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This is a very exciting issue of the $U(t)$-Mathazine. Our authors range from a highschool junior to a full professor. (Can you guess which author is which?) Many contributions came from students at UTSC, but there are also contributions from outside the University of Toronto. The magazine is growing, and we hope to see more contributions for our next issue.

Parker Glynn-Adey - Editor and Designer

Table of Contents

- On the Density of Planar Sets Avoiding Unit Distance by Veselin Jungić 2
- Ringing the Changes: A Dance of Algebra and Geometry by Parker Glynn-Adey 5
- Equidistant Sets and Almost Equidistant Sets by Zhekai Pang & Yuhong Zhang 9
- The Netflix Prize Challenge and Matrix Completion by Yuhong Zhang & Zhekai Pang 12
- Two Applications of Euler's Theorem by Zhiquan Cui 16
- Math, Musical Theatre, and Medicine: How One UofT Medical Student is Intertwining Them All by Olivia Rennie 19
- Pascal's Triangle by Yi Chen, Mengmeng Shang, Zhekai Pang 21
- Reaffirming The Validity of Cantor's Diagonal Argument Through a Systematic Approach by Artin Ghafooripour 23
- The Man Who Loved Only Numbers Book Review by Trevor Cameron 27
- An Introduction to the Continuum Hypothesis by Pourya Memarpanahi 28
- No Strangers at this Party: The Story of the Ramsey Theory Podcast by Brian Krammer 30

On the Density of Planar Sets Avoiding Unit Distance

by Veselin Jungić

The aim of this note is to mark the 100th anniversary of the birth of Leo Moser, 1921–1970, a prominent Canadian mathematician.

After graduating from the University of Manitoba with a B. Sc. degree in mathematics in 1944, Moser went on to complete an M.Sc. degree at the University of Toronto. In 1951, he completed a Ph. D. at the University of North Carolina under the supervision of Alfred Brauer. Leo Moser was a professor of mathematics at the University of Alberta from 1951 until his premature death at 49 years old in 1970. [7]

Leo Moser's mathematical interests included the fields of number theory, graph theory, and algebra. One of his lasting mathematical legacies is the so-called Moser spindle [4], a simple but in many ways a mysterious object that he co-created with his brother William in 1961. (See Figure 6 in the article Equidistant Sets and Almost Equidistant Sets in this issue for an image.) Leo Moser was known as an excellent teacher able to make mathematics "a fresh and living subject for very large classes of students just beginning their mathematical studies." [8]

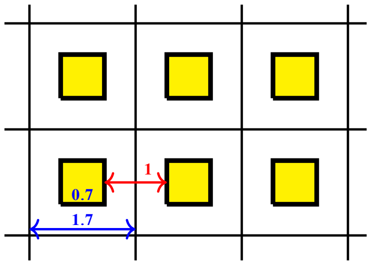
In posing new problems, Moser was a match for Paul Erdős. It is well known that the two of them were friends and collaborators. The problem of whether or not the Erdős-Moser equation $1^k + 2^k + \cdots + m^k = (m+1)^k$ has a solution other than $k = 1$ and $m = 2$ is still unsolved.

In this note, we revisit one of the early results related to the following open problem from combinatorial geometry posed by Moser in the mid 1960s [2, 3, 6]:

> What is $m_1(\mathbb{R}^2)$, the maximum density of a Lebesgue measurable set in the plane that does not contain a unit-distance pair?

In more casual terms, the question is how to determine the size (in terms of the portion of the plane) that cannot be exceeded by a set that avoids the unit distance.

For example, consider the covering of the plane with an infinite square lattice. Let the side length of each square in the lattice be 1.7. In the centre of each square we put another square with the side length 0.7. Let T be the set of all points in the plane that belong to the interior of one of the smaller squares.



We note that if two points in T belong to the same square then their distance is less than the length of the diagonal, $\sqrt{0.7^2 + 0.7^2} = \sqrt{0.98} < 1$. If two points in T belong to two different squares, then their distance is larger than 1. Hence $T \in \mathcal{T}$, where $\mathcal{T}$ is the family of all sets in the plane that avoid the unit distance.

Which portion of the plane does the set T occupy? Because all squares in the lattice are congruent, the density of the set T in the plane will be equal to the portion of a square in the lattice that a 0.7×0.7 square occupies. Hence

$$\text{Density of } T = \frac{0.7^2}{1.7^2} = \frac{0.49}{2.89} \approx 0.17.$$

Since, by definition, $m_1(\mathbb{R}^2) = \sup\{\text{Density of } U : U \in \mathcal{T}\}$, and since $T \in \mathcal{T}$, we have that $m_1(\mathbb{R}^2) \geq$ Density of $T > 0.17$. Hence, 0.17 is a lower bound for $m_1(\mathbb{R}^2)$.

The best currently known lower bound for $m_1(\mathbb{R}^2)$ is 0.2293 which was obtained by Hallard T. Croft in 1967 [2]. Interestingly enough this result – still the best known lower bound for $m_1(\mathbb{R}^2)$ – was obtained by using a common differential calculus technique. In Section 2 we give details of Croft's calculations.

The best currently known upper bound of 0.25442 was found in 2020 by Gergely Ambrus and Mátê Matolcsi [1].

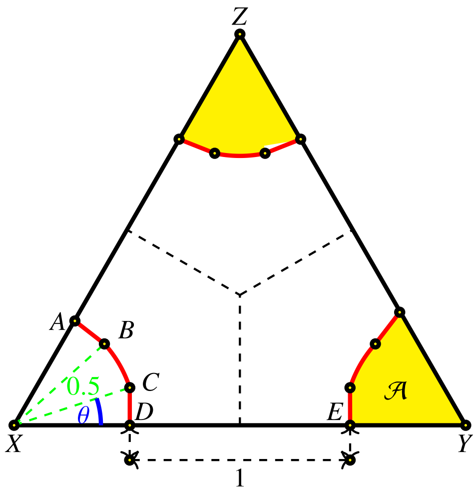
The problem was popularized by Paul Erdős, who stated that it “seems likely” that $m_1(\mathbb{R}^2) < 0.25$. [3]

The Best Known Lower Bound

In this section we follow Croft's argument and show that $m_1(\mathbb{R}^2) > 0.2293$. We start by covering the plane with an infinite equilateral triangular lattice. The length of the side of the triangles in the lattice will be determined by the following construction.

Say that $\triangle XYZ$ is one of the triangles in the lattice. We construct three mutually congruent *lumps* in $\triangle XYZ$ in the following way:

1. Fix an angle $\theta \in \left(0, \frac{\pi}{6}\right)$.
2. Let B and C be points inside of $\triangle XYZ$ such that $|\overline{XB}| = |\overline{XC}| = 0.5$ and that $\angle(YXC) = \angle(ZXB) = \theta$.
3. Draw the arc BC on the circle with the centre at X and radius equal to 0.5.
4. Let A be the point on the line segment $\overline{XZ}$ such that $\overline{AB} \perp \overline{XZ}$. Let D be the point on the line segment $\overline{XY}$ such that $\overline{CD} \perp \overline{XY}$.
5. Call the interior of the region in the plane bounded by the line segments $\overline{XA}$, $\overline{XD}$, $\overline{AB}$, $\overline{CD}$, and the arc BC, a *lump* centred at X with the angle θ.
6. Keep the angle θ the same as before and repeat the same construction for vertices Y and Z to obtain three mutually congruent lumps contained in $\triangle XYZ$.
7. Choose $|\overline{XY}|$, the length of the side of the equilateral triangle so that the distance between the corresponding points (obtained in this construction) on the line segment $\overline{XY}$ equals to 1.
8. Do the same construction for all triangles in the lattice.

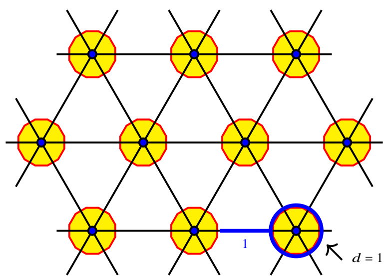


We observe that, by construction, the length of the side of each equilateral triangle is equal to

$$|\overline{XY}| = |\overline{XD}| + |\overline{DE}| + |\overline{EY}| = \frac{\cos\theta}{2} + 1 + \frac{\cos\theta}{2} = 1 + \cos\theta.$$

Let $S = S(\theta)$ be the set of all points in the plane that belong to one of the lumps obtained by the above construction for some (fixed) θ.

We note that, by construction, the distance between any two elements of S that belong to lumps with different centres is greater than 1. On the other hand, all three lumps with the same centre are contained in a circle of diameter $d = 1$. Hence, the set S does not contain a unit-distance pair.



We start our investigation of the density of the set S, by finding the area $\mathcal{A} = \mathcal{A}(\theta)$ of one lump. We consider the lump centered at the vertex X and observe that

$$\mathcal{A} = (\text{Area of } \triangle XDC) + (\text{Area of circular sector } XBC) + (\text{Area of } \triangle XAB).$$

It follows that

$$\mathcal{A} = \mathcal{A}(\theta) = 2 \cdot \frac{\sin(2\theta)}{16} + \frac{1}{8} \cdot \left(\frac{\pi}{3} - 2\theta\right) = \frac{1}{8} \cdot \left(\sin(2\theta) + \frac{\pi}{3} - 2\theta\right).$$

We recall that the question is to determine the portion of the plane that the set $S = S(\theta)$ occupies. Because all triangles in the lattice are congruent, the density of the set S in the plane will be equal to the portion of a single triangle in the lattice that the union of three lumps occupies. Hence

$$\text{Density of } S(\theta) = \delta(\theta) = \frac{3 \cdot \mathcal{A}}{\text{Area of } \triangle XYZ} = \frac{\sqrt{3}}{2} \cdot \frac{\sin(2\theta) + \frac{\pi}{3} - 2\theta}{(1 + \cos\theta)^2}.$$

From[1]

$$\frac{d\delta}{d\theta} = 2\sqrt{3} \cdot \frac{\sin\theta \cdot (\frac{\pi}{6} - \sin\theta - \theta)}{(1 + \cos\theta)^3}$$

it follows that, for $\theta \in \left(0, \frac{\pi}{6}\right)$, $\frac{d\delta}{d\theta} = 0$ if and only if $\sin\theta + \theta = \frac{\pi}{6}$.

By the Intermediate Value Theorem there is $\theta_0 \in \left(0, \frac{\pi}{6}\right)$ such that

$$\sin\theta_0 + \theta_0 = \frac{\pi}{6}.$$

Since the function $\eta(\theta) = \frac{\pi}{6} - \sin\theta - \theta$, $\theta \in \left[0, \frac{\pi}{6}\right]$, is monotone decreasing from $\eta(0) = \frac{\pi}{6}$ to $\eta(\frac{\pi}{6}) = -\frac{1}{2}$, we conclude that θ_0 is the only critical number of the function δ. WolframAlpha gives $\theta_0 \approx 0.263316$ radians.

By the First Derivative Test, the number

$$\delta(\theta_0) \approx \delta(0.263316) \approx \frac{\sqrt{3}}{2} \cdot \frac{\sin(0.526632) + \frac{\pi}{3} - 0.526632}{(1 + \cos(0.263316))^2} \approx 0.229365$$

is the absolute maximum value of $\delta = \delta(\theta)$ on the interval $(0, \frac{\pi}{6})$.

Since $S(\theta_0) \in \mathcal{T}$, it follows that

$$m_1(\mathbb{R}^2) \geq \text{Density of } S(\theta_0) = \delta(\theta_0) > 0.2293.$$

Conclusion

The history of finding the upper and lower bounds of $m_1(\mathbb{R}^2)$ is quite interesting as many different techniques are used.

The best currently known upper bound of 0.25442 was found in 2020 by Gergely Ambrus and Mátê Matolcsi [1]. Going through Ambrus and Matolsci's paper leads the reader to realize that the search for the answer to Leo Moser's problem brings together the best of what contemporary mathematics has to offer, from graph theory to Fourier analysis to algorithms and computation.

We are getting closer and closer to Erdős's conjecture that $m_1(\mathbb{R}^2) < 0.25$ from many decades ago!

[1] For more details, see Exercise 6.6.30 in [5].

References

[1] Ambrus, G. and Matolcsi, M. (2020). Density estimates of 1-avoiding sets via higher order correlations.

[2] Croft, H.T. (1967.) Incidence incidents. *Eureka* 30: 22-26.

[3] P. Erdős, Problems and results in combinatorial geometry. in: Discrete Geometry and Convexity (New York, 1982), Annals of the New York Academy of Sciences 440, New York Academy of Sciences, New York, 1985, pp. 1-11.

[4] Jungić, V. (2020.) An introduction of the problem of finding the chromatic number of the plane, Part I. *Crux Mathematicorum* 45:390-397.

[5] Jungić, V. (2021.) Introduction to Ramsey Theory: Lecture notes for undergraduate course. http://www.sfu.ca/ vjungic/RamseyNotes/Exercises-Polychromatic.html.

[6] Moser, W.O.J. (1991.) Problems, problems, problems. *Discrete Applied Mathematics* 31: 201–225.

[7] O'Connor, J. J., Robertson, F.F. Leo Moser, MacTutor History of Mathematics. University of St Andrews, Scotland, November 2006. https://mathshistory.st-andrews.ac.uk/Biographies/Moser_Leo.

[8] Wyman, M. (1971.) Biographical sketch - Leo Moser. *Rocky Mountain J. Math.* 1:255-256.

