Convection in a spherical shell heated by an isothermal core and internal sources:
implications for the thermal state of planetary mantles

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Abstract

The parallelized three-dimensional spherical convection code, TERRA, is employed to study the mean temperatures and planforms of convecting planetary mantles in spherical shell geometries. We vary the factor $f$ which controls the degree of curvature, defined as the ratio of the radii of the inner and outer bounding surfaces, the Bénard-Rayleigh number, $Ra_B$, and the dimensionless rate of internal heating, $H$. We develop parameterized expressions for predicting the mean temperature of convecting spherical shells which are heated partially from within and partially from below by a hot isothermal lower boundary. Our parameterization is fit to a data set of mean temperatures from 23 numerical model calculations for $f = 0.547$ (appropriate to Earth’s mantle). We then demonstrate that this parameterization of mean temperature in terms of $f$, $Ra_B$ and $H$ extends to other values of $f$ as well. For all values of $f$, $Ra_B$ and $H$ considered in this study, our predicted mean temperatures agree with the model calculations to within 2.4%. The scaling analysis is extended to obtain an expression for surface heat flux in terms of $Ra_B$ and $H$ for $f = 0.547$. In that case we obtain a predictive equation for surface heat flux that agrees to within 11% of the observed values. Our findings provide a useful tool for parameterizing the temperature and surface heat flux of planetary mantles of varying geometry and heating configurations.

Key words: internal heating, three-dimensional, spherical, mantle convection, planetary mantles

PACS:
1 Introduction

Some knowledge of the composition, thickness and heating rate of a terrestrial planet’s mantle is required for even the most rudimentary model of its thermal structure. The possibilities for these fundamental parameters allows for a diverse range of thermal states, even in a simple medium featuring uniform properties and boundary conditions (Bercovici et al., 1989; Schubert et al., 1993). Previous studies of uniform property fluids have shown that despite the apparent complexity of the obtainable convective planforms, simple predictive equations can be derived for bulk characteristics such as the mean temperature; both in plane layer systems, featuring heating from within and below (Sotin and Labrosse, 1999), and axi-symmetric spherical shell systems, featuring heating by an isothermal core (Vangelov and Jarvis, 1994). The findings of these studies have been extended to derive theoretical but untested predictive equations for the mean temperature of an infinite Prandtl number, uniform property, convecting fluid in a random thickness spherical shell heated by both internal sources and an isothermal core (Sotin and Labrosse, 1999). The goal of the study presented here is to test and refine these equations to provide a validated parameterization for predicting mean temperatures.

Studies (Jarvis et al., 1995) of solely bottom heated fluid spherical shells have shown that the number of convection cells and the time-dependence of the flow is dependent on the ratio, $f$, of the inner and outer shell radii. Convection planform and time-dependence are also dependent on the rate of internal heating, $H$, and the Bénard-Rayleigh number, $Ra_B$, of the system (a measure of the vigour of the flow driven by bottom heating). Because convection cell
planform also affects mean temperature (Jarvis et al., 1995) it is not obvious that a simple relation between the parameters, \( H \) and \( Ra_B \), and the global temperature should be obtainable.

We obtain numerical solutions for the temperature fields in isoviscous spherical shell models of planetary mantles heated by an isothermal boundary condition at the core and by uniformly distributed internal sources. The models are cooled from above by an isothermal surface. The specification of an isothermal condition at the core assumes that the planet has a liquid core (or outer core) and is therefore able to mix on much more rapid timescales than the mantle. We consider a range of model geometries (determined by the ratio of the inner radius to the outer radius of the spherical shell, \( f \), the curvature factor) but initially focus on models that feature the Earth’s curvature factor, \( f = 0.547 \). We compare our results in this fixed geometry to previous findings in a Cartesian geometry \( (f=1) \) in order to refine previously proposed equations for a generalized curvature factor. In addition to working towards a single equation for predicting the global thermal characteristics of a planetary mantle with arbitrary \( f \) and heating mode, we examine the effect of these parameters in causing transitions in the convective planform of the systems.

2 Modelling Method

We model infinite Prandtl number convection in a spherical shell with radially inward directed gravity using the parallelised code TERRA (Bunge and Baumgardner, 1995; Bunge et al., 1996, 1997; Phillips and Bunge, 2005). In order to compare our findings with the most relevant previous studies (e.g., Sotin and Labrosse, 1999) we employ the Boussinesq approximation and sup-
press the effects of compressibility. We focus on isoviscous convection heated by both core heat loss and internal mantle heat sources. The rate of internal heating is constant in time. Isothermal boundary conditions are specified at the inner shell radius, \( R_i \), and outer shell radius, \( R_o \). The temperature difference across the shell thickness, \( d = R_o - R_i \), is \( \Delta T \). The vigour of convection driven by the temperature difference of the bounding surfaces of the spherical shell can be measured by the Bénard-Rayleigh number,

\[
Ra_B = \frac{g\alpha \Delta T d^3}{\kappa \nu}
\]  

(1)

where \( g \) is gravitational acceleration; \( \alpha \) is thermal expansivity; \( \kappa \) is thermal diffusivity and \( \nu \) is kinematic viscosity.

Typical values of the parameters defining \( Ra_B \) indicate that the mantles of Earth and other terrestrial planets are characterised by Bénard-Rayleigh numbers that are well above the critical value (McKenzie et al., 1974; Schubert et al., 2001). However, the source of most of the heat flux from Earth’s mantle is the concentration of radiogenic elements in the mantle (Schubert et al., 1980). Significant heating in other planets with a similar make-up to the Earth likely also comes from internal mantle heat sources. The vigour of convection driven by internal sources can be specified in terms of an internal heating Rayleigh number,

\[
Ra_H = \frac{g\alpha \phi d^{5}}{\kappa k \nu}
\]  

(2)

where \( k \) is thermal conductivity and \( \phi \) is the rate of internal heat generation per unit volume (Roberts, 1967).
We nondimensionalise the system of equations governing convection in the spherical shell in terms of the diffusion time across the shell thickness, so that dimensional times are recovered from nondimensional times by multiplying by $d^2/\kappa$. In its nondimensional form the equation for the conservation of heat in the system described is thus

$$\frac{\partial T}{\partial \tau} = \nabla^2 T - \mathbf{v} \cdot \nabla T + \frac{Ra_H}{Ra_B},$$

(3)

where $T$ is nondimensional temperature, $\tau$ is non-dimensional time and $\mathbf{v}$ is nondimensional velocity. $Ra_H/Ra_B$ is equivalent to the nondimensional internal heating rate, $H$.

