Learning How to Learn Mathematics

I. There's gotta be a better way!

Think back to a time when you were taking a math course. Have you experienced thoughts like:

> *We're not even 10 minutes into the lesson and I'm already lost! I've gotta get out of here!*

> *How did I fall this far behind? I'll never catch up!*

> *I feel like my prof is speaking a foreign language. What does that word mean again?*

> *I feel even worse after seeing the solution. I would have never thought of doing that!*

> *How the heck am I supposed to remember all of this content for the exam?*

There are different reasons why these thoughts are so relatable. For example:

- Learning is often *messy* and *uncomfortable*.
- To develop proficiency in mathematics, we need **learning strategies** that *acknowledge* and *respond to the unique nature of the discipline*. However, **learning *how* to learn** math is rarely taught in schools.
- The *nature of assessments* and *traditional test-taking environments* may trigger a lot of stress and anxiety if we do not have an efficient and effective **study routine**.

II. ⚠ BEWARE of Memorization *without* Understanding

Before we discuss why index cards are important, let's first highlight what we should avoid when using them. **Memorization without understanding is *dangerous* in mathematics**. This is because of its defining characteristics, which include: exploring new concepts; deepening our understanding through struggle, failure, and success; engaging in logical and strategic reasoning; identifying patterns and relationships; and communicating ideas. Therefore, conceptual understanding is a must if we are to achieve proficiency.

Because mathematics is **cumulative** in nature, new concepts are often built on top of older ones. For example, we need to understand the meaning of *addition* before we can learn about *multiplication by a positive integer* (which is defined as *repeated addition*). Therefore, one of the goals behind using index cards is to **remember what we understand** so that we can use that prior knowledge to establish and explore future content.

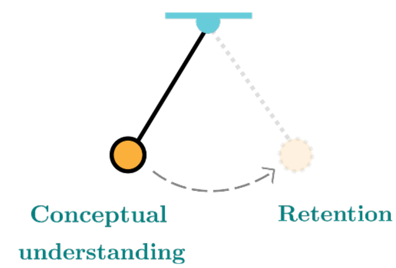
With greater retention, we reduce confusion and prevent *cognitive overload* as we continue to learn. The APA Dictionary of Psychology defines ***cognitive overload*** as "the situation in which the demands placed on a person by mental work (the cognitive load) are greater than the person's mental abilities can cope with" (VandenBos, 2015, p. 205). For example, imagine that a few days ago, your teacher presented the formal definition of *absolute value*. Today, they are solving an exercise that requires this definition, but you only vaguely remember them covering it. As your brain frantically tries to recall the definition, you get overwhelmed and have difficulty processing the rest of the solution.

Cognitive overload is a common and normal part of learning. However, it can interfere with our ability to understand and engage with the content. So, we want to take steps to minimize it as much as possible.

Finding Balance

In the math community, there have been long-standing debates about whether *retention* should be emphasized more than *conceptual understanding* (or vice versa). Teachers' beliefs have often aligned with one extreme or another, like a swinging pendulum.

Think back to your experiences while learning mathematics. Did your past teachers emphasize one more than the other?

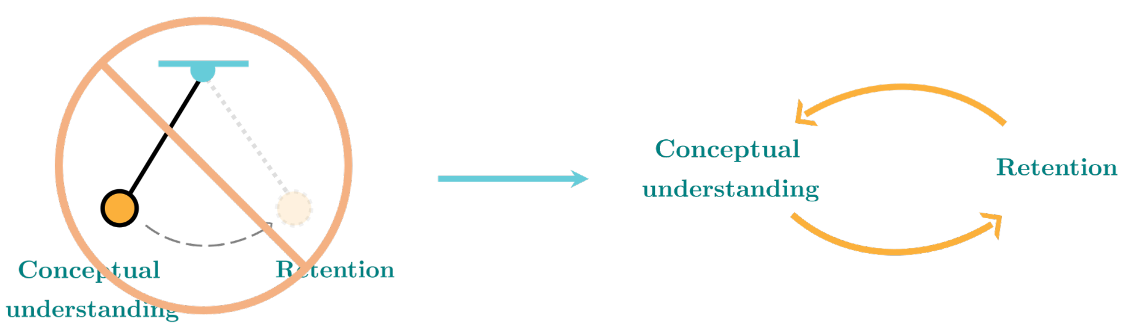


What happens when the pendulum swings *too far* in either direction?