Ringing the Changes: A Dance of Algebra and Geometry

by Parker Glynn-Adey

Introduction

The art of change ringing consists of ringing church bells in patterns called changes. It is a curious activity which inhabits a niche between mathematics, music, dance, and devotion. Bell-ringers have studied the patterns and permutations of bells for centuries, and discovered many theorems of group theory through their musical endeavours. In this article, we're going to introduce change ringing and show how a musical pattern of bells can dance through algebra and geometry.

Before we get started, note that change ringing has its own special vocabulary. A piece of music for bells is called a peal. Each peal consists of a number of rows. In each row, every bell must ring once. The operation for passing from one row to another is called a change.

Permutation Groups and Bells

Suppose that you have three bells with different tones. How many different orders could you ring them? For the first ring, you'll have three bells to choose from. For the second ring, there will only be two choices. And finally, for the last ring, there is only a single choice. So, you could ring the three bells in $6 = 3 \cdot 2 \cdot 1$ possible orders. Mathematicians write this as 3! or "three factorial".

Ringing each bell once would produce a rather short peal. Suppose that we wanted to play each of the six possible orders exactly once each. This would tour through all possible permutations of the three bells. There would be $6! = 720$ possible ways to make a peal of three bells. We want to look for some kind of order. We want a pleasingly symmetric way to ring three bells. To ensure that a peal sounds pleasant, we impose the following fundamental rule of change ringing: each change must consist of transpositions of adjacent bells. From one row to the next, a bell cannot move more than one position up or down during a change. This rule is to ensure that the resulting music is intelligible and interesting.

How many peals arise if we limit the movement of the bells to transpositions of adjacent bells? To answer that question we begin by introducing notations for the bells and where they sit in a row. Inside of a row, we specify the positions of the bells using letters. The positions of three bells within a row will be A, B, and C. In the row 321, bell 3 is in position A, bell 2 is in position B, and bell 1 is in position C. We read a row from left to right. In the row 321, bell 3 rings first, bell 2 rings second, and bell 1 rings third.

The fundamental rule of bell ringing says that a bell cannot move more than one position up or down a row during a change. For three bells, we are allowed the transpositions (AB) and (BC). Their effects on a row are illustrated below.

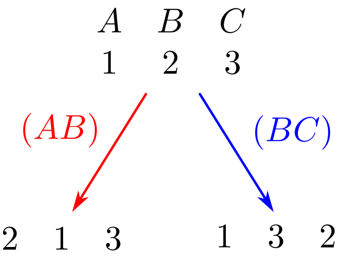


Figure 1: The Transpositions (AB) and (BC).

Notice that if we apply the transposition (AB) twice, we get back to the original row. The operation of transposing A and B is its own inverse.

$$123 \xrightarrow{(AB)} 213 \xrightarrow{(AB)} 123$$

Of course, the same is true for (BC). This has the following consequence: If we are only going to ring three bells, then we ought to alternate the transpositions (AB) and (BC). Otherwise, two consecutive transpositions of the same type will cancel out.

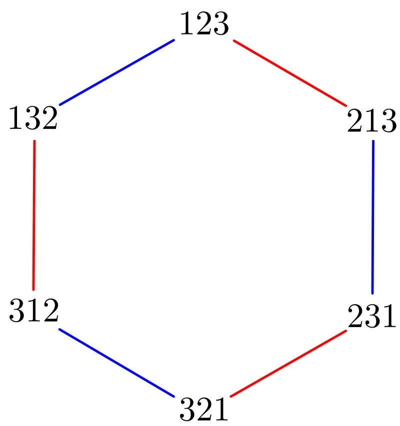


Figure 2: All the Ways to Ring Three Bells

Looking at Figure 2 we see that there are one two possible ways to ring a peal on three bells: either we travel around the hexagon clockwise or we travel around it counter-clockwise. The clockwise traversal of the hexagon is called by bell-ringers “Quick Six” and the counter-clockwise traversal is called “Slow Six”. These are the only possible ways to ring three bells. They form the building blocks for more advanced patterns.

Now we move on to the case of four bells. Here, things are much richer and more interesting, as there are many possible patterns which one can ring with four bells. There are a total of $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ possible rows. The changes allowed are (AB), (BC), (CD), and $(AB)(CD)$. Notice that this last change $(AB)(CD)$ is the product

Quick			Slow		
1	2	3	1	2	3
2	1	3	1	3	2
2	3	1	3	1	2
3	2	1	3	2	1
3	1	2	2	3	1
1	3	2	2	1	3

Table 1: The Peals "Quick Six" and "Slow Six"

of two transpositions (AB) and (CD). Bell ringers call a transposition like (AB) a plain change. To simplify the mathematics involved, we will restrict our attention to the plain changes (AB), (BC), and (CD).

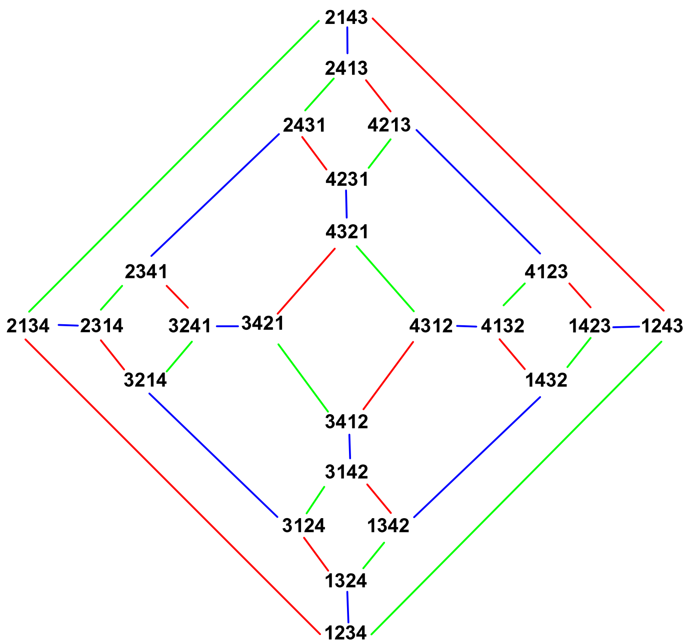


Figure 3: All the Ways to Ring Four Bells

There are some structures worth noticing in Figure 3. The vertex 1234 at the very bottom of the diagram is part of a large square with vertices 1234, 1242, 2143, 2134 going around the border of the figure. The edges of the diagram alternate in colour (CD), (AB), (CD), (AB). Each of the smaller squares has this same pattern. [For readers with some experience in group theory, these diamonds are related the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \langle (AB), (CD) \rangle \subset S_4$.] The vertex 1234 is also part of a distorted hexagon. You can see it at the bottom left hand side of the diagram. It has vertices:

$$1234 \xrightarrow{(BC)} 1324 \xrightarrow{(AB)} 3124 \xrightarrow{(BC)} 3214 \xrightarrow{(AB)} 2314 \xrightarrow{(BC)} 2134$$

This path around the hexagon is "Slow Six" rung on the first three bells. Each region of the diagram is either a square generated by (AB) and (CD), a hexagon generated by (AB) and (BC), or a hexagon generated by (BC) and (CD). The hexagons come in two varieties because we can either ring the first three bells A, B, C or the last three bells B, C, D.

In Figure 3 the squares and hexagons are distorted. It is not possible to draw the diagram in the plane without distortion in such a way that all the edges have the same length, but there is a three-dimensional polyhedron where all the edges have the same length. We describe how to construct this polyhedron. We will modify an octohedron to obtain the desired polyhedron.

An octohedron is a three dimensional solid that has eight triangular faces and six vertices. Looking at Figure 3, we notice that the figure contains exactly six squares and eight hexagons. We must modify the octohedron so that the vertices become squares and the triangular faces become hexagons. If we cut off a small neighbourhood of each vertex on the octohedron then we will introduce small squares at each vertex, because four triangles come together at each vertex of the octohedron, and the triangular faces of the octohedron will become hexagons.

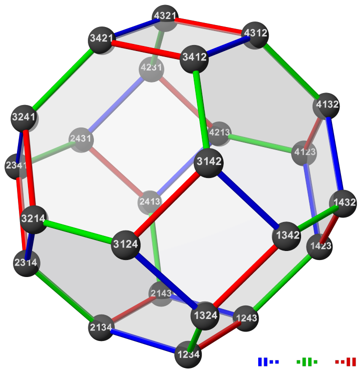


Figure 4: The Truncated Octohedron (Drawn by Tilman Piesk)

This polyhedron is called a truncated octohedron, or the S_4 permutohedron. Any path along edges which visits all the vertices of the permutohedron will be a valid peal of four bells. We encourage the interested reader to find such a path using Figure 3 or Figure 4 (See p. 32 for one solution).

And here, with this encouragement to explore further, we conclude our journey through bell ringing, algebra, and geometry. If you enjoyed this dance through bell-ringing, algebra, and geometry, then I would love to hear from you. You can contact me at parker.glynn.adey@utoronto.ca.

References

[1] Budden & Braemer (1972) The Fascination of Groups. Cambridge University Press London.
This book has a wonderful chapter (Chpt. 24) on the connection between group theory and bell-ringing. Most of the material in this article comes from that chapter. The whole book is delightfully conversational and refreshing.

[2] Batchelor & Henle (2022) The Mathematical Art of Change Ringing.
This short article contains an interview with Majorie Batchelor, a bell-ringer and mathematician.

[3] White (1987) Ringing the Cosets. The American Mathematical Monthly.
This is a heavy mathematical treatment of the relationship between bell-ringing and group theory. Recommended background for reading is MAT C01.

Equidistant Sets and Almost Equidistant Sets

by Zhekai Pang & Yuhong Zhang

Introduction

In two-dimensional space, the vertices of an equilateral triangle form a set of points where every pair has the same distance as every other pair. In three-dimensional space, we can observe that all faces of a regular tetrahedron are equilateral triangles (see Figure 5). We now generalize this to the concept of an equidistant set in d-dimensional space.

Definition. A nonempty subset X of $\mathbb{R}^d$ is called an **equidistant set** if every pair of points in X has the same distance as every other pair of points.

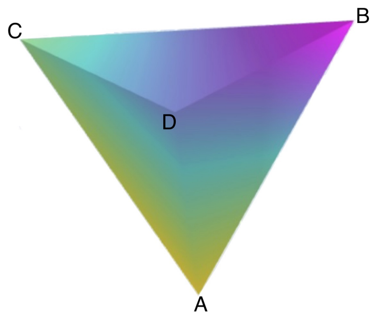


Figure 5: An equidistant set $\{A, B, C, D\}$ in $\mathbb{R}^3$

An interesting problem is to determine how large an equidistant set of points can be for a given dimension d. In fact, an equidistant set in $\mathbb{R}^2$ and $\mathbb{R}^3$ can have at most three points and four points respectively. In general, an equidistant set in $\mathbb{R}^d$ can have at most $d + 1$ points and we prove this in Proposition 1.

In this article, we will focus not only on equidistant sets, but also on a generalization of equidistant sets, almost-equidistant sets.

Definition. Let $A = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a finite subset of $\mathbb{R}^d$ with n points. We say A is **almost-equidistant** if among any three points in the set A, some two are at unit distance.

Figure 6 demonstrates an almost-equidistant set in $\mathbb{R}^2$. Each line segment represents a unit distance between the corresponding pair of points. This object is called the Moser spindle, named after the mathematicians Leo Moser and William Moser. Every triple of points in the Moser spindle has two with distance one, which shows that the maximum size of an almost-equidistant set in $\mathbb{R}^2$ is at least seven.

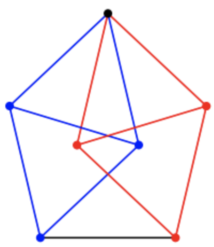


Figure 6: An almost-equidistant set of 7 points in $\mathbb{R}^2$

In the second section, we prove that the maximum size of an equidistant set in $\mathbb{R}^d$ is $d + 1$. In the third section, we state some known bounds on the maximum size of an almost-equidistant set. Our main result demonstrates that if we restrict an almost-equidistant set in $\mathbb{R}^d$ to be contained in a small ball, then its cardinality cannot exceed $2d+5$. Eigenvalues are used to accomplish this.

Equidistant Sets

Linear algebra can be used to derive bounds on the maximum cardinality of an equidistant set. To do this, we define I_n to be the $n \times n$ identity matrix and J_n the $n \times n$ matrix whose entries are all one.

The maximum cardinality of a set of mutually equidistant points is well-known and we provide a proof in the next result.

Proposition 1. Let X be an equidistant set in $\mathbb{R}^d$. Then the maximum of the cardinality of X is $d+1$.

Proof. Without loss of generality, we assume the distance between any two point is 1 since we can scale the distance. We first show that $|X| = d+2$ is not possible. We randomly pick one point in X as the origin and let other points in X be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d+1}$. Then by definition, $\|\mathbf{x}_i\| = 1$ for all $i \in \{1, \ldots, d+1\}$ and $\|\mathbf{x}_i - \mathbf{x}_j\| = 1$ for all $i \neq j$. Thus, $1 = \|\mathbf{x}_i - \mathbf{x}_j\|^2 = \|\mathbf{x}_i\|^2 + \|\mathbf{x}_j\|^2 - 2\mathbf{x}_i^T\mathbf{x}_j$, which implies $\mathbf{x}_i^T\mathbf{x}_j = \frac{1}{2}$ for all $i \neq j$.