The equations describing the conservation of mass and momentum as well as the standard linearised equation of state complete the system of equations describing flow evolution in the fluid modelled. Adopting the nondimensionalisation of time described above, these equations take the form

$$\nabla \cdot \mathbf{v} = 0,$$

(4)

$$\nabla^2 \mathbf{v} - \nabla P = Ra_B \mathbf{\hat{z}},$$

(5)

and

$$\rho = \rho_o [1 - \alpha T],$$

(6)

respectively, where $\mathbf{\hat{z}}$ is a unit vector parallel to the direction of gravitational acceleration, $P$ is the nondimensional pressure and $\rho_o$ is the density at the surface of the spherical shell which has a nondimensional temperature of zero. $Ra_B$ and $H$ are thus the sole fluid parameters governing the solution of the sys-
tem. (Note that $f$ is an independent geometric parameter which also governs the evolution of the system.)

We solve the continuity and momentum equations specifying free-slip surfaces at $R_i$ and $R_o$. The radial and lateral resolution of the numerical mesh employed in each of the calculations performed are adjusted according to the value of the specified Bénard-Rayleigh number.

The combinations of Rayleigh number, $Ra_B$, internal heating rate, $H$, and curvature factor, $f$, for all of the cases examined in this study, result in time-dependent convection. The mean temperatures and heat fluxes that we quote in the following sections are therefore temporal averages determined once each model has reached a statistically steady-state. We consider the solution to have reached such a condition once the mean-temperature of the system is no longer showing any clear long-term heating or cooling trends. The average values that we quote, along with the standard deviation of the time series, are obtained over multiple mantle overturn times once a steady condition has been attained. The initial conditions for our solutions are specified as mildly perturbed fields featuring steep temperature gradients at the solution domain boundaries and isothermal regions between the boundary regions. The temperature of the isothermal bulk of the fluid is adjusted at the start of each model run in accord with the heating rate of the calculation. We specify higher initial mean temperatures for calculations with relatively high internal heating rates. As a result, collectively, the calculations heated or cooled to their statistically steady-state much more quickly than they would have if we have used the same initial condition for all cases.
3 Results

We initially present results from 23 numerical models with a fixed ratio, $f$, of $R_i/R_o = 0.547$ (the ratio of the Earth’s outer core radius to its mean surface radius). For this geometry, we examine the thermal characteristics in calculations featuring Bénard-Rayleigh numbers ranging from $10^4$ to $10^7$ and $H$ between 2.353 and 47.064. (The upper value of this range for $H$ is approximately twice estimates of the effective current rate of heating in the Earth’s mantle that results from internal sources and secular cooling (Schubert et al., 2001). In combination, these heat sources, which mimic each other (Krishnamurti, 1968; Weinstein and Olson, 1990), may supply about 0.8 of the Earth’s mean surface heat flux.) The non-dimensional heating rate, $H$, is dependent on $d^2$; consequently, modelling planets with shallower mantles than the Earth requires specifying lower values for $H$ as well as $Ra_B$. We model planets with low values of $Ra_B$ and $H$ for an Earth-like value of $f$ in order to obtain thorough coverage of $Ra - H$ parameter space. Table 1 summarises the heating mode specification, grid resolution and nondimensional mean temperature, $\theta$, of each model solution. (Other columns of data appearing in the table are discussed in later sections). Table 1 also introduces a naming convention for our models that is based on the Rayleigh number and a multiplicative factor for the internal heating rate obtained for the aforementioned bulk silicate Earth value, 23.533.

TERRA solves equations (3)-(6) on numerical meshes constructed from concentric icosahedra (Baumgardner, 1985). The icosahedra are refined by successively adding computational nodes at the midpoints of the line segments joining neighbouring vertices. We vary the level of refinement of the grids used
in this study according to the convective vigour of the modelled flow. However, the same level of refinement is applied at all radii for a given calculation. Consequently, the spatial separation of the mesh vertices increases with distance from the centre of the numerical grid. The mean temperatures quoted in this study are true volume averages in the spherical shells and account for the change in volume of the grid elements with distance from the sphere centre.

In Figure 1 we plot temperature field snapshots from three models, Ra5e5H1, Ra1e6H1 and Ra1e7H1, in which \( f = 0.547 \) and the value of \( H \) is held fixed at 23.532. The values of the Bénard-Rayleigh numbers in these calculations are \( 5 \times 10^5, 10^6 \) and \( 10^7 \) from top to bottom, respectively. In order to enable a view of the thermal structure below the upper thermal boundary layer, we show the thermal fields only for radii < 0.98\( R_o \). The hot (orange) isosurfaces have a temperature of 0.77 and the (cool) blue isosurfaces have a temperature of 0.14. In each case the thermal field is highly time-dependent and the convection planform is characterised by downwellings with a columnar, or plume-like, morphology. Although the thermal boundary layers and plumes become finer as \( Ra_B \) is increased, the number of downwellings and the planform of the convection does not change dramatically between cases despite the overall cooling as \( Ra_B \) is increased.

Figure 2 shows a sequence of temperature field snapshots from models Ra1e6H.1, Ra1e6H.2 and Ra1e6H1, in which the Rayleigh number is held fixed (at \( Ra_B = 10^6 \)) while the internal heating rate is varied. The value of \( H \) in the snapshots shown is 2.353, 4.706 and 23.532 from top to bottom, respectively. The bottom panel in Figure 2 is the same as the middle panel in Figure 1. The isosurfaces shown in the other panels have the same values as in Figure 1 but are only displayed for radii less than 0.95\( R_o \). These figures show that the inter-
nal heating rate has a strong effect on the morphology of the downwellings as well as the temperature of the fluid interior. With relatively low internal heating rates, the downwellings are robust and principally sheet-like (top panel). In addition, a longer wavelength structure is clearly present. In comparison, the case in the bottom panel (with 10 times as much internal heating as in the top panel) features numerous but weaker downwelling plumes.

The addition of internal heating distributed throughout the convecting system destabilises the upper thermal boundary layer. Consequently, the number of downwellings that form increases as the degree of internal heating is increased. The middle panel shows an intermediate state in which the downwelling sheets have given way to vigorous downwelling plumes and illustrates how upwellings form where the hot lower thermal boundary layer is pushed aside by cold downwelling material that spreads across the surface of the hot core. A comparison of the panels in Figure 2 illustrates that because the arrival of downwellings triggers the formation of instabilities in the lower thermal boundary layer, the decrease in the dominant wavelength of the downwellings affects the wavelength of the buoyant instabilities that form at the core-mantle boundary.

