J}}{\partial d x_{i}} d x_{i} \wedge d x_{I} \wedge d x_{J} \\
& =\sum_{i=1}^{k}\left(\frac{\partial a_{I}}{\partial d x_{i}} b_{J} d x_{i} \wedge d x_{I} \wedge d x_{J}+a_{I} \frac{\partial b_{J}}{\partial d x_{i}} d x_{i} \wedge d x_{I} \wedge d x_{J}\right) \\
& =\left(\sum_{i=1}^{k} \frac{\partial a_{I}}{\partial d x_{i}} d x_{i} \wedge d x_{I}\right) \wedge\left(b_{J} d x_{J}\right)+(-1)^{p q} a_{I} d x_{I} \wedge\left(\sum_{i=1}^{k} \frac{\partial b_{J}}{\partial d x_{i}} d x_{i} \wedge d x_{J}\right) \\
& =d(\omega) \wedge \nu+(-1)^{p} \omega \wedge d(\nu) .
\end{aligned}
$$

Similarly, it suffices to prove (iv) in the case that $\omega=a_{I} d x_{I}$. Since $d\left(d x_{i}\right)=0$ (we are only taking partial derivatives of constant functions), we can use (iii) twice to write

$$
d\left(d\left(a_{I} d x_{I}\right)\right)=d\left(d a_{I}\right) \wedge d x_{I}
$$

and hence it suffices to show that $d\left(d\left(a_{I}\right)\right)=0$. But we have

$$
d\left(d\left(a_{I}\right)\right)=\sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial^{2} a_{I}}{\partial x_{i} \partial x_{j}} d x_{i} \wedge d x_{j}=\sum_{1 \leq i<j \leq k} \frac{\partial^{2} a_{I}}{\partial x_{i} \partial x_{j}}\left(d x_{i} \wedge d x_{j}+d x_{j} \wedge d x_{i}\right)=0
$$

Here we have first used that partial derivatives of smooth functions commmute, and that $d x_{i} \wedge d x_{i}=0$ and $d x_{i} \wedge d x_{j}+d x_{j} \wedge d x_{i}=d x_{i} \wedge d x_{j}-d x_{i} \wedge d x_{j}=0$.

The next goal is to prove uniqueness. Suppose that $D: \Omega^{*}(U) \rightarrow \Omega^{*+1}(U)$ satisfies the same property, then we must show that $d=D$. But if we try to prove $d$ and $D$ coincide on a general $p$-form

$$
\omega=\sum_{I} a_{I} d x_{I}
$$

then by (ii) it suffices to prove they coincide on $a_{I} d x_{I}$. By (iii) it then suffices to prove they coincide on $a_{I}$ and each $d x_{i}$. By (i), they indeed coincide on $a_{I}$. For $d x_{i}$, observe that $d x_{i}=d\left(x_{i}\right)$ which equals $D\left(x_{i}\right)$ by (i), so that by (iv) $d\left(d x_{i}\right)=0$ and $D\left(d x_{i}\right)=D\left(D\left(x_{i}\right)\right)=0$.

The exterior derivative commutes with pullback:
Proposition 19.1.5. If $g: U^{\prime} \rightarrow U$ is a smooth map between open subsets of Euclidean spaces, then $g^{*} d=d g^{*}$.

When $g$ is a diffeomorphism, there is an elegant proof by observing that $\left(g^{-1}\right)^{*} d g^{*}$ has the same properties as $d$, so by uniqueness of the exterior derivative has to be equal to it.

Proof. Recall that $g^{*}$ has the following properties: (i') $g^{*} d f=d(f \circ g)$, (ii') it is linear, (iii') it commutes with wedge product. These formal properties imply the proposition as follows: to prove that $g^{*} d$ and $d g^{*}$ coincide on a general $p$-form

$$
\omega=\sum_{I} a_{I} d x_{I},
$$

by (ii) and (ii') it suffices to prove they coincide on each $a_{I} d x_{I}$. Then by (iii) and (iii') it suffices to prove they coincide on $a_{I}$ and each $d x_{I}$. Property (i') says they coincide on $a_{I}$. For $d x_{i}$, we observe that $g^{*} d\left(d x_{i}\right)=g^{*} 0=0$ and to prove that the other side also vanishes we write $g^{*} d x_{i}=g^{*} d\left(x_{i}\right)=d g$ so $d g^{*} d x_{i}=d^{2} g=0$.

Since $d$ in particular commutes with pullback along a diffeomorphism, we can extend to smooth manifolds of dimension $k$ using charts. For $\omega \in \Omega^{p}(M), d \omega$ is defined near a point in $M$ by picking a chart $\phi: \mathbb{R}^{k} \supset U \rightarrow V \subset M$ and taking $\left(\phi^{-1}\right)^{*} d \phi^{*} \omega$. The previously established properties all generalize to manifolds, as they can be verified in a chart. This theorem serves as the definition of the exterior derivative.

Theorem 19.1.6. There is an operation $d: \Omega^{*}(-) \rightarrow \Omega^{*+1}(-)$ on differential forms on manifolds uniquely determined by the following properties:
(i) On smooth functions, i.e. 0-forms, it is the ordinary derivative.
(ii) It is linear.
(iii) It is a derivation for the wedge product.
(iv) It is a differential, $d^{2}=0$.
(v) It commutes with pullbacks along smooth maps.

We also add one useful observation from the point of view of integration: if $\omega$ is compactly-supported so is $d \omega$. Letting $\Omega_{c}^{*}(-)$ denote the compactly-supported forms, we can restrict $d$ to an operation $\Omega_{c}^{*}(-) \rightarrow \Omega_{c}^{*+1}(-)$. Note that the pullback of a compactlysupported form is not in general compactly-supported; this requires the map to the proper as $\operatorname{supp}\left(g^{*} \omega\right) \subset g^{-1}(\operatorname{supp}(\omega))$ and properness is exactly the condition that the inverse image of a compact subset is compact.

### 19.2 Stokes' theorem

Recall that last lecture we defined the integral of a compactly-supported top form over an oriented manifold, using partitions of unity.

Theorem 19.2.1 (Stokes). Let $\omega \in \Omega_{c}^{k-1}(M)$ be a compactly-supported $(k-1)$-form on an oriented smooth manifold $M$ of dimension $k$ with boundary $\partial M$, then

$$
\int_{M} d \omega=\int_{\partial M} \omega
$$

In this theorem, we need $\partial M$ to be oriented as well and to get this equation to hold, we use the conventions of Section 17.3.5 for the induced orientations on the boundary ("outward-pointing first").

Example 19.2.2. Let $M=[0,1]$ with its standard orientation. Then $\partial[0,1]=\{0,1\}$, where 1 has positive orientation and 0 has negative orientation. In this case Stokes concerns 0 -forms, i.e. functions, and says

$$
\int_{0}^{1} \frac{\partial f}{\partial x} d x=\int_{[0,1]} d f=\int_{\partial[0,1]} f=f(1)-f(0)
$$

a formula that should be quite familiar.

In fact, our proof will use the above result as input, a basic result in single-variable calculus. We also use Fubini's theorem on successive integration, e.g. [DK04b, Theorem 6.4.5].

Proof. Pick an open cover of $M$ by the codomains $V_{\alpha}$ of a collection charts $\phi_{\alpha}:[0, \infty) \times$ $\mathbb{R}^{k-1} \supset U_{\alpha} \rightarrow V_{\alpha} \subset M$. Also pick a subordinate partition of unity $\eta_{\alpha}: M \rightarrow[0,1]$. Then $\omega=\sum_{\alpha} \eta_{\alpha} \omega$, and this sum is finite because $\operatorname{supp}(\omega)$ is compact. Since both $\int_{M}$ and $d$ are linear, we may thus assume that $\omega$ is supported in $V_{\alpha}$. Then so is $d \omega$ and we have

$$
\int_{M} d \omega=\int_{U_{\alpha}} \phi_{\alpha}^{*} d \omega \quad \text { and } \quad \int_{\partial M} \omega=\int_{\partial U_{\alpha}} \phi_{\alpha}^{*} \omega .
$$

Since $\phi_{\alpha}^{*}$ commutes with $d$, we might as well replace $\phi_{\alpha}^{*} \omega$ by $\omega$ to simplify notation and extend this by 0 to a compactly-supported $(k-1)$-form on $[0, \infty) \times \mathbb{R}^{k-1}$. We have thus reduced our task to proving Stokes theorem in the special case $M=[0, \infty) \times \mathbb{R}^{k-1}$.

Since both $\int_{[0, \infty) \times \mathbb{R}^{k-1}} d \omega$ and $\int_{\{0\} \times \mathbb{R}^{k-1}} \omega$ are linear in $\omega$, it suffices to prove this for $\omega=a d x_{I}$ with $I=1<\ldots<\hat{i}<\ldots<k$. Then $d \omega=(-1)^{i-1} \frac{\partial a}{\partial x_{i}} d x_{1} \wedge \cdots \wedge d x_{k}$. There are two cases: (i) $i=1$, (ii) $i>1$.

Let's start with the latter. Then $\omega$ restricts to 0 on $\partial M$ (as it contains a $d x_{1}$ ) so we should get 0 . Pick $N$ sufficiently large so that $\operatorname{supp}\left(a_{I}\right) \subset[0, N] \times[-N, N]^{k-1}$, then

$$
\begin{aligned}
\int_{[0, \infty) \times \mathbb{R}^{k-1}} d \omega & =\int_{[0, N] \times[-N, N]^{k-1}}(-1)^{i-1} \frac{\partial a}{\partial x_{i}} d x_{1} \wedge \cdots \wedge d x_{k} \\
& =\int_{[0, N] \times[-N, N]^{k-2}}\left(\int_{[-N, N]} \frac{\partial a}{\partial x_{i}} d x_{i}\right) d x_{I} \\
& =\int_{[0, N] \times[-N, N]^{k-2}}\left(a\left(x_{1}, \ldots, N, \ldots, x_{k}\right)-a\left(x_{1}, \ldots,-N, \ldots, x_{k}\right)\right) d x_{I} \\
& =0=\int_{\{0\} \times \mathbb{R}^{k-1}} \omega .
\end{aligned}
$$

Here we have used Fubini's theorem, the fundamental theorem of analysis and that $a$ is supported in $[0, N] \times[-N, N]^{k-1}$ so that both $a\left(x_{1}, \ldots, N, \ldots, x_{k}\right)$ and $a\left(x_{1}, \ldots,-N, \ldots, x_{k}\right)$ are 0 .

The former case is similar, but has a different outcome. Pick $N$ as before, then

$$
\begin{aligned}
\int_{[0, \infty) \times \mathbb{R}^{k-1}} d \omega & =\int_{[0, N] \times[-N, N]^{k-1}} \frac{\partial a}{\partial x_{1}} d x_{1} \wedge \cdots \wedge d x_{k} \\
& =\int_{[-N, N]^{k-1}}\left(\int_{[0, N]} \frac{\partial a}{\partial x_{1}} d x_{1}\right) d x_{I} \\
& =\int_{[-N, N]^{k-1}}\left(a\left(N, x_{2}, \ldots, x_{k}\right)-a\left(0, x_{2} \ldots, x_{k}\right)\right) d x_{I} \\
& =-\int_{[-N, N]^{k-1}} a\left(0, x_{2}, \ldots, x_{k}\right) d x_{2} \wedge \cdots \wedge d x_{k} \\
& =\int_{\{0\} \times \mathbb{R}^{k-1}} \omega .
\end{aligned}
$$

Here we have used the same tools as before, as well as $a\left(N, x_{2}, \ldots, x_{k}\right)=0$. Our convention on the orientation of the boundary was chosen exactly so that the signs cancel in the last step: in the "outward-pointing first convention", a basis $\left(v_{2}, \ldots, v_{k}\right)$ of $T_{x}\left(\{0\} \times \mathbb{R}^{k-1}\right)$ is positively oriented if $\left(-e_{1}, v_{2}, \ldots, v_{k}\right)$ is, that is $\left(-e_{1}\right) \wedge v_{2} \wedge \cdots \wedge v_{k}$ equals $e_{1} \wedge \cdots e_{k}$ up to scaling by a positive real number. Hence the induced orientation on $\{0\} \times \mathbb{R}^{k-1}$ as a boundary of the upper half-plane is opposite to the usual orientation.

We now give a number of applications.

### 19.2.1 Integrating pullbacks

Suppose that $W$ is a oriented smooth manifold with boundary $\partial W$ and $f: W \rightarrow M$ is a smooth map. Then if $\omega$ is a closed $p$-form, we get that

$$
\int_{W} d f^{*} \omega=\int_{W} f^{*}(d \omega)=0
$$

but applying Stokes' formula we also get

$$
\int_{W} d f^{*} \omega=\int_{\partial W} f^{*} \omega
$$

In particular, if $\partial W$ comes divided into a disjoint union $\partial_{\text {in }} W \sqcup \partial_{\text {out }} W$ we may artificially reverse the orientation on $\partial_{\mathrm{in}} W$ (so it's "inward pointing first") and get the formula

$$
\int_{\partial_{\text {out }} W} f^{*} \omega-\int_{\partial_{\text {in }} W} f^{*} \omega=0
$$

We will use the following consequence in the next lecture:
Corollary 19.2.3. If $f_{0}$ and $f_{1}$ are homotopic smooth maps $X \rightarrow M$, then

$$
\int_{X} f_{1}^{*} \omega=\int_{X} f_{0}^{*} \omega
$$

for all closed p-forms $\omega$.

Proof. Suppose $W=X \times[0,1], \partial_{\text {in }} W=X \times\{0\}, \partial_{\text {out }} W=X \times\{1\}$. Then we can think of $f: X \times[0,1] \rightarrow M$ as a homotopy from $f_{0}:=\left.f\right|_{X \times\{0\}}$ to $f_{1}:=\left.f\right|_{X \times\{1\}}$. The orientation on $X \times\{0\}$ and $X \times\{1\}$ are now equal (instead of opposite, if we had taken the usual convention) and we get the equation

$$
\int_{X} f_{1}^{*} \omega=\int_{\partial_{\mathrm{out}} W} f^{*} \omega=\int_{\partial_{\mathrm{in}} W} f^{*} \omega=\int_{X} f_{0}^{*} \omega
$$

Thus the integral of the pullback along $f$ of a closed form only depend on the homotopy class of $f$.

### 19.3 Classical integral theorems

We now explain how the multivariable calculus theorems you have learned are special cases of Stokes' theorem. This is significantly harder than one might expect, because the classical version is harder to state precisely. In particular, we have to make precise the notions of "line element," "surface element," and "volume element."

That is, we need to explain how to integrate continuous functions $f: X \rightarrow \mathbb{R}$ over a smooth submanifold $X$ of Euclidean space. We will do following [DK04b, Chapter 7]. As for integrals of differential forms, we can not just integrate in charts due to the Jacobian term in the change-of-variables formula. To correct for this, we need a density:

Definition 19.3.1. A density for a manifold $M$ is an assignment to each chart ( $U_{\alpha}, V_{\alpha}, \phi_{\alpha}$ ) of $M$ a continuous function $\rho_{\alpha}: U_{\alpha} \rightarrow \mathbb{R}$ such that

$$
\rho_{\alpha}(x)=\rho_{\beta}\left(\phi_{\alpha \beta}(x)\right)\left|\operatorname{det} D_{x} \psi_{\alpha \beta}\right| .
$$

We then define an integral of a continuous function $f: M \rightarrow \mathbb{R}$ analogously to Theorem 18.2.5. We pick a partition of unity $\eta_{\alpha}: M \rightarrow \mathbb{R}$ with respect of the codomains $V_{\alpha}$ of charts, and set

$$
\int_{M} f d \rho:=\sum_{\alpha} \int_{U_{\alpha}} \eta_{\alpha}\left(\phi_{\alpha}(x)\right) f\left(\phi_{\alpha}(x)\right) \rho_{\alpha}(x) d x_{1} \cdots d x_{k}
$$

Definition 19.3.1 gives, by the same argument as in proof of that theorem, that this is well-defined (i.e. independent of $\eta_{\alpha}$ ).

If $X \subset \mathbb{R}^{k}$ is a $r$-dimensional smooth submanifold, then there is a canonical choice of density, the Euclidean density: in this case we can make sense of the total derivative of $D_{x} \phi_{\alpha}$ as a ( $k \times r$ )-matrix, and set

$$
\rho_{\alpha}^{\mathrm{eucl}}(x):=\sqrt{\operatorname{det}\left(\left(D_{x} \phi_{\alpha}\right)^{t}\left(D_{x} \phi_{\alpha}\right)\right)} .
$$

See [DK04b, Theorem 7.3.1] for a proof that this is a density.
The integrals of functions using "line elements," "surface elements," or "volume elements" are exactly those with respect to the Euclidean density. We will now identify integrals of differential forms as integrals of certain functions with respect to the Euclidean density.
Lemma 19.3.2. Suppose that $M$ is a compact codimension 0 submanifold of $\mathbb{R}^{k}$ with boundary $\partial M$, and $\omega \in \Omega^{k}(M)$. Define a smooth function $f: M \rightarrow \mathbb{R}$ by $\nu(x)=$ $f(x) d x_{1} \wedge \cdots \wedge d x_{k}$. We have

$$
\int_{M} \omega=\int_{M} f d \rho^{\mathrm{eucl}} .
$$

Proof. For charts given by open subsets $U$ of $M$ with inclusion $U \hookrightarrow M$, we have $\sqrt{\operatorname{det}\left(\left(D_{x} \phi_{\alpha}\right)^{t}\left(D_{x} \phi_{\alpha}\right)\right)}=1$. When we use only these charts in the definition of $\int_{M} f \omega$ in Theorem 18.2.5 we get

$$
\int_{M} f d \rho^{\mathrm{eucl}}:=\sum_{\alpha} \int_{U_{\alpha}} \eta_{\alpha}(x) f(x) d x_{1} \cdots d x_{k}
$$

The same formula falls out of Section 18.2, using the above charts as in Remark 18.2.8.

Lemma 19.3.3. Suppose that $M$ is a compact codimension 0 submanifold of $\mathbb{R}^{k}$ with boundary $\partial M$, and $\omega \in \Omega^{k-1}(M)$. Define a smooth vector field

$$
\vec{V}(x)=\left[\begin{array}{c}
a_{1}(x) \\
\vdots \\
a_{k}(x)
\end{array}\right]
$$

$\omega(x)=\sum_{i=1}^{k}(-1)^{i+1} a_{i} d x_{1} \wedge \cdots \wedge \widehat{d x_{i}} \wedge \cdots \wedge d x_{k}$. We have

$$
\int_{\partial M} \omega=\int_{\partial M} \vec{V} \cdot \vec{n} d \rho^{\text {eucl }}
$$

where $\vec{n}$ is the outward pointing unit normal vector field to $\partial M$.
Example 19.3.4 (Divergence theorem). Let $\omega=a(x) d x_{2} \wedge d x_{3}-b d x_{1} \wedge d x_{3}+c d x_{1} \wedge d x_{2}$ be a 2 -form on $\mathbb{R}^{3}$ and $M \subset \mathbb{R}^{3}$ a codimension 0 submanifold with boundary $\partial M$ with induced orientation from the standard orientation on $\mathbb{R}^{3}$. Then Stokes' theorem says that

$$
\int_{M} d \omega=\int_{\partial M} \omega .
$$

Using the above two lemma's, we get

$$
\int_{M} \operatorname{div}(\vec{V}) d \rho^{\mathrm{eucl}}=\int_{\partial M} \vec{V} \cdot \vec{n} d \rho^{\mathrm{eucl}}
$$

the classical statement of Gauss' divergence theorem [DK04b, Theorem 7.8.5].

## Chapter 20

## De Rham cohomology

Today we introduce de Rham cohomology, a construction which we will study for the next couple of lectures and is one of the basic constructions of algebraic topology. It appears in Section $4 . \S 6$ of [GP10], but I also recommend you take a look at the beginning of [BT82].

### 20.1 De Rham cohomology

### 20.1.1 Motivation from integration

Recall that Stokes' theorem says that for oriented $k$-dimensional differentiable manifolds $M$ and compactly-supported $(k-1)$-forms $\omega$ on $M$, we have

$$
\int_{M} d \omega=\int_{\partial M} \omega
$$

Thus if $M$ has no boundary, $\int_{M} d \omega=0$ for any $\omega$. This means that when computing $\int_{M} \omega$, its values only depend on $\omega$ up to addition by $d \nu$. That is, the possible values that can be obtained when integrating a $k$-form over a $k$-dimensional compact oriented manifold $M$ depend only on $\Omega^{k}(M) / d \Omega^{k-1}(M)$.

One could ask a similar question about integrals over $p$-dimensional oriented manifolds mapping to $M$ : if $X$ is such a manifold and $f: X \rightarrow M$ is a smooth map, we are interested in the integral $\int_{X} f^{*} \omega$. The argument above tells you that these integrals only depend on $\Omega^{p}(M) / d \Omega^{p-1}(M)$ : we take $p$-forms modulo those that are exterior derivatives of ( $p-1$ )-forms. A $p$-form of the latter type is said to be exact.

It seems reasonable to restrict to those $p$-forms $\omega$ with the property if $\int_{X} f^{*} \omega$ only depends on the homotopy class of $X$. As discussed at the end of the previous lecture, from Stokes' theorem applied to $\int_{X \times[0,1]} H^{*} d \omega$ with $H: X \times[0,1] \rightarrow M$ a homotopy from $f_{0}$ to $f_{1}$, it follows that $\int_{X} f_{0}^{*} \omega=\int_{X} f_{1}^{*} \omega$ if $d \omega=0$, as then $0=\int_{W} H^{*} d \omega=\int_{X} f_{1}^{*} \omega-\int_{X} f_{0}^{*} \omega$. If $d \omega=0$ then $\omega$ is said to be closed. Observe that when $\omega$ is a $k$-form then $d \omega=0$ for degree reasons, so any top form is closed.

### 20.1.2 De Rham cohomology

The previous discussion tells us that the following groups can be interpreted as encoding "all possible values of homotopy-invariant integrals over manifolds mapping to M."

However, you should not take this to be the only motivation. As you will soon see, de Rham cohomology is a very powerful invariant of differentiable manifolds and smooth maps between them.

Definition 20.1.1. Let $M$ be a manifold. The de Rham cohomology groups $H^{*}(M)$ are given by

$$
H^{p}(M):=\frac{\operatorname{ker}\left(d: \Omega^{p}(M) \rightarrow \Omega^{p+1}(M)\right)}{\operatorname{im}\left(d: \Omega^{p-1}(M) \rightarrow \Omega^{p}(M)\right)}=\frac{\text { closed } p \text {-forms }}{\text { exact } p \text {-forms }}
$$

The elements of $H^{*}(M)$ are called cohomology classes and, as indicated, are represented by closed forms up to exact forms; any two forms which differ by an exact form are said to be cohomologous.

### 20.1.3 First properties of de Rham cohomology

Let us study de Rham cohomology. Since $\Omega^{p}(M)$ are $\mathbb{R}$-vector spaces, so are the cohomology groups $H^{p}(M)$. We will soon see these $\mathbb{R}$-vector spaces contain a lot of interesting topological information about $M$. Before going into the properties that allow us to extract this information, we do a few basic examples to get some initial intuition for de Rham cohomology:

Example 20.1.2 (Vanishing above dimension). If $M$ has dimension $k$, there are no $p$-forms for $p>k$ and hence $H^{p}(M)$ vanishes for $p>k$.
Example 20.1.3 $\left(H^{0}\right)$. For $p=0$, our definition gives that $H^{0}(M)=\{f: M \rightarrow \mathbb{R} \mid d f=0\}$. The condition $d f=0$ means that $f$ is locally constant. Thus these functions have to be constant on each component of $M$, and letting $\pi_{0}(M)$ denote the set of components of $M$ we get that

$$
H^{0}(M)=\mathbb{R}^{\pi_{0}(M)}
$$

the vector space of $\mathbb{R}$-valued functions on the set $\pi_{0}(M)$.
Example 20.1.4 (Disjoint unions). Suppose that $M$ is a disjoint union of $M_{i}$. Then a $p$-form $\omega$ on $M$ is a just a collection of $p$-forms $\omega_{i}$ on each of the $M_{i}$. Then $\omega$ is closed if and only if each $\omega_{i}$ is, and exact if and only if each $\omega_{i}$ is. We conclude that

$$
H^{*}(M) \cong \prod_{i} H^{*}\left(M_{i}\right)
$$

However, in practice $M$ has finitely many components and the direct product is finite. In this case the direct product may be replaced by the more familiar direct sum.

Recall that we have defined a wedge product on differential forms, and this has the property that if $\omega \in \Omega^{p}(M)$ and $\nu \in \Omega^{q}(M)$ then $d(\omega \wedge \nu)=d(\omega) \wedge \nu+(-1)^{p} \omega \wedge d(\nu)$.

Lemma 20.1.5. The wedge product induces a graded-commutative product on $H^{*}(M)$. That is, $H^{*}(M)$ is a graded-commutative $\mathbb{R}$-algebra.

Proof. Let $\omega \in \Omega^{p}(M)$ and $\nu \in \Omega^{q}(M)$ represent cohomology classes. Then in particular $d \omega=0$ and $d \nu=0$, and we see that

$$
d(\omega \wedge \nu)=d(\omega) \wedge \nu+(-1)^{p} \omega \wedge d(\nu)=0+0=0 .
$$

Thus $\omega \wedge \nu$ represents a cohomology class. This is independent of the choice of representatives, because if $\omega-\omega^{\prime}=d \alpha$, then

$$
\omega \wedge \nu-\omega^{\prime} \wedge \nu=d(\alpha) \wedge \nu=d(\alpha \wedge \nu)
$$

and similarly in the second entry.
The properties of this induced product-unitality, associativity, and graded-commutativityfollow from those of the wedge product.

Example 20.1.6. The unit of the wedge product is the element of $H^{0}(M)$ represent by the constant function $M \rightarrow \mathbb{R}$ with value 1 .

### 20.1.4 Cohomology as a functor

Recall that we can pull back differential forms along any smooth map: given $g: M \rightarrow$ $N$ we get $g^{*}: \Omega^{*}(N) \rightarrow \Omega^{*}(M)$.

Lemma 20.1.7. The homomorphism $g^{*}$ induces a homomorphism of graded-commutative $\mathbb{R}$-algebras $g^{*}: H^{*}(N) \rightarrow H^{*}(M)$ which satisfies $(f \circ g)^{*}=g^{*} \circ f^{*}$ and $(\mathrm{id})^{*}=\mathrm{id}$.

Proof. We use that $d$ commutes with $g^{*}$, so $g^{*}$ must preserve the kernel and image of $d$. Let's check this is in detail for kernels: if $\omega \in \Omega^{p}(N)$ satisfies $d \omega=0$, then $g^{*} \omega \in \Omega^{p}(M)$ satisfies

$$
d\left(g^{*} \omega\right)=g^{*}(d \omega)=g^{*} 0=0 .
$$

The properties of pullback on cohomology follow from the corresponding properties of pullback on forms.

It is appropriate at this point to mention the foundational framework used in algebraic topology: category theory [Rie16]. A category C consists of a collection of object ob(C) and a collection of morphisms mor(C). Each of these morphisms has a source and a target, and two morphisms $f$ and $g$ can be composed to $g \circ f$ if the target of the $f$ is the source of $g$. This composition operations is associative, and every object has an identity morphism which serves as a two-sided unit for composition.

The standard way to picture a category is a collection of dots (objects) and arrows between them (morphisms). One instance of such graphic representations are the commutative diagrams we have been using (in the 40s people wrote down the formulas, and it was difficult to parse statements).
Example 20.1.8. The category of Top of topological spaces has objects given by topological spaces, and morphisms given by continuous maps.

Example 20.1.9. The category Mfd of differentiable manifolds has objects given by differentiable manifolds, and morphisms given by smooth maps.
Example 20.1.10. The category $\mathrm{GrAlg}_{\mathbb{R}}$ of graded-commutative $\mathbb{R}$-algebras has objects given by graded $\mathbb{R}$-vector spaces with a graded-commutative product, and morphisms given by grading-preserving homomorphisms.

An important application of categories is to express naturality of various constructions. For example, that they are compatible with composition is expressed through the notion of a functor. A functor $F: C \rightarrow D$ is a pair of assignments $\mathrm{ob}(F): \mathrm{ob}(\mathrm{C}) \rightarrow \mathrm{ob}(\mathrm{D})$ and $\operatorname{mor}(F): \operatorname{mor}(\mathrm{C}) \rightarrow \operatorname{mor}(\mathrm{D})$, compatible with source, target, identity and composition. The former two mean that if $f$ is a morphism from $C$ to $C^{\prime}$, then $F(f)$ is a morphism from $F(C) \rightarrow F\left(C^{\prime}\right)$, and the latter two mean that $F(\mathrm{id})=\mathrm{id}$ and $F(f \circ g)=F(f) \circ F(g)$.
Example 20.1.11. There is a forgetful functor $U: \mathrm{Mfd} \rightarrow$ Top sending each differentiable manifold to its underlying topological spaces, and regarding each smooth map as a continuous map.

It is not the case that cohomology is a functor $H^{*}: \operatorname{Mfd} \rightarrow \mathrm{GrAlg}_{\mathbb{R}}$; it would need to satisfy $(f \circ g)_{*}=f_{*} \circ g_{*}$ but instead we have $(f \circ g)_{*}=g_{*} \circ f_{*}$. This is no problem, as we can formally change the direction of morphisms in Mfd by taking the opposite category: $\mathrm{Mfd}^{\mathrm{op}}$ has the same objects and morphisms, but source and target are reversed. Then Lemma 20.1.7 says that de Rham cohomology is a functor

$$
H^{*}: \mathrm{Mfd}^{\mathrm{op}} \longrightarrow \operatorname{GrAlg}_{\mathbb{R}} .
$$

As an application of this, we make the following observation, which we will strengthen in the next lecture:

Lemma 20.1.12. If $g: M \rightarrow N$ is a diffeomorphism then $g^{*}: H^{*}(N) \rightarrow H^{*}(M)$ is an isomorphism.

Proof. The inverse $g^{-1}: N \rightarrow M$ induces a homomorphism $\left(g^{-1}\right)^{*}: H^{*}(M) \rightarrow H^{*}(N)$. The fact that cohomology is a functor tells us that this satisfies $g^{*} \circ\left(g^{-1}\right)^{*}=\left(g^{-1} \circ g\right)^{*}=$ $(\mathrm{id})^{*}=\mathrm{id}$ and similarly for the other composition.

### 20.2 First examples

Let us start with a first few computations in de Rham cohomology, before we develop the techniques that allow us to systematically compute the cohomology of many differentiable manifolds.

### 20.2.1 The real line

We already know that $H^{0}(\mathbb{R}) \cong \mathbb{R}$ by Example 20.1.3 and that $H^{*}(\mathbb{R})=0$ for $*>1$ by Example 20.1.2, so the only remaining unknown cohomology group is $H^{1}(\mathbb{R})$. Any element in it is represented by $\omega \in \Omega^{1}(\mathbb{R})$ (satisfying $d \omega=0$, but this is true for any such $\omega$ for degree reasons).

Lemma 20.2.1. $H^{1}(\mathbb{R})=0$.
Proof. We need to find an $f$ such that $\omega=d f$. Let us write $\omega=a(x) d x$ with $a: \mathbb{R} \rightarrow \mathbb{R}$ a smooth function, then

$$
\begin{aligned}
f: \mathbb{R} & \longrightarrow \mathbb{R} \\
x & \longmapsto \int_{0}^{x} a(y) d y
\end{aligned}
$$

satisfies $d f(x)=\frac{\partial f}{\partial x} d x=a(x) d x$. That is, every closed 1-form is exact.
In the next lecture we will prove the Poincaré lemma, which says that for all $n \geq 0$

$$
H^{*}\left(\mathbb{R}^{n}\right)= \begin{cases}\mathbb{R} & \text { if } *=0 \\ 0 & \text { otherwise }\end{cases}
$$

### 20.2.2 The circle

For the circle $S^{1}$, we are in a somewhat similar situation as for the real line: $H^{0}\left(S^{1}\right)=$ $\mathbb{R}$ and $H^{*}\left(S^{1}\right)=0$ for $*>1$, so only $H^{1}\left(S^{1}\right)$ remains unknown.

Lemma 20.2.2. $H^{1}\left(S^{1}\right)=\mathbb{R}$.
First proof. Let us write $\omega=a(\theta) d \theta$ with $a: S^{1} \rightarrow \mathbb{R}$ a smooth function, then the argument for the real line compels us to look at the function

$$
\begin{aligned}
f:[0,2 \pi] & \longrightarrow \mathbb{R} \\
\theta & \longmapsto \int_{0}^{\theta} a\left(e^{i \phi}\right) d \phi
\end{aligned}
$$

This gives a smooth function on $S^{1}$ if and only if $f(0)=f(2 \pi)$.
This gives an obstruction to implementing to proving that $H^{1}\left(S^{1}\right)$ vanishes along the lines of the proof for $\mathbb{R}$. But instead of giving up, we should take advantage of this and use the obstruction to define an invariant. That is, we can attempt to construct a linear functional on $H^{1}(\mathbb{R})$ by taking

$$
\begin{aligned}
& w: H^{1}\left(S^{1}\right) \longrightarrow \mathbb{R} \\
& \omega=a(\theta) d \theta \longmapsto \int_{0}^{2 \pi} a\left(e^{i \phi}\right) d \phi
\end{aligned}
$$

To check this is well-defined, we must verify it is independent of the representative $\omega$ of the cohomology class $[\omega] \in H^{1}(\mathbb{R})$. As $w$ is linear in $\omega$, so it suffices to show that $w(\omega)=0$ if $\omega=d f$ for a smooth function $f: S^{1} \rightarrow \mathbb{R}$. This is true because the integral is equal to $f(2 \pi)-f(0)=0$ by the fundamental theorem of calculus.

If $w(\omega)=0$ then $f(0)=f(2 \pi)$, and gives a smooth function $S^{1} \rightarrow \mathbb{R}$ which we can use to show that $\omega=d f$ like we did for $\mathbb{R}$. Hence the result follows once we show that $w$ is surjective. Since $w$ is linear it suffices to prove that it takes a single non-zero value, and when we evaluate on the 1 -form $\omega=d \theta$ we get $w(d \theta)=2 \pi$.

Let's give an alternative proof, which we will later generalize to the Mayer-Vietoris exact sequence for cohomology.