Consider the $(d+1) \times d$ matrix

$$A = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_{d+1}^T \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \vdots & \ddots & \vdots \\ x_{(d+1),1} & \cdots & x_{(d+1),d} \end{bmatrix},$$

where we take $\mathbf{x}_i = \begin{bmatrix} x_{i1}, \ldots, x_{id} \end{bmatrix}^T$. Then $(AA^T)_{ij} = \mathbf{x}_i^T\mathbf{x}_j$, which implies that $AA^T = \frac{1}{2}(I_{d+1} + J_{d+1})$.

All eigenvalues of I_d are 1. One eigenvalue of J_d is d and all others are 0. Thus, the eigenvalues of $\frac{1}{2}(I_{d+1}+J_{d+1})$ are $\frac{1}{2}$ with (algebraic) multiplicity d and $\frac{1}{2}(d+2)$ with (algebraic) multiplicity 1, which shows AA^T is invertible. So, $\operatorname{rank}(AA^T) = d+1$. However, $\operatorname{rank}(AA^T) = \operatorname{rank}(A^TA) \leq d$ since A^TA is a $d \times d$ matrix, which leads to a contradiction.

Next, we will show that $|X| = d+1$ is possible. Let the $d \times d$ matrix A be defined as $A = \frac{1}{\sqrt{2}}(I_d + \frac{\sqrt{d+1}-1}{d} J_d)$. Then a quick computation yields $AA^T = \frac{1}{2}(I_d + J_d)$. By taking each row of A to be an point of X, we find a set X with cardinality $d+1$. □

Almost-Equidistant Sets

Let $f(d)$ be the maximum size of an almost-equidistant set in $\mathbb{R}^d$. The Moser spindle (Figure 6) shows that $f(2) \geq 7$. Using a computer-assisted approach, Balko et al. [1] showed that equality holds: $f(2) = 7$. They also prove that $f(3) = 10$ and $f(4) \in \{12, 13\}$. It is an open problem to determine $f(4)$ and the reader is encouraged to try it out ([1] conjectures that $f(4) = 12$ but does not have a formal proof).

In higher dimensions, constructions give lower bounds on $f(d)$. The generalized Moser spindle (the construction of it is too complicated to describe here) shows that $2d+3$ is a lower bound. **Theorem 5** in [1] improves the bound to $2d+4$.

Linear algebra is useful to derive the upper bounds of $f(d)$. Let $A = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an almost-equidistant set and consider the matrix $U = (u_{ij})_{n \times n}$ defined by $u_{ij} = \|\mathbf{v}_i - \mathbf{v}_j\|^2 + (I_n)_{ij} - (J_n)_{ij}$.

In order to prove the main theorem (Theorem 5), we give several lemmas first.

Lemma 2. $\operatorname{tr}(U) = \operatorname{tr}(U^3) = 0$

Proof. By definition of the matrix U, we know $u_{ii} = 0 + 1 - 1 = 0$ for each i. Thus, $\operatorname{tr}(U) = \sum_{i=1}^{n} u_{ii} = 0$. By matrix multiplication, $(U^3)_{ii} = \sum_{j=1}^{n} \sum_{k=1}^{n} u_{ij}u_{jk}u_{ki}$. Thus, if there exists two of i, j, k that are equal, then $u_{ij}u_{jk}u_{ki} = 0$. Otherwise, i, j, k are all distinct, then $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$ are three distinct points in A. Since A is an almost equidistant set, we without loss of generality assume $\|\mathbf{v}_i - \mathbf{v}_j\|^2 = 1$. Then $u_{ij} = 1 + 0 - 1 = 0$, so $u_{ij}u_{jk}u_{ki} = 0$. Thus, $\operatorname{tr}(U) = \operatorname{tr}(U^3) = 0$. □

Lemma 3. The matrix U has at most one eigenvalue greater than 1 and at least $(n-d-2)$ eigenvalues equal to 1.

Lemma 4. Let λ_0 be the largest eigenvalue of U that is greater than 1. Let $\lambda_1 \geq \cdots \lambda_k$ be the eigenvalues of U that are less than 1. If $n \geq 2k$, then $\lambda_0^3 > \frac{(n-k)^3}{k^2} - (n-k-1)$.

Proofs of Lemmas 3 and 4 can be found in [3].

In [4], the author shows that $f(d) = O(d^{\frac{4}{3}})$. This means that some vertical stretch of $d^{\frac{4}{3}}$ dominates $f(d)$ from some d onwards. However, if we consider almost equidistant sets in small annuli, there are some better bounds because of the geometric structure as shown in [3]. We now prove our main result (by applying Lemma 4) that demonstrates if we restrict an almost-equidistant set in $\mathbb{R}^d$ to be contained in a small ball, then its cardinality cannot exceed $2d+5$.

Theorem 5. Let $A \subseteq \mathbb{R}^d$ be an almost-equidistant set with n points. Suppose for all $\mathbf{x} \in A$,

$$\|\mathbf{x}\|_2 \in \left[\sqrt{\frac{1}{2} - \frac{1}{2n}}, \sqrt{\frac{1}{2} + \frac{1}{2n}}\right].$$

Then $n \leq 2d+5$.

Proof. Let G be the Gram matrix $G_{ij} = 2\left\langle \mathbf{v}_i, \mathbf{v}_j \right\rangle$. Then G is positive semi-definite, so all eigenvalues of G are non-negative. Let V be defined as $V_{ij} = \|\mathbf{v}_i\|^2 + \|\mathbf{v}_j\|^2 - 1$. Then $U = V - G + I_n$. Since $\|\mathbf{x}\|_2 \in \left[\sqrt{\frac{1}{2} - \frac{1}{2n}}, \sqrt{\frac{1}{2} + \frac{1}{2n}}\right]$ for all $\mathbf{x} \in A$, we know $\frac{1}{2} - \frac{1}{2n} \leq \|\mathbf{x}\|^2 \leq \frac{1}{2} + \frac{1}{2n}$ for all $\mathbf{x} \in A$. Thus, $|\|v_i\|^2 + \|v_j\|^2 - 1| \leq \frac{1}{n}$ for all i, j, which implies that the absolute value of all entries of V_{ij} are bounded above by $\frac{1}{n}$.

Let μ be the largest eigenvalue of V and $\mathbf{w}$ be the corresponding eigenvector such that $w_i = 1$ and $|w_j| \leq 1$ for all $j \neq i$. Such eigenvector always exist by scaling. Then $\mu = \mu w_i = (V\mathbf{w})_i = \sum_{j=1}^{n} v_{ij} w_j \leq \sum_{j=1}^{n} |v_{ij}||w_j| \leq \sum_{j=1}^{n} \frac{1}{n} \cdot 1 = 1$. Thus, the largest eigenvalue of V is at most 1. Since all eigenvalues of I_n are 1 and the eigenvalues of G are non-negative, the largest eigenvalue of U is at most 2.

Suppose $n \geq 2d+6$. By Lemma 4,

$$8 \geq \lambda_0^3 > \frac{(n-d-1)^3}{(d+1)^2} - (n-d-2).$$

However,

$$\begin{aligned}
\frac{(n-d-1)^3}{(d+1)^2} - (n-d-2) &= (n-d-1)\left[\left(\frac{n-d-1}{d+1}\right)^2 - 1\right] + 1 \\
&\geq (d+5)\left[\left(\frac{d+5}{d+1}\right)^2 - 1\right] + 1 \\
&= (d+5)\left[\frac{16}{(d+1)^2} + \frac{8}{(d+1)}\right] + 1 \\
&\geq (d+1) \cdot \frac{8}{d+1} + 1 \\
&= 9,
\end{aligned}$$

which yields a contradiction. □

Remark 6. In the 2021 Alibaba math competition, a stronger conclusion is proposed, namely $n \leq 2d+4$, under the same assumptions.

Conclusions

In this article, we proved the maximum cardinality of equidistant sets in $\mathbb{R}^d$ and an upper bound of almost-equidistant sets in special cases by matrix analysis.

Mathematicians are still discovering the maximum size of almost-equidistant sets for general cases. In the future, we hope the bounds of the largest size of an almost-equidistant set in $\mathbb{R}^d$ will be more precise as a result of the efforts of mathematicians and the development of computational techniques.

References

[1] Balko, M., Pór, A., Scheucher, M., Swanepoel, K., and Valtr, P. (2020). Almost-equidistant sets. *Graphs and Combinatorics* 36(3):729–754.

[2] Polyanskii, A. (2019). On almost-equidistant sets. *Linear Algebra and its Applications* 563:220–230.

[3] Polyanskii, A. (2019). On almost equidistant sets II. *Electron. J. Comb.* 26(2): Article 2.14 9 pp.

[4] Kupavskii, A., Mustafa, N. H., and Swanepoel, K. J. (2019). Bounding the size of an almost-equidistant set in Euclidean space. *Combinatorics, Probability and Computing* 28(2):280–286.

The Netflix Prize Challenge and Matrix Completion

by Yuhong Zhang & Zhekai Pang

Introduction

In 2006, Netflix held an open competition, which was known as the *'Netflix Prize Challenge'*, with a grand prize of $\$1,000,000$ for improving their original algorithm for the service recommendation system called *'Cinematch'*. The following paragraphs are the details of the competition.

Netflix released the questionnaires to its clients for rating thousands of movies (users can rate a movie from 1 to 5). They received 100,480,507 results showing that 480,189 consumers rated 17,770 movies[1]. However, individual users could not watch all of the movies and give feedback to all of them. So, if Netflix was to construct a huge matrix from all this feedback, the matrix would miss a large number of elements.

	MOVIE A	MOVIE B	MOVIE C	...
USER 1	2	4	5	...
USER 2	5	?	3	...
USER 3	?	1	4	...
...	...	...	...	...

Example of Netflix Dataset

Netflix wanted to infer the ratings of the entire user population for different genres of movies (action movies, fantasy movies, romance movies, and so on) based on a very sparse dataset. Using *'Cinematch'*, the root mean squared error (RMSE, see Remark 7) is 0.9514 which is better than the trivial algorithm, however, it is able to be improved by developing new algorithms. The participants needed to improve the RMSE by 10% to win the prize. On June 26, 2009, the winning team *'BellKor's Pragmatic Chaos'*, achieved a 10.05% improvement, and the RMSE was 0.8558 [1, 3].

Definition. Standard Deviation is a statistic that measures the spread of a dataset relative to its mean.

Definition. Residual is the difference between the predicted values and the observed value of response.

Remark 7. Root Mean Squared Error (RMSE) is used to evaluate the standard deviation of the residual. A smaller RMSE is better than a larger RMSE.

Team *'BellKor's Pragmatic Chaos'* 's algorithm is based on low-rank matrix completion. The users can be categorized by different characteristics, such as gender and age. Meanwhile, the movies can be classified by various genres. Thus, we can group consumers into cliques based on their shared preference for movies, which simplifies the analysis and ensures that the problem is determined.

Mathematics Overview

In the above example, we get an $m \times n$ partially specified matrix M. However, a large portion of the entries are missing. This naturally arises in the matrix completion problem, whose goal is to fill in the missing entries to yield a matrix with as low rank as possible.

The mathematical models behind the matrix completion problem are heavily based on matrix analysis. Thus, we first give a brief overview.

Definition. A **Sparse Matrix** is a matrix such that most of the elements are zero. The sparsity of a matrix is defined as the ratio between the number of zero elements and the total number of elements.

Example 8. The matrix $\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ -1 & 0 & 5 & 12 & 3 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 \end{bmatrix}$ is a sparse matrix with sparsity of $\frac{11}{20}$.

Remark 9. Let M be a $m \times n$ matrix. The **singular value decomposition** of M is $M = U\Sigma V^T$, where U is a $m \times m$ orthogonal matrix, Σ is a $m \times n$ diagonal matrix with non-negative entries on the diagonal, and V is a $n \times n$ orthogonal matrix. The diagonal entries of Σ, denoted as $\sigma_1, \dots, \sigma_{\min\{m,n\}}$, are called the **singular values** of M and are uniquely determined by M.

Given a singular value decomposition of M as above, the following two relations hold:

$$M^T M = V\Sigma^T U^T U\Sigma V^T = V(\Sigma^T\Sigma)V^T$$
$$MM^T = U\Sigma V^T V\Sigma^T V^T = U(\Sigma^T\Sigma)U^T$$

The right-hand sides of these relations describe the eigenvalue decomposition of the left-hand sides. Consequently, the columns of V, also called right-singular vectors, are eigenvectors of $M^T M$. The columns of U, also called left-singular vectors, are eigenvectors of MM^T.

Definition. Let A be a $m \times n$ matrix. The **nuclear norm** of A, denoted as $\|A\|_*$, is defined as the sum of its singular values: $\|A\|_* = \sum_{i=1}^{min\{m,n\}} \sigma_i(A) = \mathrm{tr}(\sqrt{A^T A})$.

The nuclear norm of A can be computed from the singular value decomposition of A.

Definition. Let $x \in \mathbb{R}^n$. The L_1 **norm** of x is defined as $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

Definition. Let A be a $m \times n$ matrix. The L_1 **norm** of a matrix A is defined as $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$.

Mathematical Models

Let $\Omega \subseteq \{1, \dots, m\} \times \{1, \dots, n\}$ be the index set corresponding to the entries in M that are observed. The optimization model is

$$\begin{aligned} &\text{minimize} && \mathrm{rank}(X) \\ &\text{subject to} && X_{ij} = M_{ij} \quad \text{for all} \quad (i,j) \in \Omega \end{aligned}$$

However, this problem is NP-hard, which means that it is very difficult to solve when the size of the matrix grows larger. In [4], the authors found an alternative way as following:

$$\begin{aligned} &\text{minimize} && \|X\|_* \\ &\text{subject to} && X_{ij} = M_{ij} \quad \text{for all} \quad (i,j) \in \Omega \end{aligned}$$

This objective function is a convex relaxation of the previous one. Now it becomes a convex optimization problem, so as [6] shows, there exist standard quick algorithms to solve compared to the original non-convex optimization problem.