In Figure 3 we plot snapshots of the average non-dimensional temperature as a function of mantle depth. The plots correspond to the temperature fields shown in Figures 1 and 2 and are labelled accordingly. Figure 3a shows that, given a fixed internal heating rate, mean temperature decreases as the Rayleigh number, $Ra_B$, is increased. This behaviour has previously been noted in 2D and 3D Cartesian calculations by Jarvis and Peltier (1982) and Sotin and Labrosse (1999), respectively, and in axi-symmetric spherical geometry calculations (Butler and Peltier, 2000). Figure 3b shows that, at a fixed Rayleigh number, mean temperature increases with a greater internal heating rate, and
surface heat flux increases while basal heat flux decreases. The plots also show that the bulk of the fluid between the thermal boundary layers becomes weakly sub-adiabatic (e.g., Jeanloz and Morris, 1987) in cases featuring lower Rayleigh numbers and higher internal heating rates. This effect has been noted previously in Cartesian (e.g., McKenzie et al., 1974; Parmentier et al., 1994) and spherical geometry (e.g., Bunge et al., 2001) studies.

3.1 Mean temperature in an Earth-like geometry

In order to determine a predictive equation for the mean temperature in an internally heated convecting fluid confined to a spherical shell we start with the findings of previous studies. Jarvis et al. (1995) studied the effect of the curvature factor on thermal structure in two and three-dimenensional spherical systems heated only by isothermal cores. These authors derived a relation for the mean temperature, \( \theta_b \), of a vigorously convecting fluid bounded from below by a surface of temperature 1.0 and above by a surface of temperature 0.0. They found that mean temperature is dependent on planform and that for a given curvature factor, \( f \), the temperature can be expressed as

\[
\theta_b(f) = \frac{2}{1 + f^{-3/2}} \theta_b(1),
\]

where the subscript \( b \) refers to the bottom heated case. The dependence of the mean temperature on planform is implicit in the term \( \theta_b(1) \), corresponding to the case where \( f = 1 \). For example, it is well known that the mean temperature in a convecting plane layer heated entirely from below, varies from \( \theta_b = 0.50 \) for 2D rolls (bounded by symmetric upwelling and downwelling sheets) to \( \theta_b = 0.35 \) for axisymmetric upwelling cylindrical plumes bounded by downwelling
sheets (e.g., Kiefer and Hager, 1992). Jarvis et al. (1995) found this sensitivity of mean temperature to convective planform carries over to spherical shell geometry and that convective planform in spherical shells is in turn sensitive to $f$ for a given Rayleigh number. For $f$ less than or equal to 0.5, Jarvis et al. (1995) observe a single large plume, surrounded by a broad sheet-like downwelling, but as $f$ is increased upwellings and downwellings become more symmetric with separations comparable to the spherical shell thickness. This suggests that the appropriate value of the numerator in equation (7) varies from 0.7, when $f=0.5$, to 1.00 when $f=1$. Through comparison with the mean temperatures tabulated by Jarvis et al. (1995) for 3D spherical shell convection with $f = 0.1, 0.3, 0.5, 0.7$ and 0.9, we find that by replacing the numerator in equation (7) with $f^{1/2}$, we can parameterise the effects of planform on mean temperature to within 10% for any value of $f$.

Sotin and Labrosse (1999) argued that the mean temperature of a convecting plane-layer when heated both from below and from within can be expressed as the sum of two components. The first component is the mean temperature in the absence of internal heating. The second component is an expression for the mean temperature in the case that is entirely internally heated by uniformly distributed heat sources. Parmentier et al. (1994) found that the mean temperature in the latter case was given by a power-law relationship which, for an isothermal bottom temperature, takes the form

$$\theta_i = \alpha \frac{H^\beta}{Ra_B^\gamma},$$

(8)

where the subscript $i$ refers to the internally heated case. Combining equations (7) and (8) we obtain a prediction of the overall mean temperature, $\theta_{pre}$, of a
convecting spherical shell with partial internal heating,

\[ \theta_{\text{pre}} = \theta_b + \theta_i = \frac{f^{1/2}}{1 + f^{-3/2}} + \frac{H^\beta}{Ra_B}. \]  \hspace{1cm} (9)

Using \( \theta_b = 0.5 \), Sotin and Labrosse (1999) found \( \theta_b + \theta_i \) successfully predicted the mean temperature in 3D plane-layer convection when both internal and bottom heating were included. However, for a spherical shell the appropriate value of \( f \) must be substituted into equation (7) to obtain the value of \( \theta_b \).

Assuming a value of \( f = 0.547 \), we can proceed to test this idea by deriving an expression for \( \theta_{\text{pre}} \) for Earth-like geometry. In this case \( \theta_b = 0.213 \). Considering this value to be fixed, we can invert to find best-fitting values of the parameters \( \alpha, \beta \) and \( \gamma \) using the values of input parameters \( Ra_B \) and \( H \), and the computed model values of \( \theta \) given in Table 1. The resulting prediction only holds for the case of \( f = 0.547 \).

The system of equations obtained by substituting the values from Table 1 in equation (9) can be solved by least squares minimization, giving the result

\[ \theta_{\text{pre}} = 0.213 + 0.797 \frac{H^{0.738}}{Ra_B^{0.213}}. \]  \hspace{1cm} (10)

Figure 4 plots the mean temperatures predicted by equation (10), \( \theta_{\text{pre}} \), against the actual temperatures observed in our experiments, \( \theta \), and reveals that the inverted temperatures agree well with the data over the entire temperature range obtained. This result demonstrates that the Cartesian result of predicting the overall mean temperature, \( \theta_{\text{pre}} \), by superimposing the two components, \( \theta_b \) and \( \theta_i \), extends successfully to spherical geometry for an Earth-like value of \( f = 0.547 \). Moreover, since Cartesian geometry may be thought of as the \( f = 1 \) limit of spherical geometry, we have essentially demonstrated that this
idea works equally well for two different values of \( f \), namely \( f = 1.0 \) and \( f = 0.547 \).

