Chapter 1 – Looking at Data – Distributions

What Statistics is all about

- **Get information** from data (numbers, facts)
- Draw pictures and calculate numbers that **describe** the data
  - *can make a complicated list of numbers look very simple*
- Use data to **make decisions and conclusions** about the world.
  - *how do we know that a new drug actually works?*
  - *how do housing costs change over time?*

Data

Car magazine collected information about (38) models, like this:

Car	MPG	Weight	Cylinders	Horsepower	Country
Buick Estate Wagon	16.9	4.360	8	155	U.S.
VW Rabbit	31.9	1.925	4	71	Germany
Mercury Zephyr	20.8	3.070	6	85	U.S.
Fiat Strada	37.3	2.130	4	69	Italy
Mazda GLC	34.1	1.975	4	65	Japan
Saab 99 GLE	21.6	2.795	4	115	Sweden
BMW 320i	21.5	2.600	4	110	Germany

- each **row** describes *car* (**individual**)
- each **column** describes *feature of car* (**variable**)

Kinds of variables (chapter 1 intro)

Variables can be:

- **quantitative**: measured or counted (mpg, cylinders)
- **categorical**: name or group category (country)

Distribution of variable: list of values, how often each value occurs.

Data set

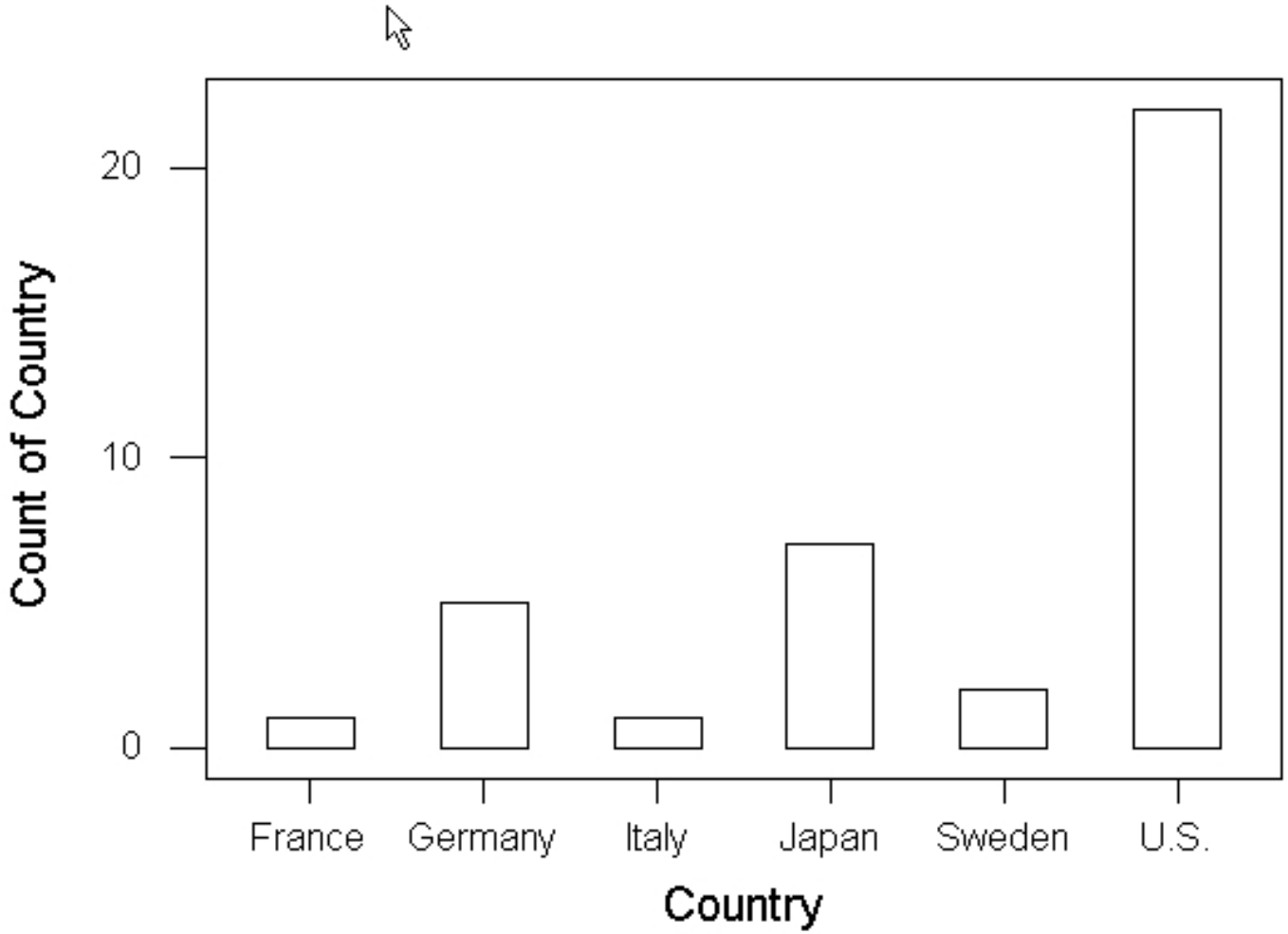
is: a number of variables for each of a number of individuals, plus a “story” of:

- **why** data were collected
- **who** is in there (which individuals)
- **what** variables measured and how

Can look at variables one at a time, or several together.
Look first at some graphs, then worry later about how to draw them.

Displaying distributions with graphs (§1.1)

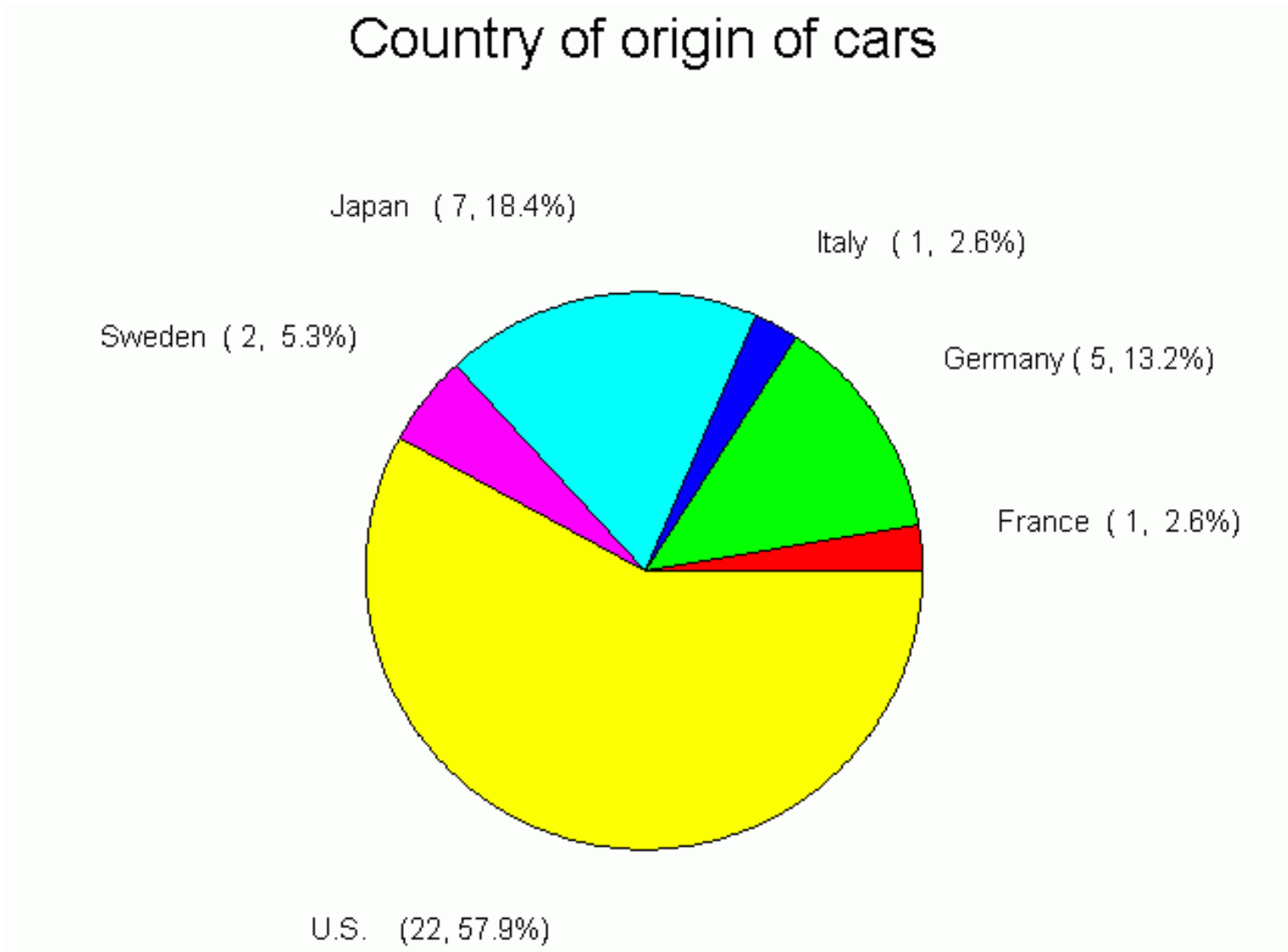
Bar chart for categorical variables:



Bar chart comments

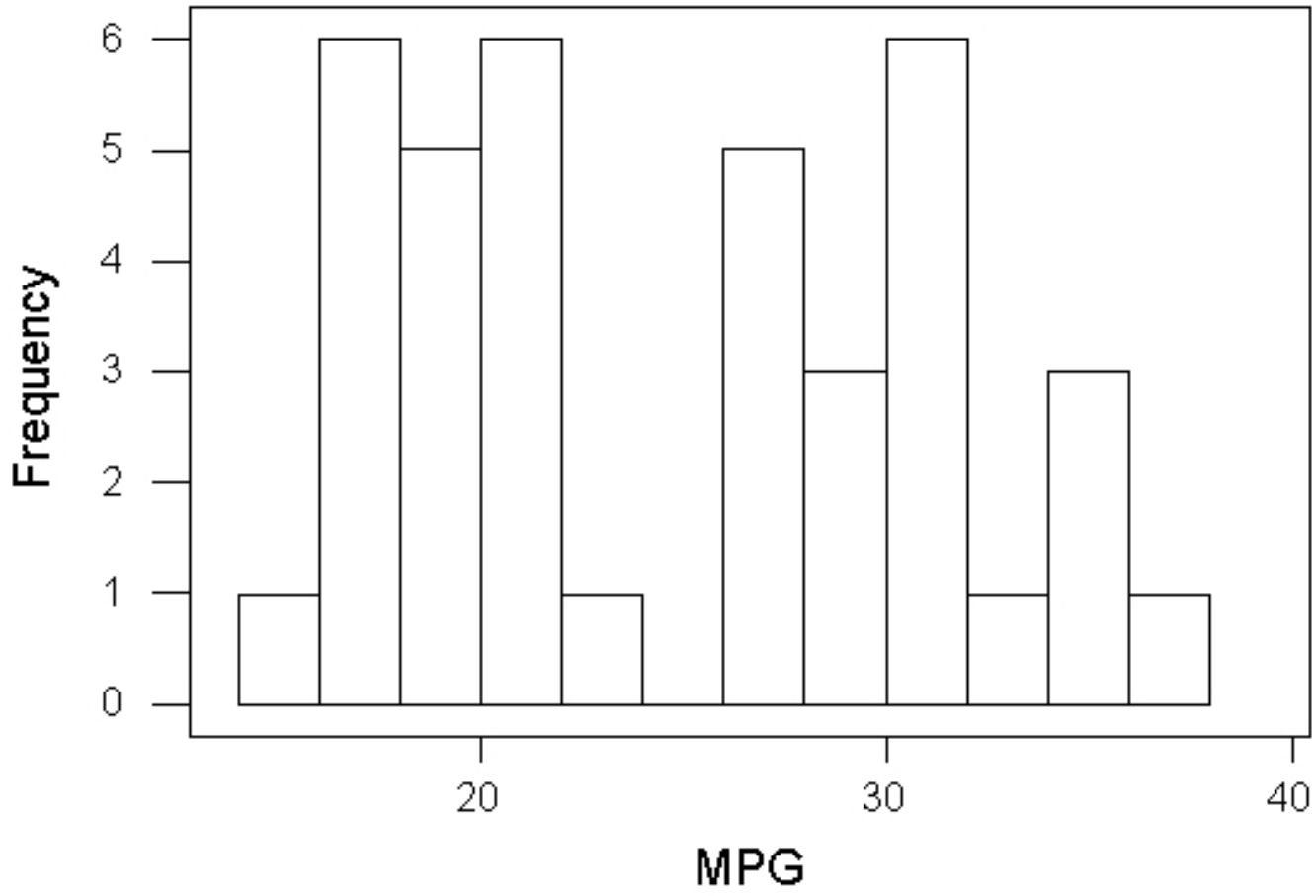
- Shows cars from each country: most from US, fewest from France, Italy.
- Very difficult to see from original list of numbers.
- (In general: how many individuals in each category.)