Proof. Let $U, V \subset S^{1}$ be an open cover by two open intervals and consider the following diagram


The left horizontal maps are induced by restrictions,

$$
\begin{aligned}
i_{0}: \Omega^{0}\left(S^{1}\right) & \longrightarrow \Omega^{0}(U) \oplus \Omega^{0}(V) \\
f & \longmapsto\left(\left.f\right|_{U},\left.f\right|_{V}\right),
\end{aligned}
$$

and similarly for $i_{1}$. The right horizontal maps are the difference of the restrictions,

$$
\begin{aligned}
j_{0}: \Omega^{0}(U) \oplus \Omega^{0}(V) & \longrightarrow \Omega^{0}(U \cap V) \\
(f, g) & \left.\longmapsto f\right|_{U \cap V}-\left.g\right|_{U \cap V}
\end{aligned}
$$

and similarly for $j_{1}$.
We start with a 1 -form $\omega \in \Omega^{1}\left(S^{1}\right)$ representing a cohomology class [ $\omega$ ], and consider $i(\omega)=\left(\left.\omega\right|_{U},\left.\omega\right|_{V}\right) \in \Omega^{1}(U) \oplus \Omega^{1}(V)$. Since $\omega$ was closed, so are both these restrictions. Since $H^{1}(U)$ and $H^{1}(V)$ vanish because both $U$ and $V$ are diffeomorphic to $\mathbb{R}$, both are exact. This gives us functions $\left(\lambda_{U}, \lambda_{V}\right) \in \Omega^{0}(U) \oplus \Omega^{0}(V)$.

Let us investigate to what extent

$$
j_{0}\left(\lambda_{U}, \lambda_{V}\right)=\left.\lambda_{U}\right|_{U \cap V}-\left.\lambda_{V}\right|_{U \cap V} \in \Omega^{0}(U \cap V)
$$

depends on the choices we made. We made two:
(a) the functions $\left(\lambda_{U}, \lambda_{V}\right)$,
(b) a representative $\omega$ of $[\omega]$.

For (a), the functions $\lambda_{U}$ and $\lambda_{V}$ are unique up to the addition of constant functions, i.e. closed 0 -forms. Adding a constant to one of these changes $j_{0}\left(\lambda_{U}, \lambda_{V}\right)$ by a constant.

For (b), a different representative is given by $\omega+d f$ with $f \in \Omega^{0}\left(S^{1}\right)$, and picking this instead leads to replacing $\lambda_{U}$ by $\Lambda_{U}+\left.f\right|_{U}$ and $\lambda_{V}$ by $\lambda_{V}+f_{\mid} V$, up to constants. When we take $j_{0}\left(\lambda_{U}+\left.f\right|_{U}, \lambda_{V}+\left.f\right|_{V}\right)$ the terms $\left.f\right|_{U \cap V}$ cancel out and we get $j_{0}\left(\lambda_{U}, \lambda_{V}\right)$

The conclusion is that the smooth function $j_{0}\left(\lambda_{U}, \lambda_{V}\right) \in \Omega^{0}(U \cap V)$ is independent of the choice of representative $\omega$, and depends on $\lambda_{U}$ and $\lambda_{V}$ only up to a constant. Since both $\left.\omega\right|_{U}$ and $\left.\omega\right|_{V}$ are equal to $\left.\omega\right|_{U \cap V}$ on $U \cap V$ and the exterior derivative is linear, we see that

$$
d\left(j\left(\lambda_{U}, \lambda_{V}\right)\right)=d\left(\left.\lambda_{U}\right|_{U \cap V}-\left.\lambda_{V}\right|_{U \cap V}\right)=\left.\omega\right|_{U \cap V}-\left.\omega\right|_{U \cap V}=0
$$

i.e. $j\left(\lambda_{U}, \lambda_{V}\right)$ is closed, or equivalently a locally constant function on $U \cap V$. Under the identification as Examples 20.1.4 and 20.1.3, it represents an element

$$
\left(a_{0}, a_{1}\right) \in H^{0}(U \cap V) \cong \mathbb{R}^{2}
$$

That is well-defined up the addition of a constant, means that we may replace ( $a_{0}, a_{1}$ ) by $\left(a_{0}+c, a_{1}+c\right)$. The elements of the form $(c, c)$ are exactly those in the image of $H^{0}(U) \oplus H^{0}(V)$ under $j_{0}$.

From this description, it follows that $a_{1}-a_{0} \in \mathbb{R}$ is independent of the choice of $\lambda_{U}$ and $\lambda_{V}$; an invariant of the original cohomology class $[\omega]$. Thus we have constructed a map

$$
\bar{w}: H^{1}\left(S^{1}\right) \longrightarrow \frac{H^{0}(U \cap V)}{\operatorname{im}\left(j: H^{0}(U) \oplus H^{0}(V) \rightarrow H^{0}(U \cap V)\right)} \cong \mathbb{R}
$$

Suppose now that $\bar{w}(\omega)=0$. Then $a_{0}=a_{1}$ and this means that though the functions $\lambda_{U}$ and $\lambda_{V}$ needs not be equal on $U \cap V$, the difference $\left.\lambda_{U}\right|_{U \cap V}-\left.\lambda_{V}\right|_{U \cap V}$ is constant. We can thus replace $\lambda_{U}$ by $\lambda_{U}-a_{0}$ to get that $\left.\lambda_{U}\right|_{U \cap V}=\left.\lambda_{V}\right|_{U \cap V}$. Hence we can glue them to obtain a function $\lambda$ on $S^{1}$, which by construction satisfies $d \lambda=\omega$.

This shows that $H^{1}\left(S^{1}\right)$ is isomorphic to the image of $\bar{w}$. To see that it is surjective, as before we can evaluate on $d \theta$.

Remark 20.2.3. The construction of $\bar{w}$ depends on a choice of isomorphism of the codomain with $\mathbb{R}$. You can pick this such that $w=\bar{w}$.

The previous proof amounts to the following: the maps $i$ and $j$ induce maps on cohomology and using partitions of unity one can produce a diagonal map to get the following diagram:

$$
\begin{aligned}
& {\left[H^{1}\left(S^{1}\right) \longrightarrow H^{1}(U) \oplus H^{1}(V) \longrightarrow H^{1}(U \cap V)\right.} \\
& H^{0}\left(S^{1}\right) \longrightarrow H^{0}(U) \oplus H^{0}(V) \longrightarrow H^{0}(U \cap V)
\end{aligned}
$$

This diagram has the special property that it is exact: the kernel of each map is the image of the previous one. Filling in what we already know, we get

$$
\begin{aligned}
& \longrightarrow H^{1}\left(S^{1}\right) \longrightarrow H^{1}(U) \oplus H^{1}(V)=0 \longrightarrow H^{1}(U \cap V)=0 \\
& H^{0}\left(S^{1}\right)=\mathbb{R} \longrightarrow H^{0}(U) \oplus H^{0}(V)=\mathbb{R}^{2} \xrightarrow{(*)} H^{0}(U \cap V)=\mathbb{R}^{2}-
\end{aligned}
$$

with starred map given by $(a, b) \longmapsto(a-b, a-b)$. This proves that $H^{1}\left(S^{1}\right)$, the kernel of the map to $H^{1}(U) \oplus H^{1}(V)=0$, is the image of the map $H^{0}(U \cap V)=\mathbb{R}^{2} \rightarrow H^{1}\left(S^{1}\right)$ whose kernel is exactly the 1-dimensional subspace spanned by $e_{1}+e_{2}$. Hence $H^{1}\left(S^{1}\right) \cong \mathbb{R}$.

What the above proof does, is construct explicitly the identification

$$
\begin{gathered}
H^{1}\left(S^{1}\right)=\operatorname{ker}\left(H^{1}\left(S^{1}\right) \rightarrow H^{1}(U) \oplus H^{1}(V)\right) \\
\bar{w} \mid \cong \\
\frac{H^{0}(U \cap V)}{\operatorname{im}\left(j: H^{0}(U) \oplus H^{0}(V) \rightarrow H^{0}(U \cap V)\right)} \cong \mathbb{R}
\end{gathered}
$$

### 20.3 Problems

Problem 20.3.1. Verify that $\bar{\omega}(d \theta) \neq 0$.
Problem 20.3.2 (Compactly-supported cohomology). Recall that $\Omega_{c}^{p}(M)$ denotes the compactly-supported $p$-forms. Since the exterior derivative $d$ preserves the condition that forms have compact support, there is also a compactly-supported variation of de Rham cohomology which is occasionally useful:

Definition 20.3.3. The compactly-supported de Rham cohomology groups $H_{c}^{*}(M)$ are given by

$$
H_{c}^{p}(M):=\frac{\operatorname{ker}\left(d: \Omega_{c}^{p}(M) \rightarrow \Omega_{c}^{p+1}(M)\right)}{\operatorname{im}\left(d: \Omega_{c}^{p-1}(M) \rightarrow \Omega_{c}^{p}(M)\right)}
$$

(a) Compute $H_{c}^{*}(\mathbb{R})$.
(b) Compute $H_{c}^{*}\left(S^{1}\right)$. (Hint: this should require no work.)

Problem 20.3.4 (Extension by zero). Prove that if $i: U \rightarrow M$ is the inclusion of an open subset, the extension of forms by zero induces a map

$$
H_{c}^{*}(U) \longrightarrow H_{c}^{*}(M)
$$

on compactly-supported cohomology.
Problem 20.3.5 (An infinitely-punctured plane). Prove that $H^{1}(\mathbb{C} \backslash \mathbb{Z})$ is not finitedimensional.

Problem 20.3.6 (Transfer maps). Let $M$ be a smooth manifold with a smooth free action of a finite group $G$, with $a_{g}: M \rightarrow M$ denoting the action of the elements $g \in G$. Recall that $M / G$ can be given the structure of a smooth manifold such that quotient $\operatorname{map} q: M \rightarrow M / G$ is a local diffeomorphism.
(a) Let $\Omega^{*}(M)^{G} \subset \Omega^{*}(M)$ be the subspace given by those differential forms that satisfy $\left(a_{g}\right)^{*} \omega=\omega$ for all $g \in G$; the invariant forms. Prove that $\Omega^{*}(M)^{G}$ is a cochain complex with differential given by exterior derivative, and prove that it is isomorphic as a cochain complex to $\Omega^{*}(M / G)$.
(b) Show that the map

$$
\Omega^{*}(M) \ni \omega \longmapsto \frac{1}{|G|} \sum_{g \in G}\left(a_{g}\right)^{*} \omega
$$

gives a map of cochain complexes $\Omega^{*}(M) \rightarrow \Omega^{*}(M)^{G}$. The induced map on cohomology is called the transfer map.
(c) What is the composition $\Omega^{*}(M)^{G} \rightarrow \Omega^{*}(M) \rightarrow \Omega^{*}(M)^{G}$ ? Show that the pullback $\operatorname{map} q^{*}: H^{*}(M / G) \rightarrow H^{*}(M)$ is injective.
(d) Let $S^{3} / I^{*}$ be the Poincare homology sphere. Prove that $H^{*}\left(S^{3} / I^{*}\right) \cong H^{*}\left(S^{3}\right)$.
(e) Explain how to obtain $H^{*}\left(\mathbb{R} P^{3}\right)$ from the above results without doing any additional computation.

## Chapter 21

## The Poincaré lemma

Last chapter we introduced de Rham cohomology, and today we prove the Poincaré lemma. This is proven in [GP10, Section 4.§6] and [BT82, Section 4].

### 21.1 The Poincaré lemma

The Poincaré lemma computes the cohomology of $\mathbb{R}^{n}$. It is the backbone of all further computations of cohomology groups.

### 21.1.1 The Poincaré lemma on $\mathbb{R}^{n}$

In the previous chapter we defined a functor

$$
H^{*}(-): \mathrm{Mfd}^{\mathrm{op}} \longrightarrow \mathrm{GrAlg}_{\mathbb{R}}
$$

sending a manifold $M$ to the graded-commutative $\mathbb{R}$-algebra $H^{*}(M)$ of de Rham cohomology groups

$$
H^{p}(M):=\frac{\operatorname{ker}\left(d: \Omega^{p}(M) \rightarrow \Omega^{p+1}(M)\right)}{\operatorname{im}\left(d: \Omega^{p-1}(M) \rightarrow \Omega^{p}(M)\right)}=\frac{\text { closed } p \text {-forms }}{\text { exact } p \text {-forms }}
$$

It sends a smooth map $f: M \rightarrow N$ to the homomorphism $f^{*}: H^{*}(N) \rightarrow H^{*}(M)$ induced by pullback of differential forms.

We also computed $H^{*}\left(S^{1}\right)$ and $H^{*}(\mathbb{R})$, obtaining the following computation in the latter case

$$
H^{*}(\mathbb{R})= \begin{cases}\mathbb{R} & \text { if } *=0 \\ 0 & \text { otherwise }\end{cases}
$$

Our immediate goal is to extend this computation to $\mathbb{R}^{n}$, by induction over $n$. A more precise statement uses the projection

$$
\begin{aligned}
\pi: \mathbb{R}^{n-1} \times \mathbb{R} & \longrightarrow \mathbb{R}^{n-1} \\
(x, t) & \longmapsto x
\end{aligned}
$$

as well as the map $s_{t_{0}}$ for $t_{0} \in \mathbb{R}$,

$$
\begin{aligned}
s_{t_{0}}: \mathbb{R}^{n-1} & \longrightarrow \mathbb{R}^{n-1} \times \mathbb{R} \\
x & \longmapsto\left(x, t_{0}\right) .
\end{aligned}
$$

These satisfy $\pi \circ s_{t_{0}}=\operatorname{id}_{\mathbb{R}^{n-1}}$. It is of course not true that $s_{t_{0}} \circ \pi$ is the identity; it is given by $(x, t) \mapsto\left(x, t_{0}\right)$.

Theorem 21.1.1 (Poincaré lemma). For each $t_{0} \in \mathbb{R}$, the map $s_{t_{0}}^{*}: H^{*}\left(\mathbb{R}^{n-1} \times \mathbb{R}\right) \rightarrow$ $H^{*}\left(\mathbb{R}^{n-1}\right)$ is an isomorphism with inverse $\pi^{*}$.

By induction over $n$, starting with the case $n=0$ where $\mathbb{R}^{0}=*$ (so in particular, we reprove the case that $n=1$ ), one can use this to prove:

Corollary 21.1.2. We have that

$$
H^{*}\left(\mathbb{R}^{n}\right)= \begin{cases}\mathbb{R} & \text { if } *=0 \\ 0 & \text { otherwise }\end{cases}
$$

Proof of Theorem 21.1.1. Let us shorten $s_{t_{0}}$ to $s$ for the sake of brevity. Above we observed that $\pi \circ s=\mathrm{id}$ so that we get $s^{*} \circ \pi^{*}=\mathrm{id}^{*}=\mathrm{id}$ on $H^{*}\left(\mathbb{R}^{n-1}\right)$ It is of course not true that $s \circ \pi=$ id. However, we will still prove that the induced maps on de Rham cohomology satisfy $\pi^{*} \circ s^{*}=\mathrm{id}$ on $H^{*}\left(\mathbb{R}^{n-1} \times \mathbb{R}\right)$.

To do so, we will prove that there is a map $K: \Omega^{*}\left(\mathbb{R}^{n-1} \times \mathbb{R}\right) \rightarrow \Omega^{*-1}\left(\mathbb{R}^{n-1} \times \mathbb{R}\right)$ satisfying

$$
\begin{equation*}
\mathrm{id}-\pi^{*} \circ s^{*}=(-1)^{p-1}(d K-K d) \tag{21.1}
\end{equation*}
$$

This tells us that on closed forms in $\Omega^{*}\left(\mathbb{R}^{n-1} \times \mathbb{R}\right)$, id and $\pi^{*} \circ s^{*}$ differ by an exact form, and hence give the same cohomology class.

To define $K$, we use coordinates $\left(x_{1}, \ldots, x_{n-1}, t\right)$ on $\mathbb{R}^{n-1} \times \mathbb{R}$ observe that every $p$-form in $\mathbb{R} \times \mathbb{R}^{n-1}$ can be uniquely written as a linear combination of $p$-forms of the following forms:
(i) $a_{I}(x, t) d x_{I}$ with $|I|=p$,
(ii) $a_{J}(x, t) d x_{J} \wedge d t$ and $|J|=p-1$.

The map $K$ will be linear, so it is uniquely determined by demanding it satisfies

$$
K\left(a_{I}(x, t) d x_{I}\right)=0 \quad \text { and } \quad K\left(a_{J}(x, t) d x_{J} \wedge d t\right)(x, t)=\left(\int_{t_{0}}^{t} a_{J}(x, s) d s\right) d x_{J}
$$

We verify that $(-1)^{p-1}(d K-K d)=\mathrm{id}-\pi^{*} \circ s^{*}$. First we do so for forms of type (i). On such forms we have that $\pi^{*} \circ s^{*}\left(a_{I}(x, t) d x_{I}\right)=a_{I}\left(x, t_{0}\right) d x_{I}$, so that

$$
\left(\mathrm{id}-\pi^{*} \circ s^{*}\right)\left(a_{I}(x, t) d x_{I}\right)=\left(a(x, t)-a\left(x, t_{0}\right)\right) d x_{I}
$$

On the other hand, we have at $(x, t)$

$$
\begin{aligned}
&(-1)^{p-1}(d K-K d)\left(a_{I}(x, t) d x_{I}\right) \\
&=(-1)^{p} K\left(\frac{\partial a_{I}(x, t)}{\partial t} d t \wedge d x_{I}+\sum_{i=1}^{n-1} \frac{\partial a_{I}(x, t)}{\partial x_{i}} d x_{i} \wedge d x_{I}\right) \\
&=(-1)^{p} K\left(\frac{\partial a_{I}(x, t)}{\partial t} d t \wedge d x_{I}\right) \\
&=K\left(\frac{\partial a_{I}(x, t)}{\partial t} d x_{I} \wedge d t\right) \\
&=\left(\int_{t_{0}}^{t} \frac{\partial a_{I}(x, s)}{\partial t} d s\right) d x_{I} \\
&=\left(a_{I}(x, t)-a_{I}\left(x, t_{0}\right)\right) d x_{I} .
\end{aligned}
$$

We conclude that (21.1) holds on forms of type (i).
For forms of type (ii), we observe that $\pi^{*} \circ s^{*}\left(a_{J}(x, t) d x_{J} \wedge d t\right)=0$ because $s^{*} d t=0$, so that

$$
\left(\mathrm{id}-\pi^{*} \circ s^{*}\right)\left(a_{J}(x, t) d x_{J} \wedge d t\right)=a_{J}(x, t) d x_{J} \wedge d t .
$$

On the other hand, for $(-1)^{p-1}(d K-K d)\left(a_{J}(x, t) d x_{J} \wedge d t\right)$ we do two separate computations

$$
\begin{aligned}
K d\left(a_{J}(x, t) d x_{J} \wedge d t\right) & =K\left(\frac{\partial a_{J}(x, t)}{\partial t} d t \wedge d x_{J} \wedge d t+\sum_{i=1}^{n-1} \frac{\partial a_{J}(x, t)}{\partial x_{i}} d x_{i} \wedge d x_{J} \wedge d t\right) \\
& =\sum_{i=1}^{n-1} K\left(\frac{\partial a_{J}(x, t)}{\partial x_{i}} d x_{i} \wedge d x_{J} \wedge d t\right) \\
& =\sum_{i=1}^{n-1}\left(\int_{t_{0}}^{t} \frac{\partial a_{J}(x, s)}{\partial x_{i}} d s\right) d x_{i} \wedge d x_{I} \\
d K\left(a_{J}(x, t) d x_{J} \wedge d t\right) & =d\left(\left(\int_{t_{0}}^{t} a_{J}(x,) d s\right) d x_{J}\right) \\
& =\frac{\partial \int_{t_{0}}^{t} a_{J}(s, x) d s}{\partial t} d t \wedge d x_{J}+\sum_{i=1}^{n-1} \frac{\partial \int_{t_{0}}^{t} a_{J}(x, s) d s}{\partial x_{i}} d x_{i} \wedge d x_{J} \\
& =a_{J}(x, t) d t \wedge d x_{J}+\sum_{i=1}^{n-1}\left(\int_{t_{0}}^{t} \frac{\partial a_{J}(x, s)}{\partial x_{i}} d s\right) d x_{i} \wedge d x_{J} \\
& =(-1)^{p-1} a_{J}(x, t) d x_{J} \wedge d t+K d\left(a_{J}(x, t) d x_{J} \wedge d t\right)
\end{aligned}
$$

Hence $(-1)^{p-1}(d K-K d)\left(a_{J}(x, t) d x_{J} \wedge d t\right)=a_{J}(x, t) d x_{J} \wedge d t$, so (21.1) also holds on forms of type (ii).

Remark 21.1.3. A map such as $K$ is called a cochain homotopy, and (21.1) says that id and $\pi^{*} \circ s^{*}$ are cochain homotopic.

### 21.1.2 The Poincaré lemma on manifolds

The proof of Theorem 21.1 .1 goes through without any modification when we replace $\mathbb{R}^{n-1}$ by any open subset $U \subset \mathbb{R}^{n-1}$. We can do even better:

Corollary 21.1.4. For each $t_{0} \in \mathbb{R}$ and smooth manifold $M$, the map $s_{t_{0}}^{*}: H^{*}(M \times \mathbb{R}) \rightarrow$ $H^{*}(M)$ is an isomorphism with inverse $\pi^{*}$.

Proof. We can describe the types (i) and (ii) in a coordinate-invariant manner: (i) are those of the form $f(x, t) \pi^{*} \omega$, (ii) are those of the form $f(x, t) \pi^{*}(\omega) \wedge d t$. Since the cotangent bundle of $M \times \mathbb{R}$ is isomorphic to $\pi^{*}\left(T^{*} M\right) \oplus \epsilon$, every form on $M \times \mathbb{R}$ can be written uniquely as a linear combination of forms of type (i) and (ii). Now the proof given above goes through with the modification that $\mathbb{R}^{n-1}$ is replaced by $M$.

Example 21.1.5. The previous corollary proves that open annulus $\mathbb{A}$ has the same cohomology as $S^{1}$, as it is diffeomorphic to $S^{1} \times \mathbb{R}$.

### 21.2 Homotopy invariance

### 21.2.1 Homotopy invariance for de Rham cohomology

Corollary 21.1.4 says that $\pi^{*}$ has as its inverse $s_{t_{0}}^{*}$ for any $t_{0}$. Since inverses are unique, this means that the maps $s_{t_{0}}^{*}: H^{*}(M \times \mathbb{R}) \rightarrow H^{*}(M)$ are all equal. Recall that $f_{0}, f_{1}: M \rightarrow N$ are homotopic if there is a map $H: M \times \mathbb{R} \rightarrow N$ such that $\left.H\right|_{M \times\{0\}}=f_{0}$ and $\left.H\right|_{M \times\{1\}}=f_{1}$, then this has the following important consequence.

Theorem 21.2.1 (Homotopy invariance). If $f_{0}, f_{1}: M \rightarrow N$ are homotopic smooth maps, then $f_{0}^{*}=f_{1}^{*}: H^{*}(N) \rightarrow H^{*}(M)$.

Proof. We can find a homotopy of the form $H: M \times \mathbb{R} \rightarrow N$. We can then factor $f_{i}$, $i=0,1$ as

and obtain equations

$$
f_{0}^{*}=s_{0}^{*} \circ H^{*}=s_{1}^{*} \circ H^{*}=f_{1}^{*}
$$

Recall that we proved that every diffeomorphism induces an isomorphism on cohomology, that is, every smooth map with an inverse does. It actually suffices that $f$ has an inverse up to homotopy.

Corollary 21.2.2. If $f: M \rightarrow N$ is a homotopy equivalence, then $f^{*}: H^{*}(N) \rightarrow H^{*}(M)$ is an isomorphism.

Example 21.2.3. If $M$ is a Moebius strip, then the inclusion $S^{1} \hookrightarrow M$ is a homotopy equivalence (the homotopy inverse is the bundle projection). Thus $H^{*}(M) \cong H^{*}\left(S^{1}\right)$.

More generally, if $E$ is the total space of a smooth vector bundle over $M$, then $H^{*}(E) \cong H^{*}(M)$. This is a generalization of 21.1.4; that corollary can be interpreted
as saying that the total spaces of trivial bundles have the same cohomology as their 0 -section.

Remark 21.2.4. At this point you can extend cohomology to a large class of spaces in a rather artificial manner. For example, if $K$ is built by gluing together finitely many simplices (i.e. vertices, edges, triangles, tetrahedra, etc.), it has an embedding into a sufficiently large Euclidean space with a small open neighborhood $U$ that is unique up to homotopy equivalence. Thus setting $H^{*}(K):=H^{*}(U)$ gives a well-defined notion of cohomology for such spaces $K$. However, algebraic topology provides an elegant definition of cohomology (with real coefficients) for any topological space. It is then a theorem that this coincides with de Rham cohomology when evaluated on a manifold; de Rham's theorem.

### 21.2.2 Applications

## Contractible manifolds

Recall that there exists contractible manifolds $M$ which are not homeomorphic to Euclidean space, such as the Whitehead manifold of Section 8.3. Nonetheless, the homotopy invariance of de Rham cohomology implies these have the same cohomology as Euclidean spaces:

$$
H^{*}(M)= \begin{cases}\mathbb{R} & \text { if } *=0 \\ 0 & \text { otherwise }\end{cases}
$$

## The interior of a manifold with boundary

If $M$ is a manifold with boundary $\partial M$, then we saw that there is an interior collar $\rho: \partial M \times[0,1] \hookrightarrow M$.

Lemma 21.2.5. The inclusion $\operatorname{int}(M) \hookrightarrow M$ is a homotopy equivalence.

Proof. The homotopy inverse $h$ is given in terms of the collar as

$$
h(p)= \begin{cases}p & \text { if } p \notin \operatorname{im}(\rho) \\ \rho(x, \eta(t)) & \text { if } p=\rho(x, t)\end{cases}
$$

where $\eta:[0,1] \rightarrow[0,1]$ is an embedding that is the identity near 1 and has image given by $[1 / 2,1]$. Intuitively, we push the manifold into itself a bit uing the collar. We leave it to reader to convince themselves that $i \circ g$ and $g \circ i$ are homotopic to the identity.

The homotopy invariance of cohomology then gives us:

Corollary 21.2.6. $H^{*}(M) \cong H^{*}(\operatorname{int}(M))$.

## Brouwer fixed point theorem

Observe that $\operatorname{int}\left(D^{2}\right)$ is diffeomorphic to $\mathbb{R}^{2}$ by $x \mapsto x /\left(1+\|x\|^{2}\right)$. By the previous corollary we obtain $H^{1}\left(D^{2}\right) \cong H^{1}\left(\mathbb{R}^{2}\right)=0$. Let us use this to give another proof of the Brouwer fixed point theorem for $D^{2}$. Recall that this follows from the following "no-retraction" theorem:

Corollary 21.2.7. There exists no smooth retraction $r: \partial D^{2} \rightarrow D^{2}$.
Proof. If such an $r$ did exist, we would have a commutative diagram

and applying the contravariant functor $H^{1}(-)$ turns this into a commutative diagram

which is obviously impossible: the identity on $\mathbb{R}$ doesn't factor over 0 .

### 21.3 Two further tricks

For later use, I will give two further tricks to compute two particular de Rham cohomology groups. For now, the reader should take this as an opportunity to get familiar with de Rham cohomology.

### 21.3.1 Top degree

Suppose that $M$ is a compact oriented $k$-dimensional manifold. That $M$ is oriented means that the top exterior power $\Lambda^{k} T^{*} M$ has an everywhere non-vanishing section $\omega$. Thus writing $\omega$ as $a d x_{1} \wedge \cdots \wedge d x_{k}$ in a chart the function $a$ is always non-vanishing. We intend to integrate this over $M$. To do so, we must use charts compatible with the orientation; in that case $a$ must be positive. Hence, when computing $\int_{M} \omega$, we get a finite sum of non-negative numbers, at least one of which is positive and hence $\int_{M} \omega>0$. Now recall that integration of forms over $M$ gives a linear functional $H^{k}(M) \rightarrow \mathbb{R}$, so that we have just shown the following:

Proposition 21.3.1. Suppose that $M$ is a compact oriented manifold of dimension $k$, then $\operatorname{dim} H^{k}(M) \geq 1$.

We will later prove that its dimension is exactly 1 under the additional hypothesis that $M$ is connected.

Example 21.3.2. For any sphere $S^{n}$, we have that $\operatorname{dim} H^{n}\left(S^{n}\right) \geq 1$. In Theorem 22.3.1 we will prove it is exactly 1-dimensional.

### 21.3.2 Degree 1

In the homeworks, you have proven that if $\gamma: S^{1} \rightarrow M$ is a smooth map then $\int_{S^{1}} \gamma^{*} \alpha$ has the following properties: (i) if $\alpha$ is exact it is zero, (ii) if $\alpha$ is closed it only depends on the homotopy class of $\gamma$. Furthermore, you have seen (iii) given a closed $\alpha$, if $M$ connected and the integrals $\int_{S^{1}} \gamma^{*} \alpha$ vanish for all $\gamma$, then $\alpha$ is exact.

If $M$ is connected and we pick a base point $p_{0} \in M$, we can define $\pi_{1}\left(M, p_{0}\right)$ to be the set of based homotopy classes of loops in $M$; this is the fundamental group of $M$ at $p_{0}$. Part (i) and (ii) say there is a map

$$
\begin{aligned}
h: \pi_{1}\left(M, p_{0}\right) & \longrightarrow\left(H^{1}(M)\right)^{*} \\
\gamma & \longmapsto \int_{S^{1}} \gamma^{*} \alpha
\end{aligned}
$$

and part (iii) says that the span of the image of $h$ is all of $\left(H^{1}(M)\right)^{*}$ (at least if it is finite-dimensional, otherwise it is dense). We didn't discuss the group structure of $\pi_{1}\left(M, p_{0}\right)$, but if you know this you will realize $h$ is a homomorphism. It is called the Hurewicz homomorphism.

Proposition 21.3.3. If $M$ is simply-connected, then $H^{1}(M)=0$.
Example 21.3.4. We used Sard's lemma to prove that $S^{n}$ is simply-connected if $n \geq 2$, and hence $H^{1}\left(S^{n}\right)=0$.
Remark 21.3.5. The Hurewicz homomorphism factors over $\pi_{1}\left(M, p_{0}\right)^{a b} \otimes \mathbb{R}$. When $M$ is compact and connected, the resulting homomorphism $\pi_{1}\left(M, p_{0}\right)^{a b} \otimes \mathbb{R} \rightarrow\left(H^{1}(M)\right)^{*}$ is in fact an isomorphism. Thus you can compute $H^{1}(M)$ knowing the fundamental group.

### 21.4 Problems

Problem 21.4.1 (The Poincaré lemma for compactly-supported cohomology). Read pages 37-39 of [BT82] about the Poincaré lemma for compactly-supported cohomology. This says in particular that

$$
H_{c}^{*}\left(\mathbb{R}^{n}\right)= \begin{cases}\mathbb{R} & \text { if } *=n \\ 0 & \text { otherwise }\end{cases}
$$

Explain why this shows that compactly-supported cohomology is not homotopy-invariant.

## Chapter 22

## The Mayer-Vietoris theorem

Last chapter we proved the Poincaré lemma, Theorem 21.1.1, computing the cohomology of $\mathbb{R}^{n}$. To exploit that computation, we now prove a "patching theorem" for de Rham cohomology. It is a generalization of the second proof we gave of $H^{1}\left(S^{1}\right)=\mathbb{R}$ in Chapter 20. This is proven in Section $4 . \S 6$ of [GP10] and Section 2 of [BT82].

### 22.1 Some homological algebra

Recall that de Rham cohomology of $M$ was constructed from the sequence of $\mathbb{R}$-vector spaces

$$
\cdots \longrightarrow \Omega^{p-1}(M) \xrightarrow{d} \Omega^{p}(M) \xrightarrow{d} \Omega^{p+1}(M) \longrightarrow \cdots .
$$

by taking the kernel of $d$ modulo the image of $d$.
This is an example of the cohomology of a cochain complex of $\mathbb{R}$-vector spaces. I will drop the $\mathbb{R}$ from now on. Let me point out that the fact that we're working with vector spaces plays no role in the arguments that follow; we can replace vector spaces by abelian groups, or modules over any ring.

A cochain complex $C^{*}$ is a collection of vector spaces with linear maps between them

$$
\cdots \longrightarrow C^{p-1} \xrightarrow{d} C^{p} \xrightarrow{d} C^{p+1} \longrightarrow \cdots
$$

satisfying $d^{2}=0$. This equation implies $\operatorname{im}\left(d: C^{p-1} \rightarrow C^{p}\right)$ is a subset of $\operatorname{ker}\left(d: C^{p} \rightarrow\right.$ $C^{p+1}$ ), and hence it makes sense to define the cohomology groups $H^{*}\left(C^{*}\right)$ as

$$
H^{p}\left(C^{*}\right):=\frac{\operatorname{ker}\left(d: C^{p} \rightarrow C^{p+1}\right)}{\operatorname{im}\left(d: C^{p-1} \rightarrow C^{p}\right)}
$$

A homomorphism of cochain complexes $f: B^{*} \rightarrow C^{*}$ is a collection of linear maps $f_{p}: B^{p} \rightarrow C^{p}$ such that $d f_{p}=f_{p+1} d$. This condition implies that $f$ induces a map on cohomology, as we have seen in the example of $g^{*}: \Omega^{*}(N) \rightarrow \Omega^{*}(M)$ for a smooth map $g: M \rightarrow N$.

### 22.1.1 Short exact sequences of cochain complexes

A long exact sequence is a sequence of vector spaces

$$
\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow \cdots
$$

such that the kernel of each map is the image of the previous one. In other words, it is a cochain complex whose cohomology vanishes at each point.

A short exact sequence is a sequence of vector spaces

$$
0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0
$$

such that the kernel of each map is the image of the previous one. That is, it is just a long exact sequence in which all but three groups vanish. Conretely having a short exact sequence means the following:

- Since the kernel of $i$ is the image of $0 \rightarrow A, i$ is injective.
- Since the image of $j$ is the kernel of the $C \rightarrow 0, j$ surjective.
- The kernel of $j$ is the image of $i$.

Example 22.1.1. Having a short exact sequence is quite useful. For example, suppose you want to compute what a particular vector spaces $A$ is isomorphic to, and you know it fits into a short exact sequence

$$
0 \xrightarrow{i} \mathbb{R} \xrightarrow{j} A \xrightarrow{k} \mathbb{R} \xrightarrow{l} 0 .
$$

Then $\mathbb{R}$ is the kernel of a surjective map $A \rightarrow \mathbb{R}$, and thus $A$ must be 2 -dimensional.
A short exact sequence of cochain complexes is a sequence of cochain complexes

$$
0 \longrightarrow A^{*} \longrightarrow B^{*} \longrightarrow C^{*} \longrightarrow 0
$$

such that each sequence

$$
0 \longrightarrow A^{p} \longrightarrow B^{p} \longrightarrow C^{p} \longrightarrow 0
$$

is a short exact sequence. The following result relates the cohomology groups $H^{*}\left(A^{*}\right)$, $H^{*}\left(B^{*}\right)$, and $H^{*}\left(C^{*}\right)$.