In recent literature, there are new methods that make improvements of the algorithm. We first introduce the rank-sparsity decomposition. Given a $m \times n$ matrix M, we want to find a low-rank matrix L and sparse matrix S such that $M = L + S$. In [2], the convex model is

$$\begin{aligned} \underset{L,S}{\text{minimize}} \quad & \|L\|_* + \gamma\|S\|_1 \\ \text{subject to} \quad & M = L + S \end{aligned}$$

Then it leads to the robust matrix completion model, which is

$$\begin{aligned} \underset{L,S}{\text{minimize}} \quad & \|L\|_* + \gamma\|S\|_1 \\ \text{subject to} \quad & M_{ij} = L_{ij} + S_{ij} \text{ for all } (i,j) \in \Omega \end{aligned}$$

This can be solved using faster algorithms such as interior-point method.

Applications

Matrix completion can be applied in different areas. For recommendation system, the popular approach in this field is collaborative filtering, which uses the known data or previous feedback to infer personal taste and promote the objects that users with similar inclinations will be interested in, such as the intrinsic problem in the Netflix Price Challenge, boosting the original collaborative filtering algorithm. The matrix completion can impute missing values, which can target clients' preferences more accurately to make more profit for the industry.

In the field of computer vision, matrix completion can be applied in graph repairing and background extraction. In low-rank matrix completion, we can always decompose the matrix into a low-rank matrix and a sparse matrix, i.e. $M = L + S$. In this case, M is the big matrix that we want to complete, L is the low-rank matrix, and S is the sparse matrix. Figures 7 and 8 are an example [5].

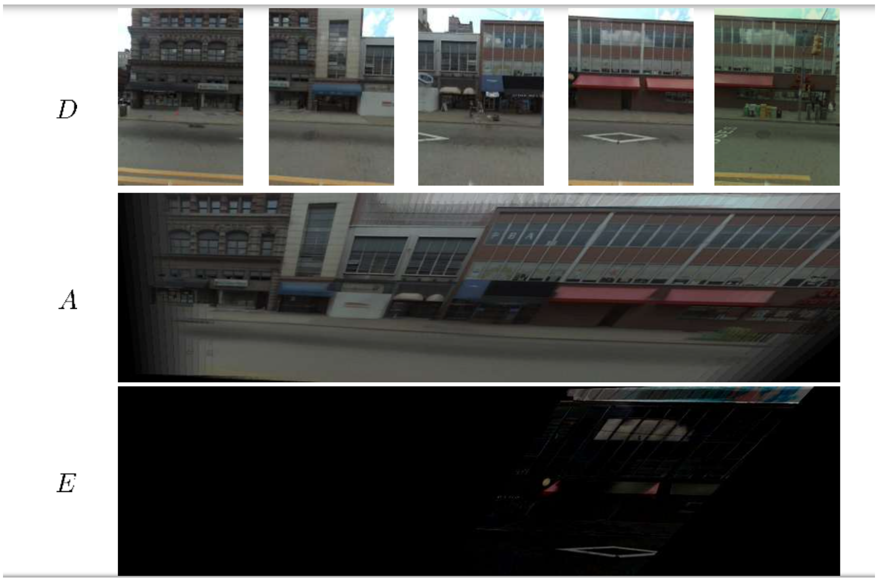


Fig 7: Example of matrix in graph repairing, D is the graph that needs to be created. A can be considered as the low-rank component. E is the error term, which is the sparse component. Picture from [5].

Comparing the development of three approaches, we can discover errors have less of a negative impact of graph repairing by using low-rank matrix completion methods. AutoStitch and Photoshop have larger recovery errors, the graph is stretched and still lacks some elements to difference severity. However, the low-rank method results in a clearer, more accurate and higher-resolution graph based on the original low-resolution and information-lost graph with some noise. Hence, the algorithm exerts significant impacts on computer vision.

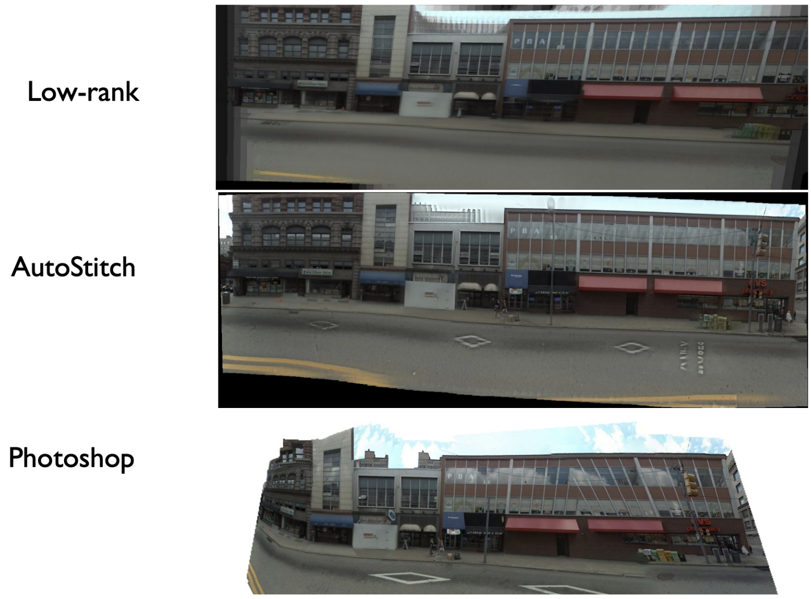


Fig 8: The results from the low-rank matrix completion method and different graphic tools. Picture from [5].

Conclusion

To sum up, the matrix completion problem not only contains rigorous mathematical theory, but also plays an important role in the industry and has many practical applications. However, since low-rank matrix completion is based on singular value decomposition, it is expensive. Thus, people have tried other approaches such as gradient descent and alternating least squares minimization to handle matrix completion problems.

References

[1] Bennett, J., and Lanning, S. (August 12, 2007). The Netflix Prize. *Proceedings of KDD Cup and Workshop 2007*. Retrieved March 30, 2022, from `http://www.cs.uic.edu/~liub/KDD-cup-2007/proceedings/The-Netflix-Prize-Bennett.pdf`

[2] Candès, E. J., Li, X., Ma, Y., and Wright, J. (2011). Robust principal component analysis?. *Journal of the ACM (JACM)* 58(3):1–37.

[3] Kincaid, J. (2009, July 26). *The Netflix Prize comes to a buzzer-beater, nailbiting finish*. TechCrunch. Retrieved March 30, 2022, from `https://techcrunch.com/2009/07/26/the-netflix-prize-comes-to-a-buzzer-beater-nailbiting-finish/`

[4] Fazel, M., Hindi, H., and Boyd, S. (2004, June). Rank minimization and applications in system theory. In *Proceedings of the 2004 American control conference* (Vol. 4, pp. 3273–3278). IEEE, 2004.

[5] Yi Ma, John Wright, and Allen Y. Yang. Sparse Representation and Low-Rank Representation in Computer Vision, ECCV 2012 Short Course. `https://people.eecs.berkeley.edu/~yima/matrix-rank/references.html#OurTutorials`

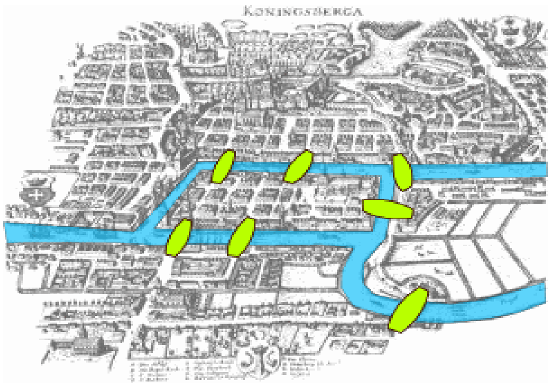
[6] Jain, P., and Kar, P. (2017). Non-convex optimization for machine learning. *Foundations and Trends® in Machine Learning* 10(3–4):142–363.

Two Applications of Euler's Theorem

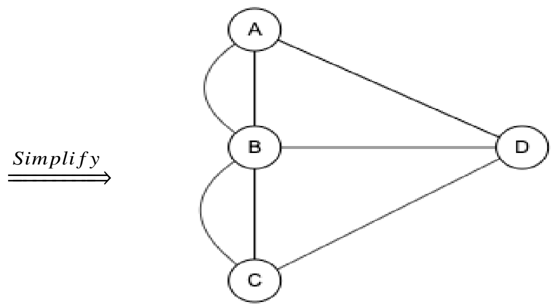
by Zhiquan Cui

Introduction

In Königsberg, a historic Prussian city that is now known as Kaliningrad in Russia, a river partitions the city into four different land masses. The City of Königsberg built seven bridges connecting these land masses to allow people to conveniently visit each land mass. Figure 9a shows what the geography of the City of Königsberg looked like in the 1700's.



(a) Map of City of Königsberg



(b) Simplified Seven Bridges Map

Fig 9: Seven Bridges of Königsberg Problem

Now imagine that you are in one of the four land masses and want to visit the other ones while traversing every bridge exactly once and ending where you started. Are you able to come up with a route that satisfies these constraints? Maybe try to draw some routes and see if any one works. After a while, you may find it impossible to design a route that traverses each bridge exactly once and has the same starting and ending point. Indeed, this is a question brought up by a mathematician named Carl Gottlieb Ehler around 1735. He sought help from Leonhard Euler who proved that it is impossible to design a route that satisfies the above constraints.

In this article, we will introduce Euler's theorem and explore its real-world applications: a type of Chinese Postman Problem in the context of COVID-19.

Graph Theory Terminology and Euler's Theorem

You may be curious about how Euler proved that it is impossible to design a route that crosses each bridge exactly once and has the same starting and ending point. It is without question that his proof is very beautiful, but more importantly it uncovered a whole new world of mathematics called *graph theory*. In his work, Euler proposed and proved a theorem that solves this problem and that is still widely used. We will give an example of one of its applications in the next part of this paper to show how powerful this theorem is.

To help you understand Euler's Theorem better, let's introduce some terminology.

A **multigraph** $G = (V, E)$ is an object consisting of a non-empty set of vertices V and a multiset E of unordered pairs of vertices. Multiple edges between a pair of vertices is permitted in a multigraph. Note that every edge has two **vertices** as its **endpoints**. For example, in Figure 9b, $V = \{A, B, C, D\}$ is the vertex set and $E = \{\{A, B\}, \{A, B\}, \{B, C\}, \{B, C\}, \{A, D\}, \{C, D\}, \{B, D\}\}$ is the edge set. An edge is **incident** to a vertex if the vertex is one of the endpoints of that edge. The **degree of a vertex** v is the number of edges incident to v. A **walk** is a finite sequence of edges that joins a sequence of vertices; when all edges are distinct, we call it a **trail**. An **Eulerian circuit** of a graph G is a trail that visits each edge of G exactly once.

We now state Euler's theorem that may be used to solve the Seven Bridges of Königsberg problem.

Euler's Theorem. An undirected and connected multigraph G has an Eulerian circuit if and only if all vertices have even degrees.

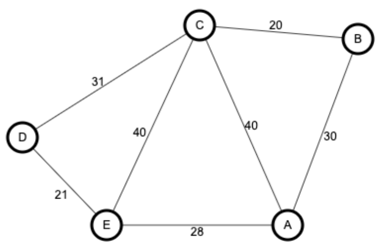
Before applying Euler's Theorem to this problem, we give an intuitive proof of the theorem. First, suppose that an undirected and connected multigraph G has an Eulerian circuit. Any vertex in G except the starting and ending point must have an even degree. This is because we need a pair of edges each time we arrive at and leave a vertex. Moreover, since we are required to have the same starting and ending vertex on the route, that vertex should also have an even degree because leaving it at the start and returning to it at the end needs two edges. Every time to visit this vertex also requires two edges. Now, suppose that all vertices of an undirected and connected multigraph of G have even degrees. Let $v \in V(G)$ be an arbitrary vertex. We claim that there is an Eulerian circuit that starts from and ends at v. Let u_1, u_2 be two vertices adjacent to v where u_1 and u_2 are not necessarily different vertices. Since the multigraph is connected, we can find a walk $W = v \to u_1 \to \cdots \to u_2$ and thus a cycle $C = v \to u_1 \to \cdots \to u_2 \to v$ as $vu_2 \in V(G)$. Now it remains for us to show that C visits every edge exactly once. Since every vertex must have an outgoing edge corresponding to an incoming edge, we can always leave a vertex via an unvisited outgoing edge and ensure every edge is visited at most once in C. Moreover, all edges in G are in C because we can always pick a pair of incoming and outgoing edges to traverse the unvisited edges at any vertices. Therefore, C visits every edge exactly once.

Now, let's see how Euler's Theorem allows us to conclude that it is impossible to find a route that visits each bridge exactly once and has the same starting and ending point. Figure 9b is a multigraph representation for the Seven Bridges of Königsberg problem. The four nodes A, B, C, D represent the four land masses and the edges represent the bridges connecting them. Note that the graph is an undirected and connected multigraph and there are four vertices each with odd degrees. By Euler's Theorem, there does not exist a trail that visits every edge exactly once while having the same starting and ending point. That means there does not exist a route that has the same starting and ending island and visits every bridge exactly once.