Model Ra1e6H2 (with \( Ra_B = 10^6 \) and \( H = \sim 47 \)) has a Rayleigh number \((Ra_B)\) that is an order of magnitude greater than Model Ra1e5H1 and an internal heating rate that is twice as great (see Table 1). However, these two systems have nearly identical mean temperatures. In order to assess the similarity in thermal structure of two systems with similar mean temperatures, but very different heating modes, we compare the thermal fields of these two models in Figure 5. The cool (green) isosurfaces in this figure have a temperature of 0.38 and the hot (orange) isosurfaces have a temperature of 1.06. (Note the temperature scale in this figure differs from the scale used in previous figures.) The temperature fields in Figure 5 are plotted only for radii less than \( 0.98 R_o \).

Figure 5 illustrates that the lateral scale of the thermal structures of these two systems differ substantially despite having almost identical mean temperatures. Model Ra1e5H1 (panel a) features thicker and less numerous downwelling plumes than Model Ra1e6H2. This is consistent with Figures 1 and 2 which show that increasing \( Ra_B \) and \( H \) both lead to a narrowing of thermal plume structures. Figure 3 also indicates that while increasing \( Ra_B \) decreases the mean temperature, \( \theta \), increasing \( H \) has the opposite effect on \( \theta \). Thus it is possible for \( \theta \) to remain constant while thermal plumes narrow due to increasing \( Ra_B \) and \( H \).
3.2 Generalisation of the predictive equation to a variable curvature factor

Using energy balance and boundary layer arguments Sotin and Labrosse (1999) proposed a generalised expression for \( \theta_i \), thereby extending equation (8) to arbitrary \( f \). Their proposed form for \( \theta_i \) is

\[
\theta_i = C \frac{((1 + f + f^2)/3)H}{{Ra_B}^{1/4}},
\]

(11)

where \( C \) is a constant which may depend on \( f \). Substituting this form for \( \theta_i \) into equation (9) yields a generalization of equation (10) to predict the mean temperature of any spherical shell heated by both an isothermal core and internal sources as

\[
\theta_{pre} = \frac{f^{1/2}}{1 + f^{-3/2}} + C \frac{((1 + f + f^2)/3)H}{{Ra_B}^{1/4}}.
\]

(12)

If \( C \) depends only on the curvature factor then substitution of the results from the numerical experiments in Table 1 into equation (12) should yield a constant value of \( C \) corresponding to \( f = 0.547 \). The value of \( C \) obtained from equation (12) is listed for each of our models with \( f = 0.547 \) in Table 1 under the heading \( C_{(12)} \). Our results show that \( C \) is not constant but instead has a clear dependence on \( Ra_B \). Table 1 indicates that for all of the internal heating rates examined, \( C \) increases as \( Ra_B \) increases. However, no dependence of \( C \) on \( H \) is indicated.

Sotin and Labrosse (1999) did not find a dependence of \( C \) on \( Ra_B \) in the Cartesian case, \( f = 1 \). Consequently, we will assume that \( C \) should have the form

\[
C(f, Ra_B) = 1.236 + \lambda Ra_B^{\nu}(1 - f)
\]

(13)
where $\lambda$ may depend on $f$. This expression is consistent with the Cartesian geometry findings regardless of the values of $\lambda$ and $\nu$. Using the data in Table 1 we can solve for the values of $\lambda$ and $\nu$ in equation (13) for the case $f = 0.547$: we find $\lambda = 0.256$ and $\nu = 0.120$. Thus, substituting the expression for $C$, in equation (13), into equation (12) we obtain

$$
\theta_{pre} = \frac{f^{1/2}}{1 + f^{3/2}} + (1.236 + 0.256Ra_B^{0.120}(1 - f)) \frac{((1 + f + f^2)/3)H^{3/4}}{Ra_B^{1/4}}.
$$

(14)

Table 1 indicates the value predicted for $\theta$ by equation (14) under the heading $\theta_{14}$, as well as the percentages by which the prediction of equation (14) differs from the observed value of $\theta$ (under the heading $\%error_{14}$). The discrepancy between the predicted and computed mean temperature is less than 2.6% in all cases. However, the predictions of equation (14) appear to be diverging from our model values for high values of $H$ (Table 1). To address this we make a small modification to equation (14). We derive a final form for a predictive equation for $\theta_{pre}$ in which we reduce the exponent on $H$ to 0.729, the value obtained by Sotin and Labrosse (1999) for a Cartesian geometry. First we solve an equation of the following form

$$
\theta = 0.5 + \sigma \frac{H^{0.729}}{Ra_B^{1/4}}
$$

(15)

using Cartesian model data. We find this equation best satisfies the Cartesian observations with $\sigma = 1.318$. Based on the results in Table 1, and emulating the form of equation (14), we obtain the following revised equation, for predicting the mean temperature of the convecting shell, $\theta_{pre}$,

$$
\theta_{pre} = \frac{f^{1/2}}{1 + f^{3/2}} + (1.318 + 0.251Ra_B^{0.123}(1 - f)) \frac{((1 + f + f^2)/3)^{3/4}H^{0.729}}{Ra_B^{1/4}}.
$$

(16)

which reduces to equation (15) in the Cartesian case.
We compare the predictions of equation (16) and the computed model temperatures by plotting contours of constant $Ra_B$, as derived from equation (16), in $\theta - H$ space in Figure 6. Each of the models used in our study has a Rayleigh number corresponding to one of the plotted curves. Thus data points that do not fall exactly on the curves indicate disagreement between the computed mean temperature and the predicted mean temperature obtained from equation (16). Figure 6 indicates close agreement between our predictions and our model calculations.

Values of $\theta_{pre}$ corresponding to the models listed in Table 1, as predicted by equation (16), are listed in Table 1 under the heading $\theta_{16}$, where the subscript 16 is used to distinguish these predictions from those of equation (14). The percentage differences between these final predictions and our computed model mean temperatures are listed in the far right-hand column in Table 1 under the heading $\%\text{error}_{16}$. The maximum disagreement we observe between $\theta_{pre}$ and $\theta$ for these models with $f = 0.547$ is 2.06%. Comparing these percentage errors with those for equation (14) indicates that, overall, equation (16) is the more successful predictive equation and has reduced the largest discrepancies at high values of $H$.