Theorem 22.1.2. If $0 \rightarrow A^{*} \rightarrow B^{*} \rightarrow C^{*} \rightarrow 0$ is a short exact sequence of cochain complexes then there exist homomorphisms $\delta: H^{p}\left(C^{*}\right) \rightarrow H^{p+1}\left(A^{*}\right)$ such that

is a long exact sequence.
The homomorphisms $\delta$ are called boundary maps, and will be constructed explicitly.

Proof. We start with construction of the homomorphism $\delta: H^{p}\left(C^{*}\right) \rightarrow H^{p+1}\left(A^{*}\right)$. To do so, consider the commutative diagram


Let $[x] \in H^{p}\left(C^{*}\right)$ be represented by $x \in C^{p}$, then since $j_{p}: B^{p} \rightarrow C^{p}$ is surjective there exists a lift $y \in B^{p}$. This satisfies $j_{p+1}(d y)=d j_{p}(y)=d x=0$. Thus $d y \in B^{p+1}$ in the kernel of $j_{p+1}$ and hence in the image of $i_{p+1}$, so there exists a lift $z \in A^{p+1}$.

We want to set $\partial[x]=[z]$. To show that this makes sense, we need to first check that $d z=0$. Since $i_{p+2}$ is injective, we might as well check that $i_{p+2}(d z)=0$. But $i_{p+2}(d z)=d\left(i_{p+1}(z)\right)=d(d y)=0$.

Next we need to prove that $[z]$ is independent of the three choices we made:
(a) the choice of representative $x \in C^{p}$ of $[x]$,
(b) the choice of lift $y \in B^{p}$ of $x$, and
(c) the choice of lift $z \in A^{p+1}$ of $d(y)$.

The last of these, (c), in fact involved no choice at all! The element $z$ is unique because $i_{p+1}$ is injective. For (b), any other choice of lift $y$ differs by an element $i_{p}(w)$, which changes $d y$ to $d\left(y+i_{p}(w)\right)=d y+i_{w}(d w)$ which has lift to $A^{p+1}$ given by $z+d w$, and hence gives rise to the same cohomology class $[z]$. Finally, for (a), any other representative of $x$ differs by $d u$ for $u \in C^{p-1}$. We may lift $u$ to $v \in B^{p-1}$ and then choose to lift of $x+d u$ to $y+d v$ (we have already shown that the end result is independent of the choice of lift). Then $d(y+d v)=d y$, so the resulting class $[z]$ is the same as before.

Let us only check exactness at the term $H^{p}\left(C^{*}\right)$, leaving the other cases for the reader. We need to prove that if $\delta([x])=0$ then $[x]$ is in the image of $H^{p}\left(B^{*}\right)$. Indeed, if $\delta([x])=0$ then $z=d a$ for some $a \in A^{p}$. Then $d i_{p}(a)=i_{p+1}(z)=d y$, so $y-i_{p}(a) \in B^{p}$ is closed. Furthermore $j_{p}\left(y-i_{p}(a)\right)=j_{p}(y)=x$ since $j_{p} \circ i_{p}=0$, so $[x]$ is the image of $\left[y-i_{p}(a)\right]$.

Remark 22.1.3. A proof as above is hard to read. You should draw the diagram and pencil in were all of the elements discussed live and are mapped. This is called diagram-chasing.

### 22.2 The Mayer-Vietoris theorem

Let $M$ be a manifold, and $U, V \subset M$ be open subsets covering $M$. Then the maps induced by restriction give rise to a pair of maps

$$
\begin{aligned}
\Omega^{p}(M) & \longrightarrow \Omega^{p}(U) \oplus \Omega^{p}(V) \\
\omega & \longmapsto\left(\left.\omega\right|_{U},\left.\omega\right|_{V}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Omega^{p}(U) \oplus \Omega^{p}(V) & \longrightarrow \Omega^{p}(U \cap V) \\
(\omega, \nu) & \left.\longmapsto \omega\right|_{U \cap V}-\left.\nu\right|_{U \cap V} .
\end{aligned}
$$

The composition of these two maps is visibly 0 , and in fact the following is true:
Lemma 22.2.1. The following is a short exact sequence of cochain complexes

$$
0 \longrightarrow \Omega^{*}(M) \longrightarrow \Omega^{*}(U) \oplus \Omega^{*}(V) \longrightarrow \Omega^{*}(U \cap V) \longrightarrow 0 .
$$

Proof. Exactness at $\Omega^{p}(M)$ amounts to the observation that a form on $M$ is uniquely determined by its restrictions to $U$ and $V$. Exactness at $\Omega^{p}(U) \oplus \Omega^{p}(V)$ amounts to the observation that a pair of forms $\omega$ on $U$ and $\nu$ on $V$ can be glued to a form on $M$ if and only if $\left.\omega\right|_{U \cap V}=\left.\nu\right|_{U \cap V}$.

It is exactness at $\Omega^{p}(U \cap V)$ that is the hardest; we must show that every form on $U \cap V$ is a difference of forms on $U$ and $V$. The problem is that a naive extension by 0 of $\omega \in \Omega^{p}(U \cap V)$ to $U$ or $V$ will not be smooth. To get around this, we will "cut off" $\omega$ appropriately before extending by 0 . Let $\rho_{U}, \rho_{V}: M \rightarrow[0,1]$ be a partition of unity subordinate to the open cover $U, V$. Then $\rho_{V} \omega$ can be extended by 0 to give a smooth $p$-form $\overline{\rho_{V} \omega}$ on $U$, and similarly $\rho_{U} \omega$ can be extended by 0 to give a smooth $p$-form $\overline{\rho_{U} \omega}$ on $V$. Then we can write $\omega$ as $\overline{\rho_{V} \omega}-\left(-\overline{\rho_{U} \omega}\right)$, which exhibits $\omega$ as being in the image of the map $\Omega^{p}(U) \oplus \Omega^{p}(V) \rightarrow \Omega^{p}(U \cap V)$.

Corollary 22.2.2 (Mayer-Vietoris). There is a long exact sequence

$$
[\underbrace{\longrightarrow H^{p+1}(M) \longrightarrow H^{p+1}(U) \oplus H^{p+1}(V) \longrightarrow H^{p}(U) \oplus H^{p}(V) \longrightarrow H^{p}(U \cap V)}]
$$

In the Mayer-Vietoris long exact sequence, the left horizontal maps

$$
H^{p}(M) \longrightarrow H^{p}(U) \oplus H^{p}(V)
$$

are given by pullback along the inclusion $U \hookrightarrow M$ and $V \hookrightarrow M$. Similarly, the right horizontal maps

$$
H^{p}(U) \oplus H^{p}(V) \longrightarrow H^{p}(U \cap V)
$$

are the difference of the pullback along the inclusion $U \cap V \hookrightarrow U$ and the pullback along the inclusion $U \cap V \hookrightarrow V$. Finally, the boundary maps can be described rather explicitly; given $[\omega] \in H^{p}(U \cap V)$, one observes that $d\left(\overline{\rho_{V} \omega}\right)$ and $d\left(-\overline{\rho_{U} \omega}\right)$ coincide on $U \cap V$ and hence glue to a well-defined $(p+1)$-form on $M$. It will in fact be supported in $U \cap V$.

### 22.3 Applications

As an application of Mayer-Vietoris, we will now compute the cohomology of three basic examples of smooth manifolds. The guidelines for its use are as follows: you need to know the cohomology of three out of the following four manifolds: $M, U, V$, and $U \cap V$. Since we don't know many examples yet, these often tend to be contractible or to be provided by an inductive hypothesis.

### 22.3.1 The cohomology groups of spheres

We start with spheres $S^{n}$.
Theorem 22.3.1. The cohomology of $S^{n}, n \geq 1$, is given by

$$
H^{*}\left(S^{n}\right)= \begin{cases}\mathbb{R} & \text { if } *=0, n, \\ 0 & \text { otherwise }\end{cases}
$$

Proof. The proof will be induction over $n$, the initial case $n=1$ having been completed two lectures ago. We can cover $S^{n}=\left\{\left(x_{0}, \ldots, x_{n}\right) \mid \sum x_{i}^{2}=1\right\}$ by two slightly enlarged hemispheres:

$$
\begin{aligned}
U & :=S^{n} \cap\left\{x \in \mathbb{R}^{n+1} \mid x_{n}>-\epsilon\right\}, \\
V & :=S^{n} \cap\left\{x \in \mathbb{R}^{n+1} \mid x_{n}<\epsilon\right\} .
\end{aligned}
$$

Then $U \cong \mathbb{R}^{n}, V \cong \mathbb{R}^{n}$ and $U \cap V \cong S^{n-1} \times \mathbb{R}$. Thus we get that both $U$ and $V$ have non-zero cohomology groups only in degree 0 , while homotopy invariance says $H^{*}(U \cap V) \cong H^{*}\left(S^{n-1}\right)$ which we know by the inductive hypothesis.

There are several cases for Mayer-Vietoris when we want to compute $H^{p}\left(S^{n}\right)$. Let us start with assume that $p>1$. In this case we have

$$
\left[\begin{array}{c}
\longrightarrow H^{p}\left(S^{n}\right) \longrightarrow H^{p}(U) \oplus H^{p}(V)=0 \longrightarrow H^{p-1}(U) \oplus H^{p-1}(V)=0 \longrightarrow H^{p-1}\left(S^{n-1}\right)
\end{array}\right]
$$

because $p-1, p \neq 0$. By exactness, we conclude that the pictured boundary map is an isomorphism, and thus

$$
H^{p-1}\left(S^{n-1}\right) \longrightarrow H^{p}\left(S^{n}\right)
$$

is an isomorphism as long as $p>1$.
To deal with $p=0,1$, we inspect the relevant part of the long exact sequence:

$$
\underbrace{}_{H^{0}\left(S^{n}\right) \longrightarrow H^{1}\left(S^{n}\right) \longrightarrow H^{0}(U) \oplus H^{1}(V)=0 \longrightarrow H^{0}(V) \cong \mathbb{R}^{2} \xrightarrow{(*)} H^{0}\left(S^{n-1}\right)=\mathbb{R}}
$$

Recalling the construction of the Mayer-Vietoris sequence the map $(*)$ is given by the difference of the restrictions, so by $\mathbb{R}^{2} \ni(x, y) \mapsto x-y \in \mathbb{R}$. This is surjective with kernel $\mathbb{R}$. From this we see that $H^{0}\left(S^{n}\right)=\mathbb{R}$ and $H^{1}\left(S^{n}\right)=0$.

### 22.3.2 The cohomology groups of punctured Euclidean spaces

We have computed $H^{*}\left(\mathbb{R}^{n}\right)$ already-it mostly vanishes-and $H^{*}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ follows from the previous computation since $\mathbb{R}^{n} \backslash\{0\} \cong S^{n-1} \times \mathbb{R}$-it has the same cohomology as $S^{n-1}$. What happens if you remove more points? It is easy for $n=1$, as then removing points just disconnects $\mathbb{R}$ into some disjoint union of copies of $\mathbb{R}$.

Theorem 22.3.2. Let $X$ be a finite subset of $\mathbb{R}^{n}, n \geq 2$, then

$$
H^{*}\left(\mathbb{R}^{n} \backslash X\right) \cong \begin{cases}\mathbb{R} & \text { if } *=0 \\ \mathbb{R}^{|X|} & \text { if } *=n-1, \\ 0 & \text { otherwise } .\end{cases}
$$

Proof. The proof is by induction over the cardinality $r$ of $X$. The initial case $r=1$ has been done above. For the induction step, we fix some $x \in X$ and cover $\mathbb{R}^{n}$ by $U=\mathbb{R}^{n} \backslash\{x\}$ and $V=\mathbb{R}^{n} \backslash(X \backslash\{x\})$. Their intersection $U \cap V$ is $\mathbb{R}^{n} \backslash X$.

We will not give the full Mayer-Vietoris sequence, but skip to the interesting part around degree $p=n-1$ :
$\rightarrow H^{n}\left(\mathbb{R}^{n}\right)=0$

$\longrightarrow H^{n-1}\left(\mathbb{R}^{n}\right)=0 \longrightarrow H^{n-1}(U) \oplus H^{n-1}(V)=\mathbb{R} \oplus \mathbb{R}^{|X|-1} \longrightarrow H^{n-1}(U \cap V) \cong \mathbb{R}-$
where we applied the inductive hypothesis to $U$ and $V$ respectively. We conclude that $H^{n-1}(\mathbb{R} \backslash X) \cong \mathbb{R}^{|X|}$.

### 22.3.3 The cohomology groups of $\mathbb{C} P^{n}$

Recall the complex projective plane $\mathbb{C} P^{n}$ from Problem 2.3.6. It is given by the quotient of the scaling action on non-zero vectors in $\mathbb{C}^{n}$ :

$$
\left(\mathbb{C}^{n+1} \backslash\{0\}\right) / \mathbb{C}^{\times} .
$$

That is, an element $\left[z_{0}: \cdots: z_{n}\right] \in \mathbb{C} P^{n}$ is described by an $(n+1)$-tuple $\left(z_{0}, \ldots, z_{n}\right)$ of complex numbers, not all zero, up to scaling.

Since $\mathbb{C} P^{1}$ is diffeomorphic to $S^{2}$, we already know its cohomology from Theorem 22.3.1. What happens for $\mathbb{C} P^{n}, n \geq 2$ ?

Theorem 22.3.3. The cohomology of $\mathbb{C} P^{n}, n \geq 1$, is given by

$$
H^{*}\left(\mathbb{C} P^{n}\right)= \begin{cases}\mathbb{R} & \text { if } 0 \leq * \leq 2 n \text { is even } \\ 0 & \text { otherwise }\end{cases}
$$

Proof. The proof is by induction over $n$, the initial case $n=1$ having been done before.
Let $U \subset \mathbb{C} P^{n}$ be the open subset consisting of those $\left[z_{0}: \ldots: z_{n}\right]$ satisfying $\left|z_{0}\right|^{2}+\ldots+\left|z_{n-1}\right|^{2}>\left|z_{n}\right|^{2}$. By scaling the last coordinate by $1-t$ with $t \in[0,1]$, this deformation retracts onto $\mathbb{C} P^{n-1}$. Let $V \subset \mathbb{C} P^{n}$ be the open subset consisting of those $\left[z_{0}: \cdots: z_{n}\right]$ with $z_{n} \neq 0$. By scaling the first $n$ coordinates by $(1-t)$ with $t \in[0,1]$, this is seen to be contractible. Then $U \cap V$ is the open subset of those $\left[z_{0}: \ldots: z_{n}\right]$ with $z_{n} \neq 0$ and $\left|z_{0}\right|^{2}+\ldots+\left|z_{n}\right|^{2}>\left|z_{n+1}\right|^{2}$. Such elements are uniquely represented by
elements of the form $\left[w_{0}: \ldots: w_{n-1}: 1\right]$ with $\left|w_{0}\right|^{2}+\ldots+\left|w_{n-1}\right|^{2}>1$. This deformation retracts onto the subspace with $\left|w_{0}\right|^{2}+\ldots+\left|w_{n-1}\right|^{2}=2$, which gives a sphere $S^{2 n-1}$.

We will not give the full Mayer-Vietoris sequence, but skip to the interesting part around degree $p=2 n-1$ :


In particular we get that $H^{*}\left(\mathbb{C} P^{n}\right)=H^{*}\left(\mathbb{C} P^{n-1}\right)$ for $*<2 n$ and $H^{2 n}\left(\mathbb{C} P^{n}\right)=\mathbb{R}$.

## Multiplicative structures

Above we computed $H^{*}\left(S^{n}\right), H^{*}\left(\mathbb{R}^{n} \backslash X\right)$, and $H^{*}\left(\mathbb{C} P^{n}\right)$ as graded $\mathbb{R}$-vector spaces. However, we actually know that these cohomology groups are a graded-commutative algebra. In the former two cases, this algebra structure is uniquely determined by the fact that it is compatible with the grading and that $H^{0}$ is generated by a unit; in both cases all products not involving a multiple of the unit vanish:

$$
H^{*}\left(S^{n}\right)=\mathbb{R}\left[x_{n}\right] /\left(x_{n}^{2}\right)
$$

the free polynomial ring on a generator $x_{n}$ of degree $n$, modulo the ideal generated by $x_{n}^{2}$. Similarly,

$$
H^{*}\left(\mathbb{R}^{n} \backslash X\right)=\mathbb{R}\left[y_{n-1}^{(x)} \mid x \in X\right] /\left(y_{n-1}^{(x)} y_{n-1}^{\left(x^{\prime}\right)} \mid x, x^{\prime} \in X\right)
$$

with a collection of generators $y_{n-1}^{(x)}$ of degree $n-1$, one for each element of $X$.
However, the algebra structure on $H^{*}\left(\mathbb{C} P^{n}\right)$ can not be determined this way. Once we establish Poincaré duality, we can prove that as an algebra

$$
H^{*}\left(\mathbb{C} P^{n}\right)=\mathbb{R}\left[x_{2}\right] /\left(x_{2}^{n+1}\right)
$$

with $x_{2}$ a generator in $H^{2}\left(\mathbb{C} P^{n}\right)$.

### 22.3.4 More examples

If you want to practice your proficiency with the Mayer-Vietoris sequence you can prove - at least additively - the following results (the convention is that a subscript on a generator denotes its degree).
Example 22.3.4. Recall the quaternionic projective plane $\mathbb{H} P^{n}$ from Problem 2.3.7. Its cohomology is given by

$$
H^{*}\left(\mathbb{H} P^{n}\right)=\mathbb{R}\left[y_{4}\right] /\left(y_{4}^{n+1}\right)
$$

Here are some computations that require more advanced techniques than we have discussed so far:

Example 22.3.5. Let $U(2)$ be the Lie group of $(2 \times 2)$-matrices with complex entries which are unitary, i.e. $A^{\dagger}=A$. Its cohomology is given by

$$
H^{*}(U(2))=\mathbb{R}\left[c_{1}, c_{3}\right] /\left(c_{1}^{2}, c_{3}^{2}\right) .
$$

Example 22.3.6. Recall the K3-manifold from Section 8.2. Its cohomology is given by

$$
H^{*}(K 3)= \begin{cases}\mathbb{R} & \text { if } *=0 \\ 0 & \text { if } *=1, \\ \mathbb{R}^{22} & \text { if } *=2, \\ 0 & \text { if } *=3, \\ \mathbb{R} & \text { if } *=4, \\ 0 & \text { otherwise }\end{cases}
$$

The multiplicative structure is determined by the bilinear map $H^{2}(K 3) \times H^{2}(K 3) \rightarrow$ $H^{4}(K 3) \cong \mathbb{R}$. In a suitable basis, it is given by the symmetric matrix

$$
\left[\begin{array}{cc}
-\mathrm{id}_{19} & 0 \\
0 & \mathrm{id}_{3}
\end{array}\right] .
$$

Remark 22.3.7. In fact, the Sullivan-Barge theorem tells you that the only restrictions on realizing a given finitely-generated graded-commutative $\mathbb{R}$-algebra $H^{*}$ with $H^{1}=0$ as the cohomology of a manifold are (i) it satisfies Poincáre duality, and (ii) if the dimension is $4 n$ it admits Pontryagin classes satisfying the congruences of the Hirzebruch signature theorem [FOT08, Theorem 3.2].

### 22.4 Problems

Problem 22.4.1 (Long exact sequence of a pair). Suppose that $M \subset N$ is a smooth submanifold.
(a) Show that the differential of $\Omega^{*}(N)$ restricts to one on $\operatorname{ker}\left[\Omega^{*}(N) \rightarrow \Omega^{*}(M)\right]$, We define the relative cohomology $H^{*}(N, M)$ as that of the cochain complex $\operatorname{ker}\left[\Omega^{*}(N) \rightarrow\right.$ $\left.\Omega^{*}(M)\right]$.
(b) Prove that there is a long exact sequence

$$
\left[\begin{array}{c}
{\left[H^{p+1}(N, M) \longrightarrow H^{p+1}(N) \oplus H^{p+1}(M) \longrightarrow H^{p}(N, M) \oplus H^{p}(N) \longrightarrow H^{p}(M)\right.}
\end{array}\right]
$$

This is known as the long exact sequence of a pair.
Problem 22.4.2 (Relative and compact-supported cohomology). Suppose that $M$ is a compact manifold with boundary $\partial M$. Prove there is an isomorphism

$$
H^{*}(M, \partial M) \cong H_{c}^{*}(M \backslash \partial M) .
$$

Problem 22.4.3 (Mayer-Vietoris for compactly-supported cohomology). Let $U, V$ be an open cover of $M$.
(a) Use extension by zero to construct a short exact sequence of cochain complexes

$$
0 \longrightarrow \Omega_{c}^{*}(U \cap V) \longrightarrow \Omega_{c}^{*}(U) \oplus \Omega_{c}^{*}(V) \longrightarrow \Omega_{c}^{*}(M) \longrightarrow 0
$$

(b) Conclude there is a Mayer-Vietoris long exact sequence in compactly-supported cohomology:

$$
\left[\begin{array}{c}
\longrightarrow H_{c}^{p+1}(U \cap V) \longrightarrow H_{c}^{p+1}(U) \oplus H_{c}^{p+1}(V) \longrightarrow H_{c}^{p}(U) \oplus H_{c}^{p}(V) \longrightarrow H_{c}^{p}(M)
\end{array}\right)
$$

Problem 22.4.4 (The compactly-supported cohomology of the Moebius strip). Let $M$ be the open Moebius strip. Use Problem 21.4.1 and Mayer-Vietoris for compactly-supported cohomology to compute $H_{c}^{*}(M)$.

Problem 22.4.5 (Cohomology of compact oriented surfaces). Recall that $\Sigma_{g}$ denote a genus $g$ surface. Use Mayer-Vietoris to compute $H^{*}\left(\Sigma_{g}\right)$.

Problem 22.4.6 (Mapping class group of the 2 -torus). Let $\Gamma_{1,0}$ denote the set of orientation-preserving diffeomorphisms of $\mathbb{T}^{2}$ up to isotopy. This forms a group under composition, called the mapping class group.

Recall that one description of the 2-torus $\mathbb{T}^{2}$ is as the quotient $\mathbb{R}^{2} / \mathbb{Z}^{2}$ of the translation action of $\mathbb{Z}^{2}$ on $\mathbb{R}^{2}$. Every isomorphism of abelian groups $\mathbb{Z}^{2} \rightarrow \mathbb{Z}^{2}$ is given by a matrix

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

with entries in $\mathbb{Z}$ and determinant $\pm 1$. This matrix gives us a linear map $A: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ which sends $\mathbb{Z}^{2}$ to itself and hence induces a diffeomorphism $\phi_{A}: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$.
(a) Show $\phi_{A}$ is orientation-preserving if and only if $\operatorname{det}(A)=1$. That is, $A$ should lie in the group $\mathrm{SL}_{2}(\mathbb{Z})$ of $(2 \times 2)$-matrices with integer entries and determinant 1 .
(b) Show that the function

$$
\begin{aligned}
\phi: \mathrm{SL}_{2}(\mathbb{Z}) & \longrightarrow \Gamma_{1,0} \\
A & \phi_{A}
\end{aligned}
$$

is a homomorphism.
(c) Use cohomology to show that $\phi_{A}$ is isotopic to $\phi_{B}$ if and only if $A=B$. Conclude that $\phi: \mathrm{SL}_{2}(\mathbb{Z}) \hookrightarrow \Gamma_{1,0}$ is injective.

Problem 22.4.7 (Rotations in topological robotics). We will look at a simplified model of a robotic arm. Ours will only have three rotational axes, given exactly by the $x^{-}, y^{-}$,
and $z$-axis. Let $R_{\alpha}^{x} \in \mathrm{SO}(3)$ denote the matrix for rotation by $\alpha$ around the $x$-axis, and similarly for $R_{\beta}^{y}$ and $R_{\gamma}^{z}$. Rotating an object is encoded by a map

$$
\begin{aligned}
r: S^{1} \times S^{1} \times S^{1} & \longrightarrow \mathrm{SO}(3) \\
(\alpha, \beta, \gamma) & \longmapsto R_{\alpha}^{x} \circ R_{\beta}^{y} \circ R_{\gamma}^{z}
\end{aligned}
$$

(a) Prove that $r$ is smooth.
(b) Indicate an argument that $r$ is surjective. You do not need to give a full proof. Part (b) means that given a desired rotation $A \in \mathrm{SO}(3)$, we can give the arm instructions $\left(\alpha_{A}, \beta_{A}, \gamma_{A}\right)$ so that $r\left(\alpha_{A}, \beta_{A}, \gamma_{A}\right)=A$. Can these instructions be chosen smoothly in $A$ ? More precisely:

Is there a smooth map $s: \mathrm{SO}(3) \longrightarrow S^{1} \times S^{1} \times S^{1}$ so that $r \circ s=\mathrm{id}_{\mathrm{SO}(3)}$ ?
First we will prove that such an $s$ needs to surjective.
(c) Prove that $H^{3}\left(\left(S^{1} \times S^{1} \times S^{1}\right) \backslash\{p\}\right)=0$ for any point $p \in S^{1} \times S^{1} \times S^{1}$.
(d) Note that if $s$ exists and is not surjective, then $s$ factors over $\left(S^{1} \times S^{1} \times S^{1}\right) \backslash\{p\}$. Derive a contradiction from this.
So if a smooth section $s$ of $r$ exists it needs to be surjective.
(e) Prove that $s$ exists and is surjective, then $r$ is a surjective submersion.
(f) Prove that if $r$ is a surjective submersion, then $S^{1} \times S^{1} \times S^{1}$ is a finite cover of $\mathrm{SO}(3)$ and derive a contradiction from this. You may use without proof that any connected manifold $M$ has a connected and simply-connected cover, which is unique up to diffeomorphism. Conclude that no $s$ as in the question exists.

Remark 22.4.8. In fact, using fundamental groups one can show that no section exists if we allow four or more rotational axes. That is, for any arrangement of axes the analogous rotation map $\left(S^{1}\right)^{n} \rightarrow \mathrm{SO}(3)$ has no section.

## Chapter 23

## Qualitative applications of Mayer-Vietoris

So far we have only used Mayer-Vietoris to compute the cohomology of specific manifolds. Today we will use it to prove finite-dimensionality of de Rham cohomology and Poincaré duality. This is proven in of [BT82, Section 5].

### 23.1 De Rham cohomology is finite-dimensional

Suppose that $M$ is a compact manifold, then we can find a finite cover by contractible subsets: take some collection of charts $\phi_{\alpha}: \mathbb{R}^{k} \supset U_{\alpha} \rightarrow V_{\alpha} \subset M$ which cover $M$, write each $U_{\alpha}$ as a union of open balls, and apply compactness. Using a trick from Riemannian geometry you can in fact do better and find a good cover in the following sense:

Definition 23.1.1. A finite open cover $U_{1}, \cdots, U_{r}$ of a topological space is good if for each non-empty subset $I \subset\{1, \ldots, r\}$, the open subset $U_{I}:=\bigcap_{i \in I} U_{i}$ is either empty or diffeomorphic to $\mathbb{R}^{n}$.

Definition 23.1.2. A smooth manifold $M$ is said to be of finite type if it admits a good open cover.

In particular, you can take $I=\{i\}$ to see that each $U_{i}$ is contractible.
Example 23.1.3. A circle is of finite type; it has a good open cover by three intervals. More generally, a $k$-sphere is a finite type; it has a good open cover by $k+2$ open subsets, by taking neighborhoods of the $k$-simplices in the boundary $\partial \Delta^{k+1}$ of a standard $(k+1)$-simplex $\Delta^{k}$ (the convex hull of the basis vectors $e_{0}, \ldots, e_{k+1}$ in $\mathbb{R}^{k+2}$ ) For example, $\Delta^{3}$ is the tetrahedron and slightly expanding the four faces of a tetrahedron gives a good open cover of $S^{2}$.
Remark 23.1.4. Definition 23.1 .1 is slightly non-standard, chosen to simplify the proof of Poincaré duality. It is more common to define a good open cover to have $U_{I}$ which are either empty or contractible. The minimal numbers of elements in a such good open cover is called the covering type [KW16]. Karoubi and Weibel used Mayer-Vietoris to prove that the $k$-sphere has no good open cover by $<k+2$ open subsets. You can prove this yourself, see Problem 23.4.1. Covering type has been largely unstudied and many open questions surrounding it; apparently the covering type of the Klein bottle is not known!

The following is proven in in [BT82, Theorem 5.1]:
Proposition 23.1.5. Every compact manifold $M$ is of finite type, i.e. admits a good open cover. Moreover, every open cover has a refinement to a good open cover.

Theorem 23.1.6. If $M$ is of finite type, $H^{*}(M)$ is finite-dimensional.
Corollary 23.1.7. If $M$ is compact, $H^{*}(M)$ is finite-dimensional.
Proof of Theorem 23.1.6. First observe that since $H^{p}(M)=0$ for $p>k$, the dimension of $M$, it suffices to prove that each $H^{p}(M)$ is finite-dimensional.

We prove the result by induction over the number $r$ of open subsets in a good open cover. In the initial case $r=1, M=U_{1}$ and $U_{1}$ is contractible, so by the homotopy invariance of de Rham cohomology $H^{0}(M)=\mathbb{R}$ and all other cohomology groups vanish.

For the induction step, suppose that $M$ has a good open cover with $r$ open subsets $U_{1}, \ldots, U_{r}$. Then $M$ can be covered by two open subsets $U:=U_{1}$ and $V:=\bigcup_{i=2}^{r} U_{i}$. Then $U$ is contractible, $V$ has a good open cover by $r-1$ open subsets (namely $U_{2}, \ldots, U_{r}$ ), and $U \cap V$ has a good open cover by $r-1$ open subsets (namely $U_{1} \cap U_{2}, \ldots, U_{1} \cap U_{r}$ ). Now consider the Mayer-Vietoris long exact sequence


We deduce from it that for each $p \geq 0, H^{p}(M)$ has a surjection onto a subspace of $H^{p}(U) \oplus H^{p}(V)$ with kernel a subspace of $H^{p-1}(U \cap V)$. Both $H^{p}(U) \oplus H^{p}(V)$ and $H^{p-1}(U \cap V)$ are finite-dimensional by the inductive hypothesis, and hence so are these subspaces. This in turn implies $H^{p}(M)$ is finite-dimensional.

Remark 23.1.8. In fact, you can bound the dimension of $H^{*}(M)$ in terms of $r$ as $\operatorname{dim} H^{*}(M) \leq 2^{r}$.

Some non-compact manifolds are of finite type, e.g. those which are the interior of a compact manifold with boundary. However, $H^{*}(M)$ is not finite-dimensional for a general non-compact manifold $M$. Problem 20.3.5 gives a counterexample..

Remark 23.1.9. Here is an alternative method to constructing a counterexample; suppose we have open subsets $U_{1} \subset U_{2} \subset \cdots$ of $M$ such that $\bigcup_{i} U_{i}=M$, then it is a fact that $H^{*}(M)$ always surjects onto $\lim _{i} H^{*}\left(U_{i}\right)$. In fact, when all $H^{*}\left(U_{i}\right)$ are finite-dimensional this is an isomorphism. This follows from the Milnor sequence and the observation that inverse systems of finite-dimensional vector spaces are Mittag-Leffler. It is easy to construct examples of $U_{i}$ where all maps $H^{*}\left(U_{i}\right) \rightarrow H^{*}\left(U_{i-1}\right)$ surjective and the dimension increases, in which case the limit will be infinite-dimensional.

### 23.2 Poincaré duality

The following is a whirlwind tour of Poincaré duality, both its proof and applications.

### 23.2.1 Statement and proof

Recall that a bilinear form $V \times W \rightarrow \mathbb{R}$ is non-degenerate if (i) $\langle v, w\rangle=0$ for all $w \in W$ if and only if $v=0$, and (ii) $\langle v, w\rangle=0$ for all $v \in V$ if and only if $w=0$. Also recall that $H_{c}^{*}(M)$ denote the compactly-supported de Rham cohomology, using compactly-supported forms instead of arbitrary forms.

Theorem 23.2.1 (Poincaré duality). If $M$ is oriented of dimension $k$ and of finite type, then the bilinear map

$$
\begin{aligned}
\langle-,-\rangle: H^{p}(M) \times H_{c}^{k-p}(M) & \longrightarrow \mathbb{R} \\
([\omega],[\nu]) & \longmapsto \int_{M} \omega \wedge \nu
\end{aligned}
$$

is non-degenerate.
Since we know from Theorem 23.1.6 that each $H^{p}(M)$ is finite-dimensional, this implies $H_{c}^{k-p}(M)$ is also finite-dimensional and the above statement is equivalent to both maps $H^{p}(M) \rightarrow\left(H_{c}^{k-p}(M)\right)^{*}$ and $H_{c}^{k-p}(M) \rightarrow\left(H^{p}(M)\right)^{*}$ being linear isomorphisms.

Remark 23.2.2. The version which says that $H^{p}(M) \rightarrow\left(H_{c}^{k-p}(M)\right)^{*}$ is an isomorphism remains true for arbitrary oriented manifold. However, the version which says that $H_{c}^{k-p}(M) \rightarrow\left(H^{p}(M)\right)^{*}$ is an isomorphism does not. To see this, take $M=\mathbb{Z}$, then $H^{0}(M)=\Pi_{\mathbb{Z}} \mathbb{N}$ and $H_{c}^{0}(M)=\oplus_{\mathbb{Z}} \mathbb{R}$. The dual of the direct sum is the product, but the converse is not true.

Since compactly-supported cohomology of a compact manifold coincides with ordinary cohomology, we get the following, making good on a promise from Section 21.3.1.

Corollary 23.2.3. If $M$ is compact oriented of dimension $k$ with empty boundary, then $H^{p}(M) \cong H^{k-p}(M)$. In particular, if $M$ is connected, $H^{k}(M) \cong \mathbb{R}$.

It is easy to deduce more consequences. Recalling from Section 21.3.2 that if $M$ is simply-connected then $H^{1}(M)=0$, we conclude that

Corollary 23.2.4. If $M$ is a compact oriented manifold of dimension $k$ and simplyconnected, then $H^{k-1}(M)=0$.

Using Problem 22.4.2, one may deduce from Theorem 23.2.1 a version for manifolds with boundary.