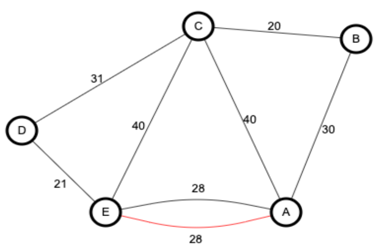
Chinese Postman Problem

The Chinese Postman Problem is a well-known application of Euler's Theorem. A postman starts from the post office and tries to deliver all the mail using the shortest distances while visiting every street at least once. To solve this type of problem, we first simplify the map into a weighted graph, where each edge is assigned a numeric "weight" representing the distance between the endpoints of the edge. Then we will find a least-weight walk connecting all the odd vertices in the graph and double the edges on the walk to form a new multigraph. Observe that all vertices in this new multigraph have even degrees. Lastly, we will try to find an Eulerian circuit of the simplified graph. When the weighted multigraph does not contain an Eulerian circuit, the Chinese mathematician Mei-Ko Kwan (1962) proposed an algorithm that resolves this problem (Kumar, 2017, p.12).

Here we use the algorithm proposed by Mei-Ko Kwan to solve a hypothetical problem under the context of COVID-19. Suppose the City of Toronto government is planning to collect the COVID-19 test specimens from four large test locations and temporary test locations on each road in the city. The government wants to get all the specimens in the shortest distance and return to the laboratory at the end.



(a) Original Map



(b) Map with Eulerian Circuit

Fig 10: COVID-19 Test Locations

In Figure 10, the node A represents the laboratory in downtown Toronto where COVID-19 specimens are tested. Vertices B, C, D and E represent the four COVID-19 test centres in Markham, Vaughan, Brampton and Mississauga, respectively. The weight of each edge represents the distance in kilometres between two test centers connected by the edge. The specimen collector will set off from the laboratory, visit the four large test locations and temporary test locations along each road, and deliver the test specimens back to the laboratory. Notice that there are odd-degree vertices in the above graph. By Euler's Theorem, it is impossible to traverse the graph and return to the starting point while visiting every edge exactly once. Since the odd-degree vertices all have degrees greater than 1, they must be visited more than once so that the edges incident to them are all visited. Our goal is to minimize the distance made to revisit the odd-degree vertices.

Here we provide a special case of Kwan's algorithm for solving the Chinese Postman Problem when the weighted graph G has exactly two odd-degree vertices (Bondy & Murty, 1977).

Step 1. Find the least-weight walk w between the two odd-degree vertices.

Step 2. Add edges in the walk w to the original graph G to form a new multigraph G'.

Step 3. Find an Eulerian circuit e of the new multigraph G' ($G' = G$ if all vertices in G are in even degrees). e is the optimal route and the total weight of e is the minimum total distance as desired.

Now let us apply this algorithm to the problem. Notice that vertices A and E have an odd degree of three.

Step 1. Find the least-weight walk to connect vertices A and E. Dijkstra's Algorithm is a powerful tool for us to achieve this goal. It takes two vertices as input and outputs a minimum weight walk in the graph between these two vertices. Using Dijkstra's algorithm, we can find the least-weight walk between E and A. However, for this example it is easy to analyze all possible walks to determine the one with minimum weight. Edge EA has the second least weight among all edges incident to vertex E. Although the edge ED has a smaller weight, it is not a candidate because the next edge that must be picked is DC which has a weight 31 and $31 + 21 > 28$. Since any walk containing edge ED, DC or EC cannot have minimum weight, EA is the least-weight walk we should choose.

Step 2. Figure 10b is the resulting multigraph after adding another edge between E and A.

Step 3. To find the Eulerian circuit e of the multigraph in Figure 10b, we can use Fleury's Algorithm. Since this graph is relatively simple, we can obtain the following Eulerian circuit by inspection

$$A \to B \to C \to D \to E \to C \to A \to E \to A$$

Therefore, the shortest travel distance needed to collect specimens from all the COVID-19 test locations is 238 kilometres.

Conclusion

In this article, we introduced a powerful theorem that can be used to solve the Seven Bridges of Königsberg and modern problems related to optimal path design using limited resources. Euler's Theorem can be applied in other problems such as street sweeper and mail delivery route design. [1] and [2] are great resources for you to learn more about this theorem and its application. In addition, in case you want to learn more about Fleury's algorithm and Dijkstra's algorithm, you may refer to [2].

References

[1] Kumar, A.(2017). A study on Euler Graph and its Applications. *International Journal of Mathematics Trends and Technology,* 43(1), 9-15.

[2] Bondy, J. A., & Murty, U. S. R. (1977). *Graph Theory with Applications.* Macmillan.

Math, Musical Theatre, and Medicine: How One UofT Medical Student is Intertwining Them All

by Olivia Rennie

For most medical students, the world of science and the world of the arts inhabit two very different spheres. Not for one UofT medical student, **Shreya Jha,** who not only continues to invest time into her artistic talents, but also creates original pieces that are produced and performed publicly. Shreya, who is currently a second-year medical student in the Temerty Faculty of Medicine, completed a combined degree in music and science prior to beginning her journey to become a physician. Her most recent production, *Statistics*, combines elements of math, music, and history into a heartwarming and thought-provoking piece. *Statistics* premiered at Toronto's Fringe Festival this summer, bringing Shreya's story to live audiences.

For context, *Statistics* depicts the story of Rose, an undergraduate student with the desire to go into medicine. The story follows her throughout the application process, but in particular, it depicts the way she draws inspiration from the historical figure who discovered the double-helix structure of DNA, Rosalind Franklin. Largely unknown due to the sexism that existed in mathematics and biology during the 1940s and 50s, this play shines a powerful light on Rosalind's story and her contribution to science. It also captures the struggle of the modern pre-medical student, as they try to make sense of who they are in the frenzy to become a physician.

To learn more about the process behind the creation of *Statistics*, I had the opportunity to interview Shreya, gaining perspective on her creative process, words of wisdom, and future projects.

What inspired you to write this play?

This show finally precipitated after about a year of wanting to break into musical theatre composition – my background was in classical composition, but I was so intrigued by the genre of musicals but didn't know how to start. That wish and urgency was kind of building up all through my second year of university, but I could never find a collaborator to work on the story and dialogue and parts I wasn't as familiar with. Then one night in April 2018, during exam season, I came up with an idea, and I realized maybe I didn't need a collaborator for this particular project. I was envisioning a story about being a science student, one that showcased all the details and strife of studying a difficult subject at a large university in a way that had never been put onstage before. I then thought about how far science and academia have come and my attention turned to this amazingly inspiring figure of the past, Rosalind Franklin. And I decided to juxtapose the two into an hour-long musical and *Statistics* was born. I spent that whole summer researching and writing and told almost nobody I was writing the show because I was terrified it wouldn't work out. The day I learned that it was chosen to premiere at the University of Toronto drama festival was probably one of the best days of my life to date, and it's amazing to think of how far the show has come.

Why do you believe it's important that the arts are integrated with the sciences?

Arts integrated with sciences kind of describes my entire career goal! I think it is so important for two reasons – one, use of art and telling of stories in the sciences brings a really human aspect to problems in medicine that can feel really granular or cut-and-dry at times. Two, I firmly believe we have barely scratched the surface of the therapeutic potential of art in the sciences. This is a really big research interest and career goal of mine – I've done investigations into the psychophysiology of optimizing piano performance, music-evoked autobiographical memories in dementia, and predictors of musical expertise and really hope to do more. A big part of why I wanted to go into medicine is I think it is one of the best pathways to come up with really targeted and creative, often arts-based interventions to improve health outcomes in different populations.

How do you find balance between being an artist and medical student?

It was honestly a really big adjustment going from a dual degree music and science student to a freelance professional composer and medical student – I consider both music and science to be part of my career. I have to do a lot of work to maintain each identity by planning ahead to make sure I am dedicating time to school and compositional projects, and also I have to have the confidence that I can balance the two sides the way I want to. I like to keep a good mix of interdisciplinary projects (music/science research or advocacy) and compositional projects alongside school. Ultimately, I have to remember that I have the power to pave my career however I want to and that there is so much ahead of me to discover and create, and that keeps me motivated!

In what way do you believe having an understanding of mathematics and statistics is important for people outside of these academic disciplines - for example, physicians?

The subject of statistics as is described in the show is very central to my belief about its importance – our protagonist very simply says, "[b]iology explains the world around us, and statistics determines which of those explanations is meaningful." Statistics and math add meaning to what we observe in the natural world, and allows us to critically appraise what we discover. I personally find the subject of statistics so fascinating and intuitive because of this.

What message do you want your audience to take away from this production?

This message of the show has changed overtime – I think in the first draft, it started as a message about you being more than your work, because that was something I needed to remember at the time. And that is very true, but I wanted to diversify from that in future drafts. The message of this show has evolved now - doing something you love is often the reward in itself, and the most successful people do what they do out of love and not due to inherent desire for success. Another message is that science is, above everything else, beautiful.

Any hints about future projects you're planning/currently working on?

So many! I'm planning to get back to editing my other two musicals and having them performed somewhere. I also have a few ideas for new shows – one is on the topic of dementia and is particularly pertinent to medicine and the research I am doing this summer. I'm hoping I'll get to do some work on that for the rest of the summer after *Statistics* closes.

Pascal's Triangle

by Yi Chen, Mengmeng Shang, Zhekai Pang

Introduction

Throughout history, many cultures have mastered 'Pascal's triangle' number pattern and consequently have arrived at the binomial theorem [6, 188-198, 247, 270-278, 354-355, 438-439]. Here we will investigate briefly about the artistic beauty hidden under the Pascal's triangle.

Pascal's triangle

Construction of Pascal's triangle [5, 49-50]

Pascal's triangle has a long history. It can be considered as a game of filling numbers. The rules are as follows:

1) Pascal's triangle has an infinite number of floors. We need to fill each floor with n numbers for each $n \in \mathbb{N}$. Then we arrange them from left to right, which gives the triangular shape.

2) The leftmost number and the rightmost number of each floor is 1.

3) Each number is equal to the sum of its 'ceiling' from the previous floor. That is $(n, k) = (n-1, k-1) + (n-1, k)$, where (n, k) denotes the k^{th} element of the floor n. This equality is usually referred to as the 'Pascal's Identity'.

The conditions 2 and 3 give a unique way to fill in numbers to Pascal's triangle, as shown in figure 1. It also turns out that Pascal's triangle is filled by binomial coefficients. And one can easily check that Pascal's triangle is also symmetric, as the symmetry property from binomial coefficients.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

The Pascal's triangle in number format [5, 49-50]

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

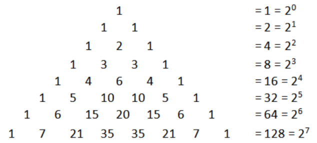
The Pascal's triangle in the binomial coefficients format [5, 49-50]

Numerical Patterns of Pascal's triangle

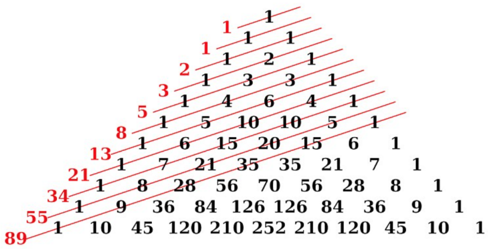
There are some interesting patterns in the Pascal's triangle by taking sum of entries. First of all, let us look at the 'diagonals'. The sum of the numbers on the diagonals turn out to be Fibonacci numbers as shown in the figure 2. That is:

$$\sum_{\substack{x+y=n \\ x \geq y}} \binom{x}{y} = F_n$$

To see this result, notice that the two previous diagonal exactly cover the ceiling of the next diagonal. From Pascal's identity, we know that the sum of two previous diagonals is equal to the sum of the next diagonal. This relationship coincides with the Fibonacci recursive relation.



Row sum are powers of 2



Fibonacci Sequence

Also, if we sum up each row in the Pascal's triangle, we will get powers of 2 as shown in figure 3. And this is a direct result by applying binomial theorem on $(1+1)^n$.

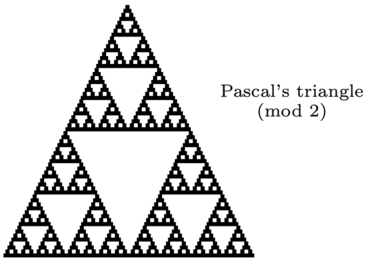
$$\sum_{m=0}^{n} \binom{n}{m} = 2^n$$

Hidden Patterns under Field $\mathbb{Z}_2$ [7][8]

We could also obtain some beautiful patterns on Pascal Triangle by taking the quotient field.

1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1
1 0 1 0 1 0 1

The Pascal Triangle mod 2 [7]

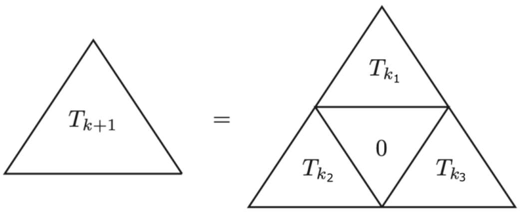


The Pascal Triangle mod 2 [7]

We take modulo 2 to each entry of the original pascal triangle. The new triangle we get is called Pascal's Triangle modulo 2. We can visualize it by using black to represent $[1]_2$ and white to represent $[0]_2$. Then we will obtain the beautiful diagram as above. We will explain the beauty in this section.

Reasoning behind the Beauty

First of all, we observe that this pattern can be repeatedly divided into smaller pieces. We can divide the large triangle into four pieces with the middle one in white and the remaining three identically. Let us clarify this property.