As $H$ is increased both the $f = 0.547$ and $f = 1$ cases evolve into systems dominated by closely spaced columnar downwellings (Houseman, 1988; Travis et al., 1990). Transitions between different planforms at lower values of $H$ (e.g., Figure 2a) occur abruptly as $H$ is increased for a fixed value of $f$. Consequently, some deviation between observation and the predictive equation might be expected below the threshold where the columnar downwelling planform occurs. However, we find equation (16) remains robust for all of the convective planforms encountered.
In this section we test the generality of equation (16) by examining two sets of models with values of $f$ that are different from the value of $f = 0.547$ upon which the parameterization in equation (16) is based. In the first set we consider four models with the same Rayleigh number, $Ra_B = 10^6$, but shells with four different curvature factors, $f$, and internal heating rates, $H$. Specifically, we examine models with $f = 0.3, 0.4, 0.7$ and 0.8. Three views of a single snapshot of the temperature fields are shown in Figure 7 for each of the four models. It is clear from this figure that both radial and lateral scales of the thermal structures can vary widely from $f = 0.3$ to $f = 0.8$.

Characteristics of these four models, including the heating parameters, $\theta_{pre}$ as per equation (16) and the computed model values of $\theta$ are summarized in Table 2. In Figure 8 we present a visual comparison of $\theta$ versus $\theta_{pre}$ which illustrates close agreement in all cases. The actual percentage differences between the values of $\theta_{pre}$ and $\theta$ are listed in Table 2. These indicate agreement at the two to three percent level. This close agreement over such a wide range of values of $f$, and $H$, and scales of thermal structures (as illustrated in Figure 7), strongly suggests that equation (16) generalizes to other values of $f$, and that the parameter $\lambda$ in equation (13) is, therefore, essentially independent of $f$.

As a second check on the generality of equation (16) we consider four models all with the same value of $f = 0.4$. Different values of the model input parameters, $Ra_B$ and $H$, were selected such that the value of $\theta_{pre}$ predicted by equation (16) is the same in every case. Characteristics of these four models
are listed in Table 3, including the computed values of the mean temperatures of the convecting shells, \( \theta \). While the common value of \( \theta_{\text{pre}} \) was 0.751, Table 3 indicates that the computed model values of \( \theta \) ranged from 0.755 to 0.739. The maximum discrepancy was 1.62\%. Again, close agreement is found between the computed and predicted mean temperature over a wide range of \( Ra_B \) and \( H \) at a value of \( f \) which is different from that at which the parameterisations were developed.

### 3.4 Surface heat flow scaling

Given the success of the predictive equations derived in the previous section for mean temperature, we conclude by determining and testing a predictive equation for surface heat flow, \( Q_s(\text{pre}) \). We follow the arguments presented by Sotin and Labrosse (1999) and invert the observed temporally averaged surface heat fluxes from our calculations listed in Table 1, to satisfy an equation of the form

\[
Q_s(\text{pre}) = \Gamma Ra^\zeta \theta^\xi.
\]  

We find that our observations of surface heat flux are best fit with equation (17) when \( \Gamma = 1.192 \), \( \zeta = 0.274 \) and \( \xi = 1.022 \). Our findings do not agree as closely with the exponents predicted by theory, \( \zeta = 1/3 \) and \( \xi = 4/3 \), as the values obtained by Sotin and Labrosse (1999), who used data from Cartesian geometry experiments. However, the agreement between our observed and predicted heat fluxes is comparable to the agreement between the observed and predicted heat fluxes in the Cartesian geometry study. Figure 9 plots the surface heat flux predicted by equation (17), \( Q_s(\text{pre}) \), against the actual surface
heat flux observed in our experiments, $Q_s$, and indicates the good agreement between the inverted heat flux values and the data for the majority of the models. Table 4 gives the observed average surface heat flux, $Q_s$, the standard deviation of the observed surface heat flux, the surface heat flux predicted by an equation of the form of equation (17) with the aforementioned values of $\Gamma$, $\zeta$ and $\xi$, and the difference between the observed and predicted values for the surface heat flux as a percentage of the latter.

The observed values and the values predicted by our equation (17) differ by less than 10% in all of the cases listed in Table 4. However, the heat fluxes obtained from equation (17) generally disagree with observed values by 10-25% in the experiments we performed with $f \neq 0.547$. In particular, we find that equation (17) gives higher surface heat flows than observed values for $f > 0.547$ and lower surface heat flows for $f < 0.547$. Given this finding, and the difference between the exponents in our equation (17) and those in the Cartesian geometry study, it must be that the parameters in equation (17) feature a dependence on the curvature factor, $f$, that is in addition to the intrinsic dependence on $f$ incorporated in the value of $\theta$. Consequently, while useful for a planet with an Earth-like core/planet radii ratio, equation (17) should be applied with caution for other geometries.

We can derive an alternate form for equation (17) that expresses the surface heat flux in terms of the boundary layer Rayleigh number,

$$Ra_\delta = Ra \left( \frac{\delta}{d} \right)^3 \frac{\Delta}{\Delta T}$$

where $\delta$ is the depth at which the temperature extrapolated from the temperature gradient at the surface is equal to the mean temperature of the convecting
fluid, $\Delta$. With these definitions we can write the dimensional surface heat flux as

$$q = k \frac{\Delta}{\delta}.$$  \hspace{1cm} (19)

Combining equations (18) and (19) and nondimensionalising the heat flux by a scaling heat flux (expressed in terms of characteristic parameters of the system), $k\Delta T/d$, leads to the result

$$Q_s = \left( \frac{Ra}{Ra_\delta} \right)^{\frac{1}{2}} \theta^4.$$  \hspace{1cm} (20)

With this equation we can determine estimates for the boundary layer Rayleigh number in our calculations by using observations of the mean temperature and surface heat flux. Our results indicate that $Ra_\delta$ increases weakly with Rayleigh number and strongly with internal heating rate from 1.7 to 7.0. The Rayleigh numbers in our experiments range from $10^4$ to $10^7$. In a 2D numerical study of convection in a cylindrical geometry, that did not feature any internal heating, Jarvis (1993) found values of $Ra_\delta$ ranging from 5.5 to 10.2 for Rayleigh numbers varying from $10^5$ to $10^7$. However, in a 3D Cartesian geometry, Sotin and Labrosse (1999) found values for $Ra_\delta$ between 18 and 33 in calculations featuring internal heating and isothermal boundaries. We suggest that the disagreement between the boundary layer Rayleigh numbers inferred from the calculations in this study and those of the Cartesian study is consistent with the disagreement between the exponents determined for equation (17) and the Cartesian results. These values appear to depend on the geometry of the system. The agreement between the results of Jarvis (1993) and our own findings, both carried out in systems featuring curvature, appear to support this argument.
4 Discussion and conclusions

We set out to develop a parameterization for the mean temperature of a convecting spherical shell, with a curvature factor, \( f \), defined by the ratio of the radii of the inner and outer spherical boundaries. We took our lead from the study in plane layer Cartesian geometry by Sotin and Labrosse (1999) who speculated on how to extend their approach to spherical geometry. We have tested and revised this approach through comparisons with a set of thirty three-dimensional, time-dependent, numerical models of convection in spherical shells. We considered various Rayleigh numbers, curvature factors and rates of internally heating.