In mathematics, the hard truth is that:

- *Retention* ***without*** *conceptual understanding is dangerous* AND
- *Conceptual understanding* ***without*** *retention is also dangerous*

Therefore, it's best to avoid the *extremes* of the swinging pendulum and consider a model that more accurately reflects the relationship between *retention* and *conceptual understanding*.



The model on the right shows that *conceptual understanding* and *retention* **enhance** each other. The retention of *prior knowledge* supports our conceptual understanding of new concepts. Meanwhile, having a conceptual understanding of content makes retention easier and empowers us to reconstruct concepts if we happen to forget them. This is an amazing relationship, and it's wise for us to capitalize on it!

III. Why are index cards important in *mathematics*?

When we think about studying mathematics, we don't usually hear about **index cards** as a useful study tool. However, they can really transform our learning experience! To get a solid understanding of *why* index cards are helpful, let's look carefully at:

(i) *the nature of mathematics* and

(ii) *the nature of assessments*

"Knowledge is power" and this knowledge will inform how we can develop a study routine that allows us to **meet the challenges** that arise and ***thrive* in spite of them**.

The Nature of Mathematics

(a) ***Mathematics is a language*** and learning different concepts requires the understanding of new *vocabulary*. Unfortunately, the volume of technical and specialized terms can be overwhelming. So, it is important that we intentionally incorporate *language-learning practices* into our study routines.

Examples include regularly attending classes, office hours, tutorials, and study groups so that we can engage with mathematics by speaking, listening, reading, and writing. This offers important moments of immersion, which are critical when learning a new language.

Despite these efforts, we require much more repeated exposure in order to develop high levels of retention, fluency, and confidence when learning a lot of content.

Good news! **Index cards** can help!

(b) ***Learning mathematics is analogous to building a house.*** This highlights the cumulative nature of mathematics, along with the importance of having a strong foundation. Because math builds, we need *prior knowledge* to learn new material. Prior knowledge consists of all of the 'tools' that we must collect, understand, and store in order to keep up with the inevitable building processes. Some of these tools include definitions, properties, theorems, strategies, and procedures.

Unfortunately, as the content mounts and the required math skills become progressively more complex, our working memory may become overloaded, making it difficult for us to understand, retrieve, use, and store course material. Relating the learning of mathematics to the building of a house also reveals the dangers of falling behind in a course or having gaps in our knowledge. This is because gaps (in the form of 'missing tools' and 'building blocks') will inevitably compromise the structural integrity of our "math houses", and prevent us from building consistently and effectively in our courses. The comparison also helps to explain why cramming before an evaluation is extremely risky and taxing.

So, how do we prevent cognitive overload and stay on top of constant building that is required in mathematics?

There are **index cards** for this too!

(c) ***Mathematics is a game of logic, strategy, and efficiency.*** This challenges the notion that mathematics is random, nonsensical, and impossible. By thinking of mathematics as a strategic game that is rooted in logical thought, we can better understand why making mistakes is natural and inevitable, and that resilience is needed to transform our mistakes into teachable moments. This explains why *productive* struggles and failures can lead to greater success.

The analogy of a game also highlights why *consistent* and *deliberate practice* plays a vital role in making mathematics possible. Through deliberate practice, we develop a lot of strategies and make mistakes that provide opportunities to deepen our learning. With our strategies and learned lessons, we can move through the mathematics more *efficiently* and *effectively*, while also reducing the frequency of errors. Remember that the goal is *progress*, not perfection (which is impossible).

The big question is: how do we remember all of these important lessons and strategies when there are so many?

Can you guess the answer?

There are **index cards** for this as well! The possibilities are endless!

The Nature of Assessments

Ok... It's time for us to have a difficult, but really important conversation about **high-stakes assessments**. I won't have to elaborate nearly as much as I did in the previous sections because you have probably *experienced* these kinds of assessments first-hand in grade school, college, or university.