Corollary 23.2.5 (Poincaré-Lefschetz duality). If $M$ is compact oriented of dimension $k$ with boundary $\partial M$, then $H^{p}(M) \cong H^{k-p}(M, \partial M)$.

### 23.2.2 The proof of Poincaré duality

Before we start the proof we give a fundamental example:

Example 23.2.6. We know that $H^{*}\left(\mathbb{R}^{k}\right)$ is non-zero except for $*=0$, in which case it is $\mathbb{R}$ generated by the class [1] represented by the constant function with value 1 . Similarly, $H_{c}^{*}\left(\mathbb{R}^{k}\right)$ is non-zero except for $*=k$, Problem 21.4.1, in which case it is $\mathbb{R}$ generated by the class $\left[\lambda(x) d x_{1} \wedge \cdots d x_{k}\right]$ represented by any compactly-supported $k$-form $\lambda(x) \cdot d x_{1} \wedge \cdots \wedge d x_{k}$ with $\lambda: \mathbb{R}^{k} \rightarrow \mathbb{R}$ a compactly-supported smooth function satisfying $\int_{\mathbb{R}^{k}} \lambda(x) d x_{1} \cdots d x_{k}=1$.

Then the computation $\left\langle[1],\left[\lambda(x) d x_{1} \wedge \cdots d x_{k}\right]\right\rangle=\int_{\mathbb{R}^{k}} \lambda(x) d x_{1} \cdots d x_{k}=1$ exhibits the bilinear form as being non-degenerate.

Proof of Theorem 21.3.2. We will prove that the slightly-modified map (we have added a sign)

$$
\begin{aligned}
\rho_{M}: H^{p}(M) & \longrightarrow\left(H_{c}^{k-p}(M)\right)^{*} \\
\omega & \longmapsto\left(\nu \mapsto \epsilon(p) \int_{M} \omega \wedge \nu\right),
\end{aligned}
$$

is an isomorphism. Here, $\epsilon(p)=1$ if $p \equiv 0,1(\bmod 4)$ and $\epsilon(p)=-1$ if $p \equiv 2,3(\bmod 4)$. The proof will be by induction over the number of elements $r$ in the finite good cover $U_{1}, \ldots, U_{r}$. The initial case $r=1$ has been done in Example 23.2.6.

For the induction step, we write $M$ as the union of the two open subsets $U:=U_{1}$ and $V:=U_{2} \cup \cdots \cup U_{r}$. Each of $U, V$ and $U \cap V$ is oriented with a good open cover with either 1 or $r-1$ elements, and thus the inductive hypothesis applies to them.

There are Mayer-Vietoris long exact sequences in cohomology and compactly-supported cohomology, the latter being reversed in direction with the maps not induced by pullback but by extension by 0 :

$$
\cdots \longrightarrow H^{p}(M) \longrightarrow H^{p}(U) \oplus H^{p}(V) \longrightarrow H^{p}(U \cap V) \longrightarrow H^{p+1}(M) \longrightarrow \cdots
$$

and

$$
\cdots \longleftarrow H_{c}^{p}(M) \longleftarrow H_{c}^{p}(U) \oplus H_{c}^{p}(V) \longleftarrow H_{c}^{p}(U \cap V) \longleftarrow H_{c}^{p+1}(M) \longleftarrow \cdots
$$

The latter may be dualized to a long exact sequence

$$
\cdots \longrightarrow H_{c}^{p}(M)^{*} \longrightarrow H_{c}^{p}(U)^{*} \oplus H_{c}^{p}(V)^{*} \longrightarrow H_{c}^{p}(U \cap V)^{*} \longrightarrow H_{c}^{p-1}(M)^{*} \longrightarrow \cdots
$$

We can now write down integration maps from the long exact sequence for cohomology to the dual of that for compactly-supported cohomology (Problem 22.4.3), a representative part of which is given by


We claim this commutes. This is easy to see in the left two squares. For example, for the leftmost one it amounts to verifying that for each pair ( $\nu_{U}, \nu_{V}$ ) of compactly-supported
$k$ - $p$-forms and each $p$-form $\Omega$, we have that

$$
\begin{aligned}
\left(\rho_{U} \oplus \rho_{V}\right)\left(j^{*} \omega\right)\left(\nu_{U}, \nu_{V}\right) & =\left.\epsilon(p) \int_{U} \omega\right|_{U} \wedge \nu_{U}+\left.\epsilon(p) \int_{V} \omega\right|_{V} \wedge \nu_{V} \\
& =\epsilon(p) \int_{M} \omega \wedge\left(\nu_{U}+\nu_{V}\right) \\
& =j^{*}\left(\rho_{M}\right)\left(\nu_{U}, \nu_{V}\right)
\end{aligned}
$$

It is the right square that is harder, as it involves boundary maps. For a $(p+1)$-form $\omega$ on $U \cap V, \partial(\omega)$ is given by picking a partition of unity $\eta_{U}, \eta_{V}: M \rightarrow[0,1]$ subordinate to $U, V$ and taking the $(p+1)$-form which coincides with $d\left(\left.\eta_{U} \omega\right|_{U}\right)$ on $U$ (which has support in $U$ ) and $d\left(-\left.\eta_{V} \omega\right|_{V}\right)$ on $V$ (which has support in $V$ ). Similarly, the boundary map on compactly-supported cohomology sends a $(k-p-1)$-form $\nu$ to $d\left(\rho_{U} \nu\right)-d\left(\rho_{V} \nu\right)$, which is supported in $U \cap V$.

Then we compute

$$
\begin{aligned}
\rho_{M}(\partial(\omega))(\nu) & =\epsilon(p+1) \int_{M} \partial(\omega) \wedge \nu \\
& =\epsilon(p+1) \int_{U} d\left(\left.\eta_{U} \omega\right|_{U}\right) \wedge \nu+\epsilon(p+1) \int_{V} d\left(\left.\eta_{V} \omega\right|_{V}\right) \wedge \nu \\
& =\left.\epsilon(p+1) \int_{U} d\left(\eta_{U}\right) \wedge \omega\right|_{U} \wedge \nu+\left.\epsilon(p+1) \int_{V} d\left(\eta_{V}\right) \wedge \omega\right|_{V} \wedge \nu
\end{aligned}
$$

where the second step uses that $d$ is a derivation and $\omega$ is closed. We can in turn write this as

$$
\begin{aligned}
& \left.(-1)^{p} \epsilon(p+1) \int_{U} \omega\right|_{U} \wedge d\left(\eta_{U}\right) \wedge \nu+\left.(-1)^{p} \epsilon(p+1) \int_{V} \omega\right|_{V} \wedge d\left(\eta_{V}\right) \wedge \nu \\
& \quad=\left.(-1)^{p} \epsilon(p+1) \int_{M} \omega\right|_{U} \wedge\left(d\left(\eta_{U}\right) \wedge \nu-d\left(\eta_{V}\right) \wedge \nu\right) \\
& =(-1)^{p} \epsilon(p+1) \int_{M} \omega \wedge\left(d\left(\eta_{U} \wedge \nu\right)-d\left(\eta_{V} \wedge \nu\right)\right) \\
& =(-1)^{p} \epsilon(p+1) \int_{M} \omega \wedge \partial(\nu) .
\end{aligned}
$$

Now we observe that there are two cases: if $p$ is even then $\epsilon(p+1)=\epsilon(p)$, and if $p$ is odd then $\epsilon(p+1)=-\epsilon(p)$, so this is exactly $\rho_{M}(\omega)(\partial(\nu))$.

Thus we have a commutative diagram of long exact sequences with two-thirds of the vertical maps isomorphisms


It follows from Lemma 23.2.7 that the $\rho_{M}$ must also be an isomorphism.
The following is a standard result in homological algebra (there is a much more general version):

Lemma 23.2.7 (5-lemma). If in a commutative diagrams of vector spaces

all vertical maps except $f_{3}$ are known to be isomorphisms, then $f_{3}$ must also be an isomorphism.

Proof. We shall prove that $f_{3}$ is injective, leaving the proof that it is surjective to the reader. Suppose that $f_{3}(x)=0$, then in particular its image in $B_{4}$ vanishes. Since $f_{4}$ is an isomorphism, the image of $x$ in $A_{4}$ must also vanish. By exactness, this means that $x$ is the image of some $y \in A_{2}$. We know that $f_{2}(y)$ is mapped to 0 in $B_{3}$, so by exactness $f_{2}(y)$ is the image of some $z \in B_{1}$. Since $f_{1}$ and $f_{2}$ are isomorphisms, this means that there is some $w \in A_{1}$ which maps to $y \in A_{2}$. The element $x$ must vanish, being in the image of a composition of two maps in an exact sequence, which is the zero map by exactness.

### 23.3 Applications of Poincaré duality

### 23.3.1 The Poincaré dual to a submanifold

Every closed oriented submanifold $X \subset M$ of dimension $p$ gives rise to a linear functional

$$
\begin{aligned}
i_{X}: H_{c}^{p}(M) & \longrightarrow \mathbb{R} \\
\omega & \longmapsto \int_{X} \omega .
\end{aligned}
$$

By Poincaré duality $\left(H_{c}^{p}(M)\right)^{*} \cong H^{k-p}(M)$ there is a closed $(k-p)$-form $\eta_{X}$ such that $\int_{M} \eta_{X} \wedge \omega=i_{X}(\omega)$. Its cohomology class $\left[\eta_{X}\right] \in H^{k-p}(M)$ is the Poincaré dual to $X$.

If $X$ is compact, we can integrate any $p$-form over it and use the isomorphism $\left(H^{p}(M)\right)^{*} \cong H_{c}^{k-p}(M)$ to see that we may pick a compactly-supported $\eta_{X}^{\prime}$. The compactly-supported supported cohomology class $\left[\eta_{X}\right] \in H_{c}^{k-p}(M)$ is the compact Poincaré dual to $X$.

Example 23.3.1. If we take $M=\mathbb{R}^{n}$ and $X=*$, the homomorphism $i_{X}: H_{c}^{0}(M) \rightarrow \mathbb{R}$ is necessarily trivial because there are no non-zero compactly-supported functions. Hence the Poincaré dual is 0 . However, if we take instead the linear functional $i_{X}: H^{0}(M) \rightarrow \mathbb{R}$, we get an isomorphism and the compact Poincaré dual is represented by $\lambda(x) d x_{1} \wedge \cdots \wedge d x_{n}$ with $\lambda: \mathbb{R}^{n} \rightarrow \mathbb{R}$ a compactly-supported smooth function with integral 1.

### 23.3.2 Multiplicative structure of the cohomology of $\mathbb{C} P^{n}$

In Section 22.3 .3 we computed $H^{*}\left(\mathbb{C} P^{n}\right)$ additively; it is $\mathbb{R}$ in degrees $*=2 i$ for $0 \leq i \leq n$ and vanishes otherwise. We now explain how to obtain the algebra structure.

Proposition 23.3.2. As a graded-commutative $\mathbb{R}$-algebra, $H^{*}\left(\mathbb{C} P^{n}\right)=\mathbb{R}\left[x_{2}\right] /\left(x_{2}^{n+1}\right)$.
Proof. We prove this by induction over $n$, the case $n=1$ being obvious as $H^{*}\left(\mathbb{C} P^{1}\right)=$ $\mathbb{R}\left[x_{2}\right] /\left(x_{2}^{2}\right)$ for degree reasons. (Alternatively, you can use that $\mathbb{C} P^{1}$ is diffeomorphic to $S^{2}$.)

During the Mayer-Vietoris computation of the additive structure of $H^{*}\left(\mathbb{C} P^{n}\right)$ we learned that the inclusion $\mathbb{C} P^{n-1} \hookrightarrow \mathbb{C} P^{n}$ induces an isomorphism on de Rham cohomology in degrees $*<2 n$. Thus $H^{2 i}\left(\mathbb{C} P^{n}\right)$ is generated by $x_{2}^{i}$ for $i<n$, and it remains to prove that $x_{2}^{n}$ is non-zero, as then it necessarily generates the 1-dimensional group $H^{2 n}\left(\mathbb{C} P^{n}\right)$.

But that $x_{2}^{n}$ is non-zero follows from Poincaré duality: there must exist a class $y$ in $H^{2}\left(\mathbb{C} P^{n}\right)$ such that $y \cdot x_{2}^{n-1} \in H^{2 n-2}\left(\mathbb{C} P^{n}\right)$ is a non-zero element of $H^{2 n}\left(\mathbb{C} P^{n}\right)$ otherwise $x_{2}^{n-1}$ would be the pairing as being non-degenerate. But $y$ must be a non-zero multiple of $x_{2}$ and hence $x_{2} \cdot x_{2}^{n-1} \neq 0$.

The multiplicative structure of cohomology groups can be used to prove results which can not be proven if you just know the additive structure. For example, the additive structure of $H^{*}\left(\mathbb{C} P^{n}\right)$ does not rule out that there may exist smooth maps $S^{2 n} \rightarrow \mathbb{C} P^{n} \rightarrow S^{2 n}$ whose composition is the identity. However, the multiplicative structure does:

Corollary 23.3.3. If $n \geq 2$, there is no smooth map $S^{2 n} \rightarrow \mathbb{C} P^{n}$ of non-zero degree.
Proof. Such a map would need to be non-zero on $H^{2 n}$, but since the map $H^{*}\left(\mathbb{C} P^{n}\right)=$ $\mathbb{R}\left[x_{2}\right] /\left(x_{2}^{n+1}\right) \rightarrow H^{*}\left(S^{2 n}\right)=\mathbb{R}\left[y_{2 n}\right] /\left(y_{2 n}^{2}\right)$ is a homomorphism, the value on the generator $x_{2}^{n}$ of $H^{2 n}\left(\mathbb{C} P^{n}\right)$ is the $n$th power of the value on the generator of $x_{2}$ of $H^{2}\left(\mathbb{C} P^{n}\right)$. But this is necessarily 0 .

### 23.4 Problems

Problem 23.4.1 (Bounds on non-zero cohomology groups).
(a) Suppose that $M$ has a finite good open cover with $r$ subsets. Prove that the largest $p$ such that $\widetilde{H}^{p}(M) \neq 0$ must be $\leq r-2$. (Hint: induct over $r$ ).
(b) Prove that a finite good open cover of compact oriented $k$-dimensional manifold with $k \geq 1$ contains of at least $r+2$ subsets. (What happens if $k=0$ ?)

Problem 23.4.2 (Strengthening the 5 -lemma). How much can you weaken the assumptions on $f_{1}, f_{2}, f_{4}, f_{5}$ in Lemma 23.2.7 such that the conclusion still holds?

Problem 23.4.3 (The Künneth map). In this problem, all cochain complexes are over $\mathbb{R}$.
(a) Prove that given two cochain complexes $C^{*}$ and $D^{*}$,

$$
\left(C^{*} \otimes D^{*}\right)^{p}=\bigoplus_{k+l=p} C^{k} \otimes D^{l}, \quad d(x \otimes y)=d(x) \otimes y+(-1)^{|x|} x \otimes d(y)
$$

is again a cochain complex. (The sign is another instance of the Koszul sign rule.)
(b) Prove that the map

$$
\begin{align*}
H^{*}\left(C^{*}\right) \otimes H^{*}\left(D^{*}\right) & \longrightarrow H^{*}\left(C^{*} \otimes D^{*}\right) \\
{[x] \otimes[y] } & \longmapsto[x \otimes y] \tag{23.1}
\end{align*}
$$

is well-defined.
Problem 23.4.4 (The Künneth theorem). In this problem you will use the techniques of this chapter to prove the Künneth theorem. Let $M, N$ be smooth manifolds.
(a) Let $\pi_{1}: M \times N \rightarrow M$ and $\pi_{2}: M \times N \rightarrow N$ be the projections. Show that

$$
\begin{align*}
\Omega^{*}(M) \otimes \Omega^{*}(N) & \longrightarrow \Omega^{*}(M \times N) \\
\omega \otimes \nu & \longmapsto \pi_{1}^{*}(\omega) \wedge \pi_{2}^{*}(\nu) \tag{23.2}
\end{align*}
$$

is a map of cochain complexes.
Combining (23.1) and (23.2) we get maps

$$
\begin{equation*}
H^{*}(M) \otimes H^{*}(N) \longrightarrow H^{*}(M \times N) \tag{23.3}
\end{equation*}
$$

We will prove it is an isomorphism when $N$ is of finite type.
(b) Prove (23.3) is an isomorphism when $N=\mathbb{R}^{k}$.
(c) Prove that $H^{p}(M) \otimes$ - preserves kernel and images. Conclude it preserves long exact sequences.
(d) Construct a map of long exact sequences

$$
\begin{gathered}
\underset{p+q=n}{\oplus} H^{p}(M) \otimes H^{q}(U \cup V) \longrightarrow \underset{p+q=n}{\oplus} H^{\oplus}(M) \otimes H^{q}(V) \\
\bigoplus_{p+q=n} H^{p}(M) \otimes H^{q}(U) \longrightarrow \underset{p+q=n}{\bigoplus} H^{p}(M) \otimes H^{q}(U \cap V) \\
H^{n}(M \times(U \cup V)) \longrightarrow H^{n}(M \times(U \cap V))
\end{gathered}
$$

with vertical maps given by (23.3).
(e) Prove using induction over the number of elements in a good open cover of $N$ and the 5 -lemma, that map (23.3) induces an isomorphism

$$
H^{*}(M) \otimes H^{*}(N) \stackrel{\cong}{\cong} H^{*}(M \times N)
$$

when $N$ is of finite type.
(f) Compute $H^{*}\left(\mathbb{T}^{n}\right)$.

Problem 23.4.5 (The cohomology of sphere bundles). Suppose that $\pi: E \rightarrow B$ is a manifold bundle whose fibers are diffeomorphic to $S^{k}$ and whose base $B$ is compact.
(a) Prove that there exists a finite good open cover $\left\{U_{i}\right\}_{i=1}^{n}$ of $B$ such that $\left.\pi\right|_{U_{i}}$ is a trivial bundle.
(b) Do an induction over the number of elements in a good open cover as in (a) to show that the map $\pi^{*}: H^{*}(B) \rightarrow H^{*}(E)$ is an isomorphism for $*<k$ and injective for $*=k$.
(c) Does it suffices that fibers $M$ satisfy $H^{*}(M)=0$ for $0<*<k$ for (b) to be true?

## Chapter 24

## Invariant forms in de Rham cohomology

In this chapter we prove that when a compact Lie group $G$ acts on a manifold $M$, we can compute the cohomology of $M$ using only those differential forms which are invariant under the action of $G$. This will be particularly useful in the case that $G$ acts on itself by left multiplication.

### 24.1 Invariant forms

Let $G$ be a Lie group. For each $g \in G$ there is a smooth map

$$
\begin{array}{rl}
l_{g}: G & G \\
h & \longmapsto g h
\end{array}
$$

given by left multiplication with $g$.

Definition 24.1.1. We say that a $p$-form $\omega \in \Omega^{p}(G)$ is left-invariant if for all $g \in G$, $l_{g}^{*} \omega=\omega$.

As $l_{g}^{*}$ is linear, the left-invariant forms form a linear subspace of $\Omega^{p}(G)$ which we denote by $\Omega^{p}(G)^{G}$. If $G$ is $k$-dimensional, this is $\binom{k}{p}$-dimensional:

Lemma 24.1.2. There is an isomorphism $\Omega^{p}(G)^{G} \cong \Lambda^{p}\left(T_{e}^{*} G\right)$.

Proof. The evaluation of a left-invariant $p$-form $\omega$ at $e \in G$ gives a linear map ev: $\Omega^{p}(G) \rightarrow$ $\Lambda^{p}\left(T_{e}^{*} G\right)$. We claim this is a linear isomorphism when restricted to $\Omega^{p}(G)^{G}$. Indeed, its inverse is given by

$$
\Lambda^{p}\left(T_{e}^{*} G\right) \ni \alpha \longmapsto\left(g \mapsto l_{g}^{*} \alpha\right) \in \Omega^{p}(G)^{G}
$$

with $l_{g}^{*}: \Lambda^{p}\left(T_{e}^{*} G\right) \rightarrow \Lambda^{p}\left(T_{g}^{*} G\right)$ induced by $l_{g}$.

Remark 24.1.3. This shows again that a Lie group admits an orientation: any non-zero element of $\Lambda^{k}\left(T_{e}^{*} G\right)$ gives an everywhere non-zero section of the orientation line bundle.

Example 24.1.4. There is also a right multiplication map

$$
\begin{aligned}
r_{g}: G & \longrightarrow G \\
h & \longmapsto h g^{-1}
\end{aligned}
$$

and a $p$-form is right-invariant if for all $g \in G, r_{g}^{*} \omega=\omega$.
For top forms on a compact connected Lie group, left invariance is equivalent to right invariance. To prove this, we start with the observation that $r_{h}^{*}$ preserves the left-invariant forms of any degree: if $\omega$ is left-invariant then

$$
l_{g}^{*} r_{h}^{*} \omega=r_{h}^{*} l_{g}^{*} \omega=r_{h}^{*} \omega,
$$

where we used that $l_{g}$ and $r_{h}$ commute because left and right multiplication do.
Suppose now $G$ is $k$-dimensional and restrict your attention to top forms. Because $\Omega^{k}(G)^{G}$ is one-dimensional, there is a unique smooth function $f: G \rightarrow \mathbb{R}$ such that $\operatorname{ev}\left(r_{h}^{*} \omega\right)=f(h) \operatorname{ev}(\omega)$ for all $\omega \in \Omega^{k}(G)^{G}$. This is a homomorphism by

$$
\operatorname{ev}\left(r_{g h}^{*} \omega\right)=\operatorname{ev}\left(r_{h}^{*} r_{g}^{*} \omega\right)=f(h) \operatorname{ev}\left(r_{g}^{*} \omega\right)=f(h) f(g) \operatorname{ev}(\omega)=f(g) f(h) \operatorname{ev}(\omega) .
$$

Because $G$ is compact, any homomorphism $G \rightarrow \mathbb{R}$ must be take values in $\{ \pm 1\}$. Because $G$ is connected, it must then always be 1 .

More generally, suppose that a Lie group $G$ acts smoothly on a manifold $X$. Then for each $g \in G$ the map

$$
\begin{aligned}
l_{g}: X & \longrightarrow X \\
x & \longmapsto g x
\end{aligned}
$$

is smooth, where $g x$ is shorthand for the action of $g \in G$ on $x \in X$.
Definition 24.1.5. We say that a $p$-form $\omega \in \Omega^{p}(X)$ is invariant if for all $g \in G$, $l_{g}^{*} \omega=\omega$.

As before, invariant forms are a linear subspace of $\Omega^{p}(X)$, which we denote by $\Omega^{p}(X)^{G} \subset \Omega^{p}(X)$.
Example 24.1.6. Left-invariant forms are the special case of $G$ acting on itself by $h \mapsto g h$, and right-invariant that of $G$ acting on itself by $h \mapsto g h^{-1}$.

Invariant forms are preserved by the exterior derivative:
Lemma 24.1.7. $d\left(\Omega^{p}(X)^{G}\right) \subset \Omega^{p+1}(X)^{G}$.
Proof. Pullback along a map commutes with $d$, so if $\omega \in \Omega^{p}(X)^{G}$ then we have

$$
l_{g}^{*} d \omega=d\left(l_{g}^{*} \omega\right)=d \omega .
$$

Thus we can restrict the differential $d$ to $\Omega^{*}(X)^{G}$, and get an inclusion of cochain complexes

$$
\Omega^{*}(X)^{G} \longrightarrow \Omega^{*}(X)
$$

Theorem 24.1.8. If $G$ is compact and connected, then

$$
\Omega^{*}(X)^{G} \longrightarrow \Omega^{*}(X)
$$

induces an isomorphism on cohomology.
In particular, one can compute the cohomology of $X$ using only invariant forms!
Remark 24.1.9. Suppose that $G$ has dimension $k$. Through the isomorphism in Lemma 24.1.2, we can use the differential on the left-invariant forms on $G$ to make

$$
0 \rightarrow \Lambda^{0}\left(T_{e}^{*} G\right) \rightarrow \Lambda^{1}\left(T_{e}^{*} G\right) \rightarrow \cdots \rightarrow \Lambda^{k}\left(T_{e}^{*} G\right) \rightarrow 0
$$

into a cochain complex. This computes the cohomology of $G$ and gives another proof that this is finite-dimensional. However, it is not so useful for computations because we have not given an explicit description of the differential; it is the Lie algebra cohomology of the Lie algebra $\mathfrak{g}$ with trivial coefficients [CE48].

To prove Theorem 24.1.8 we will construct an "averaging map"

$$
\Omega^{*}(G) \longrightarrow \Omega^{*}(G)^{G}
$$

which is the identity on $\Omega^{*}(G)^{G}$ and whose composition with the inclusion $\Omega^{*}(G)^{G} \rightarrow$ $\Omega^{*}(G)$ induces the identity on cohomology.

Intuitively, the averaging map is

$$
\omega \longmapsto \int_{G} l_{g}^{*} \omega .
$$

Unfortunately, this does not parse since we only defined integrals of top forms over a manifold. To make sense of it, we generalize our notion of integration over a manifold to integration along the fibers of certain smooth maps. Let $\alpha: G \times X \rightarrow X$ be action map $(g, x) \mapsto g x$, and define

$$
\begin{aligned}
\bar{\alpha}: G \times X & \longrightarrow G \times X \\
(g, x) & \longmapsto\left(g, g^{-1} x\right) .
\end{aligned}
$$

Then our averaging map is given by

$$
\begin{aligned}
& \Omega^{r}(G \times X) \stackrel{\bar{\alpha}^{*}}{\longrightarrow} \Omega^{r}(G \times X) \xrightarrow{\pi_{1}^{*} \rho \wedge-} \Omega^{r+k}(G \times X) \\
& \pi_{2}^{*}(-) \uparrow \\
& \quad \Omega^{r}(X)
\end{aligned}
$$

for an left-invariant top form $\rho$ on the $k$-dimensional oriented Lie group $G$ such that $\int_{G} \rho=1$. Here $\pi_{2}^{*}$ is pullback, and $\left(\pi_{2}\right)_{*}$ is the aforementioned "fiber integration." The left-invariance of $\rho$ will guarantee that the resulting $r$-form is left-invariant. We will then interpolate $\rho$ to something akin to a $\delta$-function at the identity.

This proof strategy is implemented in Section 24.3. First we need to develop integration of differential forms along the fibers of a manifold bundle.

### 24.2 Fiber integration

### 24.2.1 Manifold bundles

The data of a smooth vector bundle in particular is a smooth map $p: E \rightarrow X$ whose fibers are diffeomorphic to $\mathbb{R}^{k}$. It must be locally trivial, in the sense that each point $x \in X$ admits an open neighborhood $V$ and a commutative diagram

the horizontal maps are diffeomorphisms. There is nothing special about $\mathbb{R}^{k}$ here, and we can replace it with any other smooth manifold $M$ :

Definition 24.2.1. Suppose that either $\partial M=\varnothing$ or $\partial X=\varnothing$. A smooth manifold bundle with fiber $M$ is a smooth map $p: E \rightarrow X$ such that for each point $x \in X$ there is an open neighborhood $V$ and a commutative diagram

with horizontal maps diffeomorphisms.
Usually both $\partial M$ and $\partial X$ will be empty. We will denote the fiber $p^{-1}(x)$ by $E_{x}$; by definition it is diffeomorphic to $M$.
Example 24.2.2. There is always a trivial manifold bundle $\pi_{1}: X \times M \rightarrow X$. In our proof of Theorem 24.1.8 we will only need these bundles.
Example 24.2.3. Suppose that $\partial M \neq \varnothing$ (hence we assume $\partial X=\varnothing$ ), then $\left.p\right|_{\partial E}: \partial E \rightarrow X$ is a smooth manifold bundle with fiber $\partial M$. Indeed, the local trivializations in Definition 24.2.1 restrict to local trivializations


Example 24.2.4. If a compact Lie group $G$ acts freely and smoothly on $M$ (see Section 7.3 for these definitions), then $M \rightarrow M / G$ is a smooth manifold bundle with fiber $G$, see Corollary 25.3.3. A famous example of a manifold bundle arises this way: the Hopf fibration

$$
S^{3} \longrightarrow S^{2}
$$

with fibers $S^{1}$ is obtained by taking the quotient of the unit quaternions (diffeomorphic to $S^{3}$ ) by the unit complex numbers (diffeomorphic to $S^{1}$ ). Figure 24.1 illustrates how the circle fibers fit together in $\mathbb{R}^{3}$; removing one point from $S^{3}$ means that one of the circles becomes a line, here the vertical axis.


Figure 24.1 Some fibers of the Hopf fibration pictured in $\mathbb{R}^{3} \subset S^{3}$ (from [Fra07]).

The fibers of a smooth manifold bundle, being smooth manifolds, each have a tangent bundle. These assemble to a vector bundle over $E$, as can be seen from the following construction:

Definition 24.2.5. The vertical tangent bundle $T_{v} E$ of the manifold bundle $p: E \rightarrow X$ is the smooth vector bundle on $E$ given by $\operatorname{ker}(d p: T E \rightarrow T X)$.

The inclusion $E_{x} \hookrightarrow E$ of a fiber is a smooth embedding, so its derivative is an injective map of smooth vector bundles

$$
\begin{equation*}
T E_{x} \longrightarrow T_{v} E \tag{24.1}
\end{equation*}
$$

This is an isomorphism onto $\left.T_{v} E\right|_{E_{x}}$, by observing its composition with $d p$ is zero and doing a dimension count. That is, we identify $T_{e} E_{x}$ with $\operatorname{ker}\left(d_{e} p\right)$.

Definition 24.2.6. A fiberwise orientation of a manifold bundle $p: E \rightarrow X$ is an orientation of $T_{v} E$.

In particular, through (24.1) this gives an orientation of each of the fibers of $p$.

### 24.2.2 Fiber integration

We start with some multilinear algebra. Linear maps $\omega: \Lambda^{p} V \rightarrow W$ are in natural bijection with alternating multilinear maps $w: V^{k} \rightarrow W$. Given such a multilinear map $w$ and a vector $v \in V$, we form a new alternating multilinear map $V^{k-1} \rightarrow W$ as follows:

$$
\left(v, \ldots, v_{k-1}\right) \longmapsto w\left(v, v_{1}, \ldots, v_{k-1}\right) .
$$

The corresponding linear map $\Lambda^{p-1} V \rightarrow W$ is denoted $\iota_{v} \omega$, and called the interior product.

As usual, this generalizes to a definition on manifolds. A p-form can be interpreted as a section of the smooth vector bundle with fiber over $m$ given by the vector space of linear maps $\Lambda^{p} T_{m} M \rightarrow \mathbb{R}$. Given a section $X$ of $T M$ (also known as a vector field) we can thus form an interior product

$$
\iota_{X}: \Omega^{p} M \longrightarrow \Omega^{p-1} M
$$

Lemma 24.2.7. The interior product has the following properties:

- $\iota_{X+Y} \omega=\iota_{X} \omega+\iota_{Y} \omega$ and $\iota_{c X} \omega=c \iota_{X} \omega=\iota_{x}(c \omega)$.
- $\iota_{X}(\omega+\rho)=\iota_{X} \omega+\iota_{X} \rho$ and $\iota_{X}(c \omega)=c \iota_{X} \omega /$
- $\iota_{X} \iota_{Y} \omega=-\iota_{Y} \iota_{X} \omega$.
- $\iota_{X}(\omega \wedge \rho)=\left(\iota_{X} \omega\right) \wedge \rho+(-1)^{|\omega|} \omega \wedge\left(\iota_{X} \rho\right)$.

Proof. These properties that may be verified pointwise, and then follow from the formula given above.

Suppose $p: E \rightarrow X$ is a manifold bundle with $m$-dimensional fibers $M$ with a fiberwise orientation. Then from an $(r+m)$-form $\omega \in \Omega^{r+m}(E)$ we will produce an $r$-form $p_{*} \omega$ on $X$ by integrating over the fibers. We start with a lemma:

Lemma 24.2.8. For any $v \in T_{x} X$, there is a smooth vector field $Y$ on $E$ such that for all $e \in E_{x}$ the image of $d_{e} p(Y(e))=v \in T_{x} X$.

Proof. It suffices to produce one locally in the base $X$, since we can always multiply the vector field with $\eta \circ p$ for $\eta: X \rightarrow[0, \infty)$ a bump function which takes the value 1 near $x$, and extend by zero. Thus without loss of generality the manifold bundle is the trivial one $\pi_{2}: M \times X \rightarrow X$

There exists a smooth vector field $Y^{\prime}$ on $X$ extending $v$ by Problem 17.5.3, so we need to produce the lift to $E$ and we may take $Y$ to be

$$
(m, x) \longmapsto\left(x \mapsto Y^{\prime}(x) \in T_{x} X \subset T_{m} M \oplus T_{x} X \cong T_{(m, x)}(M \times X)\right)
$$

We can only integrate with compact support, but did not assume the fibers of $p$ are compact. Hence we will need to impose the condition that $\operatorname{supp}(\omega) \cap E_{x}$ is a compact subset of $E_{x}$ for all $x \in X$. Such a differential form $\omega$ is said to be have vertically compact support. This condition is preserved by $d$, so the vertically compact supported forms give a sub-cochain complex $\Omega_{v c}^{*}(E) \subset \Omega^{*}(E)$ and associated cohomology groups $H_{v c}^{*}(E)$.

Definition 24.2.9. Given $\omega \in \Omega_{v c}^{m+r}(E)$, we define a $r$-form $p_{*} \omega$ on $X$ by taking its corresponding alternating multilinear form at $x \in X$ to be

$$
\left(T_{x} X\right)^{r} \ni\left(v_{1}, \ldots, v_{r}\right) \longmapsto p_{*} \omega\left(v_{1}, \ldots, v_{r}\right):=\int_{E_{x}} \iota_{Y_{r}} \cdots \iota_{Y_{1}} \omega \in \mathbb{R}
$$

where $Y_{i}$ is a choice of vector field lifting $v_{i}$ as in Lemma 24.2.8.

Of course, the reason we took $r$ vectors is that $\iota_{Y_{r}} \cdots \iota_{Y_{1}} \omega$ is a compactly-supported $m$-form on $E_{x} \cong M$ and hence may be integrated. Note this requires an orientation, which is why we assume $p: E \rightarrow X$ has a fiberwise orientation. Let us check this definition makes sense:

Lemma 24.2.10. This is multilinear alternating, depends smoothly on $x$, and is independent of the choice of $Y_{i}$.

Proof. The first statement follows from Lemma 24.2.7 and the fact that integration is linear. The second statement follows from the fact that if a compactly-supported smooth function depends smoothly on some parameters, so does its integral.