Our study focussed, initially, on a set of 23 models with \( f = 0.547 \), appropriate for whole mantle convection in the Earth. The general approach suggested by Sotin and Labrosse (1999) worked extremely well for the case of \( f = 0.547 \), as illustrated in Figure 4. However, the best-fitting values of the scaling parameters were different from those for Cartesian plane layers (\( f = 1 \)). This lead us to attempt to determine appropriate scaling parameters which were, themselves, functions of \( f \), \( Ra_B \) and \( H \) and which would allow successful predictions of mean temperatures for any curvature factor, Rayleigh number or heating mode. We found that a functional dependence of one scaling parameter on \( Ra_B \) and \( f \) was necessary, and sufficient, to obtain a generalized predictive equation, equation (16). This parameter has the form, \( C = 1.318 + 0.251Ra_B^{0.123}(1 - f) \).

Our extension of the scaling analysis to surface heat flux, \( Q_s \), suggests that this quantity has a more complex dependence on \( f \) than that featured in the scaling equation for \( \theta \). However, we find that for the geometry we investigated in
detail (Earth-like) scaling parameters can be determined that yield a successful predictive equation of the form of equation (17) for the surface heat flux in terms of $Ra_B$ and $\theta$. Sotin and Laborosse (1999) similarly found a successful parameterisation in the form of equation (17) for a Cartesian geometry.

For all values of $f$, $Ra_B$ and $H$ considered here, equation (16) successfully predicts, to within 2.4%, the mean temperature of infinite Prandtl number fluids convecting in spherical shells that are heated both by internal heat sources and a hot bottom boundary. This approach may prove useful to estimate mean temperatures of the planetary mantles of Earth and other terrestrial planets, for any given values of the input parameters $f$, $Ra_B$ and $H$. It may also be useful in thermal history modelling to provide an evolving estimate of the mean mantle temperature as planetary cooling causes a gradual reduction of $Ra_B$ and $H$ over geological time. However, our models neglect many complications of realistic planetary mantles, such as compressibility, temperature-dependent viscosity, surface plates, depth-dependent viscosity and thermodynamic properties and mineral phase transitions. Nevertheless, our initial success in obtaining a general parameterization for $\theta$ which accounts for the curvature factor, heating mode and Rayleigh number is an encouraging first step. Butler and Peltier (2000) previously generalized scaling theories for heat flow, characteristic velocities and thermal boundary layer thicknesses in fixed geometry, axi-symmetric, mantle convection calculations featuring the effects of compressibility and depth-dependent thermodynamic properties. Those findings suggest that the inclusion of additional effects may be possible in future 3D calculations.

Another possible limitation of our study is that it is based on fitting a parameterization scheme to mean temperatures that are generated by numerical mod-
els with free-slip upper and lower boundaries. These are appropriate boundary conditions for the Earth’s mantle, since its upper boundary is dominated by mobile lithospheric plates and its lower boundary is the liquid outer core. On other planets, where plate tectonics is absent, rigid upper boundaries may be a more appropriate boundary condition. The combination of a free-slip inner boundary and a rigid outer boundary introduces an asymmetry that affects heat flux at the boundaries. Consequently, the findings presented here may not extend to systems featuring any other combinations of mechanical boundary conditions. Further work will be required to determine the quantitative sensitivity of the parameters in equation (16) to the mechanical boundary conditions of these systems.

5 Acknowledgements

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FIGURE CAPTIONS:

**Figure 1:** Temperature field variation as a function of Rayleigh number, $Ra_B$. The Rayleigh numbers in the three models are $5 \times 10^5$, $10^6$ and $10^7$ from top to bottom, respectively. The ratio of the inner shell radius to the outer shell radius is 0.547 in all three models and the internal heating rate, $H$, is fixed at 23.532. The core temperature has a nondimensional value of 1.0 and the surface temperature has a nondimensional value of 0.0. The panels in this figure only show regions with radii less than $0.98R_o$. The orange isosurface has a value of 0.77 and the blue isosurface has a value of 0.14.

**Figure 2:** Temperature field variation as a function of internal heating rate, $H$. The internal heating rates in the three models are 2.353, 4.706 and 23.532 from top to bottom, respectively. The ratio of the inner shell radius to the outer shell radius is 0.547 in all three models and the Rayleigh number, $Ra_B$, is fixed at $10^6$. The core temperature has a nondimensional value of 1.0 and the surface temperature has a nondimensional value of 0.0. The panels in this figure only show regions with radii less than $0.95R_o$ except for the bottom panel which is rendered as in Fig. 1 (top panel). The isosurface values and colour palette are the same as in Figure 1.

**Figure 3:** Snapshots of mean temperature as a function of depth. Panel a corresponds to the three model snapshots depicted in Figure 1 and panel b corresponds to the three models depicted in Figure 2. The solid curve in each panel corresponds to the same model, Ra1e6H1.
Figure 4: A plot of the mean temperature, $\theta$ as computed with TERRA versus predicted mean temperature, $\theta_{pre}$ as given by equation (10) for each of the 23 models listed in Table 1 for $f = 0.547$.

Figure 5: Temperature field variation in two models with approximately equal globally averaged mean temperatures ($\theta \sim 0.922$). The ratio of the inner shell radius to the outer shell radius is 0.547 in both models. The model in the left panel has a Rayleigh number of $10^5$ and an internal heating rate of 23.532. The model in the right panel features a Rayleigh number of $10^6$ and an internal heating rate of 47.064. The panels in this figure only show regions with radii less than $0.98R_o$. The colour palette differs from that used in Figures 1 and 2 and the isothermal core with a nondimensional temperature of 1.0 is not rendered. The orange isosurface has a value of 1.06 and the green isosurface has a value of 0.38.

Figure 6: Contours of $Ra_B$ in $H - \theta$ space, as predicted from equation (16) using model values of $H$ and $\theta$ from Table 1. Solid circles indicate the values of $Ra_B$ in our numerical models. The right hand frame is a blow up of the lower left corner in the left hand frame.