When I say "high-stakes assessment", I am referring to assignments, tests, or exams that can significantly impact our ability to pass a course or receive a grade that is high enough to advance certain goals.

One of the main concerns is that these assessments can *indirectly* **prevent** us from:

- receiving a diploma or degree,
- accessing a particular program or career path, or
- being able to pay for our education due to the financial consequences of dropping or retaking a course

Unfortunately, **traditional test-taking environments** exacerbate these already stressful circumstances.

(a) ***Timed assessments*** require us to recall and work through course content *relatively quickly*. However, being timed can also be anxiety-provoking, and anxiety is notorious for making it difficult to recall information that would have been easily accessible if we were calm.

(b) ***Limited or NO aids*** further compound the challenge of being timed. For example, without aids, we are expected to remember *most* of the course content that is being assessed *and* be able to recall things rather quickly.

The Common Thread: Active (*not* passive) in Nature

We can now see that mathematics and assessments have one thing in common: both are very **active** in nature. Therefore, it's not surprising that *passive* study strategies (e.g. watching a lecture, reading a textbook, absent-mindedly taking notes, etc.) may not provide enough preparation for learning mathematics well or excelling on assessments. Research has shown that **some of the *most popular* study strategies**, including *highlighting, rereading,* and *summarizing notes*, **are actually the *least* effective methods** when compared to others that are available. Are these the strategies that you use most often?

To see the ***active* learning practices** that are *more effective*, check out the following resources:

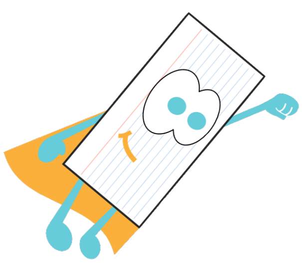
– An article by John Dunlosky (a professor of psychology) titled, "Strengthening the Student Toolbox: Study Strategies to Boost Learning" at: https://files.eric.ed.gov/fulltext/EJ1021069.pdf

– A 20-minute video by a university student in their final year.
In this video, the student offers a *research-informed* list of study strategies that are most effective. Among the papers cited, they also refer to a different paper by John Dunlosky, which contains some of the research that was used to back the claims in the article above.

https://www.youtube.com/watch?v=ukLnPbIffxE&t=0s&ab_channel=AliAbdaal

> In this handout, we will put the spotlight on *creating and reviewing index cards*. However, note that it is ***only one part*** of a productive study routine. Check out the resources above to learn more about what else is needed!

IV. Index Cards to the Rescue!



Creating and *reviewing index cards* can **empower** us to overcome some of the challenges that arise while learning mathematics.

(i) The **process** of making quality index cards (physical or digital) can support us in solidifying content by pushing us to revisit, comprehend, and summarize information. Essentially, we are preparing this material in bite-size chunks so that we can transfer it to our long-term memory.

(ii) When our index cards are well made *(see examples in the next section)* and we review them regularly, they train us to quickly and **actively recall** content *without* exerting a lot of mental effort. This helps us to stay focused on overarching tasks and prevent cognitive overload while learning or completing assessments. We also develop increased confidence with the material, which helps us to ward off debilitating levels of stress and anxiety.

If done right, you will be amazed at how much index cards improve your learning experience!

⚠️ BEWARE! Index cards will *not* be very effective if. . .

(i) *we try to make them all at the last minute* (e.g. just before the final exam).
The idea is to make them daily, as the course progresses.

(ii) *we rarely review them.* The purpose of making the index cards daily is so that we can also review them daily (e.g. first thing in the morning, while commuting, or before bed).

V. Getting the Index Cards

(i) For ***physical* index cards**:

Buy Them! You may be able to purchase index cards that are already on a ring. Alternatively, you can buy the index cards and rings separately, and then use a hole-puncher to create the holes.

(ii) For ***digital* index cards**:

Search for an app! Do a web search for "best flashcard app" or "best *free* flashcard app". Many of these apps are free and can be downloaded on phones, laptops, tablets, or desktop computers.