For the third statement, it suffices to prove that if $Z_{1}, \ldots, Z_{r}$ is a collection of alternative choices of lifts then the multilinear alternating forms

$$
\begin{aligned}
& \left(T_{e} E_{x}\right)^{m} \ni\left(w_{1}, \ldots, w_{m}\right) \longmapsto \omega\left(Y_{1}(e), \ldots, Y_{r}(e), w_{1}, \ldots, w_{m}\right) \\
& \left(T_{e} E_{x}\right)^{m} \ni\left(w_{1}, \ldots, w_{m}\right) \longmapsto \omega\left(Z_{1}(e), \ldots, Z_{r}(e), w_{1}, \ldots, w_{m}\right)
\end{aligned}
$$

are equal. By induction and antisymmetry, it suffices to prove we may replace $Y_{1}(e)$ in the first formula by $Z_{1}(e)$ without changing its value. By construction $Y_{1}(e)-Z_{1}(e)$ lies in $\operatorname{ker}\left(d_{e} p\right)=T_{e} E_{x}$, so $\left(w_{0}, \ldots, w_{m}\right)=\left(Y_{1}(e)-Z_{1}(e), w_{1}, \ldots, w_{m}\right)$ is a collection of $(m+1)$ vectors in the $m$-dimensional vector space $T_{e} E_{x}$. This means that the alternating multilinear form

$$
\left(w_{0}, \ldots, w_{m}\right) \longmapsto \omega\left(w_{0}, Y_{2}(e), \ldots, Y_{r}(e), w_{1}, \ldots, w_{m}\right)
$$

must vanish on it, and hence we have

$$
\omega\left(Y_{1}(e), Y_{2}(e), \ldots, Y_{r}(e), w_{1}, \ldots, w_{m}\right)=\omega\left(Z_{1}(e), Y_{2}(e), \ldots, Y_{r}(e), w_{1}, \ldots, w_{m}\right)
$$

Thus, we get a well-defined $r$-form $p_{*} \omega \in \Omega^{r}(X)$. This is called the push-forward of $\omega$ along $p$ or fiber intergration of $\omega$ along $p$.

Lemma 24.2.11. Push-forward $p_{*}: \Omega_{v c}^{m+r}(E) \rightarrow \Omega^{r}(X)$ has the following properies:

- It is linear.
- It commutes with $d$ when $\partial M=\varnothing$.

Proof. The first statement follows from Lemma 24.2.7 and the fact that integration is linear. The second statement is proven analogously to Stokes' theorem and left to Problem 24.4.1.

In particular, it induces a map on cohomology

$$
p_{*}: H_{v c}^{m+r}(E) \longrightarrow H^{r}(X)
$$

We need two further properties:
Lemma 24.2.12 (Push-pull formula). For $\rho \in \Omega_{v c}^{*}(E)$ and $\omega \in \Omega^{*}(X), p_{*}\left(\rho \wedge p^{*} \omega\right)=$ $p_{*}(\rho) \wedge \omega$.

Proof. Assume that $\rho \in \Omega_{v c}^{p+r}(E)$ and $\omega \in \Omega^{s}(X)$. We simply compute the value of both sides on $v_{1}, \ldots, v_{r+s}$ :

$$
\begin{aligned}
p_{*}\left(\rho \wedge p^{*} \omega\right)\left(v_{1}, \ldots, v_{r+s}\right) & =\int_{E_{x}} \iota_{Y_{r+s}} \cdots \iota_{Y_{1}}\left(\rho \wedge p^{*} \omega\right) \\
& =\int_{E_{x}} \sum_{\sigma \in \mathfrak{S}_{r+s}}(-1)^{\epsilon(\sigma)}\left(\iota_{Y_{\sigma(r)}} \cdots \iota_{Y_{\sigma(1)}} \rho\right)\left(\iota_{Y_{\sigma(r+s)}} \cdots \iota_{Y_{\sigma(r+1)}} \omega\right) \\
& =\int_{E_{x}} \sum_{\sigma \in \mathfrak{S}_{r+s}}(-1)^{\epsilon(\sigma)}\left(\iota_{Y_{\sigma(r)}} \cdots \iota_{Y_{\sigma(1)}} \rho\right)\left(\omega\left(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}\right)\right) \\
& =\sum_{\sigma \in \mathfrak{S}_{r+s}}(-1)^{\epsilon(\sigma)}\left(p_{*} \rho\right)\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right) \omega\left(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}\right) \\
& =\left(\left(p_{*} \rho\right) \wedge \omega\right)\left(v_{1}, \ldots, v_{r+s}\right) .
\end{aligned}
$$

Here the second equality uses that unless we take $s$ interior product with $\omega$, the restriction to $E_{x}$ will vanish.

A map of manifold bundles is a commutative diagram

where $F$ induces a diffeomorphism on fibers.
Lemma 24.2.13 (Naturality). Given an map of fiberwise oriented manifold bundles

which preserves orientations of the fibers, we have $f^{*} p_{*}^{\prime}=p_{*} F^{*}$.

### 24.3 Invariant forms compute de Rham cohomology

We now prove Theorem 24.1.8. Fixing a left-invariant top form $\rho$ on $G$ with $\int_{G} \rho=1$, we can now make sense of the diagram

defining the bottom averaging map. As $G$ is compact, all forms on $G \times X$ have vertically compact support. Let us verify ave $(\omega)$ is left-invariant.

Lemma 24.3.1. We have that $l_{g}^{*} \circ$ ave $=$ ave.
Proof. We start by moving $l_{g}^{*}$ past the first term $\left(\pi_{2}\right)_{*}$ in

$$
\text { ave }=\left(\pi_{2}\right)_{*} \circ\left(\pi_{1}^{*} \rho \wedge \bar{\alpha}^{*} \pi_{2}^{*}(-)\right)
$$

To do so, note there is a commutative diagram

so by naturality of pushforward we have $l_{g}^{*}\left(\pi_{2}\right)_{*}=\left(\pi_{2}\right)_{*}\left(l_{g} \times l_{g}\right)^{*}$.
Since pullback commutes with wedge product, it suffices to prove that $\left(l_{g} \times l_{g}\right)^{*}$ fixes $\pi_{1}^{*} \rho$ and $\bar{\alpha}^{*} \pi_{2}^{*} \omega$. For the first, we have

$$
\left(l_{g} \times l_{g}\right)^{*} \pi_{1}^{*} \rho=\pi_{1}^{*} l_{g}^{*} \rho=\pi_{1}^{*} \rho
$$

because the following diagram commutes

and $\rho$ is left-invariant.
For the second, we have

$$
\left(l_{g} \times l_{g}\right)^{*} \bar{\alpha}^{*} \pi_{2}^{*} \omega=\bar{\alpha}^{*}\left(l_{g} \times \mathrm{id}\right)^{*} \pi_{2}^{*} \omega=\pi_{2}^{*} \omega
$$

because the following diagrams commute


In particular, the right diagram says that $(h, x) \mapsto(g h, g x) \mapsto\left(g h, h^{-1} g^{-1} g x\right)=$ $\left(g h, h^{-1}(x)\right)$ is the same as $(h, x) \mapsto\left(h, h^{-1} x\right) \mapsto\left(g h, h^{-1} x\right)$.

We claim that the inclusion map and averaging map

$$
i: \Omega^{*}(X)^{G} \longrightarrow \Omega^{*}(X) \quad \text { ave }: \Omega^{*}(X) \longrightarrow \Omega^{*}(X)^{G}
$$

induce mutually inverse map on cohomology. In fact, in one direction this is already true on the level of differential forms.

Lemma 24.3.2. ave $\circ i=\mathrm{id}$.

Proof. If $\omega$ is $G$-invariant, then $\bar{\alpha}^{*} \pi_{2}^{*} \omega=\pi_{2}^{*} \omega$ as the value at $(g, x)$ of the left hand side is the value at $(g, x)$ of $\pi_{2}^{*} l_{g^{-1}}^{*} \omega$. Thus on the image of $i$, ave is given by

$$
\omega \longmapsto\left(\pi_{2}\right)_{*}\left(\pi_{1}^{*} \rho \wedge \pi_{2}^{*} \omega\right)=\left(\pi_{2}\right)_{*}\left(\pi_{1}^{*} \rho\right) \wedge \omega=\omega
$$

by the push-pull formula and the assumption that $\int_{G} \rho=1$.
Lemma 24.3.3. On cohomology, $i \circ$ ave $=\mathrm{id}$.
Proof. Fix a contractible open neighborhood $U$ around $e \in G$. Using bump functions, we may construct a $k$-form $\bar{\rho}$ in $G$ with compact support in $U$ and integral equal to 1 . As $\int_{G}-: H^{k}(G) \rightarrow \mathbb{R}$ is a isomorphism because $G$ is compact and connected, $[\bar{\rho}]=[\rho] \in H^{k}(G)$. The induced map on cohomology by ave is equal to

$$
\omega \longmapsto\left(\pi_{2}\right)_{*}\left(\pi_{1}^{*} \bar{\rho} \wedge \bar{\alpha}^{*} \pi_{2}^{*} \omega\right) .
$$

Since $\bar{\rho}$ is supported in $U$, this is equal to the map induced on cohomology by


Note the map $i_{e}: X \rightarrow U \times X$ given by $x \mapsto(e, x)$ is a homotopy equivalence. Thus we can verify $\left.\bar{\alpha}\right|_{U \times X} ^{*} \circ \pi_{2}^{*}=\pi_{2}^{*}$ by composing with $i_{e}^{*}$ :

$$
\left.i_{e}^{*} \circ \bar{\alpha}\right|_{U \times X} ^{*} \circ \pi_{2}^{*}=i_{e}^{*} \circ \pi_{2}^{*}
$$

because the diagram

commutes.
Thus map is given by

$$
\omega \longmapsto\left(\pi_{2}\right)_{*}\left(\pi_{1}^{*} \rho \wedge \pi_{2}^{*} \omega\right)=\left(\pi_{2}\right)_{*}\left(\pi_{1}^{*} \rho\right) \wedge \omega=\omega,
$$

by the same argument as in Lemma 24.3.2.
This completes the proof of Theorem 24.1.8.

### 24.4 Problems

Problem 24.4.1 (Stokes' theorem for fiber integration). Adapt the proof of Stokes' theorem in local coordinates to prove that

$$
d p_{*} \omega=p_{*} d \omega \pm\left.\left(\partial p_{*}\right) \omega\right|_{\partial E} .
$$

Problem 24.4.2 (Compact Lie groups acting on manifolds). Suppose that $G$ is finite group acting on a cochain complex $C^{*}$ of $\mathbb{R}$-vector spaces (so it acts on each $C^{i}$ by linear maps and the differential $C^{i} \rightarrow C^{i+1}$ is equivariant).
(a) Prove that $G$ acts on $H^{*}\left(C^{*}\right)$.
(b) Prove that the invariants $\left(C^{i}\right)^{G}$ assemble to a cochain complex $\left(C^{*}\right)^{G}$.
(c) Prove that $H^{*}\left(C^{*}\right)^{G} \cong H^{*}\left(\left(C^{*}\right)^{G}\right)$.

Now suppose that $H$ is a compact Lie group acting smoothly on $X$, with identity component $H_{0}$. We saw that the inclusion $\Omega^{*}(X)^{H_{0}} \rightarrow \Omega^{*}(X)$ induces an isomorphism on cohomology.
(d) Prove that the action of $H$ on $\Omega^{*}(X)^{H_{0}}$ and $H^{*}(X)$ factors over the group of path-components $\pi_{0}(H)$.
(e) Prove that $H^{*}\left(\Omega^{*}(X)^{H}\right) \cong H^{*}(X)^{\pi_{0}(H)}$.

Problem 24.4.3 (Manifolds with transitive actions). Suppose that a compact connected Lie group $G$ acts smoothly and transitively on a manifold $X$ of dimension $k$. Prove that $\operatorname{dim} H^{p}(X) \leq\binom{ k}{p}$.

## Chapter 25

## Flows along vector fields

Even though we are now familiar with de Rham cohomology, a nagging question remains: what is its geometric significance? For the remainder of these notes, our goal is to connect Morse theory to de Rham cohomology. Today we start the technical preparations. This material can be found in Section 1.4 of [Wal16].

### 25.1 Flows along vector fields

When we do Morse theory on a manifold $M$ in the next chapter, we will deform subsets of $M$ by flowing them along the gradient vector field of a Morse function $f: M \rightarrow \mathbb{R}$ (to define the gradient we will need to pick a Riemannian metric). Thus, we have to define flows along vector fields on manifolds. As usual, we take a known result on open subsets of $\mathbb{R}^{k}$ and extend it to $k$-dimensional manifolds using charts.

### 25.1.1 Flows on $\mathbb{R}^{k}$

The result we use is the existence and uniqueness theorem for solutions to ordinary differential equations, cf. [Wal16, Theorem 1.4.1]:

Theorem 25.1.1. Let $U \subset \mathbb{R}^{k}$ be open, $K \subset U$ be compact, and $\mathcal{X}$ a smooth vector field on $U$. Then there exists an $\epsilon>0$, an open neighborhood $U^{\prime} \subset U$ of $K$, and a unique smooth map $g: U^{\prime} \times(-\epsilon, \epsilon) \rightarrow U$ such that

$$
\frac{d}{d t} g(x, t)=\mathcal{X}(g(x, t)) \quad \text { and } \quad g(x, 0)=x .
$$

Let us restate this using the following notion:
Definition 25.1.2. An integral curve for $\mathcal{X}$ through $x$, is a smooth map $\gamma:(-\epsilon, \epsilon) \rightarrow U$ such that $\gamma(0)=x$ and $\frac{d}{d t} \gamma(t)=\mathcal{X}(\gamma(t))$.

Theorem 25.1.1 says that integral curves exist, are unique, and depend smoothly on the initial condition. For $t \in(-\epsilon, \epsilon)$, let us denote by $\psi_{t}$ the map $x \mapsto g(x, t)$. This is called a flow, because it has the following properties:

Proposition 25.1.3. $\psi_{0}=\mathrm{id}$ and $\psi_{t}\left(\psi_{s}(x)\right)=\psi_{s+t}(x)$ whenever both are defined.

Proof. The first property is clear. The second property uses that $g$ is unique. The map $t \mapsto g(x, s+t)$ at $t=0$ is equal to $g(x, s)$ and has derivative $\frac{d}{d t} g(x, s+t)=\mathcal{X}(g(x, s+t))$. That is, it has the properties uniquely defining $g\left(x^{\prime}, t\right)$ with $x^{\prime}=g(x, s)$. Thus we see that

$$
\psi_{s+t}(x)=g(x, s+t)=g(g(x, s), t)=\psi_{t}\left(\psi_{s}(x)\right)
$$

You can recover $\mathcal{X}$ from the flow as the derivative of $\psi_{t}$ with respect to $t$ at $t=0$.

### 25.1.2 Flows on manifolds

To extend these results to manifolds, we study the behavior of solutions to ordinary differential equations under diffeomorphisms. Given a diffeomorphism $\phi: \mathbb{R}^{k} \supset U \rightarrow$ $V \subset \mathbb{R}^{k}$, we can push forward $\mathcal{X}$ along $\phi$ to get a vector field $\phi_{*} \mathcal{X}$ on $V$. In fact, the pushforward of vector fields is defined on arbitrary manifolds, and is given by using the applying total derivative of the diffeomorphism to the vector field:

Definition 25.1.4. If $\varphi: M \rightarrow N$ is a diffeomorphism and $\mathcal{X}$ is a vector field on $M$, then the pushforward of $\mathcal{X}$ along $\varphi$ is given by

$$
\varphi_{*} \mathcal{X}(p):=d_{\varphi^{-1}(p)} \varphi\left[\mathcal{X}\left(\varphi^{-1}(p)\right)\right] .
$$

For open subsets of $\mathbb{R}^{k}$ the derivative is given by total derivative and we write

$$
\phi_{*} \mathcal{X}=D_{\phi^{-1}(x)} \phi\left[\mathcal{X}\left(\phi^{-1}(x)\right)\right] .
$$

On the one hand we can apply Theorem 25.1.1 to $\phi_{*} \mathcal{X}$ on $V$ using the compact subset $K^{\prime}:=\phi(K)$. The result is a solution $g^{\prime}: U^{\prime} \times\left(-\epsilon^{\prime}, \epsilon^{\prime}\right) \rightarrow V$ to the differential equation

$$
\begin{equation*}
\frac{d}{d t} g^{\prime}\left(x^{\prime}, t\right)=\phi_{*} \mathcal{X}\left(g^{\prime}\left(x^{\prime}, t\right)\right) \quad \text { and } \quad g^{\prime}\left(x^{\prime}, 0\right)=x^{\prime} \tag{25.1}
\end{equation*}
$$

On the other hand we can transport the solution $g$ to $V$ using $\phi$ :

$$
\begin{aligned}
g^{\prime \prime}: \phi\left(U^{\prime}\right) \times(-\epsilon, \epsilon) & \longrightarrow V \\
\left(x^{\prime}, t\right) & \longmapsto \phi\left(g\left(\phi^{-1}\left(x^{\prime}\right), t\right)\right) .
\end{aligned}
$$

I claim that this is a solution to (25.1). To prove this, observe it satisfies

$$
g^{\prime \prime}\left(x^{\prime}, 0\right)=\phi\left(g\left(\phi^{-1}\left(x^{\prime}\right), 0\right)\right)=\phi\left(\phi^{-1}\left(x^{\prime}\right)\right)=x^{\prime}
$$

and that we can use the chain rule to deduce that

$$
\begin{aligned}
\frac{d}{d t} g^{\prime \prime}\left(x^{\prime}, t\right) & =D_{g\left(\phi^{-1}\left(x^{\prime}\right), t\right)} \phi\left[\frac{d}{d t} g\left(\phi^{-1}\left(x^{\prime}\right), t\right)\right] \\
& =D_{\phi^{-1}\left(g^{\prime \prime}\left(x^{\prime}, t\right)\right)} \phi\left[\mathcal{X}\left(g\left(\phi^{-1}\left(x^{\prime}\right), t\right)\right)\right] \\
& =\left(\phi_{*} \mathcal{X}\right)\left(g^{\prime \prime}\left(x^{\prime}, t\right)\right)
\end{aligned}
$$

By uniqueness, any other solution of (25.1) has to coincide on $g^{\prime \prime}(x, t)$ on the intersection of their domain of definition: hence $g^{\prime \prime}=g^{\prime}$ on $U^{\prime} \times\left(-\min \left(\epsilon, \epsilon^{\prime}\right), \min \left(\epsilon, \epsilon^{\prime}\right)\right)$.

We will use this result to extend the technique of flowing along vector fields to manifolds.

Theorem 25.1.5. Let $M$ be a manifold and $\mathcal{X}$ be a vector field on $M$. Then there exists a smooth map $\eta: M \rightarrow \mathbb{R}_{>0}$ and a unique smooth map $g:\{(p, t) \in M \times \mathbb{R}| | t \mid<\eta(p)\} \rightarrow M$ such that

$$
\begin{equation*}
\frac{d}{d t} g(p, t)=\mathcal{X}(g(p, t)) \quad \text { and } \quad g(p, 0)=p \tag{25.2}
\end{equation*}
$$

Proof. We can find a collection of charts $\phi_{\alpha}: \mathbb{R}^{k} \supset U_{\alpha} \rightarrow V_{\alpha} \subset M$ and compact subsets $K_{\alpha} \subset V_{\alpha}$ such that the $K_{\alpha}$ cover $M$.

Every point $p \in M$ lies in some compact subset $K_{\alpha}$ of $M$. We can push forward the restriction $\left.\mathcal{X}\right|_{V_{\alpha}}$ to $U_{\alpha}$ along $\phi_{\alpha}^{-1}$ and apply Theorem 25.1.1 to the resulting vector field $\left(\phi_{\alpha}^{-1}\right)_{*} \mathcal{X}$. This gives us a smooth map $\tilde{g}_{\alpha}: U_{\alpha}^{\prime} \times\left(-\epsilon_{\alpha}, \epsilon_{\alpha}\right) \rightarrow U_{\alpha}$ with $U_{\alpha}^{\prime}$ an open neighborhood of $\phi_{\alpha}^{-1}\left(K_{\alpha}\right)$. As above, we get a solution $g$ to (25.2) on an open neighborhood of $K_{\alpha} \times\left(-\epsilon_{\alpha}, \epsilon_{\alpha}\right)$, by setting its value on $(p, t) \in K_{\alpha} \times\left(-\epsilon_{\alpha}, \epsilon_{\alpha}\right)$ to be

$$
g(p, t):=\phi_{\alpha}\left(\tilde{g}_{\alpha}\left(\phi_{\alpha}^{-1}(p), t\right)\right)
$$

We must check that combining these local solutions to (25.2) give rise to a well-defined smooth map. That is, if $p \in K_{\alpha} \cap K_{\beta}$, then we should have

$$
\phi_{\alpha}\left(\tilde{g}_{\alpha}\left(\phi_{\alpha}^{-1}(p), t\right)\right)=\phi_{\beta}^{\prime}\left(\tilde{g}_{\beta}\left(\left(\phi_{\beta}^{\prime}\right)^{-1}(p), t\right)\right)
$$

as long as $t$ is small enough so that both are defined. This is guaranteed by the previous discussion applied to the diffeomorphism $\left(\phi_{\beta}\right)^{-1} \circ \phi_{\alpha}: \phi_{\alpha}^{-1}\left(V_{\alpha} \cap V_{\beta}\right) \rightarrow\left(\phi_{\beta}\right)^{-1}\left(V_{\alpha} \cap V_{\beta}\right)$; pushing forward the vector field $\left(\phi_{\alpha}^{-1}\right)_{*} \mathcal{X}$ along this diffeomorphism gives $\left(\phi_{\beta}^{-1}\right)_{*} \mathcal{X}$.

The result is a solution to (25.2) defined on an open neighborhood $V$ of $M \times\{0\}$ in $M \times \mathbb{R}$ given by $\bigcup_{\alpha} K_{\alpha} \times\left(-\epsilon_{\alpha}, \epsilon_{\alpha}\right)$. Such an open subset always contains one of the type mentioned in the theorem.

Remark 25.1.6. This proof is one of the places where it is important that manifolds are Hausdorff: on the line with doubled origin the flow along $\frac{\partial}{\partial x}$ exists but is not unique (you have to decide which of the origins to go into). This Hausdorffness assumption is hidden in the proof: it is used to see that $K_{\alpha} \cap K_{\beta}$ is compact.

As in the local case, we can extract a flow out of $g$ by setting $\psi_{t}(p)=g(p, t)$ for $(p, t) \in V$. This satisfies $\psi_{0}(p)=p$ and $\psi_{t}\left(\psi_{s}(p)\right)=\psi_{s+t}(p)$ as long as both are defined, and one can recover $\mathcal{X}$ from the flow by taking the derivative of $\psi_{t}$ with respect to $t$ at $t=0$.

What can we say about the domain of definition? By uniqueness any two solutions to (25.2) agree on the overlap of their domains of definition, so by combining these we can extend the domain. In particular, there is a solution with maximal domain of definition. However, even when domain of definition is maximal, $t \mapsto g(p, t)$ might still only be defined on some proper open interval $\left(a_{p}, b_{p}\right) \subset \mathbb{R}$ with $a_{p}<0$ and $b_{p}>0$ :
Example 25.1.7. Let $M=\mathbb{R} \backslash 0$ and $\mathcal{X}=\frac{\partial}{\partial x}$. Then the maximal domain of definition of $g$ is given by those $(x, t) \in \mathbb{R} \times \mathbb{R}$ such that $t>x$ if $x<0$, and $t<x$ if $x>0$.

However, this can only occur if the integral curve through $p$ leaves all compact subsets of $M$ eventually. Lemma 1.4.3 of [Wal16] says:

Lemma 25.1.8. Suppose $g$ has maximal domain and fix $p \in M$. Either $b_{p}=\infty$ or the map $g(p,-):\left[0, b_{p}\right) \rightarrow M$ is proper. Similarly, either $a_{p}=-\infty$ or the map $g(p,-):\left(a_{p}, 0\right] \rightarrow M$ is proper.

Corollary 25.1.9. Suppose $M$ is compact. If a solution to (25.2) has maximal domain then its domain is $M \times \mathbb{R}$.

Proof. As $M$ is compact, no map $\left[0, b_{p}\right) \rightarrow M$ or $\left(a_{p}, 0\right] \rightarrow M$ is proper.
Remark 25.1.10. When $M$ is compact, this corollary implies there is a one-to-one correspondence between 1-parameter groups of diffeomorphisms and smooth vector fields.

Remark 25.1.11. There are of course other conditions under which the maximal domain is all of $M \times \mathbb{R}$, e.g. if $\mathcal{X}$ is compactly-supported or more generally, if $\mathcal{X}$ coincides outside of a compact subset with a vector field $\mathcal{Y}$ whose maximal domain is $M \times \mathbb{R}$.

### 25.2 Isotopy extension

We will now give the first of several important applications of flows along vector fields, a very important geometric tool called isotopy extension.

### 25.2.1 The isotopy extension theorem

It is based on the following idea: if you imagine your manifold $M$ as being made from a stretchy fabric, then you can use your finger to move one point $p \in M$ to some other point $p^{\prime} \in M$ and deform the rest of the manifold along to produce a diffeomorphism $M \rightarrow M$ which moves $p$ to $p^{\prime}$.

In other words, imagining $M$ as being made out of a stretchy fabric suggests that any isotopy of embeddings $* \rightarrow M$ (starting at the map with value $p$ and ending at the map with value $p^{\prime}$ ) can be extended to an isotopy of diffeomorphisms $M \rightarrow M$. An isotopy of diffeomorphisms is also called an ambient isotopy, suggesting the following interpretation: you do not just move the objects in question but also their surrounding environment.

The isotopy extension theorem says that an isotopy extends to an ambient isotopy under mild assumptions.

Theorem 25.2.1 (Isotopy extension). Suppose that $M$ and $X$ smooth manifolds without boundary, and that $X$ is compact. Then any isotopy of embeddings $e_{t}: X \times[0,1] \rightarrow M$ can be extended to an isotopy of diffeomorphisms, in the following sense: there exists a family of diffeomorphisms $\phi_{t}: M \times[0,1] \rightarrow M$ satisfying $\phi_{0}=\operatorname{id}$ and $\phi_{t}\left(e_{0}\right)=e_{t}$. Furthermore, each $\phi_{t}$ will be compactly-supported (that is, equal to the identity outside a compact subset).

Proof. Here $M$ is $k$-dimensional and $X$ is $\ell$-dimensional. Let us define $e: X \times[0,1] \rightarrow$ $M \times \mathbb{R}$ by $e(p, t)=e_{t}(p)$. The smooth vector field on $X \times[0,1]$ given by $\frac{\partial}{\partial t}$ can be pushed forward along the embedding $e$ to obtain a vector field $\mathcal{X}$ on $e(X \times[0,1]) \subset M \times[0,1]$. Suppose we could extend this to a vector field $\mathcal{X}^{\prime}$ on all of $M \times \mathbb{R}$. Then I claim that if


Figure 25.1 The end result of pushing the origin to the red point, depicted by its effect on vertical lines in $\mathbb{R}^{2}$. The dashed line gives the boundary of the support.
we flow along $\mathcal{X}^{\prime}$ for time $t$ with initial condition $\left(e_{0}(p), 0\right)$, we end up at $\left(e_{t}(p), t\right)$. To see this, we must prove that

$$
t \longmapsto\left(e_{t}(p), t\right)
$$

is an integral curve for $\mathcal{X}^{\prime}$. To see this, takes its derivative with respect to $t$ and apply the chain rule

$$
\frac{d}{d t}\left(e_{t}(p), t\right)=d_{(p, t)} e\left[\frac{d}{d t}(t \mapsto(p, t))\right]=e_{*}\left[\frac{\partial}{\partial t}(p, t)\right]=\mathcal{X}^{\prime}(e(p, t))
$$

In other words, flowing $e_{0}$ with image in $M \times\{0\}$ along $\mathcal{X}^{\prime}$ for time $t$ produces $e_{t}$ wth image in $M \times\{t\}$. We can try to produce $\phi_{t}$ by flowing the identity map of $M \times\{0\}$ along $\mathcal{X}$ for time $t$. There are two problems:
(i) the flow may not exist,
(ii) it is not necessarily the case that the flow sends $M \times\{0\}$ to $M \times\{t\}$.

Problem (ii) is solved by extending $\mathcal{X}$ not just to any smooth vector field $\mathcal{X}^{\prime}$ on $M \times \mathbb{R}$, but one that projects to $\frac{\partial}{\partial t}$ under $d \pi$ for $\pi: M \times \mathbb{R} \rightarrow \mathbb{R}$. If so, we get the differential equation

$$
\frac{d}{d t}\left(\pi \circ \psi_{t}(p, s)\right)=d \pi \circ \mathcal{X}^{\prime}\left(\psi_{t}(p, s)\right)=\frac{\partial}{\partial t},
$$

and the initial condition $\pi \circ \psi_{0}(p, s)=s$ guarantees that $\pi \circ \psi_{t}(p, s)=s+t$.
If we make sure that $\mathcal{X}^{\prime}$ is equal to $\frac{\partial}{\partial t}$ outside of a compact set, this will solve problem (i). It guarantees that the flow exists, because $\mathcal{X}^{\prime}$ coincides outside of a compact set with a vector field whose maximal domain of solution is all of $M \times \mathbb{R} \times \mathbb{R}$.

Having imposed theses conditions, we can thus prove the theorem by taking

$$
\begin{aligned}
\phi: M \times[0,1] & \longrightarrow M \times[0,1] \\
(p, t) & \longmapsto \psi_{t}(p, 0),
\end{aligned}
$$

or in other words, $\phi_{t}=\psi_{t}(p, 0)$.
So it remains to construct an extension $\mathcal{X}^{\prime}$ with the desired properties. Firstly, it suffices to construct a smooth vector field $\mathcal{X}^{\prime}$ which
(a) coincides with $\mathcal{X}$ on $e(X \times[0,1])$,
(b) coincides with $\frac{\partial}{\partial t}$ outside a compact subset of $M \times \mathbb{R}$,
(c) satisfies the property $d \pi \circ \mathcal{X}^{\prime}$ is a positive multiple of $\frac{\partial}{\partial t}$ everywhere.

We may then afterwards modify $\mathcal{X}^{\prime}$ by scaling it with smooth function that is 1 on $X \times \mathbb{R}$, to get that $d \pi \circ \mathcal{X}^{\prime}=\frac{\partial}{\partial t}$ on all of $M \times \mathbb{R}$.

Since $X$ is compact, we may find a finite collection of charts $\phi_{i}: \mathbb{R}^{k+1} \supset U_{i} \rightarrow V_{i} \subset$ $M \times \mathbb{R}$ covering the image of $e$, satisfying $\phi_{i}^{-1}\left(V_{i} \cap e(X \times[0,1])\right)=U_{i} \cap\left(\mathbb{R}^{\ell-1} \times[0, \infty) \times\{0\}\right)$ and $\pi_{k+1} \circ \phi_{i}=\pi_{k+1}$. Let $\mathcal{X}_{i}^{\prime}$ be the vector field on $V_{i}$ given as follows:
Step (i): first extend $\left(\phi_{i}\right)_{*}\left(\frac{\partial}{\partial t}\right)$ on $U_{i} \cap\left(\mathbb{R}^{\ell-1} \times[0, \infty) \times\{0\}\right)$ to $U_{i} \cap\left(\mathbb{R}^{\ell} \times\{0\}\right)$,
Step (ii): then extend it in constant manner to the remaining $(k-m)$ coordinate directions of $U_{i}$,
Step (iii): apply $\left(\phi_{i}^{-1}\right)_{*}$.
This extends $\left.\mathcal{X}\right|_{V_{i}}$, so in particular has the property that $d \pi \circ \mathcal{X}_{i}^{\prime} \frac{\partial}{\partial t}$ on $V_{i} \cap e(X \times[0,1])$. Hence by possibly shrinking $V_{i}$ to a smaller open neighborhood of $V_{i} \cap e(X \times[0,1])$, we may assume that $\pi_{*}\left(\mathcal{X}_{i}^{\prime}\right)$ is a positive multiple of $\frac{\partial}{\partial t}$.

Let $V_{0}$ be an open subset of $M \times \mathbb{R}$ satisfying $V_{0} \cap e(X \times[0,1])=\varnothing$ and $V_{0} \cup \bigcup_{i=1}^{k} V_{i}=$ $M \times \mathbb{R}$, and let $\eta_{i}$ be smooth partition of unity subordinate to this open cover. The desired vector field is

$$
\mathcal{X}^{\prime}:=\eta_{0} \cdot \frac{\partial}{\partial t}+\sum_{i=1}^{k} \eta_{i} \cdot \mathcal{X}_{i}^{\prime}
$$

By construction this extends $\mathcal{X}$ and the condition that $d \pi \circ \mathcal{X}^{\prime}$ is a multiple of $\frac{\partial}{\partial t}$ by a positive smooth function is preserved by taking convex linear combinations such as those that appear when using partitions of unity.

### 25.2.2 Transitivity of diffeomorphisms

We have previously asserted that there exists a diffeomorphism of $\mathbb{R}^{k}$ mapping the origin to any specified point $x \in \mathbb{R}^{k}$. Let us use isotopy extension to generalize this to all connected manifolds:

Corollary 25.2.2. Suppose that $M$ is a $k$-dimensional connected manifold and $p, p^{\prime} \in M$, then there exists a compactly-supported diffeomorphism $\varphi: M \rightarrow M$ which is isotopic to the identity such that $\varphi(p)=\varphi\left(p^{\prime}\right)$.


An embedding of $\mathbb{R}$ into $\mathbb{R}^{3}$ given by knot $X$ centered at the origin for $t=0$ moving rightwards to $\infty$ as $t$ increases.

Figure 25.2 A family of embeddings to which isotopy extension does not apply, because at $t=0$ and $t=\infty$ the complements are not diffeomorphic. It does not satisfy the assumption that $X$ is compact.

Proof. Since $M$ is connected, there exists a path $\gamma$ from $p$ to $p^{\prime}$. Defining

$$
\begin{aligned}
e: * \times[0,1] & \longrightarrow M \\
(*, t) & \longmapsto \gamma(t),
\end{aligned}
$$

this can be interpreted as an isotopy of embeddings from $e_{0}$ to $e_{1}$. Applying the isotopy extension theorem to $e$, we find an isotopy $\phi_{t}: M \times[0,1] \rightarrow M$ such that $\phi_{0}=\mathrm{id}$ and $\phi_{t} \circ e_{0}=e_{t}$. Then $\phi_{1}$ is the desired diffeomorphism.