Self-similarity [7]

Regarding the diagram, we can make the following assumptions:

1) The triangle T_{k_1} on the top has height 2^k(contains 2^k rows) and is symmetric. Since each time we divide a large triangle into small triangles with half of its height, it is natural to assume each has a height of power of 2. By the symmetry property of Pascal's triangle, it is symmetric.

2) For $0 \leq x < 2^k$, we have $\binom{x}{y} \equiv \binom{x+2^k}{y} \mod 2$. This illustrates that the top triangle T_{k_1} is identical to the left triangle T_{k_2}. So T_{k_2} is also symmetric and its reflection is identical to itself. Therefore, by symmetry property of pascal triangle, the right triangle T_{k_3} is reflection of T_{k_2} also identical to T_{k_1}. So all 3 triangles $T_{k_1}, T_{k_2}, T_{k_3}$ are identical under this assumption. What's a practical amoun 3) For $0 \leq x < 2^k$ and $x < y < 2^k$, $\binom{x+2^k}{y} \equiv 0 \mod 2$. This illustrates the triangle in the middle is in white.

Note that 2) and 3) characterize the property. However, it is very complicated to check all conditions. Recall that Pascal's identity gives us the power to determine the entire triangle once we are given the numbers on two 'diagonals'. We can use this fact to simplify our conditions.

2') For $0 \leq x < 2^k$, $\binom{x+2^k}{x} \equiv \binom{x}{x} \equiv 1 \mod 2$. We have number 1 on all diagonals of T_{k_2} and T_{k_1}, so they must generate the same triangle.

3') For $0 < y < 2^k$, $\binom{2^k}{y} \equiv 0 \mod 2$. Suppose we have a segment of 0, by Pascal's identity, the numbers next floor beneath this segment is also 0. This process eventually generates a triangle area of 0 as we desired in assumption 3).

Also, notice that 3') implies 2') since 3') implies 3) shows $\binom{x+2^k}{x+1} \equiv 0 \mod 2$ for all $0 \leq x < 2^k$. Together with $\binom{2^k}{0} = 1$ and Pascal's identity, we verify 2').

Therefore, our goal is to prove the following proposition:

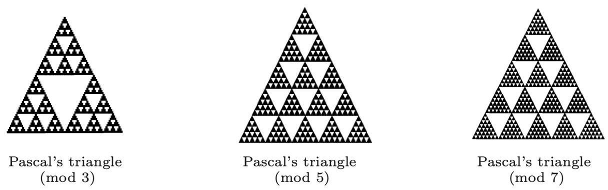
Proposition 10. For $k \in \mathbb{N}^+$ and $0 < y < 2^k$, we have $\binom{2^k}{y} \equiv 0 \mod 2$.

Proof. Note that $(1+x)^{2^k} = \sum_{y=0}^{2^k} \binom{2^k}{y} x^y$. So it is enough to show $(1+x)^{2^k} \equiv 1 + x^{2^k} \mod 2$ [3]

We prove by induction. The base case ($k = 1$) is trivial. Now suppose the equation holds for fixed $k \in \mathbb{N}^+$. Then $(1+x)^{2^{k+1}} = ((1+x)^{2^k})^2 = (1+x^{2^k})^2 \equiv 1 + 2x^{2^k} + x^{2^{k+1}} \equiv 1 + x^{2^{k+1}} \mod 2$, where the second equality uses the induction hypothesis. □

Hidden Patterns under Field $\mathbb{Z}_p$

Furthermore, we consider Pascal's Triangle modulo p, where p is a prime. We can visualize it by using white to represent $[0]_p$ and black to represent $[k]_p$, where $[k]_p \neq [0]_p$.



Pascal's Triangle mod p

The condition we need to build up patterns of prime p is as follows:

Proposition 11. For $k \in \mathbb{N}^+$, $0 < a < p$ and $0 \leq y \leq ap^k$, we have $\binom{ap^k}{y} \not\equiv 0 \mod p \iff y \equiv 0 \mod p^k$

Why does this proposition generalize the pattern? Because it produces several segments of 0's at $\binom{ap^k}{bp^k+1}$ to $\binom{ap^k}{(b+1)p^k-1}$, which generate the small white triangles. It also guarantees the two diagonals $\binom{ap^k}{bp^k}$ to $\binom{(a+1)p^k}{bp^k}$ and $\binom{ap^k}{bp^k}$ to $\binom{(a+1)p^k}{(b+1)p^k}$ are in the same equivalent class.

Therefore, it generates a non-zero multiple of a smaller Pascal's triangle of size p^k, which has an identical pattern.

We will need Lucas's theorem to prove the proposition 11.

Theorem 12. Let $m, n \in \mathbb{Z}_{\geq 0}$ and p be a prime. Let $m = \sum_{i=0}^{l} m_i p^i$ and $n = \sum_{i=0}^{l} n_i p^i$ be the base p expansions of m and n, respectively. Then

$$\binom{m}{n} \equiv \prod_{i=0}^{l} \binom{m_i}{n_i} \mod p$$

Proof. (of proposition 11)

Let $m = ap^k$ and $y = \sum_{i=0}^{k} y_i p^i$, then apply Lucas's theorem, we have $\binom{ap^k}{y} \equiv \binom{a}{y_k}\binom{0}{y_{k-1}} \cdots \binom{0}{y_0} \mod p$. As $y_k \leq a < p$, we have $\binom{a}{y_k} \not\equiv 0 \mod p$. Therefore, $\binom{ap^k}{y} \not\equiv 0 \iff y_0, y_1, \ldots, y_{k-1} = 0 \iff y \equiv 0 \mod p^k$ □

Conclusion

We have discussed the hidden number patterns in the Pascal's triangle, the link between Pascal's triangle and the fields $\mathbb{Z}_2$ and $\mathbb{Z}_p$. This has been an interesting topic since ancient time. The unified rigorous mathematics and the artistic beauty hidden behind the mathematics had attracted both artists and mathematicians to investigate topics such as Pascal's triangle.

References

[1] N. Fine, *Binomial Coefficients Modulo a Prime*, The American Mathematical Monthly, 1947.

[2] F. S. Roberts, B. Tesman, 2009, *Applied Combninatorics*. 2nd edition. Tayler and Francis Group, CRC Press.

[3] Meinrenken E.(2019-2020), MAT157Y1Y Analysis I. Retrieved March 12th, 2022.

[4] Cavers M.(2021), MATC44H3 Introduction to Combinatorics. Retrieved March 13th, 2022.

[5] Lovász, L. (2003) Discrete Mathematics: Elementary and Beyond",

[6] Joseph, G.G. (2011) The Crest of the Peacock: Non-European Roots of Mathematics (Third Edition)

[7] Granville, A. (2021) Number Theory Revealed: An Introduction and Number Theory Revealed: A masterclass - Appendix 3A. Factoring binomial coefficients and Pascal's triangle modulo p https://dms.umontreal.ca/~revealed/Appendix3A.pdf, Accessed: 2022-03-13

[8] Riddle, L. (2022) Classic Iterated Function Systems: Mathematical Behavior of Pascal's Triangle (mod 2)

Reaffirming The Validity of Cantor's Diagonal Argument Through a Systematic Approach

by Artin Ghafooripour

History and Introduction to Cantor's Diagonal Argument

Georg Cantor, the founder of set theory and a renowned mathematician, made massive breakthroughs in math with his work on set theory. Within Cantor's work lies the groundbreaking assertion that differently sized infinities exist. A prominent example of such differently sized infinities is Cantor's theorem that the set of all natural numbers (including 0, represented as $\mathbb{N}$) is smaller than the set of all real numbers between 0 and 1 (represented as $\mathbb{R}$), despite both sets being infinite - in short, there is no bijection from $\mathbb{N}$ to $\mathbb{R}$. Despite how controversial this theorem was, Cantor's work definitively proved it in numerous ways,[1] revolutionizing math in the process.

Cantor's most famous proof, the diagonalization proof[2] (also known as the diagonal argument) was published in 1891 and has broad applications, including a proof for the theorem that $\mathbb{N}$ and $\mathbb{R}$ are not bijective. Cantor's diagonal argument is a simple and elegant proof by contradiction that compares the cardinality of two infinite sets by trying to match the members of each set one-to-one, based on the fact that bijective sets will match exactly[3]. The proof is as follows: begin by assuming $\mathbb{N}$ and $\mathbb{R}$ have the same cardinality. Then, create an infinite table, containing in one column the set of all natural numbers from 0 to infinity in ascending order, and in the other column, a complete set of real numbers between 0 to 1. An issue immediately arises when trying to include $\mathbb{R}$ — members of $\mathbb{R}$ can't be arranged in ascending order, as there is no first real number after 0. To address this, Cantor's diagonal argument allows $\mathbb{R}$ to be arbitrarily ordered, as long as all members of $\mathbb{R}$ are present at some point in the column. In this table, members of $\mathbb{N}$ act as index numbers to the corresponding real number. Cantor then diagonalizes the $\mathbb{R}$ column, taking the first digit of the first real number and changing it, the second digit of the second real number and changing it as well, and so on. Through this process, a new real number is created that cannot be found anywhere on the original list of real numbers, revealing it was never indexed despite all of $\mathbb{N}$ already indexing something. This proves $\mathbb{R}$ has a higher cardinality than $\mathbb{N}$, therefore contradicting the initial assumption and proving that there is no bijection between $\mathbb{N}$ and $\mathbb{R}$.

Natural Numbers (Index Numbers)	Real Numbers Between 0 and 1
0	0.**8**723468276387...
1	0.8**6**54785762376...
2	0.52**4**6376654378...
3	0.098**9**283848321...
4	0.9574**6**25637265...
5	0.93758**2**9756746...
$\vdots$	$\vdots$

Digits to be diagonalized (highlighted in red): 864962 . . . New number $N_1 = 0.975073\ldots$ The red numbers must change in some way; in this case, they increase by 1.

Fig. 1. Demonstration of diagonalization.

Here it should be noted that Cantor's diagonal argument has numerous applications as a proof, and so can also be misapplied to produce false or misleading results. There are certain rules when applying any proof, which, if violated, may yield faulty results. Very briefly, some of the rules in Cantor's diagonal argument are the existence of "a side, a top, an array, a diagonal, a value, and a countervalue."[4]

Furthermore, it is important to note that when looking at a bijection, the order of members of a set is arbitrary, as any order should give the same result; comparing the cardinality of two sets needs nothing more than counting and matching their members. Misapplications, like the subject of this paper, can lead to contradictory and/or paradoxical results, so one must be careful to ensure the rules are followed.

Arriving At The Misleading Contradiction To Cantor's Theorem

The very first obstacle when trying to demonstrate a bijection between $\mathbb{N}$ and $\mathbb{R}$ is that, as mentioned before, there is no first real number after 0. But what if there was a way to force $\mathbb{R}$ into a specific, countable order? Such an arrangement would have a clear pattern, so each member of $\mathbb{R}$ should have a predictable location on the table. This, in turn, means that they should also be indexable by members of $\mathbb{N}$, as all N also have predictable locations. Since this modification doesn't change the cardinality of $\mathbb{R}$, the end result of Cantor's proof should stay the same, so let's try it and see if that holds.

To create a simple approach to this problem, the binary (base-2) number system will be put in place of the base-10 positional notation (later, the connection between the two number systems in relation to Cantor's theorem will be made clear). In order to determine what specific order to force the base-2 members of $\mathbb{R}$ into, truth tables prove a helpful tool. Commonly used when designing circuits, truth tables document every possible combination of inputs in any given circuit, as well as their desired output. These combinations are arranged in ascending order and can each be given an index number (which also corresponds to their value when converted to base-10).

Index Number	Input A	Input B	Input C	Output
0	0	0	0	x
1	0	0	1	x
2	0	1	0	x
3	0	1	1	x
4	1	0	0	x
5	1	0	1	x
6	1	1	0	x
7	1	1	1	x

Fig. 2. Basic truth table. Note that outputs are irrelevant in this case, and so, are all assigned a "don't care" value (denoted by x).

Finally, to solve the problem that no smallest real number exists after 0 and create order, the order of the truth table inputs can be reversed. For example, under normal circumstances in a truth table, 110_2 would both be equal and be indexed by 6_{10}. The difference now will be that 110_2, while still being equal to 6_{10}, will now be indexed by 3_{10}. Put simply, the base-2 numbers are now being inversely written from right to left, and while they still maintain their expected values, their indexing order will be different. Applying this inverse counting logic to Cantor's diagonal argument, a table with one index column and infinite input columns can be created, each input representing one column of digits and collectively forming the $\mathbb{R}$ column. The inputs need not be individually noted, so they can be combined into the $\mathbb{R}$ column. Fig. 3 demonstrates what this diagonalization table would look like and how it would function under diagonalization rules.

Base-10 Natural Numbers (Index Numbers)	Base-2 Real Numbers Between 0 and 1
0	0.000000...
1	0.100000...
2	0.010000...
3	0.110000...
4	0.001000...
5	0.101000...
⋮	⋮

Digits to be diagonalized (highlighted in red): 000000 New number $N_2 = 0.111111\ldots$.

Fig. 3. Demonstration of diagonalization following the inverse counting logic.