You can also see an example of a digital app in the video that was provided on the previous page:
(i.e. https://www.youtube.com/watch?v=ukLnPbIffxE&t=0s&ab_channel=AliAbdaal)

VI. Making the Index Cards (the possibilities are endless!)

- **Definition Cards**

Front

Def'n

Absolute Value
$|a|$, where $a \in \mathbb{R}$

prompts:
① geometric
② algebraic
③ example

Back

① Geometrically:
$|a|$ is the DISTANCE of $a \in \mathbb{R}$ from zero on the real number line.

e.g. $|-5| = 5$ since -5 is 5 units from zero

5 UNITS
$-5 \quad 0 \quad \mathbb{R}$

② Algebraically, $|a| = \begin{cases} a, & \text{if } a \geq 0 \quad \text{(Case 1)} \\ -a, & \text{if } a < 0 \quad \text{(Case 2)} \end{cases}$

e.g. $|-5| = -(-5) = 5$ since $-5 < 0$ (Case 2)

- **<u>Theorem</u> Cards (excluding *properties*)**

Front

Theorem

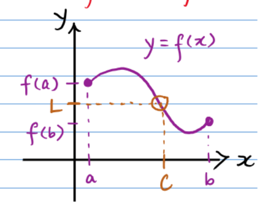
Intermediate Value Theorem
(IVT)

prompts:
① hypothesis
② conclusion

Back

① Hypothesis
IF (i) f is continuous on the closed interval $[a,b]$
(ii) L is some number between $f(a)$ and $f(b)$

② Conclusion
THEN, there is at least one number $c \in (a,b)$ satisfying $f(c)=L$



- **<u>Properties</u> Cards**

Properties are very common and important in mathematics. They are usually presented as theorems, meaning that the results have been rigorously proven. From time to time, we may also see them given as axioms, which are results that we assume to be true without proof.

Front

Theorem

Properties of Limits
(a.k.a. Limit Laws)

prompts:
Hypotheses? ① involving sums/differences
② involving products
③ involving division

Back

Hypotheses: IF $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

THEN: ① $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

② $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

③ Additional Hypothesis: $\lim_{x\to a} g(x) \neq 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} g(x)\right)}$$

- **<u>Strategy</u> Cards** (for clarity and efficiency)

Strategy cards usually arise while practicing. You may notice that certain kinds of exercises involve strategies that help to simplify your solutions and minimize anxiety, confusion, and errors. Once you identify them, use them whenever possible and put them on an index card to help you remember them!

Front

Strategies

Taking Derivatives

Back

$\boxed{S1}$ Ask yourself, "Can I MANIPULATE?"
e.g. $y = \frac{x}{\sqrt[3]{x^2}}$; Can you simplify to get $y = x^{1/3}$?

$\boxed{S2}$ Determine the outermost operation (e.g. addition/subtraction, multiplication, exponentiation, division, or composition) and apply the derivative rules one at a time as you peel back the layers.

e.g. $f(x) = e^{2x} \cdot \sin(3x+1)$
OUTERMOST: multiplication $\Longrightarrow$ Product Rule
Then: Compositions $\Longrightarrow$ Use Chain Rule

- **Question Cards** (for lessons learned)

  Question cards are fantastic for holding on to your lightbulb/teachable moments or mistakes that you want to avoid in the future. These may come up when you are discussing concepts with others (e.g. in study groups) or after making a critical error. Misconceptions are quite common when we are learning mathematics and one memorable way to address them is by making index cards like these. Examples include TRUE/FALSE index cards or cards that ask very direct questions like, "Am I allowed to ______?" or "What are the conditions under which ______ is applicable?" There are no limits! Ask yourself anything that seems important!

Front

Question

For which functions, if any, is Power Rule applicable? Explain why or why not.

(a) $y = x^x$ (b) $f(x) = 2^x$ (c) $y = \sqrt[6]{7-2x}$

Back

ONLY **(C)**

Why? Power Rule is only applicable for the following form:

$h(x) = C \cdot [F(x)]^n$

NUMERICAL exponent

some CONSTANT

some NON constant function

Note: the derivative may also require other derivative rules depending on $F(x)$.

References

VandenBos, G. R. (Ed.). (2015). *APA dictionary of psychology* (2nd ed.). American Psychological Association.