Example 25.2.3. Engulfing theorems give an answer to the following question:
If $X \subset M$ is a submanifold and $U \subset M$ is an open subset, under which
conditions does there exist a diffeomorphism $h: M \rightarrow M$ so that $X \subset h(U)$ ?
We will answer this question when $M$ is path-connected of dimension $k \geq 2, X$ is a finite set of points, and $U$ is an open ball (i.e. diffeomorphic to $\mathbb{R}^{k}$ ).

Recall that the configuration space of $r$ ordered points in $M$ is given by

$$
\operatorname{Conf}_{r}(M):=\left\{\left(m_{1}, \ldots, m_{r}\right) \mid m_{i} \neq m_{j} \text { if } i \neq j\right\} \subset M^{r}
$$

In other words, it is the complement in $M^{r}$ of the thick diagonal $\Delta=\left\{\left(m_{1}, \ldots, m_{r}\right) \mid\right.$ $m_{i}=m_{j}$ for some $\left.i \neq j\right\}$. In Problem 13.4.6 we proved that if $M$ is path-connected of dimension $\geq 2$ then so is $\operatorname{Conf}_{r}(M)$.

Now suppose we are given a collection of distinct points $p_{1}, \ldots, p_{r}$ in $M$. This gives a configuration $p \in \operatorname{Conf}_{r}(M)$. We can connect this by a smooth path $\gamma$ from $p$ to $q \in \operatorname{Conf}_{r}(U)$. Consider $\gamma$ as an isotopy of embeddings $\{1, \ldots, r\} \hookrightarrow M$, we can apply isotopy extension to find a compactly-supported diffeomorphism $h$ isotopic to the identity such that $\left\{p_{1}, \ldots, p_{r}\right\} \subset h(U)$. In particular, this implies that the diffeomorphisms of $M$ act $r$-transitively for all $r \geq 2$.

### 25.2.3 Knot complements

It follows from Corollary 25.2 .2 that $M \backslash p$ and $M \backslash p^{\prime}$ are diffeomorphic; the restriction of $\varphi$ gives this diffeomorphism. This can be generalized as follows.

Recall that a knot is an embedding $e: S^{1} \rightarrow \mathbb{R}^{3}$ up to isotopy. One might think of trying to distinguish a knot by its complement $\mathbb{R}^{3} \backslash e\left(S^{1}\right)$. However, it is not obviously
clear this is well-defined, because its diffeomorphism type may depend on the choice of $e$ within the isotopy class. However, the isotopy extension theorem tells us that for any two representatives $e, e^{\prime}: S^{1} \rightarrow \mathbb{R}^{3}$ of a knot, there exists a diffeomorphism $\varphi: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ such that $\varphi \circ e=e^{\prime}$. This restrict to a diffeomorphism

$$
\left.\varphi\right|_{\mathbb{R}^{3} \backslash e\left(S^{1}\right)}: \mathbb{R}^{3} \backslash e\left(S^{1}\right) \longrightarrow \mathbb{R}^{3} \backslash e^{\prime}\left(S^{1}\right)
$$

Thus we conclude that the complements are diffeomorphic.

### 25.3 The Ehresmann fibration theorem

In Section 24.2.1 we discussed manifold bundles.
Theorem 25.3.1 (Ehresmann fibration theorem). Suppose that $X$ is connected, then a surjective proper submersion $p: E \rightarrow X$ is a manifold bundle.

Proof. It remains to check that $p: E \rightarrow X$ is locally trivial. That is, we need to find for each point $x \in X$ a local trivialization: an open neighborhood $U$ of $x$ and a commutative diagram

with horizontal maps diffeomorphisms. By restricting to a chart in $X$, we thus may assume without loss of generality that $X=\mathbb{R}^{k}$ and $x$ is the origin.

By induction over $k$, it suffices to prove that a proper submersion $p: E \rightarrow \mathbb{R}^{k}$ whose restriction to $\mathbb{R}^{k-1} \times\{0\}$ has a local trivialization near the origin, has a local trivialization near the origin. To do so, it suffices to find an open neighborhood $U \subset \mathbb{R}^{k-1}$ of the origin, an $\epsilon>0$, and a commutative diagram

with horizontal maps diffeomorphisms.
To do so, we use a vector field $\mathcal{X}$ on $E$ such that $d p \circ \mathcal{X}=\frac{\partial}{\partial x_{k}}$. Such a vector field can be constructed locally in $E$ using charts provided by the submersion theorem, Lemma 6.3.3, and these can be combined using a partition of unity as in the proof of Theorem 25.2.1. Now we apply Theorem 25.1 .5 to $\mathcal{X}$; the open subset $\{(p, t) \in M \times \mathbb{R}| | t \mid<\eta(p)\}$ contains an open subset of the form $U \times(-\epsilon, \epsilon)$ and the map $G$ is given by

$$
\begin{aligned}
p^{-1}(U) \times(-\epsilon, \epsilon) & \longrightarrow p^{-1}(U \times(-\epsilon, \epsilon)) \\
(p, t) & \longmapsto g(p, t) .
\end{aligned}
$$

As before, the uniqueness clause for solutions of ODE's guarantees that this is a diffeomorphism and the fact that $d p \circ \mathcal{X}=\frac{\partial}{\partial x_{k}}$ guarantees that its composition with $p$ is ( $p \times \mathrm{id}$ ) .

Remark 25.3.2. The assumption that $X$ is connected is only used to guarantee that the fibers of $p$ are all diffeomorphic.

Using Theorem 7.3.1, this implies the following:
Corollary 25.3.3. If a compact Lie group $G$ acts freely and smoothly on $M$, then the quotient map $M \rightarrow M / G$ is a manifold bundle with fibers diffeomorphic to $G$.

### 25.4 Problems

Problem 25.4.1. Let $G$ be a compact connected Lie group.
(a) Show that there is an isomorphism between the tangent space $T_{e} G$ and the vector space left-invariant vector fields on $G$.
(b) For $X \in T_{e} G$, let $\varphi_{t}^{X}$ be the flow generated by the left-invariant vector field corresponding to $X$. Prove that its maximal domain is $\mathbb{R}$.

Problem 25.4.2 (Diffeomorphisms of the 2-disk). In this problem we give the argument that every diffeomorphism $f$ on $\mathbb{R}^{2}$ that is the identity outside a compact subset of $\operatorname{int}\left([0,1]^{2}\right)$ is isotopic to the identity. This is a famous result of Smale [Sma59].
(a) Prove that every diffeomorphism $g$ of $\mathbb{R}$ that is the identity outside a compact subset of $\operatorname{int}([0,1])$ is isotopic to the identity. (Hint: it is just a strictly increasing function.)
Consider the vector field $\frac{\partial}{\partial x}$ on $\mathbb{R}^{2}$. We can push it forward along $f$ to get a vector field $\mathcal{X}=f_{*} \frac{\partial}{\partial x}$.
(b) Prove that $\mathcal{X}$ is $\frac{\partial}{\partial x}$ outside a compact subset of $\operatorname{int}([0,1])$. Conclude that it has a flow $\phi_{t}^{\mathcal{X}}$ with maximal domain $\mathbb{R}^{2} \times \mathbb{R}$.
(c) Show we can recover $f$ within the square $[0,1]^{2}$ from this as follows: for $(x, t) \in$ $[0,1]^{2}$ we have

$$
f(t, y)=\phi_{t}^{\mathcal{X}}(0, y) .
$$

We say a smooth vector field $\mathcal{Y}$ on $\mathbb{R}^{2}$ is nice if

- it is equal to $\frac{\partial}{\partial x}$ outside a compact subset of $\operatorname{int}\left([0,1]^{2}\right)$, and
- it is everywhere non-zero.

The Poincaré-Bendixson theorem in dynamics implies that a nice vector field $\mathcal{Y}$ has a flow with the following property: there exists a smooth function $\tau:[0,1]^{2} \rightarrow \mathbb{R} \geq 0$ such that $\phi_{\tau(x, y)}^{\mathcal{Y}}(x, y) \in\{1\} \times[0,1]$ for any $(x, y) \in \operatorname{int}\left([0,1]^{2}\right)$. You may use this without proof.
(d) Use uniqueness of flows to prove that for nice $\mathcal{Y}$, the map

$$
\begin{aligned}
{[0,1]^{2} } & \longrightarrow[0,1]^{2} \\
(t, y) & \longmapsto \phi_{t \cdot \tau(0, y)}^{\mathcal{Y}}(0, y)
\end{aligned}
$$

is a bijection which is the identity on $\{0\} \times[0,1] \cup[0,1] \times\{0,1\}$.

This map is however not the identity on the side $\{1\} \times[0,1]$ of the square. This can be fixed by invoking part (a). If we do so, we obtain that every smooth path of nice vector fields $\mathcal{Y}$ gives an isotopy of diffeomorphisms of $\mathbb{R}^{2}$ that are the identity outside a compact subset of $\operatorname{int}\left([0,1]^{2}\right)$. You may use this without proof.
(e) Any nice vector field $\mathcal{Y}$ can be considered as a smooth map $Y: \mathbb{R}^{2} \rightarrow \mathbb{C} \backslash\{0\}$ that takes the value 1 outside a compact subset of $\operatorname{int}([0,1])$. You may use without proof that any such $Y$ is of the form $\exp (\bar{Y})$ with $\bar{Y}: \mathbb{R}^{2} \rightarrow\{z \mid \operatorname{Re}(z)>0\} \subset \mathbb{C}$ smooth and taking the 0 outside a compact subset of $\operatorname{int}\left([0,1]^{2}\right)$. Prove that $\mathcal{X}$ is homotopic to $\frac{\partial}{\partial x}$ through nice vector fields $\mathcal{Y}$.
(f) Conclude that $f$ is isotopic to the identity through diffeomorphisms that are the identity outside a compact subset of $\operatorname{int}\left([0,1]^{2}\right)$.


Figure 25.3 The red line is the flow-line along $\mathcal{Y}$ starting at $(0, y)$, and the $\tau(y)$ is the first and only time where this flowline hits the side $\{1\} \times[0,1]$ of $[0,1]^{2}$.

Remark 25.4.3. The result proved here is equivalent to the statement that the topological group $\operatorname{Diff}_{\partial}\left(D^{2}\right)$ is path-connected. Smale proved it is in fact contractible in [Sma59]. For other dimensions, $\operatorname{Diff}_{\partial}\left(D^{d}\right)$ path-connected when $d \leq 3$ and not path-connected in general for $d \geq 5$. The case $d=4$ is an open problem.

Problem 25.4.4 (On Ehresmann's fibration theorem). Give a counterexample to the statement that any surjective submersion $p: E \rightarrow X$ with connected $X$ is a manifold bundle. (That is, we drop the assumption that $p$ is proper from Ehresmann's fibration theorem.)

Problem 25.4.5 (Examples of manifold bundles).
(a) Suppose that $G$ is a compact Lie group which acts smoothly and freely on a smooth manifold $M$ and $H \subset G$ is a closed Lie subgroup. Prove that there is a
unique smooth map $p$ which fits in a commutative diagram

with $q_{G}$ and $q_{H}$ the quotient maps.
(b) Prove that $p$ is a surjective submersion. Use Ehresmann's fibration theorem to conclude it is a manifold bundle with fibers $G / H$.

Problem 25.4.6 (Cohomology of certain sphere bundles).
(a) Prove that if $\mathrm{O}(n)$ acts acts smoothly and freely on a smooth manifold $M$, then $M / \mathrm{O}(n-1) \rightarrow M / \mathrm{O}(n)$ is a smooth manifold bundle with fiber $S^{n-1}$.
(b) Use Problem 23.4.5 to prove that if $M$ is also assumed compact, then $H^{*}(M / \mathrm{O}(n)) \rightarrow$ $H^{*}(M / \mathrm{O}(n-1))$ is an isomorphism for $*<n-1$ and injective for $*=n-1$.

## Chapter 26

## First fundamental theorem of Morse theory

In this chapter, we discuss that part of Morse theory which does not involve critical points. We define Morse functions, prove they exist, and show that if $[a, b] \subset \mathbb{R}$ contains no critical values of $f$, then $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times[a, b]$. This can be found in Section 1.7 of [GP10], Section 5.1 of [Wal16], and Section 3 of [Mil63].

### 26.1 Morse functions

Recall that for a smooth function $f: M \rightarrow \mathbb{R}$, a point $p \in M$ so that $d_{p} f$ is not surjective is called a critical point. Given a critical point and local coordinates $\left(x_{1}, \ldots, x_{k}\right)$, one can define the Hessian. For simplicity, suppose that $p$ is the origin in these local coordinates, then we have a $(k \times k)$-matrix with $(i, j)$ th entry given by

$$
\operatorname{Hess}_{0}(f)_{i j}:=\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}(0) .
$$

Remark 26.1.1. By Taylor's theorem, in these local coordinates $f$ is near the origin given by

$$
f(x)=f(0)+\frac{1}{2} \sum_{i, j=1}^{k} \frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}(0) x_{i} x_{j}+O\left(x^{3}\right) .
$$

We say that $p$ is a non-degenerate critical point if the Hessian matrix as described above is invertible. Though the Hessian itself depends on a choice of coordinates, that it is invertible does not, by the following lemma which is an easy consequence of the chain rule:

Lemma 26.1.2. Suppose $\phi: \mathbb{R}^{k} \supset U \rightarrow U^{\prime} \subset \mathbb{R}^{k}$ is a diffeomorphism such that $\phi(0)=0$. Then the origin is a non-degenerate critical point $f: U^{\prime} \rightarrow \mathbb{R}$ if and only if it is a non-degenerate critical point of $f \circ \phi$.

Definition 26.1.3. A smooth function $f: M \rightarrow \mathbb{R}$ is a Morse function if all its critical points are non-degenerate.

Example 26.1.4. It follows from the expression in Remark 26.1.1 that non-degenerate critical points are isolated. In particular, a Morse function on a compact manifold only has finitely many critical points.

### 26.1.1 Existence of Morse functions

Morse functions are generic among smooth maps $f: M \rightarrow \mathbb{R}$, in the sense of Definition 11.3.1. This follows from the following theorem, which depends on a choice of embedding $e: M \hookrightarrow \mathbb{R}^{N}$ (this exists by Theorem 12.1.6). Let $e_{1}, \ldots, e_{N}: M \rightarrow \mathbb{R}$ denote the coordinates of $e$.

Theorem 26.1.5. For a dense set of $\left(a_{1}, \ldots, a_{N}\right) \in \mathbb{R}^{N}$, the smooth map

$$
\begin{aligned}
f_{a}: M & \longrightarrow \mathbb{R} \\
p & \longmapsto f(p)+a_{1} e_{1}(p)+\cdots+a_{N} e_{N}(p)
\end{aligned}
$$

is a Morse function.
Proof. We shall denote the map in the statement of the theorem as $f_{a}$.
We first consider the local situation; suppose $U \subset \mathbb{R}^{k}$ is an open subset and $g: U \rightarrow \mathbb{R}$ is a smooth function. Then we claim that for outside of a set of $b \in \mathbb{R}^{k}$ of measure zero, the map

$$
\begin{aligned}
g_{b}: U & \longrightarrow \mathbb{R} \\
\left(x_{1}, \ldots, x_{k}\right) & \longmapsto g\left(x_{1}, \ldots, x_{k}\right)+b_{1} x_{1}+\cdots+b_{k} x_{k}
\end{aligned}
$$

is a Morse function.
To do so, we start with the observation that $p$ is a critical point of $g_{b}$ if and only if $D_{p} g=-b$. Since we working on $\mathbb{R}^{k}$, the Hessian is also well-defined at points which are not critical point. Thus it makes sense to say that $g$ and $g_{b}$ have the same Hessians; this is true because $g_{a}$ is obtained by adding a linear perturbation to $g$. We next consider the function

$$
\begin{aligned}
G: U & \longrightarrow \mathbb{R}^{k} \\
\left(x_{1}, \ldots, x_{k}\right) & \longmapsto\left(\frac{\partial g}{\partial x_{1}}\left(x_{1}, \ldots, x_{k}\right), \ldots, \frac{\partial g}{\partial x_{k}}\left(x_{1}, \ldots, x_{k}\right)\right)
\end{aligned}
$$

because $-b$ is a critical value of $G$ if and only if the Hessian of $g$ (or equivalently $g_{b}$ ) at $p$ is non-degenerate. Thus $g_{b}$ is Morse if and only if $-b$ is a regular value of $G$. By Sard's theorem, Theorem 11.3.4, these critical values have measure zero.

Having established this local statement, we use a prove the global one. To do so, we use Problem 9.3.3 to find a countable open cover $\left\{U_{\alpha}\right\}$ of $M$ such that for each $U_{\alpha}$ there exist $k$ integers $i_{1}, \ldots, i_{k}$ in $\{1, \ldots, N\}$ such that coordinate functions $e_{i_{1}}, \ldots, e_{i_{k}}: U_{\alpha} \rightarrow \mathbb{R}$ give local coordinates on $U_{\alpha}$. We can use $e_{i_{1}}, \ldots, e_{i_{k}}$ as local coordinates $x_{i_{1}}, \ldots, x_{i_{k}}$ on $U_{\alpha}$. Then for values $c_{j_{k+1}}, \ldots, c_{j_{N}} \in \mathbb{R}$ corresponding to the complementary coordinates we consider the smooth function

$$
\begin{aligned}
f^{c}: U_{\alpha} & \longrightarrow \mathbb{R} \\
\left(x_{i_{1}}, \ldots, x_{i_{1}}\right) & \longmapsto f\left(x_{i_{1}}, \ldots, x_{i_{k}}\right)+c_{j_{k+1}} e_{j_{k+1}}\left(x_{i_{1}}, \ldots, x_{i_{k}}\right)+\cdots+c_{j_{N}} e_{j_{N}}\left(x_{i_{1}}, \ldots, x_{i_{k}}\right) .
\end{aligned}
$$

By the above local argument, the set of $b \in \mathbb{R}^{k}$ such that $\left(f^{c}\right)_{b}$ is not Morse, has measure zero. By Lemma 11.4.1 the set of $\left(b_{1}, \ldots, b_{k}, c_{k+1}, \ldots, c_{N}\right) \in \mathbb{R}^{N}$ such that $\left(f^{c}\right)_{b}$ is not Morse, also has measure zero.

Thus, for each each $\alpha$ there is a measure zero set of $a=(b, c) \in \mathbb{R}^{N}$ so that $f_{a}$ is not Morse on $U_{\alpha}$. Since a countable union of measure zero subset still has measure zero, there is a dense set of $a \in \mathbb{R}^{N}$ so that $f_{a}$ is Morse on all of $M$.

### 26.2 The first fundamental theorem of Morse theory

Let $M$ be a manifold without boundary and $f: M \rightarrow \mathbb{R}$ be a Morse function. We shall study $M$ by studying the (sub)level sets

$$
M_{\leq a}:=f^{-1}((-\infty, a]) \quad \text { and } \quad M_{a}:=f^{-1}(\{a\})
$$

By the submersion theorem, if $a$ is a regular value, $M_{\leq a} \subset M$ is a codimension zero submanifold with boundary $\partial M_{\leq a}=M_{a}$ given by a level set.

### 26.2.1 Gradients

Given a smooth function $f: \mathbb{R}^{k} \rightarrow \mathbb{R}$, its gradient is the vector field

$$
\nabla f=\left[\begin{array}{c}
\frac{\partial f}{\partial x_{1}} \\
\vdots \\
\frac{\partial f}{\partial x_{k}}
\end{array}\right]
$$

That is, the component in the direction of the standard basis vector $e_{i}$ is given by $\frac{\partial f}{\partial x_{i}}$. Using the standard Riemannian metric, we can identify each basis vector of $\mathbb{R}^{k}$ with a basis vector of its dual $\left(\mathbb{R}^{k}\right)^{*}: e_{i}$ corresponds to the linear functional $\left\langle e_{i},-\right\rangle$. In other words, the Riemannian metric provides an isomorphism of the tangent spaces to points in $\mathbb{R}^{k}$ with the corresponding cotangent spaces. From a vector field, a section of the tangent bundle, we thus get a 1 -form, a section of the cotangent bundle. In this particular case, the Riemannian metric sends $e_{i}$ to $d x_{i}$, and we see that $\nabla f$ gets sent to

$$
d f=\sum_{i=1}^{k} \frac{\partial f}{\partial x_{i}} d x_{i}
$$

This discussion extends to manifolds with a Riemannian metric $g$. This Riemannian metric is given by a smoothly varying non-degenerate bilinear form on the tangent space $T_{p}(M)$,

$$
T_{p}(M) \times T_{p}(M) \ni(v, w) \longmapsto g(v, w) \in \mathbb{R}
$$

and thus provides an isomorphism of vector bundles $T M \rightarrow T^{*} M$

$$
T_{p}(M) \in v \longmapsto g(v,-) \in T_{p}^{*}(M)
$$

In particular, it sends sections of $T M$ to sections of $T^{*} M$ and vice versa: every vector field corresponds to a unique 1-form.

Given a smooth function $f: M \rightarrow \mathbb{R}$, its derivative is a 1 -form $d f \in \Omega^{1}(M)$. The Riemannian metric sends this to a vector field, which we call the gradient of $f$ and denote by $\nabla f$. This notation and terminology is not ideal, as the gradient depends on the choice of Riemannian metric.

### 26.2.2 Gradient flow without critical points

Suppose that $M$ is compact, then using the techniques of Chapter 25 we can flow along $\nabla f$. The result is a smooth family of diffeomorphisms $\phi_{t}: M \rightarrow M$ for $t \in \mathbb{R}$, satisfying $\phi_{0}=\mathrm{id}, \phi_{s} \circ \phi_{t}=\phi_{s+t}$ and $\left.\frac{d}{d t} \phi_{t}\right|_{t=0}=\nabla f$.

To understand this flow, let us see how $f$ varies over an integral curve $\phi_{t}(p)$. Let $\|-\|^{2}$ denote the norm on $T^{*} M$ coming from the Riemannian metric, then we compute that

$$
\begin{aligned}
\left.\frac{d}{d t} f\left(\phi_{t}(p)\right)\right|_{t=0} & =d_{p} f\left(\left.\frac{\partial \phi_{t}(p)}{\partial t}\right|_{t=0}\right) \\
& =d_{p} f(\nabla f(p)) \\
& =\left\|d_{p} f\right\|^{2}
\end{aligned}
$$

Since $\phi_{t}$ is a flow, this implies that $\left.\frac{d}{d t} f\left(\phi_{t}(p)\right)\right|_{t=s}=\left\|d_{\phi_{s}}(p) f\right\|^{2}$. We conclude that:
Lemma 26.2.1. $f\left(\phi_{t}(p)\right)$ is non-decreasing with $t$ and strictly increasing with $t$ when $\phi_{t}(p)$ is not a critical point.

We shall use this to study the subset

$$
M_{[a, b]}:=f^{-1}([a, b]),
$$

for $a<b$ regular values. This is a codimension zero submanifold of $M$ with boundary $M_{a} \sqcup M_{b}$. Let us take $p \in M_{[a, b]}$ and consider the integral curve $\phi_{t}(p)$. When does this leave $M_{[a, b]}$ ?

Lemma 26.2.2. Fix $p \in M_{a}$. Let $(0, c)$ for $c>0$ be the maximal interval such that $\phi_{t}(p) \in \operatorname{int}\left(M_{[a, b]}\right)$ for $t \in(0, c)$. Then if $c$ is finite, $\phi_{c}(p) \in M_{b}$, and if $c=\infty$ then there are $t_{i} \rightarrow \infty$ such that $\phi_{t_{i}}(p)$ converges to a critical point.

Proof. Suppose that $c$ is finite. Then we know that $\phi_{c}(p)$ is defined but not in $\operatorname{int}\left(M_{[a, b]}\right)$ (or we could extend the interval $(0, c)$ ). Thus it is either in $M_{a}$ or $M_{b}$, and since $a$ is not a critical value, $f\left(\phi_{t}(p)\right)$ is strictly increasing with $t$ at $t=0$. It is non-decreasing afterwards, so we must have that $\phi_{c}(p) \in M_{b}$.

If $c=\infty$, then since $f\left(\phi_{t}(p)\right)$ increases at $t \rightarrow \infty$ but remains strictly smaller than $b$,

$$
\int_{0}^{N}\left\|d_{\phi_{t}(p)} f\right\|^{2} d t=\int_{0}^{N} \frac{d}{d t} f\left(\phi_{t}(p)\right) d t=f\left(\phi_{N}(p)\right)-f\left(\phi_{0}(p)\right)
$$

converges as $N \rightarrow \infty$. Thus $\left\|d_{\phi_{t}(p)} f\right\|$ must decrease to 0 as $t$ increases. This means that $\phi_{t}(p)$ will eventually be contained in any open neighborhood of the critical points in $M_{[a, b]}$. Since the $M_{[a, b]}$ is compact, this means that we can find a subsequence which converges to a critical point.

We are interested in the case that there is no critical point in $M_{[a, b]}$, and thus the second case in the above lemma can't occur. The same argument then tells us that when we start any $p \in M_{[a, b]}$, there is some maximal finite interval $\left(c^{\prime}, c\right)$ with $c^{\prime}<c$ such that $\phi_{t}(p) \in \operatorname{int}\left(M_{[a, b]}\right)$ for $t \in\left(c^{\prime}, c\right)$ and $\phi_{c^{\prime}}(p) \in M_{a}$ and $\phi_{c}(p) \in M_{b}$.


Figure 26.1 An example of a proper map $f: M \rightarrow \mathbb{R}$ such that $M_{[a, b]}$ contains no critical point. note that $M_{(-\infty, a]}$ contains 7 critical points.

Theorem 26.2.3 (First fundamental theorem of Morse theory). If the interval $[a, b]$ contains no critical values, then there is a diffeomorphism $M_{[a, b]} \rightarrow M_{a} \times[0,1]$ which restricts to the map $M_{a} \rightarrow M_{a} \times\{0\}$ given by $p \mapsto(p, 0)$.

Proof. By the previous lemma, for each $p \in M_{a}$ there is a $c(p)>0$ such that $\phi_{c(p)}(p) \in M_{b}$. This is unique because $f\left(\phi_{t}(p)\right)$ is non-decreasing and strictly increases at $t=c(p)$. By smooth dependence of solutions of ordinary differential equations on initial conditions, $c: M_{a} \rightarrow(0, \infty)$ is smooth. Now consider the map

$$
\begin{aligned}
\Psi: M_{a} \times[0,1] & \longrightarrow M_{[a, b]} \\
(p, t) & \longmapsto \phi_{t c(p)}(p) .
\end{aligned}
$$

In other words, it is the composition of the diffeomorphism $(p, t) \mapsto(p, t c(p))$ between $M_{a} \times[0,1]$ and $N:=\left\{(p, t) \in M_{a} \times \mathbb{R} \mid 0 \leq t \leq c(p)\right\}$ and the smooth map $\phi: M_{a} \times \mathbb{R} \rightarrow M$.

It has an inverse given as follows: given by $p \in M_{[a, b]}$ take $\left(c^{\prime}, c\right)$ as above and define $\Phi(p):=\left(\phi_{c^{\prime}}(p),-c^{\prime}\right)$. This is smooth using the smooth dependence of solutions of ODE's on initial conditions and smoothness of $\phi$. It is an inverse by uniqueness of solutions to ODE's.

Corollary 26.2.4. If the interval $[a, b]$ contains no critical values, then $M_{\leq a}$ is diffeomorphic to $M_{\leq b}$.

Proof. $M_{\leq b}$ is obtained from $M_{\leq a}$ by gluing on $M_{[a, b]}$. Recall that the existence of collars tells us that $M_{\leq a}$ contains a neighborhood $C$ of $M_{a}$ with a diffeomorphism
$M_{a} \times[-1,0] \rightarrow C$. Since $M_{[a, b]} \cong M_{a} \times[0,1]$ by the previous theorem, we see that $M_{\leq b}$ is diffeomorphic to $M_{\leq a}$ via

$$
\begin{aligned}
M_{\leq b} & \longrightarrow M_{\leq a} \\
& p \longmapsto \begin{cases}c(q, \eta(t)) & \text { if } p=c(q, t) \in C \\
c(q, \eta(t)) & \text { if } p=\Psi(q, t) \in M_{[a, b]} \\
p & \text { if } p \in M_{\leq a} \backslash C\end{cases}
\end{aligned}
$$

with $\eta:[-1,1] \rightarrow[-1,0]$ a diffeomorphism which is the identity near -1 .

### 26.3 Reeb's theorem

If $M$ is compact, then any smooth $f: M \rightarrow \mathbb{R}$ has to have a minimum and a maximum. Thus any Morse function on $M$ has at least two critical points. What happens if it has exactly two critical points?

Theorem 26.3.1 (Reeb). If a compact $k$-dimensional manifold $M$ admits a Morse function with exactly two critical points, then $M$ is homeomorphic to $S^{k}$.

The proof uses the Morse lemma, Lemma 27.1.1, which we will prove in the next chapter. In particular, this says that if $p$ is a minimum there are coordinates $x_{1}, \ldots, x_{k}$ around $p$ in which $p$ is the origin and $f$ is given by $f(x)=f(p)+\sum_{i=1}^{k} x_{i}^{2}$. Similarly, if $p$ is a maximum there are such coordinates in which $f$ is given by $f(x)=f(p)-\sum_{i=1}^{k} x_{i}^{2}$.

Proof. Let $p$ be such that $f(p)=a$ is the minimum and $q$ be such that $f(q)=b$ is the maximum. By the Morse lemma, we can find an $\epsilon>0$ small enough so that the following is true: $M_{\leq a+\epsilon}$ is diffeomorphic (using the coordinates $x_{1}, \ldots, x_{k}$ ) to a little disk $D_{\epsilon}^{k}(a)=\left\{\left(x_{1}, \ldots, x_{k}\right) \mid \sum_{i=1}^{k} x_{i}^{2} \leq \epsilon\right\}$, and similarly $M_{\geq b-\epsilon}$ is diffeomorphic to a little disk $D_{\epsilon}^{k}$. Hence their boundaries $M_{a+\epsilon}$ and $M_{b+\epsilon}$ are diffeomorphic to $(k-1)$-spheres. The region $M_{[a+\epsilon, b-\epsilon]}$ contains no critical points, so is diffeomorphic to $M_{a+\epsilon} \times[0,1]$.

Thus $M$ is obtained by gluing a cylinder $M_{[a+\epsilon, b-\epsilon]}=S^{k-1} \times[0,1]$ to two disks $D^{k}$ given by $M_{\leq a+\epsilon}$ and $M_{\geq b-\epsilon}$ (using the technique of Problem 13.4.5). The diffeomorphism produced by Theorem 26.2.3 is such that

$$
S^{k-1} \times\{0\}=M_{a+\epsilon} \times\{0\} \longrightarrow M_{a+\epsilon}=\partial M_{\leq a+\epsilon}=S^{k-1}
$$

is the identity, so doing this first gluing we see that there are diffeomorphisms

$$
\sigma: M_{\leq b-\epsilon} \cong D^{k} \cup\left(S^{k-1} \times[0,1]\right) \cong D^{k}
$$

This is a particular instance of Corollary 26.2.4.
However, we have no control over the diffeomorphism

$$
g: S^{k-1} \times\{1\}=M_{a+\epsilon} \times\{1\} \longrightarrow M_{b-\epsilon}=\partial M_{\geq b-\epsilon}=S^{k-1}
$$

The best we can do is the following: by Proposition 26.3.2 there exists a homeomorphism $G: D^{k} \rightarrow D^{k}$ extending this diffeomorphism. That is, we can find a homeomorphism

$$
\rho: M_{\geq b-\epsilon} \longrightarrow D^{k}
$$

which is compatible with $\sigma$.
Then we can write a homeomorphism $M \rightarrow S^{k}$ as follows:

$$
\begin{align*}
M=M_{\leq b-\epsilon} \cup M_{\geq b-\epsilon} & \longrightarrow S^{k}=D^{k} \cup D^{k} \\
& p
\end{align*}>\left\{\begin{array}{ll}
\sigma(p) & \text { if } p \in M_{\leq b-\epsilon}  \tag{26.1}\\
\rho(p) & \text { if } p \in M_{\geq b-\epsilon}
\end{array} .\right.
$$

Proposition 26.3.2 (Alexander trick). Every homeomorphism (so in particular diffeomorphism) $g: S^{k-1} \rightarrow S^{k-1}$ extends to a homeomorphism $G: D^{k} \rightarrow D^{k}$.

Proof. In radial coordinates, it is given by $G(r, \theta):=(r, g(\theta))$.
Remark 26.3.3. For later use, we point out that if $g: S^{k-1} \rightarrow S^{k-1}$ extended to $D^{k}$ as a diffeomorphism, then the formula (26.1) shows that $M$ is diffeomorphic to $S^{k}$.

### 26.4 Problems

Problem 26.4.1 (Morse functions are stable).
(a) Suppose that $f: \mathbb{R}^{k} \supset U \rightarrow \mathbb{R}$ is a smooth function. Prove that $f$ is Morse if and only if

$$
\operatorname{det}\left(\operatorname{Hess}_{x} f\right)^{2}+\sum_{i=1}^{k}\left(\frac{\partial f}{\partial x_{i}}(x)\right)^{2}>0
$$

for all $x \in U$.
(b) Prove that if $M$ is compact, the class of Morse functions on $M$ is stable, in the sense of Definition 11.2.1.

## Second fundamental theorem of Morse theory

In this chapter we discuss the part of Morse theory which does involve critical points and show if $f^{-1}([a, b])$ contains a single critical point, then $f^{-1}((-\infty, b])$ is obtained by attaching a single handle to $f^{-1}((-\infty, a])$. This can be found in Section 5.1 of [Wal16] and Chapter I.§3 of [Mil63].

Remark 27.0.1. Throughout this chapter we shall ignore the issue of "smoothing corners." If you want to understand these technical details, see Section 2.6 of [Wal16].

### 27.1 The second fundamental theorem of Morse theory

Let $M$ be a compact manifold and $f: M \rightarrow \mathbb{R}$ be a Morse function. We recall some notation from the previous chapter:

$$
M_{a}:=f^{-1}(\{a\}), \quad M_{\leq a}:=f^{-1}((-\infty, a]) \quad \text { and } \quad M_{[a, b]}:=f^{-1}([a, b]) .
$$

In the previous chapter we saw that if there is no critical value in $[a, b]$ or equivalently no critical point in $M_{[a, b]}$ - then there is a diffeomorphism $M_{[a, b]} \rightarrow M_{a} \times[a, b]$ that is the identity on $M_{a}$.