Figure 7: Sample temperature fields for models with different values of $f$. The Rayleigh number has the same value of $Ra_B = 10^6$ in each frame. The internal heating rates, $H$, are 56.037, 41.170, 10.292 and 13.723 in frames (a), (b), (c) and (d), respectively. Values of $f$ are 0.3, 0.4, 0.7 and 0.8 in frames (a), (b), (c) and (d), respectively. Three views of the same temperature field are shown for each model. The left panel is a plot of the temperature field in a cross sectional view, the centre and left-hand panels are plots of temperature isosurfaces. The isosurfaces plotted are 17% cooler and 17% warmer than the
mean temperature of the fluid in the middle and right-hand panels, respectively.

**Figure 8:** Plot of $\theta$ versus $\theta_{pre}$ for the four models depicted in Figure 7. Numerical values of the plotted data are listed in Table 2.

**Figure 9:** A plot of the observed mean surface heat flux, $Q$ as computed with TERRA versus predicted mean surface heat flux, $Q_{pre}$ as given by equation (17) for each of the 23 models listed in Table 4.
### Table 1

Summary of experiments and results for models with internal heating and $f=0.547$

<table>
<thead>
<tr>
<th>Model</th>
<th>$Ra_B$</th>
<th>$H$</th>
<th>Radial Layers</th>
<th>$\theta$</th>
<th>$C_{(12)}$</th>
<th>$\theta_{14}$</th>
<th>%error$_{14}$</th>
<th>$\theta_{16}$</th>
<th>%error$_{16}$</th>
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</thead>
<tbody>
<tr>
<td>Ra1e5H.1</td>
<td>$10^5$</td>
<td>2.3533</td>
<td>64</td>
<td>0.342(0.00452)</td>
<td>1.737</td>
<td>0.339</td>
<td>-0.93</td>
<td>0.343</td>
<td>0.36</td>
</tr>
<tr>
<td>Ra2e5H.1</td>
<td>$2 \times 10^5$</td>
<td>2.3533</td>
<td>64</td>
<td>0.319(0.00341)</td>
<td>1.698</td>
<td>0.321</td>
<td>0.72</td>
<td>0.325</td>
<td>1.87</td>
</tr>
<tr>
<td>Ra5e5H.1</td>
<td>$5 \times 10^5$</td>
<td>2.3533</td>
<td>64</td>
<td>0.305(0.00279)</td>
<td>1.853</td>
<td>0.302</td>
<td>-0.99</td>
<td>0.305</td>
<td>0.03</td>
</tr>
<tr>
<td>Ra1e6H.1</td>
<td>$10^6$</td>
<td>2.3533</td>
<td>64</td>
<td>0.297(0.00048)</td>
<td>2.012</td>
<td>0.290</td>
<td>-2.47</td>
<td>0.293</td>
<td>-1.53</td>
</tr>
<tr>
<td>Ra1e5H.2</td>
<td>$10^5$</td>
<td>4.7066</td>
<td>64</td>
<td>0.420(0.00474)</td>
<td>1.658</td>
<td>0.425</td>
<td>1.09</td>
<td>0.429</td>
<td>2.06</td>
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<tr>
<td>Ra2e5H.2</td>
<td>$2 \times 10^5$</td>
<td>4.7066</td>
<td>64</td>
<td>0.396(0.00284)</td>
<td>1.743</td>
<td>0.395</td>
<td>-0.22</td>
<td>0.399</td>
<td>0.69</td>
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<tr>
<td>Ra5e5H.2</td>
<td>$5 \times 10^5$</td>
<td>4.7066</td>
<td>64</td>
<td>0.362(0.00143)</td>
<td>1.784</td>
<td>0.363</td>
<td>0.18</td>
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<td>0.99</td>
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<td>1.880</td>
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<td>-0.81</td>
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<tr>
<td>Ra1e5H.5</td>
<td>$10^5$</td>
<td>11.766</td>
<td>32</td>
<td>0.644(0.00290)</td>
<td>1.736</td>
<td>0.634</td>
<td>-1.62</td>
<td>0.634</td>
<td>-1.59</td>
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<tr>
<td>Ra5e6H.5</td>
<td>$5 \times 10^6$</td>
<td>11.766</td>
<td>128</td>
<td>0.398(0.00026)</td>
<td>1.982</td>
<td>0.397</td>
<td>-0.30</td>
<td>0.397</td>
<td>-0.27</td>
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<tr>
<td>Ra1e7H.5</td>
<td>$10^7$</td>
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<td>128</td>
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<td>2.025</td>
<td>0.373</td>
<td>0.14</td>
<td>0.373</td>
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<tr>
<td>Ra1e4H1</td>
<td>$10^4$</td>
<td>23.533</td>
<td>32</td>
<td>1.371(0.00845)</td>
<td>1.560</td>
<td>1.389</td>
<td>1.31</td>
<td>1.373</td>
<td>0.18</td>
</tr>
<tr>
<td>Ra2e4H1</td>
<td>$2 \times 10^4$</td>
<td>23.533</td>
<td>32</td>
<td>1.210(0.00587)</td>
<td>1.597</td>
<td>1.221</td>
<td>0.89</td>
<td>1.207</td>
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<tr>
<td>Ra5e4H1</td>
<td>$5 \times 10^4$</td>
<td>23.533</td>
<td>32</td>
<td>1.036(0.00256)</td>
<td>1.658</td>
<td>1.036</td>
<td>0.03</td>
<td>1.025</td>
<td>-1.09</td>
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<tr>
<td>Ra1e5H1</td>
<td>$10^5$</td>
<td>23.533</td>
<td>32</td>
<td>0.921(0.00141)</td>
<td>1.696</td>
<td>0.921</td>
<td>-0.05</td>
<td>0.911</td>
<td>-1.14</td>
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<tr>
<td>Ra2e5H1</td>
<td>$2 \times 10^5$</td>
<td>23.533</td>
<td>64</td>
<td>0.821(0.00048)</td>
<td>1.732</td>
<td>0.822</td>
<td>0.11</td>
<td>0.813</td>
<td>-0.94</td>
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<tr>
<td>Ra5e5H1</td>
<td>$5 \times 10^5$</td>
<td>23.533</td>
<td>64</td>
<td>0.716(0.00085)</td>
<td>1.802</td>
<td>0.713</td>
<td>-0.37</td>
<td>0.706</td>
<td>-1.37</td>
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<tr>
<td>Ra1e6H1</td>
<td>$10^6$</td>
<td>23.533</td>
<td>64</td>
<td>0.646(0.00101)</td>
<td>1.844</td>
<td>0.645</td>
<td>-0.15</td>
<td>0.639</td>
<td>-1.10</td>
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<tr>
<td>Ra5e6H1</td>
<td>$5 \times 10^6$</td>
<td>23.533</td>
<td>128</td>
<td>0.516(0.00008)</td>
<td>1.930</td>
<td>0.522</td>
<td>1.17</td>
<td>0.518</td>
<td>0.36</td>
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<tr>
<td>Ra1e7H1</td>
<td>$10^7$</td>
<td>23.533</td>
<td>128</td>
<td>0.475(0.00040)</td>
<td>1.985</td>
<td>0.481</td>
<td>1.30</td>
<td>0.478</td>
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<tr>
<td>Ra1e5H1.5</td>
<td>$10^5$</td>
<td>35.298</td>
<td>64</td>
<td>1.163(0.00200)</td>
<td>1.679</td>
<td>1.172</td>
<td>0.77</td>
<td>1.151</td>
<td>-1.08</td>
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<tr>
<td>Ra1e6H2</td>
<td>$10^6$</td>
<td>47.064</td>
<td>64</td>
<td>0.923(0.00053)</td>
<td>1.798</td>
<td>0.940</td>
<td>1.77</td>
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<tr>
<td>Ra1e7H2</td>
<td>$10^7$</td>
<td>47.064</td>
<td>128</td>
<td>0.647(0.00034)</td>
<td>1.955</td>
<td>0.664</td>
<td>2.58</td>
<td>0.652</td>
<td>0.72</td>
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</table>
### Table 2