On the surface, the fig. 3 table follows all the rules Cantor laid out. However, in this case, when applying diagonalization, it appears that no new numbers can be produced. The number produced through diagonalization is guaranteed to be $0.11\overline{1}_2$. What's interesting is that this number is exactly equal to $0.99\overline{9}_{10}$, which is, in turn, equal to 1, a fact rigorously proven with methods such as Dedekind cuts[5] and Cauchy sequences[6]. Because the range of $\mathbb{R}$ is from 0 to 1, the diagonalized number marks the end of the set of base-2 numbers between 0 and 1. In fact, this number would not even truly be new, as the method of counting used ensures a clear order. If fig. 3 is continued, note that any number with a certain number of digits will, no matter how many combinations of 1s and 0s possible, always conclude with only 1s present in those digits. For example, a number with 3 digits will increment as follows: .000, .100, .010, .110, .001, .101, .011, .111. The next number after .111 will have to add another digit. As the number of digits increases, the value of the final increment will continue to approach 1; so, when extended to infinity, the $\mathbb{R}$ column will conclude with 1. Even if diagonalization is applied once more, as it typically can be in Cantor's diagonalization proof, this still fails to make a new number, and it again produces $0.11\overline{1}_2$. The same concept can now be extended to base-10 numbers. A completely base-10 table may not be analogous to a truth table, but importantly, the inverse counting still applies.

Natural Numbers (Index Numbers)	Real Numbers Between 0 and 1
0	0.000000...
1	0.100000...

Natural Numbers (Index Numbers)	Real Numbers Between 0 and 1
2	0.200000...
3	0.300000...
4	0.400000...
5	0.500000...
⋮	⋮
10	0.010000...
11	0.110000...
⋮	⋮

Digits to be diagonalized (highlighted in red): 000000 . . . . New number N_3 : 0.999999 . . ..
Fig. 4. Demonstration of diagonalization on a base-10 table following the pattern in fig. 3.

Since fig. 4's table is in base-10, diagonalization is no longer a matter of simply toggling the digits; there are 9 digits different from the original 0 to choose from. However, the base-2 table has already established the set's final number is $0.99\overline{9}_{10}$, because $0.99\overline{9}_{10}$ is equivalent to $0.11\overline{1}_2$ and therefore 1. Since 1 is also the largest number in the set, this fact means any number before the final must also be smaller than it, and therefore already indexed previously, like in the base-2 table. Taken together, it appears that diagonalizing the zeros to any number besides 9 will produce a number already indexed, giving the impression that Cantor's diagonal argument has failed the proof by contradiction, proving its original assumption.

Why then does this approach fail in contradicting Cantor's diagonal argument? Why is it misleading? And what does it really prove? Cantor's diagonal argument requires that **all** numbers between 0 and 1 be indexed by $\mathbb{N}$ to prove that the two sets have the same cardinality, and further shows that such is not the case. The numbers in figs. 3 and 4, at first glance, all appear to be indexed on the tables presented. In reality, this cannot be the case, as demonstrated by a property of fig. 4. Note the pattern of fig. 4, whereby it appears to mirror $\mathbb{R}$ onto $\mathbb{N}$ in every row. It is quite predictable and easy to find any real number on any row simply by mirroring the desired number across the decimal point to find its index. This simple fact leads to an immediate problem when irrational numbers are considered. This is because all irrational numbers contain an infinite - unending - number of digits, and those digits share no discernible pattern; in other words, they are random. For example, the mirror function should still apply to $\pi/10$, equal to 0.31415. . . , to provide its index number. However, mirroring this number is impossible. The mirrored number can't have a beginning digit because there is no last digit to $\pi/10$, nor can a final digit ever be identified. Therefore, even though the tables appear to successfully index all of $\mathbb{R}$, closer inspection reveals their misleading nature, showing that no irrational number at all can be represented in either table. Indeed, $\mathbb{N}$ has a lesser cardinality than $\mathbb{R}$, even when trying to force $\mathbb{N}$ and $\mathbb{R}$ to correspond. However, these tables unintentionally prove a different fact that Cantor also proved himself: $\mathbb{N}$ can index all rational numbers. All rational numbers, even if non-terminating, have a pattern, and should they be mirrored, their index would have a knowable "final" number, despite being non-terminating. This mathematical thought experiment serves both to reaffirm Cantor's diagonal argument through its failure to disprove it, and beyond that, provides one more proof for Cantor's idea that $\mathbb{N}$ can be placed in a one-to-one correspondence with all rational numbers.

References

[1] Ewald, W. (1996). *From Kant to Hilbert: A source book in the foundations of mathematics* (Vol. 2). Oxford University Press.

[2] Rudin, Walter (1976). *Principles of Mathematical Analysis* (3rd ed.). New York: McGraw-Hill.

[3] Johnson, P. E. (1972). "The Genesis and Development of Set Theory." The Two-Year College Mathematics Journal, 3(1), 55–62.

[4] Simmons, K. (1993). *Universality and the liar: An essay on truth and the diagonal argument*. Cambridge University Press.

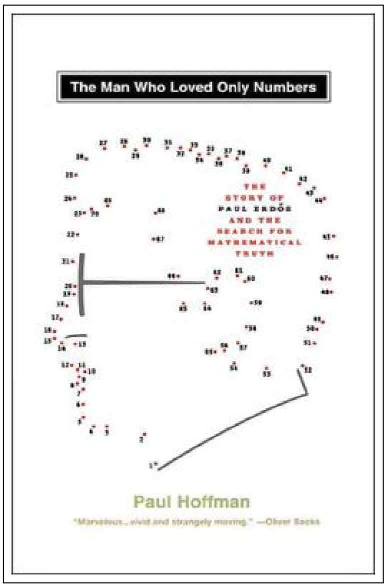
[5] Richman, F. (1999, December). "Is 0.999.. = 1?" *Mathematics Magazine*, 396–400.

[6] Griffiths, H. B., & Hilton, P. J. (1979). *A comprehensive textbook of classical mathematics: A contemporary interpretation*. Springer.

The Man Who Loved Only Numbers Book Review

by Trevor Cameron

Paul Hoffman’s 1998 book "The Man Who Loved Only Numbers" is a wide-ranging yet accessible read with just as much to offer the mathematician as the layman–of which I am firmly the latter. While ostensibly a biography of the infamously eccentric Hungarian mathematician Paul Erdős (pronounced "air-dish"), it’s much more than just that. It’s also a brief history of mathematics, mathematicians and Erdős’s native Hungary.



I’m somewhat ashamed to admit that I had never heard of Erdős before picking up this book. If you’re in the same boat, do yourself a favour and search "Paul Erdős interview" on YouTube. While the book goes into considerable detail in describing Erdős’s singular mannerisms and way of speaking, words on a page can only go so far–you really have to see (and hear) him for yourself.

Erdős was, for lack of a better way of putting it, a lifelong child prodigy. He specialized in prime numbers and number theory, which Hoffman describes as fields of mathematics in which young prodigies tend to cut their teeth, but ultimately outgrow. This is hardly the only thing Erdős can be said never to have outgrown, however; old age is sometimes described as a sort of second infancy, but Erdős seems to have skipped over the bit in the middle. His mother took care of his day-to-day necessities until her death. Erdős could barely dress himself, and admits to having first buttered his own bread at the age of 21. A man more often than not of no fixed address, he spent the majority of his life living out of a suitcase and imposing himself on his collaborators. He was a notoriously bad house guest too, known for waking his hosts up soon after dawn and guilting them into making him breakfast (which he was naturally incapable of doing himself). Much of the story is given to us second-hand via the late mathematician Ronald Graham, Erdős’s long-time collaborator and personal assistant of sorts, as well as his most frequent and tolerant host.

The book’s title may be taken as close to literally as possible; Erdős was a lifelong celibate who had no interests other than mathematics. He was an inveterate amphetamine user who once described mathematicians as "machines for turning coffee into theorems". He spoke using a sort of contagious private lingo that his collaborators often found themselves adopting in spite of themselves. God was the "Supreme Fascist", lecturing was "preaching", and men, women and children were "slaves", "bosses" and "epsilons" respectively. I can only imagine that modern psychiatrists would have a field day diagnosing his various idiosyncrasies, and I find it rather refreshing that Hoffman wastes none of his 268 pages attempting to. Erdős is presented simply as he was. However, the book is about much more than just Erdős. There’s also a fair bit of the history of Hungary’s Jewish community through the World Wars, which was a hotbed of strong mathematicians, including Paul Turán, Vera Sós, and John von Neumann. It’s riddled with minibiographies of a wide range of different mathematicians, some household names and others lesser known. Naturally, most of them were Erdős’s collaborators; Erdős was so prolific and worked with so many different mathematicians that those who worked with him were said to have an "Erdős number" of 1, those who worked with someone who worked with him had an "Erdős number" of 2, etc., etc.

If you find your curiosity piqued by Erdős himself, but you’re worried about being overburdened by complex mathematics, don’t be. Yes, there are proofs and theorems sprinkled throughout, but you won’t lose the thread of the narrative if you simply gloss over them–I did myself. The math is there, but it’s never the focus. Although you will come across the occasional ten dollar word (mathematical jargon in particular), the book is written in such a way that just about any generically educated person with an interest in the subject matter could enjoy it. In other words, don’t let the math stop you from reading this book. In fact, I would go so far as to say that the tasteful, non-distracting incorporation of mathematics into the text is a large part of what makes "The Man Who Loved Only Numbers" such an expertly crafted piece of work. Hoffman never intellectually flexes on you; instead, he peels back the curtain just enough to entice you into a world you may have never known was there.

An Introduction to the Continuum Hypothesis

by Pourya Memarpanahi

In my article Russell's Paradox from last year's issue of $U(t)$-Mathazine, we discussed the axioms of set theory, often referred to as Zermelo-Fraenkel Set Theory with Axiom of Choice or ZFC for short. In order to realize our ultimate goal of elucidating the Continuum Hypothesis, this time we're going to delve more into the basics of set theory. To get there, we'll investigate an open problem about the size of a set we have all seen and encountered numerous times: our favourite set $\mathbf{x} = \mathbb{R}$. There have been many discussions and debates revolving around the size of this set. It's known as-you guessed it–the Continuum Hypothesis.

This is both an independent and consistent mathematical statement. An independent statement of *ZFC* is a statement if it can be neither proved nor refuted by *ZFC*. A mathematical statement is said to be consistent with *Axioms of ZFC* elementarily speaking if its addition as an axiom does not lead to any contradiction. You should not be able to prove a formula and its negation simultaneously.

Now *CH* is an independent statement, so $ZFC \nvdash CH$(reads as *ZFC* does not prove *CH*), hence $ZFC + \neg CH$ is consistent, assuming the consistency of ZFC itself. Also $ZFC \nvdash \neg CH$(ZFC does not prove its negation) which is equivalent to saying $ZFC + CH$ is consistent.

Before we delve into the actual statement of *CH*, let us explore some more definitions. A set $\mathbf{x}$ is said to be *countable*, if there is a bijective map from ω into it. Equivalently, we could enumerate all its elements using the natural numbers. If there is no such bijective map, then we define the set to be uncountable. Therefore, such a set should have a larger cardinality, since we had already established that ω is a set with the *"smallest"* infinite size, *countable*, and we denote its cardinality by $\aleph_0 = |\omega|$. However, there are other sets of that cardinality! Namely $\mathbb{Z}$.

You might wonder that since the set of integers $\mathbb{Z}$ contains ω, it should have a lager size, but that is not how we defined two sets with the same cardinality! It was not in terms of sets containment. We can construct a natural bijective map among those two sets. Map 0 to 0, all positive integers to even numbers and negative numbers to odd numbers. There are other sets with cardinality $\aleph_0$, such as the set of rationals $\mathbb{Q}$, or the Cartesian product $\mathbb{Q} \times \mathbb{Q} = \{(a, b) : a, b \in \mathbb{Q}\}$. Any finite product of countable sets is also countable.

Let's consider an infinite product now. In this case, even if you take an infinite product of FINITE sets, the product will not be countable! As an example, consider the set $2^\omega = 2 \times 2 \times 2 \times ..$ It is just the following set: $\{f : f : \omega \to 2\}$. You can view it as the set of all possible sequences consisting of the binary digits 0 and 1. As a matter of fact, the size of this set is as big as the set of reals $\mathbb{R}$. But let us see why this set is not countable. Suppose it is countable, and therefore we could enumerate all its elements indexed by the naturals, $< f_n : n \in \omega >$. Now let us construct a function $g : \omega \to 2$ where it is not equal to f_n for any $n \in \omega$ and hence contradicts its countability. Define $g(n) = 1 - f_n(n)$. These types of arguments are called *Cantor Diagonalization Arguments*. Hence $|2^\omega| > \aleph_0$, but why is it equal to $|\mathbb{R}|$? To see that, first we should observe that $|\mathbb{R}| = |(-1, 1)|$. A bijection could be $\frac{x^3}{x^2-1}$ mapping $(-1, 1)$ into $\mathbb{R}$, and consequently $|\mathbb{R}| = |(a, b)| = |[a, b]|$ for any two reals a, b. To see the latter equality, we can use the *Cantor-Bernstein Theorem* and one direction should be trivial.