### 27.1.1 The Morse lemma

What happens when there is a unique non-degenerate critical point $p$ in $M_{[a, b]}$ ? Pick a chart $\phi: \mathbb{R}^{k} \supset U \rightarrow V \subset M$ such that $\phi(0)=p$, and in terms of coordinates $\left(x_{1}, \ldots, x_{k}\right) \in U, f$ is given by

$$
f\left(x_{1}, \ldots, x_{k}\right)=c-\sum_{i=1}^{\lambda} x_{i}^{2}+\sum_{i=\lambda+1}^{k} x_{i}^{2} .
$$

This is possible by the Morse lemma, and we provide a proof below that is different from the one in [GP10]:

Lemma 27.1.1. If a critical point $p \in M$ of $f: M \rightarrow \mathbb{R}$ is non-degenerate then there exists a chart as above.

Proof. Without loss of generality we may assume that $f(p)=0$, and fix a chart $\phi: \mathbb{R}^{k} \supset$ $U \rightarrow V \subset M$ such that $\phi(0)=p$. Let $x=\left(x_{1}, \ldots, x_{k}\right)$ denote the coordinates near $p$ coming from this chart, defined on $U \subset \mathbb{R}^{k}$.

Let $\operatorname{Sym}\left(\mathbb{R}^{k}\right)$ denote the space of symmetric $(k \times k)$-matrices over $\mathbb{R}$; we give this a smooth structure by using the entries to identify it with a Euclidean space. The multi-variable version of Taylor approximation says that there is a smooth map $Q: W \rightarrow$ $\operatorname{Sym}\left(\mathbb{R}^{k}\right)$ such that $f(x)=\langle Q(x) x, x\rangle$, which satisfies $Q(0)=\operatorname{Hess}_{0}(f)$. We first want to change coordinates from $x$ to $y$ so that $Q$ is independent of $y$. To do this, make the ansatz that $y=A(x) x$ for a smooth map $A: U \rightarrow \mathrm{GL}_{k}(\mathbb{R})$. In that case we need to solve the equation

$$
\langle Q(0) A(x) x, A(x) x\rangle=\langle Q(x) x, x\rangle,
$$

or equivalently $A^{t}(x) Q(0) A(x)=Q(x)$. We then consider the smooth map $G: \operatorname{Sym}\left(\mathbb{R}^{k}\right) \times$ $U \rightarrow \operatorname{Sym}\left(\mathbb{R}^{k}\right)$ given by

$$
(B, x) \mapsto\left(\mathrm{id}+\frac{1}{2} Q(0)^{-1} B\right)^{t} Q(0)\left(\mathrm{id}+\frac{1}{2} Q(0)^{-1} B\right)-Q(x)
$$

This is equal to 0 at $(B, x)=(0,0)$ and its derivative with respect to $B$ at $B=0$ is the identity:

$$
\begin{aligned}
\frac{\partial}{\partial B} G(B, 0) & =\left(\frac{1}{2} Q(0)^{-1}\right)^{t} Q(0)+Q(0)\left(\frac{1}{2} Q(0)^{-1}\right) \\
& =\frac{1}{2} \mathrm{id}+\frac{1}{2} \mathrm{id}=\mathrm{id}
\end{aligned}
$$

By the implicit function theorem, there exists a neighborhood $U^{\prime}$ of 0 in $U$ and a smooth $\operatorname{map} \beta: U \rightarrow \operatorname{Sym}\left(\mathbb{R}^{k}\right)$ such that $G(\beta(x), x)=0$. Taking

$$
A(x):=\mathrm{id}+\frac{1}{2} Q(0)^{-1} \beta(x)
$$

we obtain that $\langle Q(0) A(x) x, A(x) x\rangle=\langle Q(x) x, x\rangle$. So we shall use coordinates $y=A(x) x$. Since $x \mapsto A(x) x$ has derivative id at 0 , by the inverse function theorem there exists some smaller neighborhood $U^{\prime \prime}$ on which this map is a diffeomorphism.

Now that in $y$-coordinates we have that $f(y)=\langle Q(0) y, y\rangle$, it is a matter finding a matrix $A$ such that $A^{t} Q(0) A$ diagonal with entries $\pm 1$ and using the coordinates $z=A y$ instead. This is possible by Gram-Schmidt.

### 27.1.2 A single critical point

Let $f$ and $p$ be as before, and pick coordinates as in the Morse lemma (Lemma 27.1.1) around it.

Let $\epsilon>0$ be small enough such that $U$ contains the ball $B_{\sqrt{2 \epsilon}}(0)$ and $a<c-2 \epsilon<$ $c+2 \epsilon<b$. Then we shall describe the difference between $f^{-1}([a, c-\epsilon])$ and $f^{-1}([a, c+\epsilon])$, at first up to homotopy and then as a manifold. To do so, define the subset $C \subset B_{\sqrt{2 \epsilon}}(0)$ by $\left\{\left(x_{1}, \ldots, x_{\lambda}, 0, \ldots, 0\right) \mid \sum_{i=1}^{\lambda} x_{i}^{2} \leq \epsilon\right\}$, where $C$ stands for core. This is of course a $\lambda$-dimensional disk, whose boundary $(\lambda-1)$-sphere lies in $f^{-1}(c-\epsilon)$.

The description of $M_{[a, c+\epsilon]}$ up to homotopy equivalence is as follows, and along the way we will in fact obtain a description up to diffeomorphism.

Proposition 27.1.2. $M_{[a, c+\epsilon]}$ is homotopy equivalent, as a topological space, to the union $M_{[a, c-\epsilon]} \cup C$.

We shall use the notion of a deformation retraction: for $A \subset X$ closed, a deformation retraction of $X$ onto $A$ is a homotopy $H: X \times[0,1] \rightarrow X$ such that $H(x, 1) \in A$ for all $x \in X$ and $H(a, t)=a$ for all $a \in A$ and $t \in[0,1]$. If there is a deformation retraction of $X$ onto $A$, then $i: A \hookrightarrow X$ is a homotopy equivalence; its homotopy inverse is $H(-, 1)$.

To prove the proposition, we shall find a neighborhood $V$ of $M_{[a, c-\epsilon]} \cup C$ which is a deformation retract of $M_{[a, c+\epsilon]}$ and itself deformation retracts onto $M_{[a, c-\epsilon]} \cup C$ :

$$
M_{[a, c-\epsilon]} \cup C \stackrel{\sim}{\leftrightarrows} V \stackrel{\sim}{\leftrightarrows} M_{[a, c+\epsilon]} .
$$

The idea is that $V$ is obtained by adding a little tube around $C$ to $M_{[a, c-\epsilon]}$ into $U$, and then use a flow argument to deform $M_{[a, c+\epsilon]}$ onto $V$.

To produce $V$, we modify $f$ to another function $F$ with some special properties. We will only change $f$ on the subset $M_{[c-\epsilon, c+\epsilon]}$, using a smooth function $\phi:[0, \infty) \rightarrow[0, \infty)$ satisfying
(i) $\phi(0) \in(\epsilon, 2 \epsilon)$,
(ii) $\phi(t)=\phi(0)$ for $t$ near 0 ,
(iii) $\phi(t)=0$ for $t \in[2 \epsilon, \infty)$, and
(iv) $\phi^{\prime}(t) \in(-1,0]$ for all $t \in[0, \infty)$.


Figure 27.1 The function $\phi$.

Then the function $F$ shall be given by

$$
\begin{aligned}
F: M & \rightarrow \mathbb{R} \\
x & \mapsto \begin{cases}f(x)-\phi\left(\sum_{i=1}^{\lambda} x_{i}^{2}+2 \sum_{i=\lambda+1}^{k} x_{i}^{2}\right) & \text { if } x \in V \\
f(x) & \text { otherwise }\end{cases}
\end{aligned}
$$

This is a smooth function because $\phi\left(\sum_{i=1}^{\lambda} x_{i}^{2}+2 \sum_{i=\lambda+1}^{k} x_{i}^{2}\right)$ has compact support in $V$.

Lemma 27.1.3. $F$ has the following properties:
(1) $M_{[a, c+\epsilon]}=F^{-1}([a, c+\epsilon])$.
(2) $F$ has the same critical points as $f$.
(3) $\operatorname{In} B_{\sqrt{2 \epsilon}}(0) \subset U, F^{-1}([a, c-\epsilon])$ is described by Figure 27.2. More precisely, $V$ is diffeomorphic to $M_{[a, c-\epsilon]} \cup\left(D^{\lambda} \times D^{k-\lambda}\right)$ attached along an embedding $\partial D^{\lambda} \times D^{k-\lambda}$ (up to smoothing corners), with $C$ corresponding to $D^{\lambda} \times\{0\}$.


Figure 27.2 The set $U$ is the union of the red and purple parts. The set is $f^{-1}([a, c+\epsilon])$ is the union of the red, purple and dashed parts.

Proof. Let us write $x=(y, z)$ when $x \in U$, with $y=\left(y_{1}, \ldots, y_{\lambda}\right)$ denoting the first $\lambda$ coordinates and $z=\left(z_{1}, \ldots, z_{k-\lambda}\right)$ denoting the remaining $k-\lambda$.

Part (1) follows by noting that since $F \leq f$ (since $\phi$ is non-negative), we have that $f^{-1}([a, c+\epsilon]) \subset F^{-1}([a, c+\epsilon])$. For the converse, if $x \in F^{-1}([a, c+\epsilon])$ and $\phi\left(\|y\|^{2}+2\|z\|^{2}\right)>0$, then $\|y\|^{2}+2\|z\|^{2}<2 \epsilon($ since $\phi(t)=0$ when $t \geq 2 \epsilon)$, so that

$$
f(x)-f(c)=-\|y\|^{2}+\|z\|^{2} \leq \frac{1}{2}\|y\|^{2}+\|z\|^{2}<\epsilon
$$

and thus $x \in f^{-1}([a, c+\epsilon])$ as well.
For part (2) there is only something to check when $p \in V$. Working in local coordinates, we have that $\frac{1}{2} \nabla F(x)=\left(-y-\phi^{\prime}(x) y, z-\phi^{\prime}(x) 2 z\right)$. This certainly vanishes at 0 , so $p$ is a critical point. To see this is the only critical point, note that since $\phi^{\prime}(x)>-1$, we must have $y=0$ and since $\phi^{\prime}(x) \leq 0$, we must have $z=0$.


Figure 27.3 The gray part consists of those disks $D_{y}$ in the proof of Lemma 27.1.3 that do not coincide with those for the original function $f$.

The precise proof of part (3) is a rather long computation, as we need to produce an explicit diffeomorphism; details can be found in Chapter 3 of [Mil63] or Section VII.2.2 of [Kos93]. The main observation is that upon fixing the first $\lambda$-coordinates to be equal to $y=\left(y_{1}, \ldots, y_{\lambda}\right)$ with $\|y\|^{2} \leq \epsilon$, the intersection of $F^{-1}([a, c-\epsilon])$ with the $(k-\lambda)$ dimensional plane $\{y\} \times \mathbb{R}^{k-\lambda}$ is given by a disk whose radius depends smoothly on $y$. Of course, as soon as $\|y\|^{2}+2\|z\|^{2}$ reaches $T_{0}:=\inf \{t \mid \phi(t)=0\}$, then this disk coincides with the intersection of the original set $f^{-1}([a, c-\epsilon])$ with the $(k-\lambda)$-dimensional plane $\{y\} \times \mathbb{R}^{k-\lambda}$.

To check this, note that this intersection is given by the set $(y, z) \in \mathbb{R}^{\lambda} \times \mathbb{R}^{k-\lambda}$ with $z$ satisfying

$$
c-\|y\|^{2}+\|z\|^{2}-\phi\left(\|y\|^{2}+2\|z\|^{2}\right) \leq c-\epsilon .
$$

The condition may be rewritten in terms of $\alpha(y, z):=\|y\|^{2}+2\|z\|^{2}$ as

$$
\begin{equation*}
\phi(\alpha(y, z))-\alpha(y, z) / 2 \geq \epsilon-\frac{3}{2}\|y\|^{2} . \tag{27.1}
\end{equation*}
$$

Since $\phi(t)-t / 2$ is decreasing on the interval $[0,2 \epsilon]$ from $\phi(0)>\epsilon$ to $-\epsilon$, there is a unique $t_{0}>0$ such that $\phi\left(t_{0}\right)-t_{0} / 2=\epsilon-\frac{3}{2}\|y\|^{2}$. In terms of $t_{0}$, the inequality (27.1) is equivalent to

$$
\begin{equation*}
\|z\|^{2} \leq \frac{1}{2}\left(t_{0}-\|y\|^{2}\right) \tag{27.2}
\end{equation*}
$$

Since $\phi(0)>\epsilon$ and $\phi^{\prime}(t)>-1$, we have that $\phi\left(t_{0}\right)>\epsilon-t_{0}$, so that we have $\phi\left(t_{0}\right)-t_{0} / 2>\epsilon-\frac{3}{2} t_{0}$ and thus that $t_{0}>\|y\|^{2}$, so the right hand side of (27.2) is strictly positive. The set $D_{y}:=\left\{(y, z) \left\lvert\,\|z\|^{2} \leq \frac{1}{2}\left(t_{0}-\|y\|^{2}\right)\right.\right\}$ is the desired disk.

We shall then define $V=F^{-1}([a, c-\epsilon])$, which is diffeomorphic to $M_{[a, c+\epsilon]}$. To see this, apply the first fundamental theorem of Morse theorem using the observation that there is no critical point in $M_{[a, c+\epsilon]} \backslash V$. From this observation and part (3) of the Lemma, we not only obtain the homotopy-theoretic description also the stronger statement that $M_{[a, c+\epsilon]}$ is diffeomorphic to $\left(M_{a} \times[a, c-\epsilon]\right) \cup\left(D^{\lambda} \times D^{k-\lambda}\right)$. Thus we have proven:

Theorem 27.1.4 (Second fundamental theorem of Morse theory). If $M_{[a, b]}$ contains a unique non-degenerate critical point in its interior, which has index $\lambda$, then there is a diffeomorphism (up to smoothing corners)

$$
M_{(-\infty, b]} \xrightarrow{\cong} M_{(-\infty, a]} \cup_{\partial D^{\lambda} \times D^{k-\lambda}}\left(D^{\lambda} \times D^{k-\lambda}\right) .
$$

### 27.1.3 Handle decompositions

The construction which takes a manifold $W$ with boundary $\partial W$ and an embedding $e: \partial D^{\lambda} \times D^{k-\lambda} \hookrightarrow \partial W$ to the manifold obtained by smoothing the corners in

$$
W \cup_{\partial D^{\lambda} \times D^{k-\lambda}} D^{\lambda} \times D^{k-\lambda},
$$

is called a handle attachment of index $\lambda$.
The second fundamental theorem of Morse theory says that each critical point of index $\lambda$ corresponds to a handle attachment of index $\lambda$, as long as all critical points have distinct critical values. This is a minor restriction, as by a small perturbation we may assume this is the case, cf. Exercise 1.§7.19 of [GP10].

Since every manifold admits a Morse function and Morse singularities are isolated, we conclude that every compact manifold $M$ can be obtained by a finite number of handle attachments. We say it admits a handle decomposition.
Example 27.1.5. The height function

$$
\begin{aligned}
S^{k} & \longrightarrow \mathbb{R} \\
\left(x_{0}, \ldots, x_{k}\right) & \longmapsto x_{0}
\end{aligned}
$$

is a Morse function with a minimum at $(-1,0, \ldots, 0)$ (so index 0 ) and a maximum at $(1,0, \ldots, 0)$ (so index $k$ ). Thus we see that $S^{k}$ has a handle decomposition with a single 0 - and $k$-handle. This is just the decomposition

$$
S^{k}=\left(D^{0} \times D^{k}\right) \cup_{\partial D^{k} \times D^{0}}\left(D^{k} \times D^{0}\right)
$$

into two hemispheres.

### 27.2 Morse functions and de Rham cohomology

The relationship between de Rham cohomology and Morse functions will be the following:


Figure 27.4 A 3-dimensional 1-handle $D^{1} \times D^{2}$ attached to $\mathbb{R}^{2}=\partial\left(\mathbb{R}^{2} \times(-\infty, 0]\right)$. The red line is $D^{1} \times\{0\}$, the orange disk is $\{0\} \times D^{2}$.

Proposition 27.2.1 (Morse inequalities). Let $f: M \rightarrow \mathbb{R}$ be a Morse function on a $k$-dimensional compact manifold $M$, then for each $0 \leq \lambda \leq k$ there is an inequality

$$
\#\{\text { critical points of } f \text { of index } \lambda\} \geq \operatorname{dim} H^{\lambda}(M)
$$

Proof. We may assume without loss of generality that $f$ has critical points with distinct critical values, and shall ignore the smoothing of corners in this proof. Pick $a_{0}<\cdots<a_{n}$ such that $f(M) \subset\left[a_{0}, a_{n}\right]$ and each subset $M_{\left[a_{i-1}, a_{i}\right]}$ contains a unique critical point.

We shall prove by induction over $i$ that there is an inequality

$$
\#\left\{\text { critical points of }\left.f\right|_{M_{\left(-\infty, a_{i}\right]}} \text { of index } \lambda\right\} \geq \operatorname{dim} H^{\lambda}\left(M_{\left(-\infty, a_{i}\right]}\right)
$$

The initial case is $i=0$, and then $M_{\left(-\infty, a_{0}\right]}=\varnothing$ and the statement is clearly true. For the induction step, we use the second fundamental theorem of Morse theory:

$$
M_{\left(-\infty, a_{i}\right]} \cong M_{\left(-\infty, a_{i-1}\right]} \cup_{\partial D^{\lambda} \times D^{k-\lambda}} D^{\lambda} \times D^{k-\lambda}
$$

Let us apply Mayer-Vietoris to the open cover

$$
\begin{gathered}
U=\operatorname{int}\left(D^{\lambda}\right) \times D^{k-\lambda} \\
V=M_{\left(-\infty, a_{i-1}\right]} \cup_{\partial D^{\lambda} \times D^{k-\lambda}}\left(D^{\lambda} \backslash D_{1 / 2}^{\lambda}\right) \times D^{k-\lambda} .
\end{gathered}
$$

Then $U$ is contractible, $V$ is homotopy equivalent to $M_{\left(-\infty, a_{i-1}\right]}$ and $U \cap V$ is homotopy equivalent to $S^{\lambda-1}$.

From the Mayer-Vietoris long exact sequence we conclude that

$$
H^{i}\left(M_{\left(-\infty, a_{i}\right]}\right) \longrightarrow H^{i}\left(M_{\left(-\infty, a_{i-1}\right]}\right)
$$

is an isomorphism unless $i=\lambda, \lambda-1$. In those cases, we get an exact sequence (for convenience we assume $\lambda \geq 3$, dealing with $H^{0}$ 's requires a bit of additional care)


Two things can happen to the $\mathbb{R}$ in $H^{\lambda-1}(U \cap V)$ : either it adds to $H^{\lambda}$

$$
\begin{aligned}
& \operatorname{dim} H^{\lambda}\left(M_{\left(-\infty, a_{i}\right]}\right)=\operatorname{dim} H^{\lambda}\left(M_{\left(-\infty, a_{i-1}\right]}\right)+1 \\
& \quad \text { and } \operatorname{dim} H^{\lambda-1}\left(M_{\left(-\infty, a_{i}\right]}\right)=\operatorname{dim} H^{\lambda-1}\left(M_{\left(-\infty, a_{i-1}\right]}\right)
\end{aligned}
$$

or it subtracts from $H^{\lambda-1}$,

$$
\begin{aligned}
& \operatorname{dim} H^{\lambda}\left(M_{\left(-\infty, a_{i}\right]}\right)=\operatorname{dim} H^{\lambda}\left(M_{\left(-\infty, a_{i-1}\right]}\right) \\
& \quad \text { and } \operatorname{dim} H^{\lambda-1}\left(M_{\left(-\infty, a_{i}\right]}\right)=\operatorname{dim} H^{\lambda-1}\left(M_{\left(-\infty, a_{i-1}\right]}\right)-1
\end{aligned}
$$

In both cases the inequalities to be proven are satisfied. (Indeed, it may be helpful to observe that equality occurs only if all critical points add cohomology and never subtract cohomology).

Example 27.2.2. We know the cohomology of the 2-torus: $H^{0}\left(\mathbb{T}^{2}\right)=\mathbb{R}, H^{1}\left(\mathbb{T}^{2}\right)=\mathbb{R}^{2}$, $H^{2}\left(\mathbb{T}^{2}\right)=\mathbb{R}$. Thus every Morse function on $\mathbb{T}^{2}$ has at least one minimum, one maximum, and two saddle points. We leave it to you to find an example of such a Morse function.
Example 27.2.3. It is not true that you can always find a Morse function with exactly $\operatorname{dim} H^{\lambda}$ critical points of index $\lambda$. For example, only $H^{0}\left(\mathbb{R} P^{2}\right)=\mathbb{R}$ is non-zero, but since $\mathbb{R} P^{2}$ is compact every Morse function on it has a maximum.
Remark 27.2.4. Given a Morse function $f: M \rightarrow \mathbb{R}$, there is a chain complex $C_{*}^{f}$ with $C_{p}^{f}$ given by the free $\mathbb{R}$-vector space on the critical points of $f$ of index $p$, and differential given by counting flowlines. Its homology is the Morse homology $H_{*}(M ; f)$. It turns out to be independent of $f$ and for compact $M$ there is an isomorphism $H_{p}(M ; f)^{*} \cong H^{p}(M)$.

### 27.3 Problems

Problem 27.3.1 (Morse inequalities for surface). Draw an embedded $\Sigma_{g}$ in $\mathbb{R}^{3}$ such that the projection on the $z$-axis gives through Proposition 27.2 .1 the bounds

$$
\operatorname{dim} H^{0}\left(\Sigma_{g}\right) \leq 1 \quad \operatorname{dim} H^{1}\left(\Sigma_{g}\right) \leq 2 g \quad \operatorname{dim} H^{2}\left(\Sigma_{g}\right) \leq 1
$$

## Chapter 28

## Exotic 7-spheres

In this chapter we will construct a smooth manifold which is homeomorphic to $S^{7}$ but not diffeomorphic to it, an exotic sphere. We can prove the first statement, but the latter we can only outline. It is based on the signature theorem, which in turn relies on a computation of the rational oriented cobordism ring.

### 28.1 The signature theorem

### 28.1.1 Unoriented cobordism

Instead of trying to classify smooth manifolds up to diffeomorphism, one may first try to classify them up to the following weaker equivalence relation:

Definition 28.1.1. Two compact $k$-dimensional smooth manifolds $M_{0}$ and $M_{1}$ with empty boundary are said to be cobordant if there is a compact ( $k+1$ )-dimensional smooth manifold $W$ such that $\partial W=M_{1} \sqcup M_{0}$.

We call $W$ a cobordism from $M_{0}$ to $M_{1}$. Here the "equation" $\partial W=M_{1} \sqcup M_{0}$ means that the boundary of $W$ comes with a diffeomorphism to the disjoint union of $M_{0}$ and $M_{1}$. In particular, if $M_{1}$ is diffeomorphic to $M_{0}$ we can interpret the cylinder $M_{0} \times[0,1]$ as a cobordism from $M_{0}$ to $M_{1}$. Note that we can equally well interpret the cylinder as a cobordism from $M_{1}$ to $M_{1}$, from $M_{0} \sqcup M_{1}$ to $\varnothing$. or from $\varnothing$ to $M_{0} \sqcup M_{1}$.
Example 28.1.2. If $W \rightarrow \mathbb{R}$ is a proper smooth map without critical values then the Ehresmann fibration theorem, Theorem 25.3.1 says $\left.W\right|_{[a, b]}$ is a cylinder between the fibers $\left.W\right|_{a}$ and $\left.W\right|_{b}$. When $W \rightarrow \mathbb{R}$ is just a proper smooth map with regular values $a, b \in \mathbb{R}$, then $\left.W\right|_{[a, b]}$ is a cobordism between $\left.W\right|_{a}$ and $\left.W\right|_{b}$.

Lemma 28.1.3. Cobordism is an equivalence relation.
Proof. To see it is reflexive, note that the cylinder $M_{0} \times[0,1]$ exhibits $M_{0}$ as cobordant to $M_{0}$. For symmetry, note that $W$ as a cobordism from $M_{0}$ to $M_{1}$ can also be interpreted as a cobordism from $M_{1}$ to $M_{0}$. Finally, for associativity, note that if $W_{0}$ is a cobordism from $M_{0}$ to $M_{1}$ and $W_{1}$ is a cobordism from $M_{1}$ to $M_{2}$, then $W_{0} \cup_{M_{1}} W_{1}$ (obtained by the technique of Problem 13.4.5) is a cobordism from $M_{0}$ to $M_{2}$.

Definition 28.1.4. We let the $k$ th unoriented cobordism group $\Omega_{k}^{\mathrm{O}}$ denote the set of $k$-dimensional compact manifolds up to cobordism. We denote the cobordism class of $M$ by $[M]$.

Lemma 28.1.5. Disjoint union makes $\Omega_{k}^{\mathrm{O}}$ into an abelian group:

$$
[M]+[N]:=[M \sqcup N] .
$$

Proof. It is straightforward to show that $\sqcup$ is compatible with the equivalence relation of cobordism, and gives an associative and commutative binary operation on $\Omega_{k}^{O}$ with identity given by $\varnothing$. It remains to see why there are inverses. To do so, we interpret $M \times[0,1]$ not as a cobordism from $M$ to $M$ but as a cobordism from $M \sqcup M$ to $\varnothing$, so

$$
[M]+[M]=[M \sqcup M]=[\varnothing]=0
$$

and thus $[M]$ is its own inverse.
It is a consequence of the proof of this lemma that $\Omega_{k}^{O}$ is a 2-torsion abelian group. Example 28.1.6 ( $\left.\Omega_{0}^{\mathrm{O}}\right)$. A compact $d$-dimensional manifold $M$ represents the identity in $\Omega_{k}^{\mathrm{O}}$ if and only if it bounds a compact manifold. By the classification of 0-dimensional compact manifolds, these are given by a finite disjoint union of points. By the classification of 1-dimensional compact manifolds, a finite disjoint union of points is a boundary if and only if it consists of an even number of points. We conclude that the homomorphism

$$
\begin{aligned}
\Omega_{0}^{\mathrm{O}} & \longrightarrow \mathbb{Z} / 2 \\
\{r \text { points }\} & \longmapsto r \quad \bmod 2
\end{aligned}
$$

is an isomorphism.
Example 28.1.7 ( $\left.\Omega_{1}^{\mathrm{O}}\right)$. Similarly, the classification of 1-dimensional compact manifolds says that every such manifold without boundary is a finite disjoint union of circles. This is the boundary of a finite disjoint union of 2-dimensional disks, so $\Omega_{1}^{\mathrm{O}}=0$.

Let us assemble all $\Omega_{k}^{O}$ into a single graded abelian group $\Omega_{*}^{O}$. In addition to disjoint unions, we can take cartesian products. We will leave the proof of the following lemma to the reader:

Lemma 28.1.8. Cartesian product makes $\Omega_{*}^{\mathrm{O}}$ into a graded-commutative algebra:

$$
[M] \cdot[N]:=[M \times N] .
$$

The following is a deep result of Thom [Tho54], with addendum by Dold [Dol56]; its proof uses a lot of algebraic topology.

Theorem 28.1.9 (Thom, Dold). There is an isomorphism of graded-commutative algebras

$$
\Omega_{*}^{\mathrm{O}} \cong \mathbb{F}_{2}\left[x_{i} \mid i>0 \text { and } i \neq 2^{k}-1\right]
$$

where $x_{i}$ in degree $i$ is represented the following manifolds: for $i$ even $\mathbb{R} P^{i}$ and for $i=2^{r}(2 s+1)-1$ the Dold manifold $D\left(2^{r}-1, s 2^{r}\right)$ of Problem 7.4.2.

This should be surprising, as it is a complete classification of smooth manifolds up to an equivalence relation that does not seem very weak. It is also quite useful, as invariants obtained by taking inverse images of regular values are often only well-defined up to cobordism and hence take values in $\Omega_{*}^{\mathrm{O}}$.

### 28.1.2 Oriented cobordism

As the terminology suggests, we want to modify unoriented cobordism to take into account orientations.

Definition 28.1.10. Two compact oriented $k$-dimensional smooth manifolds $M_{0}, M_{1}$ with empty boundary are said to be oriented cobordant if there is a compact oriented $(k+1)$-dimensional smooth manifold $W$ such that $\partial W=M_{1} \sqcup-M_{0}$ (recall that $-M_{0}$ denotes $M_{0}$ with opposite orientation).

The following is proven for oriented cobordism by taking into account orientations in the proofs for unoriented cobordism:

Lemma 28.1.11. Oriented cobordism is an equivalence relation.
Definition 28.1.12. We let the $k$ th oriented cobordism group $\Omega_{k}^{\mathrm{SO}}$ be the set of $k$ dimensional compact oriented manifolds up to oriented cobordism. We denote the cobordism class of $M$ by $[M]$.

Lemma 28.1.13. Disjoint union makes $\Omega_{k}^{\mathrm{O}}$ into an abelian group, and cartesian product makes the graded abelian group $\Omega_{*}^{\mathrm{SO}}$ into a graded-commutative algebra.

If you go through the proof of this lemma, you'll learn that the inverse of $[M]$ is $[-M]$, i.e. $M$ with the opposite orientation. In particular, it is not the case that $\Omega_{*}^{S O}$ consists of 2 -torsion groups. The graded-commutativity comes from the fact that $M \times N$ is oriented by appending to the orientation of $T_{m} M$ that of $T_{n} N$, so if one reverses the order the orientation changes if and only if both $M$ and $N$ are odd-dimensional.

Example 28.1.14 ( $\Omega_{0}^{\mathrm{SO}}$ and $\Omega_{1}^{\mathrm{SO}}$ ). The classification of compact oriented 0 - and 1dimensional manifolds says that these are a finite disjoint union of oriented points or a finite disjoint union of circles. This can be used to prove that

$$
\left.\begin{array}{rl}
\Omega_{0}^{\mathrm{SO}} & \longrightarrow \mathbb{Z} \\
\{\text { positively oriented points } \\
\text { and } s \text { negatively oriented points }
\end{array}\right\} \longmapsto r-s
$$

is an isomorphism, and that $\Omega_{1}^{\mathrm{SO}}=0$.
Example 28.1.15 ( $\left.\Omega_{2}^{\mathrm{SO}}\right)$. The classification of compact oriented surfaces says that each of these is a disjoint union of $\Sigma_{g}$ for some $g \geq 0$. Each of these bounds a solid handlebody, so $\Omega_{2}^{S O}=0$.

The oriented cobordism ring is harder to describe, so we settle for its rationalization $\Omega_{*}^{\mathrm{SO}} \otimes \mathbb{Q}$. The following is again a deep result of Thom [Tho54]:

Theorem 28.1.16 (Thom). There is an isomorphism of graded-commutative algebras

$$
\Omega_{*}^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}\left[z_{4 i} \mid i>0\right]
$$

where $z_{4 i}$ in degree $4 i$ is represented by $\mathbb{C} P^{2 i}$.
Example 28.1.17. That this is not the full story can be seen by the computation that $\Omega_{5}^{\mathrm{SO}}=\mathbb{Z} / 2$, generated by $[\mathrm{SU}(3) / \mathrm{SO}(3)]$. All torsion is 2 -torsion, and $\Omega_{*}^{\mathrm{SO}} /$ tors is a free polynomial ring generated by the Milnor manifolds as in Problem 8.4.2 [Mil60].

It is outside the scope of this course, but for each oriented manifold there are invariants

$$
p_{i}(T M) \in H^{4 i}(M) \quad \text { for } i \geq 0,
$$

called Pontryagin classes. As the notation suggests, these make sense for any oriented vector bundle and here we just apply them to $T M$. They record to what extent a vector bundle is a non-trivial vector bundle. For example, if $T M$ is trivial (e.g. if $M$ is a Lie group) they all vanish.

For a compact oriented $4 k$-dimensional manifold with empty boundary, one can extract from these cohomology classes a number as follows: if $1 \leq i_{1} \leq i_{2} \leq \cdots \leq i_{s} \leq k$ is a consequence of integers (possibly repeated) such that $i_{1}+\cdots+i_{s}=k$, we take

$$
\int_{M} p_{i_{1}}(T M) \cdots p_{i_{s}}(T M) \in \mathbb{R}
$$

It is a non-trivial fact that these numbers are in fact integers, and give homomorphisms

$$
\int_{M} p_{I}: \Omega_{4 k}^{\mathrm{SO}} \longrightarrow \mathbb{Z}
$$

called Pontryagin numbers.
Tensoring with the rationals, we get linear maps $\Omega_{4 k}^{\mathrm{SO}} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$. Thom proved that these are linearly independent. As the number of sequences $I$ is the same as dimension of $\mathbb{Q}\left[z_{4 i} \mid i>0\right]$ in degree $4 k$, equal to the number of partitions $p(r)$ or $r$, we get:
Proposition 28.1.18 (Thom). The linear map $\oplus_{I} \int_{M} p_{I}: \Omega_{4 k}^{\mathrm{SO}} \otimes \mathbb{Q} \rightarrow \mathbb{Q}^{p(r)}$ is an isomorphism.

Example 28.1.19. If $\int_{M} p_{I}(M)=0$ for all sequences $I$, then there exists some $N \geq 1$ such that $\bigsqcup_{N} M$ bounds a compact oriented manifold. This for example holds whenever $G$ is a compact Lie group.

### 28.1.3 The signature

Suppose that $M$ is a compact oriented even-dimensional manifold, say of dimension $k=2 r$. Then there is a bilinear form

$$
\begin{aligned}
\langle-,-\rangle: H^{r}(M) \otimes H^{r}(M) & \longrightarrow \mathbb{R} \\
{[\omega] \otimes[\nu] } & \longmapsto \int_{M} \omega \wedge \nu .
\end{aligned}
$$

By graded-commutativity of the wedge product, this is anti-symmetric if $r$ is odd and symmetric if $r$ is even. By Poincaré duality, Theorem 23.2.1, it is non-degenerate.

For $r$ odd, by Problem 28.3.2 there exists a sympletic basis $e_{1}, \ldots, e_{s}, f_{1}, \ldots, f_{s}$ of $H^{r}(M)$. This means that it satisfies

$$
\left\langle e_{i}, e_{j}\right\rangle=0=\left\langle f_{i}, f_{j}\right\rangle, \quad\left\langle e_{i}, f_{j}\right\rangle= \begin{cases}1 & \text { if } i=j \\ 0 & \text { otherwise }\end{cases}
$$

That is, in this basis it is given by the skew-symmetric matrix

$$
\left[\begin{array}{cc}
0 & \mathrm{id}_{s} \\
-\mathrm{id}_{s} & 0
\end{array}\right] .
$$

In particular, we can't obtain any information from it that Betti numbers don't already tell us.