Summary of experiments and results for models with variable $f$

<table>
<thead>
<tr>
<th>Model</th>
<th>$f$</th>
<th>$Ra_B$</th>
<th>$H$</th>
<th>Radial Layers</th>
<th>$\theta$</th>
<th>$\theta_{pre}$</th>
<th>$(\theta_{pre} - \theta)/\theta$ (%)</th>
</tr>
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<tbody>
<tr>
<td>f0.3Ra1e6</td>
<td>0.3</td>
<td>$10^6$</td>
<td>56.037</td>
<td>64</td>
<td>0.819(0.00067)</td>
<td>0.839</td>
<td>2.34</td>
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<tr>
<td>f0.4Ra1e6</td>
<td>0.4</td>
<td>$10^6$</td>
<td>41.170</td>
<td>64</td>
<td>0.744(0.00039)</td>
<td>0.751</td>
<td>0.91</td>
</tr>
<tr>
<td>f0.7Ra1e6</td>
<td>0.7</td>
<td>$10^6$</td>
<td>10.293</td>
<td>64</td>
<td>0.551(0.00080)</td>
<td>0.545</td>
<td>-1.05</td>
</tr>
<tr>
<td>f0.8Ra1e6</td>
<td>0.8</td>
<td>$10^6$</td>
<td>13.723</td>
<td>64</td>
<td>0.651(0.00049)</td>
<td>0.664</td>
<td>1.96</td>
</tr>
</tbody>
</table>

### Table 3

Summary of experiments and results for models with $f = 0.4$
and $\theta_{pre} = 0.751$

<table>
<thead>
<tr>
<th>Model</th>
<th>$Ra_B$</th>
<th>$H$</th>
<th>Radial Layers</th>
<th>$\theta$</th>
<th>$(\theta_{pre} - \theta)/\theta$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f0.4Ra1e5</td>
<td>$10^5$</td>
<td>21.428</td>
<td>64</td>
<td>0.755(0.00665)</td>
<td>-0.53</td>
</tr>
<tr>
<td>f0.4Ra5e5</td>
<td>$5 \times 10^5$</td>
<td>33.913</td>
<td>64</td>
<td>0.745(0.00073)</td>
<td>0.80</td>
</tr>
<tr>
<td>f0.4Ra1e6</td>
<td>$10^6$</td>
<td>41.170</td>
<td>64</td>
<td>0.744(0.00039)</td>
<td>0.93</td>
</tr>
<tr>
<td>f0.4Ra2e6</td>
<td>$2 \times 10^6$</td>
<td>49.863</td>
<td>64</td>
<td>0.739(0.00045)</td>
<td>1.60</td>
</tr>
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Table 4

Surface heat flux data for calculations with \( f = 0.547 \)

<table>
<thead>
<tr>
<th>Model</th>
<th>Ra_B</th>
<th>H</th>
<th>Q_s</th>
<th>Q_s(pre)</th>
<th>(Q_s(pre) - Q_s)/Q_s(pre) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ra1e5H.1</td>
<td>10^5</td>
<td>2.3533</td>
<td>8.68(0.1390)</td>
<td>9.32</td>
<td>6.90</td>
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<tr>
<td>Ra2e5H.1</td>
<td>2 \times 10^5</td>
<td>2.3533</td>
<td>10.21(0.2020)</td>
<td>10.65</td>
<td>4.20</td>
</tr>
<tr>
<td>Ra5e5H.1</td>
<td>5 \times 10^5</td>
<td>2.3533</td>
<td>13.47(0.2250)</td>
<td>12.83</td>
<td>-5.01</td>
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<tr>
<td>Ra1e6H.1</td>
<td>10^6</td>
<td>2.3533</td>
<td>16.42(0.2280)</td>
<td>14.86</td>
<td>-10.54</td>
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<tr>
<td>Ra1e5H.2</td>
<td>10^5</td>
<td>4.7066</td>
<td>10.58(0.1500)</td>
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<td>9.56</td>
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<td>Ra2e5H.2</td>
<td>2 \times 10^5</td>
<td>4.7066</td>
<td>12.32(0.2300)</td>
<td>13.13</td>
<td>6.16</td>
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<tr>
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<td>5 \times 10^5</td>
<td>4.7066</td>
<td>15.30(0.2210)</td>
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<td>18.01(0.3020)</td>
<td>17.58</td>
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</tr>
<tr>
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<td>10^5</td>
<td>11.766</td>
<td>15.59(0.1830)</td>
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<td>10.64</td>
</tr>
<tr>
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<td>11.766</td>
<td>31.83(0.2090)</td>
<td>31.52</td>
<td>-0.99</td>
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<td>10^7</td>
<td>11.766</td>
<td>36.99(0.2630)</td>
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<td>Ra1e4H1</td>
<td>10^4</td>
<td>23.533</td>
<td>22.11(0.3340)</td>
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<td>23.533</td>
<td>22.70(0.3950)</td>
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<td>23.72(0.3410)</td>
<td>23.58</td>
<td>-0.59</td>
</tr>
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