First let us denote the cardinality of $\mathbb{R}$ by $\mathfrak{c}$. To understand why $|2^\omega| = \mathfrak{c}$, we should realize that basic counting arguments should "naively" convince us of the following equality: $|\mathscr{P}(\omega)| = |2^\omega|$.
Given any finite set $\mathbf{x}$, in order to compute the cardinality of its power set,(the set of all of its subsets), we need to count the number of its subsets. Now to constitute a certain subset, we either pick an element of our set or we do not select it. Therefore, you have 2 choices for each element in terms of composing a subset of our given set. For example, if $|\mathbf{x}| = n$ then $|\mathscr{P}(\mathbf{x})| = 2^n$, including the $\emptyset$, for the *null set* is a subset of any set. Equivalently we could have defined an injection from $\mathscr{P}(\omega)$ into 2^ω using the characteristic function χ. Hence, a subset of ω, E, will be mapped to $\chi_E \in 2^\omega$. ($ie.$ $\chi_E(n) = 1$ if $n \in E$ and its value would be zero if otherwise). In a similar fashion, an injection from 2^ω into $\mathscr{P}(\omega)$. Also note that there is natural injection from 2^ω to $(0, 1)$ (any real number $x \in (0, 1)$ has a decimal representation). Now consider the following set $\mathscr{B} = \{(a, b) : a, b \in \mathbb{Q}\}$. Note that $\mathbb{Q}$ is countable, and since the definition of $\mathscr{B}$ depends on the parameters in $\mathbb{Q}$, $|\mathscr{B}| = \aleph_0$. So, using similar type of arguments, we should be able to convince ourselves that $|\mathscr{P}(\omega)| = |\mathscr{P}(\mathscr{B})|$. Now take an arbitrary $x \in \mathbb{R}$ and we can have it mapped into this set $(-\infty, x) \cup (x, \infty)$. This preceding set is called an *"Open-Set"*, and this set is actually composed of countable unions of elements of $\mathscr{B}$. Hence we can easily see that we have constructed an injection from $\mathbb{R}$

into $\mathscr{P}(\mathscr{B})$. Combining all these observations and facts, and invoking the *Cantor-Bernstein Theorem* gives us the desired result: $|2^{\omega}| = \mathfrak{c} = |\mathbb{R}|$

Now, what is CH? It is the following statement: There is no subset $A \subset \mathbb{R}$ such that $\aleph_0 < |A| < \mathfrak{c}$. Hence, CH states that $\mathfrak{c} = \aleph_1$. $\aleph_1$ is the first uncountable cardinal. A introductory definition of an ordinal would be a transitive set $\mathbf{x}$, ($\forall a \in \mathbf{x}, a \subset \mathbf{x}$; so it contains the elements of its element), which is *well ordered* with respect to $\in$ relation.

A set $\mathbf{x}$ is well ordered, if it is linearly ordered by a relation R, and every non-empty subset of it has a minimal element with respect to the relation. A linear order is a binary relation, which is reflexive (any element of the set in that relation to itself), transitive, antisymmetric(if the relation between a pair of elements commutes, then those two elements must be the same element), and any two elements are comparable. As previously mentioned, for ordinals we take our binary relation $R = \in$, so the relation here is just the usual membership as an element. Also note that no set $\mathbf{x}$ is an element of itself: $\mathbf{x} \notin \mathbf{x}$, so $\in$ is irreflexive. Hence given any two ordinals, α and β, we either have $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$
ω or the set 7 are two examples of ordinals. A cardinal is an ordinal κ, which is the minimum of all the ordinals α for which there is a bijection map among κ and the ordinal α.
Now, naively speaking think of $\aleph_1$ as the second "real" big infinity. What do I mean by "real" here? Note $\omega = \{0, 1, 2, 3, ..\}$, the first infinite type of sets. Then using the appropriate axioms, we can have the successor of it, and then build the respective successor of that, and so forth.

For example $\omega + 1 = \omega \cup \{\omega\}$ is the immediate successor of ω, then $\omega + 2 = \omega + 1 \cup \{\omega + 1\}$, which is the successor of $\omega + 1$. And we can keep going. Now we can put an ordinal on top of all these consecutive successors. That would be $\omega + \omega = \omega.2$. This ordinal is just the limit of all these ordinals of this format $\omega + n$. $\omega.2$ just represents two copies of ω stacked on top of each other. But then the procedure continues. We can have the following *successor ordinals*: $\omega.2 + 1$, $\omega.2 + 2, \ldots$, and ultimately consider another *limit ordinal* $\omega + \omega + \omega = \omega.3 = \lim(\omega.2 + n)$, so three copies of ω stacked on top of each other, then have $\omega.4, \ldots, \omega.5, ..$ and we can continue this process and go higher, and "eventually" reach ω copies of ω: $\omega + \omega + \omega + \ldots = \omega.\omega = \omega^2$. Let us go up the ladder, this hierarchy of ordinals: the next set of ordinals will be $\omega^2 + 1, .., \omega^2 + \omega$, $\omega^2 + \omega + 1, .., \omega^2 + \omega + \omega = \omega^2 + \omega.2$, $\omega^2 + \omega.2 + 1, ..., \omega^2 + \omega.3, \ldots, \omega^2 + \omega^2 = \omega^2.2$, $\omega^2.2 + 1, ..., \omega^2.2 + \omega$, $\omega^2.2 + \omega + 1, ..., \omega^2.2 + \omega + \omega = \omega^2.2 + \omega.2$, $\omega^2.2 + \omega.2 + 1, \ldots, \omega^2.3, \ldots, \omega^2.\omega = \omega^3$.
Then we can have $\omega^3 + 1, \ldots, \omega^4, \omega^4 + 1, .., \omega^5, ..$, and ultimately reach ω^{ω}. This ordinal has no immediate predecessor, so that is what we meant by "eventually"; we just put ω^{ω} on top of ω^n for all $n \in \omega$. As previously mentioned, it is a *limit ordinal*. It is actually the limit(supremum) of the ordinals below it. The process does not stop there. We can have the successor of that, which would be ω copies of ω copies of ω copies of....ω copies of ω (ω many times), plus an additional element on top of it, $\omega^{\omega} + 1 = \omega^{\omega} \cup \{\omega^{\omega}\}$. We can keep on going; however, the main point is not the existence of such ordinals, but to understand that all these types of infinite sets have the same cardinality as $\aleph_0$, so they are not that "big". That is why we called $\aleph_1$ the second "real" infinity in this article. Then we can have $\aleph_2$ and as a matter of fact we can investigate and talk about the ω^{th} uncountable cardinal. Believe it or not, $\aleph_{\omega}$ is still not that "big".
Also note that addition of ordinals is not commutative! $2 + \omega \neq \omega + 2$. Think of the first one as just ω, and the latter one is the following set: $\omega + 2 = \{0, 1, 2......, \omega, \omega + 1\}$. Hence, this ordinal contains ω and $\omega + 1$. 2 elements on top of the ordinal ω. (ie. $\forall n \in \mathbb{N}$, $n < \omega < \omega + 1$).

Okay, let's relax for a moment, take a few deep breaths, and let all of these definitions sink in. So far, we've explored the axioms of set theory, and how we can end up with larger and larger sets–far larger than we may initially have expected. By now, we understand the basics of what the always contentious Continuum Hypothesis is about. But we're not done just yet! In the next issue of $U(t)$-Mathazine, we'll be exploring some of the more interesting implications of the Continuum Hypothesis.

No Strangers at this Party: The Story of the Ramsey Theory Podcast

by Brian Krammer

"There are almost as many types of mathematicians as there are types of human beings. Among them are technicians, there are artists, there are poets, there are dreamers, men of affairs, and many more."
– Richard Rado (1906-1989)

I believe there is a moment experienced by every undergraduate math student at some point in their degree. It comes when studying a proof, theorem, or problem of some kind, and it can be summed up by a few simple questions: *Who* came up with this!? *How* did they come up with this!? So many of these proofs seem completely obvious in hindsight, but they rely on some key insight, some stroke of genius, that it baffles to consider how they were ever discovered in the first place. In completing my minor in mathematics, and studying a deluge of similar proofs, I developed a distinct impression of mathematicians as the incomprehensible eccentric-genius type, the kind immortalized in pop culture. Mathematicians were less like people and more akin to the mythic heroes of the Odyssey and Beowulf. Myriad students like myself, I'm sure, have gotten the same impression.

Thankfully, this false image of "The Mathematician" would not survive my time in **Math 301: Ramsey Theory** taught by Simon Fraser University's own Dr. Veselin Jungic. For the uninitiated, Ramsey Theory may seem an alien subject. It is, after all, infamously hard to define. As Dr. Jungic shared with us in our first class, the most well-known definition comes from Theodore S. Motzkin who said: "Complete disorder is impossible." Motzkin's statement certainly captures the breadth of results implied by Ramsey Theory, but it does little to clarify exactly *what* the field is. What problems does it address? What branches is it related to? What are its foundations? Some of my classmates helped me to sum it up as follows:

"Ramsey Theory, more explicitly described, is a branch of combinatorics that deals with the emergence of patterns within substructures. It is about understanding what kinds of patterns are guaranteed to exist if we were to sort a class of objects into defined subclasses. Even if we know little or nothing about what is being sorted, we can usually identify the qualities of the resultant patterns. Stated simply, Ramsey Theory is the study of patterns and the conditions under which they arise."

It was not, however, the math itself that impacted me so greatly in Dr. Jungic's Math 301 class. It was rather the final project. The class was designed to appeal not just to mathematics majors, but to the broad range of people who sought a minor in the subject – I was such a student, completing a maths minor alongside a major in English Literature. With the diversity of students and skillsets in mind, Dr. Jungic gave us the freedom to choose from a vast array of final projects, from artistic works to in-depth explorations of specific theorems. I filled out the project form hoping I would be put in a group to design a game based on Ramsey Theory (after all, I am an avid Boardgamer), but I needed a backup project idea, and another caught my eye: A podcast comprising a series of interviews with prominent Ramsey theorists. This project idea was proposed by Dr. Jungic himself, and he offered to help us connect with a variety of mathematicians if we were interested. I figured that if I didn't end up working on a game, recording the podcast could add some nice variety to the semester. I have never been so glad not to get my first option for a project topic.

When the dust settled there were 3 groups, each of 3 or 4 students, working on the podcast. Dr. Jungic gave each group a list of names and we got to work. We read about the different professors, prepared specific questions for them based on their areas of study and developed a general script to help the interview run smoothly. We didn't expect a ton of responses, but to our astonishment we had 14 professors agree to interviews across the three groups! The interviews were recorded through a series of zoom calls, both due to the pandemic, and the fact that many of the professors were located on completely different continents from us. This was the experience that so deepened my understanding, and respect for mathematicians.

In speaking with Dr. Steve Butler of Iowa State University, I witnessed contagious joviality and a nigh unmatched joy for mathematical discovery.

Dr. Joel Spencer, one of the co-authors of the seminal textbook on Ramsey Theory, spoke with unbelievable care and attention. Before the interview even began, he was asking us about our interests and studies. Despite his incredible accomplishments, it was clear that he cared more for us than the math.

Dr. Jaroslav Nešetřil of Charles University in Prague demonstrated incredible humility and authenticity that would never be apparent to one who holds the "eccentric-genius" image of a mathematician such as I had.

Dr. Fan Chung, whose late husband Dr. Ron Graham also worked extensively in the field, shared with us several anecdotes and stories of her time working on Ramsey Theory with her husband and with the enigmatic Dr. Paul Erdős. Her openness to sharing with us, despite our being complete strangers, was astonishing.

I was forced, upon finishing the last interview, to come up with one simple conclusion. The image I had in my head of "the mathematician" was wrong. These people were people first, and mathematicians second. I saw an incredibly diverse, tight-knit, and yet open group of people. They were like a massive extended mathematical family and one which they were happy to have you join. There was no sense of elitism, no "us versus them," no animosity. They were simply excited to study some fascinating mathematics, and they were happy to have them along with any who would join.

That is why we chose "No Strangers at This Party" as the title for our podcast. The "party problem" is foundational in Ramsey Theory. It begins with a simple observation: Among any 6 people gathered at a party, there must either be a group of at least 3 who are all mutual acquaintances, or there is a group of at least 3 who are all mutual strangers. The challenge is to prove it. Frank P. Ramsey, for whom the field has its name, proved the general case of this problem with what is now known as Ramsey's Theorem. Our guests have proven a very specific case. In the world of Ramsey Theory, there are no strangers, only people.

To learn more about these incredible mathematicians, listen to all 14 episodes of *Ramsey Theory: No Strangers at This Party* on *Spotify*, *Apple Podcasts*, or *Google Podcasts.*

Special thanks to Dr. Veselin Jungic for facilitating the project, and to my group members Amritha Raj, Anmol Singh, and Manan Sood whose thoughts helped formulate some of this article.

Solution to the Permutation Puzzle

N	Permutation	Transposition
1	1234	(CD)
2	1243	(BC)
3	1423	(AB)
4	4123	(CD)
5	4132	(AB)
6	1432	(BC)
7	1342	(CD)
8	1324	(AB)
9	3124	(CD)
10	3142	(BC)
11	3412	(AB)
12	4312	(CD)
13	4321	(AC)
14	3421	(BC)
15	3241	(CD)
16	3214	(AB)
17	2314	(CD)
18	2341	(BC)
19	2431	(AB)
20	4231	(CD)
21	4213	(AB)
22	2413	(BC)
23	2143	(CD)
24	2134	

Call for Submissions

Have a topic in mathematics and/or statistics that you're passionate about? Want to have your voice heard? Then look no further, because this is your chance to contribute to the next edition of the $U(t)-$Mathazine! The process is simple:

- Contact us saying that you have an idea for a piece.
- We will connect you with a mentor to help prepare your piece.
- Prepare an early draft of the piece by February.
- Respond to edits and comments throughout the summer.
- Your piece will be published in the late summer.

We welcome a variety of different submission types - illustrations, articles, drawings, comics, creative writing, research, book reviews, etc. Have an idea but not sure where to start? Please get in touch, and we will be happy to help you with the writing and editing process.

Articles and inquiries should be directed to our email address:

`mathstats.utsc@utoronto.ca`

We look forward to receiving your submissions!

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