For $r$ even, we can use Sylvester's theorem-a direct consequence of the spectral theorem for symmetric matrices-which says that there exists a basis $e_{1}, \ldots, e_{s}, f_{1}, \ldots, f_{t}$ of $H^{r}(M)$ such that

$$
\left\langle e_{i}, f_{j}\right\rangle=0, \quad\left\langle e_{i}, e_{j}\right\rangle=\left\{\begin{array}{ll}
1 & \text { if } i=j, \\
0 & \text { otherwise },
\end{array} \quad\left\langle f_{i}, f_{j}\right\rangle= \begin{cases}-1 & \text { if } i=j \\
0 & \text { otherwise }\end{cases}\right.
$$

That is, in this basis it is given by the symmetric matrix

$$
\left[\begin{array}{cc}
\mathrm{id}_{s} & 0 \\
0 & -\mathrm{id}_{t}
\end{array}\right]
$$

The numbers $s$ and $t$ are unique, and from them we extract the following invariant:
Definition 28.1.20. If $M$ is a compact oriented $4 r$-dimensional manifold, then its signature $\sigma(M)$ is given by $s-t$.

Example 28.1.21. By Example 22.3.6, the signature of the $K 3$-manifold is -16 .
By construction, the signature is additive in disjoint unions and reserving the orientation multiplies it by -1 . Using the following example, any integer can be realized as the signature of a $4 r$-dimensional manifold.
Example 28.1.22. The signature of $\mathbb{C} P^{2 i}$ is 1 .

## The signature is a cobordism-invariant

We will now prove that the signature only depends on the oriented cobordism class of $M$. To do so, it suffices to prove that if a $4 r$-dimensional compact oriented manifold $M$ bounds a $(4 r+1)$-dimensional compact oriented manifold $W$ then $\sigma(M)=0$. Indeed, if $M_{0}$ is oriented cobordant to $M_{1}$, then this implies $\sigma\left(M_{0} \sqcup-M_{1}\right)=0$ or equivalently $\sigma\left(M_{0}\right)-\sigma\left(M_{1}\right)=0$.

Lemma 28.1.23. Let $i: M \hookrightarrow W$ denote the inclusion and take $[\omega] \in H^{4 r}(N)$. Then $\int_{M} i^{*} \omega=0$.

Proof. By Stokes' theorem, Theorem 19.2.1, we have $\int_{M} i^{*} \omega=\int_{N} d \omega=0$ because $\omega$ is closed.

We will use the following algebraic observation.
Lemma 28.1.24. Suppose we have a $\mathbb{R}$-vector space $V$ of dimension $2 n$ with nondegenerate symmetric bilinear form $\langle-,-\rangle: V \otimes V \rightarrow \mathbb{R}$ which has an n-dimensional subspace $W \subset V$ such that the restriction $\left.\langle-,-\rangle\right|_{W}: W \otimes W \rightarrow \mathbb{R}$ is identically zero. Then we have $\sigma(V)=0$.

Proof. The proof is by induction over $n$. Fix $e \in W$, then by non-degeneracy there is an $f \in V$ such that $\langle e, f\rangle=1$. By replacing $f$ by $f-\frac{1}{2}\langle f, f\rangle e$ we may assume $\langle f, f\rangle=0$. Then on the linear subspace $U=\operatorname{span}(e, f)$, the bilinear form $\langle-,-\rangle$ has signature 0 , and $V=U \oplus U^{\perp}$. As $U^{\perp}$ is $2(n-1)$-dimensional, $W \cap U^{\perp}$ is $(n-1)$-dimensional, and $\langle-,-\rangle$ vanishes identically on it, we may invoke the induction hypothesis.

Proposition 28.1.25. If a $4 r$-dimensional compact oriented manifold $M$ bounds a $(4 r+1)$-dimensional compact oriented manifold $W$ then $\sigma(M)=0$.

Proof. It suffices to prove that $H^{2 k}(M)$ is of dimension $2 n$ and contains an $n$-dimensional subspace on which $\langle-,-\rangle$ vanishes identically. We claim that the image of $i^{*}: H^{2 k}(W) \rightarrow$ $H^{2 k}(M)$ has the desired property. By Lemma 28.1.23 the bilinear form $\langle-,-\rangle$ vanishes on it, so it suffices to prove that its dimension is half of that $H^{2 k}(M)$.

The long exact sequence of a pair and Poincaré-Lefschetz duality assemble to a commutative diagram


Our starting point is the tautological equation:

$$
\operatorname{dim} H^{2 k}(M)=\operatorname{dimim}\left(i^{*}\right)+\operatorname{dimim}\left(i^{*}\right)^{\perp} .
$$

On the one hand, the isomorphism of the top row to the bottom row and exactness gives

$$
\operatorname{dim} \operatorname{im}\left(i^{*}\right)=\operatorname{dim} \operatorname{ker}\left(\left(i^{*}\right)^{*}\right)
$$

On the other hand, we have

$$
\operatorname{dimim}\left(i^{*}\right)^{\perp}=\operatorname{dim} \operatorname{ker}\left(\left(i^{*}\right)^{*}\right)
$$

because $\lambda: H^{2 k}(M) \rightarrow \mathbb{R}$ is in the kernel of $\left(i^{*}\right)^{*}$ if and only if it annihilates the image of $i^{*}$. We thus get $\operatorname{dim} H^{2 k}(M)=2 \operatorname{dim} \operatorname{im}\left(i^{*}\right)$ and the result follows.

## The signature theorem

What we have just proved implies that the signature gives a surjective homomorphism

$$
\sigma: \Omega_{4 k}^{\mathrm{SO}} \longrightarrow \mathbb{Z}
$$

which upon rationalization gives a linear functional $\sigma: \Omega_{4 k}^{\mathrm{SO}} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$.
By Proposition 28.1 .18 this is a linear combination with rational coefficients of Pontryagin numbers. Hirzebruch determined what these coefficients are in terms of the coefficients of the Taylor series expansion of $\frac{\sqrt{z}}{\tanh (\sqrt{z})}$ around $z$. We shall not describe this procedure, but will remark that is easily implemented on a computer.

Theorem 28.1.26 (Hirzebruch). The signature of a $4 k$-dimensional compact oriented manifold is given by

$$
\sigma(M)=\int_{M} L_{k}\left(p_{1}(T M), \ldots, p_{k}(T M)\right)
$$

where

$$
\begin{aligned}
& L_{0}=1 \\
& L_{1}=\frac{1}{3} p_{1} \\
& L_{2}=\frac{1}{45}\left(7 p_{2}-p_{1}^{2}\right) \\
& L_{3}=\frac{1}{945}\left(62 p_{3}-13 p_{1} p_{2}+2 p_{1}^{3}\right) \\
& L_{4}=\frac{1}{14175}\left(381 p_{4}-71 p_{1} p_{3}-19 p_{2}^{2}+22 p_{1}^{2} p_{2}-3 p_{1}^{4}\right) \\
& \quad \text { etc. }
\end{aligned}
$$

This is a quite remarkable theorem. A priori, all we know about the Pontryagin numbers is that they are integers. However, as the signature is by definition an integer, the signature theorem imposes intricate arithmetic conditions on these numbers.

### 28.2 Milnor's construction

We will now describe some 7-dimensional manifolds and prove that they are homeomorphic to $S^{7}$. We will then give a brief explanation why these are not diffeomorphic to $S^{7}$, a result due to Milnor [Mil56a]. The uses the aforementioned arithmetic conditions imposed on Pontryagin numbers.

The unit norm quaternions $S(\mathbb{H})$ on $S(\mathbb{H})$ by multiplication on the left and the right. Thus we can write down for each pair of integers $(i, j)$ a diffeomorphism

$$
\begin{aligned}
S(\mathbb{H}) \times S(\mathbb{H}) & \longrightarrow S(\mathbb{H}) \times S(\mathbb{H}) \\
(x, y) & \longmapsto\left(x, x^{i} y x^{j}\right) .
\end{aligned}
$$

We can use this to construct 7-dimensional manifolds $X_{i, j}$ as follows: we start with two copies $D^{4} \times S(\mathbb{H})$. Now we recall that $S(\mathbb{H}) \cong S^{3}$, so each of these has boundary $S^{3} \times S(\mathbb{H}) \cong S(\mathbb{H}) \times S(\mathbb{H})$. We identify these using the above diffeomorphism. Each of these is a 3 -sphere bundle over $S^{4}$.

To endow this topological space with a smooth structure, we use the existence of collars. We can avoid the use of these technical tools by gluing along open subsets in the base instead, thinking of the base as a one-point compactified $\mathbb{H}$. To do so, take two copies of $\mathbb{H} \times S(\mathbb{H})$ and identify the open subsets $(\mathbb{H} \backslash 0) \times S(\mathbb{H})$ using the diffeomorphism

$$
\begin{aligned}
(\mathbb{H} \backslash 0) \times S(\mathbb{H}) & \longrightarrow(\mathbb{H} \backslash 0) \times S(\mathbb{H}) \\
(x, y) & \longmapsto\left(\frac{x}{\|x\|^{2}}, \frac{x^{i} y x^{j}}{\|x\|^{i+j}}\right) .
\end{aligned}
$$

Here $\|x\|^{2}=\|a+b i+c j+d k\|^{2}=a^{2}+b^{2}+c^{2}+d^{2}$ is the (squared) quaternion norm. Example 28.2.1. $X_{0,0} \cong S^{4} \times S^{3}$ and $X_{1,0} \cong S^{7}$.

Proposition 28.2.2. If $i+j=1, X_{i, j}$ admits a Morse function with two critical points.
Proof. We start in the first chart $\mathbb{H} \times S(\mathbb{H})$, extending to the remaining 3 -sphere $\{\infty\} \times$ $S(\mathbb{H})$ later. The idea is to take the real part $\Re(x)=\Re(a+b i+c j+d k)=a$ on the fibers, scaled by a suitable function of norm of the base $\mathbb{H} \cup \infty$ to localize all critical points over 0 :

$$
f(x, y)=\frac{\Re(y)}{\sqrt{1+\|x\|^{2}}}
$$

For its derivative to vanish, certainly the partial derivatives of $\Re(y)$ with respect to the coordinates of $y$ have to vanish. The function $\Re$ on $S(\mathbb{H})$ is just the height function on $S^{3}$, so this occurs only if $y= \pm 1$. A further condition is then that the partial derivatives of $1 / \sqrt{1+\|x\|^{2}}$ with respect to the coordinates of $x$ has to vanish, and this only happens when $x=0$. We leave to the reader to check that the maximum at $(x, y)=(0,1)$ and the minimum at $(x, y)=(0,-1)$ are non-degenerate.

We claim that in the other chart, the Morse function is given by

$$
f\left(x^{\prime}, y^{\prime}\right):=\frac{\Re\left(x^{\prime}\left(y^{\prime}\right)^{-1}\right)}{\sqrt{1+\left\|x^{\prime}\right\|^{2}}}
$$

Indeed, when substituting the coordinate change

$$
\left(x^{\prime}, y^{\prime}\right)=\left(\frac{x}{\|x\|^{2}}, \frac{x^{i} y x^{j}}{\|x\|}\right)
$$

we get, using cyclic invariance of $\Re$ and the fact that $\Re\left(y^{-1}\right)=\Re(y)$,

$$
\begin{aligned}
f\left(x^{\prime}, y^{\prime}\right) & =\frac{\Re\left(x^{\prime}\left(y^{\prime}\right)^{-1}\right)}{\sqrt{1+\left\|x^{\prime}\right\|^{2}}} \\
& =\frac{\|x\|}{\|x\|^{2}} \frac{\Re\left(x x^{-j} y^{-1} x^{-i}\right)}{\sqrt{1+\|x\|^{2} /\|x\|^{4}}} \\
& =\frac{1}{\|x\|} \frac{\Re\left(y^{-1}\right)}{\sqrt{1+1 /\|x\|^{2}}} \\
& =\frac{\Re(y)}{\sqrt{1+\|x\|^{2}}} .
\end{aligned}
$$

We know already know that $f\left(x^{\prime}, y^{\prime}\right)$ has no critical points unless possibly when $x^{\prime}=0$. But fixing $y^{\prime}=1$ and restricting to real $x^{\prime}=a$, we get $f^{\prime}(a, 0)=\frac{a}{\sqrt{1+a^{2}}}$ which has no critical point at $a=0$. Hence $f\left(x^{\prime}, y^{\prime}\right)$ has no critical points.

Thus Theorem 26.3.1 gives:
Corollary 28.2.3. If $i+j=1, X_{i, j}$ is homeomorphic to $S^{7}$.
We will combine this with the following fact:
Theorem 28.2.4 (Milnor). $X_{i, j}$ can not be diffeomorphic to $S^{7}$ unless $(i-j)^{2} \equiv 1$ $(\bmod 7)$.

We do not have the tools to fill in the details of the following proof. That would require at least a course in algebraic topology. The idea is straightforward though: $X_{i, j}$ bounds a 4-disk bundle $W_{i, j}$ over $S^{4}$ and if it were diffeomorphic to $S^{7}$ then we can glue a $D^{8}$ along it to get a compact oriented manifold which contradicts the signature theorem unless the condition in the theorem is satisfied.

Proof sketch. The $X_{i, j}$, given by 3 -sphere bundles over $S^{4}$, naturally bound an 8dimensional manifold $W_{i, j}$; the corresponding 4-disk bundle over $S^{4}$.

Associated to any oriented compact 7 -dimensional $M$ which bounds a compact oriented 8 -dimensional manifold $W$, there are three invariants $\sigma(W, \partial W), \int_{W, \partial W} p_{1}^{2}$, and $\int_{W, \partial W} p_{2}$. We will not define these, but they are relative versions of the signature and Pontryagin numbers which we discussed before, and in particular are all integers.

If we have two such $W$ 's, say $W_{1}$ and $W_{2}$, we can form the closed oriented manifold $V:=W_{1} \cup_{M} W_{2}$. Its invariants are related to the relative ones by the equations

$$
\begin{aligned}
\sigma(V) & =\sigma\left(W_{1}, \partial W_{1}\right)-\sigma\left(W_{2}, \partial W_{2}\right), \\
\int_{V} p_{1}^{2}(T V) & =\int_{W_{1}, \partial W_{1}} p_{1}^{2}\left(T W_{1}\right)-\int_{W_{2}, \partial W_{2}} p_{1}^{2}\left(T W_{2}\right), \\
\int_{V} p_{2}(T V) & =\int_{W_{1}, \partial W_{1}} p_{2}\left(T W_{1}\right)-\int_{W_{1}, \partial W_{1}} p_{2}\left(T W_{2}\right) .
\end{aligned}
$$

The Hirzebruch signature theorem tells for closed $V$

$$
45 \sigma(V)=7 \int_{V} p_{2}(T V)-\int_{V} p_{1}^{2}(T V)
$$

Thus we see that

$$
\lambda(M):=45 \sigma(W, \partial W)-\int_{W, \partial W} p_{1}^{2}(T W) \quad(\bmod 7) \in \mathbb{Z} / 7
$$

is independent of $W$. It is an invariant of $M$.
Let us return to the task at hand. On the one hand, one may use the construction of $W_{i, j}$ to compute

$$
\sigma\left(W_{i, j}, \partial W_{i, j}\right)=1 \quad \text { and } \quad \int_{W_{i, j}, \partial W_{i, j}} p_{1}^{2}\left(T W_{i, j}\right)=4(i-j)^{2} .
$$

Since $45 \equiv 3(\bmod 7)$ and $4^{-1}=2(\bmod 7)$, we get $\lambda\left(W_{i, j}\right)=(i-j)^{2}-1$.
On the other hand, if $W_{i, j}$ is diffeomorphic to $S^{7}$ it bounds $D^{8}$ and one may use this to compute

$$
\sigma\left(D^{8}, \partial D^{8}\right)=0 \quad \text { and } \quad \int_{D^{8}, \partial D^{8}} p_{1}^{2}\left(T D^{8}\right)=0
$$

Since $\lambda\left(X_{i, j}\right)$ is independent of the bounding manifold, this implies that $\lambda\left(W_{i, j}\right)=0$. Comparing these values we see that a necessary condition for $W_{i, j}$ to be diffeomorphic to $S^{7}$ is that $(i-j)^{2} \equiv 1(\bmod 7)$.

Taking $i=2$ and $j=-1$, we get $(i-j)^{2}=3^{2} \equiv 2(\bmod 7)$ and we have found an exotic sphere! In fact, Kervaire and Milnor proved that there 28 oriented exotic 7 -spheres up to orientation-preserving diffeomorphism [KM63].

Let us now return our attention to Reeb's theorem. Observe that the diffeomorphism $g$ we obtained in its proof is orientation preserving, as it is the restriction of an obviously orientation-preserving diffeomorphism $M_{a} \times[0,1] \rightarrow M_{[a, b]}$.

Corollary 28.2.5. There exist orientation-preserving diffeomorphisms of $S^{6}$ which are not isotopic to the identity.

Proof. Suppose that in the case of $X_{2,-1}$, the orientation-preserving diffeomorphism $g$ of $S^{6}$ obtained in Theorem 26.3.1 is isotopic to the identity, say by a family of diffeomorphisms $g_{t}$ starting at the identity and ending at $g$. Think of $S^{6}$ as sitting inside of $\mathbb{R}^{7}$ via the standard embedding $\iota$ and apply the isotopy extension theorem, Theorem 25.2.1, to the family of embeddings

$$
\iota \circ g_{t}: S^{6} \longrightarrow \mathbb{R}^{7}
$$

We then obtain a family of compactly-supported diffeomorphisms $\varphi_{t}$ of $\mathbb{R}^{7}$ such that $g_{t}=\varphi_{t} \circ \iota$. Since $g_{t} \operatorname{maps} S^{6}$ to $S^{6}, \varphi_{t} \operatorname{maps} D^{7}$ to $D^{7}$. Then $\rho:=\left.\varphi_{1}\right|_{D^{7}}$ is a diffeomorphism of $D^{7}$ extending $g$. As suggested in Remark 26.3.3, using it in the last part of the proof of Theorem 26.3 .1 would prove that $X_{2,-1}$ is diffeomorphic to $S^{7}$, and we get a contradiction. Thus $g$ was not isotopic to the identity.

Remark 28.2.6 (The Gromoll-Meyer sphere). One of Milnor's exotic spheres-in fact, $X_{2,-1}$ - can be obtained explicitly up to diffeomorphism as a quotient of a Lie group [GM74]. Let $\operatorname{Sp}(n)$ denote the group of $(n \times n)$-matrices with quaternion entries satisying $Q^{\dagger} Q=\mathrm{id}=Q Q^{\dagger}$ where $Q^{\dagger}$ denotes the transpose conjugate of $Q$. There is an action of $\operatorname{Sp}(1)$ on $\operatorname{Sp}(2)$, where $q \in \operatorname{Sp}(1)$ acts on $Q \in \operatorname{Sp}(2)$ by

$$
\left[\begin{array}{ll}
q & 0 \\
0 & q
\end{array}\right] Q\left[\begin{array}{ll}
\bar{q} & 0 \\
0 & 1
\end{array}\right] .
$$

Then there is a diffeomorphism $X_{2,-1} \cong \operatorname{Sp}(2) / \operatorname{Sp}(1)$.

### 28.3 Problems

Problem 28.3.1 (Cobordism is an algebra). Prove Lemma 28.1.8.
Problem 28.3.2 (Symplectic bases). Prove that $V$ is a finite-dimensional $\mathbb{R}$-vector space with non-degenerate anti-symmetric bilinear form $\langle-,-\rangle: V \otimes V \rightarrow \mathbb{R}$, then it admits a symplectic basis.

Problem 28.3.3 (Signature is multiplicative). Use the Künneth theorem of Problem 23.4.4 to prove that

$$
\sigma(M \times N)=\sigma(M) \sigma(N)
$$

## Chapter 29

## Outlook

Let me end with an overview of the progress made in differential topology after the material covered at the end of these notes. Along the way I will point out some references you might want to look at in the future. We will stop at the end of the 2000's, as it is hard to tell at this point what the themes of differential topology in the 2010's were, being so close to it still.

## $29.1 \quad 50 ’$ s

Most of the results that we covered in these notes had been proven by the middle of the 50 's. Two important topics we only briefly touched on are cobordism theory and characteristic classes, which we did in Section 28.1.

Cobordism theory concerns the classification of manifolds up to cobordism, often taking into account additional structure such as an orientation or a map to a fixed topological space $X$ (see e.g. Chapter 8 of [Wal16]). It was developed by Thom [Tho54], whose main achievement was the reduction of the determination of these cobordism rings to a problem in algebraic topology. He then solved this algebro-topological problem completely for unoriented cobordism and rationally for oriented cobordism; this is Theorem 28.1.9 and 28.1.16. You can find an account in a number of textbooks on advanced algebraic topology, such as [Swi02].

One way to state the conclusion of these computations is a manifold is up to (oriented) cobordism determined by certain characteristic numbers. These assign to the tangent bundle of $M$ some elements of $\mathbb{Z} / 2$ for unoriented cobordism, or $\mathbb{Z}$ and $\mathbb{Z} / 2$ for oriented cobordism. We saw the $\mathbb{Z}$-valued ones in Section 28.1.2, the Pontryagin numbers. The $\mathbb{Z} / 2$-valued ones are the Stiefel-Whitney numbers. Two manifolds are (oriented) cobordant if and only if their characteristic numbers are equal.

The characteristic numbers are obtained from characteristic classes of the tangent bundle, by integration over the manifold. These characteristic classes more generally serve to distinguish finite-dimensional vector bundles over topological spaces, and play an important role in a lot of early algebraic topology [MS74].

### 29.2 60's

The techniques developed in the 50 's were refined to the following meta-theorem: manifolds of dimension $\geq 5$ are controlled by homotopy theory and algebraic $K$-theory.

Let me outline this by stating a result that gives a condition for two manifolds to be diffeomorphic. By Morse theory, all compact manifolds have a finite handle decomposition: they are built by gluing pieces $D^{\lambda} \times D^{k-\lambda}$ along $\partial D^{\lambda} \times D^{k-\lambda}$. Smale made the striking observation that a geometric trick of Whitney allowed you to simplify handle decompositions, as soon as certain invariants in topology and algebra vanish. To make this more concrete, let me state the s-cobordism theorem [Mil65] (or Section 5.5 of [Wal16]):

Definition 29.2.1. A compact manifold $W$ with boundary $\partial W=M_{0} \sqcup M_{1}$ is an $h$ cobordism from $M_{0}$ to $M_{1}$ if both inclusions $M_{0} \hookrightarrow W$ and $M_{1} \hookrightarrow W$ are homotopy equivalences.

Theorem 29.2.2 (s-cobordism theorem). Let $k \geq 5$, and $W$ be a $(k+1)$-dimensional $h$-cobordism from $M_{0}$ to $M_{1}$. Then $W$ is diffeomorphic to $M_{0} \times[0,1]$ rel $M_{0}$ if and only if an invariant $\tau(W) \in \mathrm{Wh}_{1}\left(\pi_{1}(M)\right)$ vanishes.

Here we see a homotopy-theoretic conditions-the inclusions $M_{0} \hookrightarrow W$ and $M_{1} \hookrightarrow W$ are homotopy equivalences- and an algebraic condition-the group Whitehead group $\mathrm{Wh}_{1}\left(\pi_{1}(M)\right)$ is obtained by imposing an equivalence relation on the invertible matrices with entries in the group ring $\mathbb{Z}\left[\pi_{1}(M)\right]$ (e.g. Example III.1.8 of [Wei13] and its references).

Example 29.2.3. Row reduction and the Euclidean algorithm imply that $\mathrm{Wh}_{1}\left(\pi_{1}(M)\right)=$ 0 . So every $h$-cobordism between simply-connected manifolds of dimension $\geq 6$ is diffeomorphic to a cylinder.

### 29.3 70's

Throughout the 60's and 70's the above theory was refined to a classification of all manifolds of dimension $\geq 5$, up to occasionally very hard computations in homotopy theory and algebraic $K$-theory. The resulting theory is known as surgery theory [L0̈2, CLM].

This name may become clearer when I outline how it proves existence: you start with a space $X$ satisfying some kind of Poincaré duality; a Poincaré complex of dimension $k$. Let us pick a map into $X$ from any manifold $M$, subject to the condition that it has degree 1 (and some minor tangential condition); a degree one normal map. Then we add and remove handles to $M$ with the goal of making the map $f: M \rightarrow X$ more and more like a homotopy equivalence (this is the "surgery step"). Some obstructions will occur, valued in a group similar to the Whitehead group mentioned above.

To prove uniqueness, you use a relative version: you start with $M_{0}$ and $M_{1}$ with homotopy equivalences to $X$, connect them by a cobordism $W$ with a map to $X$ and try to make the map $W \rightarrow X$ into a homotopy equivalence without modifying its boundary $\partial W=M_{0} \sqcup M_{1}$. If you succeed to do so (e.g. when $X$ is simply-connected) $W$ will be an $h$-cobordism and we exactly know when it is cylinder. Now we see the intended
application of the $s$-cobordism theorem: once we know $W \cong M_{0} \times[0,1]$ we can restrict to $M_{1}$ to get a diffeomorphism $M_{1} \cong M_{0}$.

Let me give a precise statement when $X$ is simply-connected. Let $\mathcal{S}_{k}(X)$ denote the set of $k$-dimensional compact oriented manifolds with a homotopy equivalence to $X$, up to orientation-preserving diffeomorphism.

Theorem 29.3.1 (Surgery exact sequence). Suppose $X$ is simply-connected Poincaré complex of dimension $k \geq 5$. Then there is an exact sequence

$$
\mathcal{N}_{n+1}(X \times[0,1] ; X \times\{0,1\}) \longrightarrow L_{k+1}(\mathbb{Z}) \longrightarrow \mathcal{S}_{k}(X) \longrightarrow \mathcal{N}_{n}(X) \longrightarrow L_{k}(\mathbb{Z})
$$

Here $\mathcal{N}_{k}(X)$ is the set of $k$-dimensional compact oriented manifolds with a degree one normal map to $X$ up to cobordism. This can be computed by the cobordism-theoretic techniques of Thom. Finally, $L_{k}(\mathbb{Z})$ is a symmetric $L$-theory group explicitly given by

$$
L_{k}(\mathbb{Z})= \begin{cases}\mathbb{Z} & \text { if } k=4 i \\ \mathbb{Z} / 2 & \text { if } k=4 i+2 \\ 0 & \text { otherwise }\end{cases}
$$

Example 29.3.2. You can use surgery theory to classify exotic spheres, cf. Chapter 11 off [CLM]. In fact, the classification was done first in [KM63] and then developed into surgery theory as above.

I do not mean to give the impression that is the end of the story. The computation of the relevant invariants is a hard and interesting problem, drawing surprisingly much on geometric group theory.

## $29.4 \quad 80$ 's

So far we have not commented on low dimensions, i.e. $k \leq 4$. The classification of the manifolds of dimension $k=0,1,2$ was completed in the 19th century, though they lacked the definitions and tools to give rigorous proofs. Steady progress was made throughout the 20th century on 3 -dimensional manifolds (more on that later), but the 1980's were the decade of 4-dimensional manifolds.

Around the same time two results appeared. Individually they were not so surprising, but combined they showed that the world of 4 -manifolds was wildly different from anything encountered before. On the one hand, Freedman used some ingenious infinite iterations of the high-dimensional techniques to prove that the classification outlined above goes through in dimension 4 if you are only interested in studying manifolds up to homeomorphism [Sco05].

On the other hand, Donaldson exhibited new invariants of 4-dimensional manifolds through gauge theory. Let me explain this by analogy with the 2 -dimensional case. Here is a method to distinguish surfaces of different genus: you can study the space of complex structures on $\Sigma$, as the dimension of this space can be expressed in terms of the genus of $\Sigma$. Putting a complex structure on a surface amounts to solving a partial differential equation. The idea behind gauge theory is that there exist some partial
differential equations on geometric objects on 4-manifolds-inspired by physics, such as Yang-Mills equations-whose spaces of solutions can serve as very strong invariants of these manifolds [DK90].

Using these invariants and Freedman's results, it was found that there are examples of 4 -manifolds that are homeomorphic but not diffeomorphic. This is not so surprising-the same thing happens in higher dimensions-but the surprise was that are not just some examples but extremely many examples. As an extreme example, consider the following theorem:

Theorem 29.4.1. There is a unique smooth structure on $\mathbb{R}^{n}$ up to diffeomorphism when $n \neq 4$, but infinitely many when $n=4$.

Research into the subtlety of dimension 4 continues to this day, using an interesting mix of hard analysis to define invariants and abstract technology to organize the computation of these invariants.

### 29.5 90's

As the term "gauge theory" suggests, the partial differential equations used to study 4-manifolds were inspired by theoretical physics. The 90 's saw a great flourishing of the interaction between differential topology and theoretical physics, due to the discovery of a new knot invariant by Jones and its subsequent reinterpretation in terms of quantum field theory by Witten [Wit89].

As a demonstration of the relationship between differential topology and physics, let me discuss the most elementary definition of a topological quantum field theory (e.g. [Koc04] in dimension 2).

Definition 29.5.1. A $k$-dimensional topological quantum field theory assigns to each $k$-dimensional compact oriented manifold $M$ a vector space $V(M)$, taking disjoint union to tensor product: $Z\left(M_{1} \sqcup M_{2}\right) \cong Z\left(M_{1}\right) \otimes Z\left(M_{2}\right)$. It assigns to each cobordism $W$ from $M_{1}$ to $M_{2}$ a linear map $Z(W): Z\left(M_{1}\right) \rightarrow Z\left(M_{2}\right)$, compatible with composition of cobordisms: $Z\left(W \cup_{M_{1}} W^{\prime}\right)=Z\left(W^{\prime}\right) \circ Z(W)$.

Remark 29.5.2. A better but more abstract definition is that $Z$ is a symmetric monoidal functor $\left(\mathrm{Cob}_{k}, \sqcup\right) \rightarrow($ Vect, $\otimes)$. This makes clear we can replace the target by any symmetric monoidal category.
Remark 29.5.3. The above definition fails to capture that quantum field theories are supposed to be local. This is added in the notion of an extended topological quantum field theory, which were classified by Lurie [Lur09].

## $29.6 \quad 00$ 's

The main applications of topological field theories are in dimension $k \leq 3$. As mentioned before, dimensions $\leq 2$ have long been well-understood. However, it was only in the 00 's that the classification of 3 -manifolds was completed, with Perelman's proof of the Poincaré conjecture [MT07]:

Theorem 29.6.1 (Perelman). If a 3-manifold $M$ is homotopy equivalent to $S^{3}$, it is diffeomorphic to $S^{3}$.

More accurately, his techniques not only resolved the Poincaré conjecture, but completed the Hamilton-Thurston geometrization program. Like surfaces, you can cut 3-manifolds into simpler pieces. In the case of surfaces these simple pieces are just sphere with disks removed, but in the 3 -dimensional case they remain complicated. The geometrization program asserted that each of these simple pieces admitted one of 7 particular geometric structures. When a piece has such a structure, we can use geometric methods tailored to the particular geometric structure to classify it [Thu02].

As in the high-dimensional case, this classification is not a classification in the sense of "a complete list of all manifolds." It just reduces the geometry question to algebraic questions, still difficult to solve but hopefully easier. In particular, hyperbolic structures are one of afore-mentioned geometric structures and it is a difficult problem to classify hyperbolic manifolds.

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[^0]:    ${ }^{1}$ The more well-known figure is that the group $\Theta_{7}$ of oriented exotic spheres up to orientationpreserving homeomorphism is isomorphic to $\mathbb{Z} / 28$ (see Chapter 28). In this group, inverse is given by reversing the orientation, so that when we allow (not necessarily orientation-preserving) diffeomorphisms there are 15 elements, corresponding $\{0\},\{14\}$ and $\{a, 28-a\}$ for $1 \leq a \leq 13$.
    ${ }^{2}$ It is more accurate to think of this as there being many distinct 4 -dimensional smooth manifolds that for a magical reason happen to be homeomorphic to $\mathbb{R}^{4}$.

[^1]:    ${ }^{3}$ You can read it at https://www.emis.de/classics/Riemann/Geom.pdf. More about the history of manifolds can be found in [Sch99].

[^2]:    ${ }^{4}$ There is even an octionic projective space $\mathbb{O} P^{2}$, also known as the Cayley projective plane, but no $\mathbb{O} P^{k}$ for $k>2$. This is harder to construct.

[^3]:    ${ }^{1}$ These are called elliptic curves. Note that since $\tau \mathbb{Z}+\mathbb{Z}$ is closed under addition and inverses, the $\mathbb{C}$-vector space structure on $\mathbb{C}$ descends to a group structure on $\mathbb{C} /(\tau \mathbb{Z}+\mathbb{Z})$, which is the addition on the corresponding elliptic curve. Elliptic curves have automorphisms; there is always negation, and for those coming from particular lattices (the square and hexagonal ones) there are additional automorphisms. In the actual moduli space we remember these automorphism groups, and so it is not a manifold but an orbifold.

[^4]:    ${ }^{2}$ For more about the link between elliptic curves and modular forms see [Kob93].

[^5]:    ${ }^{3}$ The more advanced reader may be surprised; the moduli space of genus $g$ Riemann surfaces with $n$ marked points is supposed to have fundamental group given by mapping class group $\Gamma_{g, n}$ of isotopy-classes of orientation-preserving diffeomorphisms of such a surface; for $g=1$ and $n=1$ this is $\mathrm{SL}_{2}(\mathbb{Z})$. In this statement we are supposed to take the orbifold fundamental group, which takes into account the finite automorphisms we ignored when discussing the coarse moduli space. In the quotient one point has automorphism group $\mathbb{Z} / 6$, one has $\mathbb{Z} / 4$, and the remainder $\mathbb{Z} / 2 ; \mathrm{SL}_{2}(\mathbb{Z})$ is the amalgated product $\mathbb{Z} / 6 *_{\mathbb{Z} / 2} \mathbb{Z} / 4$.

[^6]:    ${ }^{1}$ See https://www.ams.org/notices/200310/fea-milnor.pdf for the history and context of this problem.
    ${ }^{2}$ See e.g. http://www.ams.org/notices/200406/fea-weeks.pdf and https://mathoverflow.net/ a/9717/798.

[^7]:    ${ }^{3}$ The name is due to Andre Weil, who motivated it by "In the second part of my report, we deal with the Kähler varieties known as K3, named in honor of Kummer, Kähler, Kodaira and of the beautiful mountain K2 in Kashmir."

[^8]:    ${ }^{1}$ See https://www.ams.org/notices/200810/tx081001266p.pdf.

[^9]:    ${ }^{1}$ See http://www.lassp. cornell.edu/sethna/pubPDF/OrderParameters.pdf.