Pie chart


Country of origin of cars
Japan (7, 18.4%)
Italy (1, 2.6%)
Sweden (2, 5.3%)
Germany (5, 13.2%)
France (1, 2.6%)
U.S. (22, 57.9%)

Graphs for quantitative variables

For *quantitative* variables, can use **histogram**.
Example: miles per gallon.



Learning from histogram

- Cars divide into 2 groups: those with MPG around 20, those with MPG around 30.
- Again, hard to see without picture.

An alternative to histogram: stemplot

MPGs, rounded:

```
28   31   21   37   16   32   34   34
22   32   29   27   19   27   18   21
22   27   16   22   34   30   17   17
17   20   19   19   31   27   35
22   32   18   17   18   28   30
```

Make list of tens digits, write units on correct row (first 3 cars)

```
1 |
2 | 8 1
3 | 1
```

Continue until all cars done:

Stemplot continued

```
1 | 6 9 8 6 7 7 7 9 9 8 7 8
2 | 8 1 2 7 9 7 7 1 2 7 2 0 7 2 8
3 | 1 7 2 4 4 2 4 0 1 5 2 0
```

Optional last step: sort the leaves, lowest to highest, on each line:

```
1 | 6 6 7 7 7 7 8 8 8 9 9 9
2 | 0 1 1 2 2 2 2 7 7 7 7 7 8 8 9
3 | 0 0 1 1 2 2 2 4 4 4 5 7
```

- Advantages: easy to do by hand; easy to see lowest (16) and highest (37) values
- Disadvantage (here): hard to see shape, compared to histogram: too few “stems”.

Choosing stems

- Have to pick what to use as stems. Might be 10's, so leaves are units (as here). Might be units, so leaves are first decimal place (0.1). Depends on data.
- Example: guinea pig survival times (Table 1.8): 73, 102, 121, 137, 214, 403, 598 (plus a bunch of other values). Try using 100s as stems, 10s as leaves, discard last digit:

```
0 | 7
1 | 0 2 3
2 | 1
3 |
4 | 0
5 | 9
```

- Seems to give about right number of stems.

Splitting stems

Recall MPG stemplot:

```
1 | 6 6 7 7 7 7 8 8 8 9 9 9
2 | 0 1 1 2 2 2 2 7 7 7 7 7 8 8 9
3 | 0 0 1 1 2 2 2 4 4 4 5 7
```

To get more lines on a stemplot (and better picture of shape of distribution), split each line into two parts: “leaves” 0–4 and leaves 5–9. For MPGs this gives:

```
1 | 6 6 7 7 7 7 8 8 8 9 9 9
2 | 0 1 1 2 2 2 2
2 | 7 7 7 7 8 8 9
3 | 0 0 1 1 2 2 2 4 4 4
3 | 5 7
```

Like the histogram, shows many cars with MPGs in high teens, and in low 30s. Look carefully also to see “gap” between 22 and 27.

Drawing stemplots

- Choose digit to be *stem* (eg. tens digit). Next digit is leaf, and discard any digits beyond that.
- Separate each value (in your head) into *stem* and *leaf*.
- Write stems in column, smallest at top, vertical line to right.
- Write each leaf in row to right of its stem.
- (optional) arrange leaves in increasing order.

Fixing up a stemplot

Aim of stemplot is to show right number of stems to see *shape* of distribution. Stemplot as drawn sometimes seems to have wrong number of stems.
Also, sometimes not clear which digit should be stem, so pick one, draw stemplot and then re-draw if necessary:

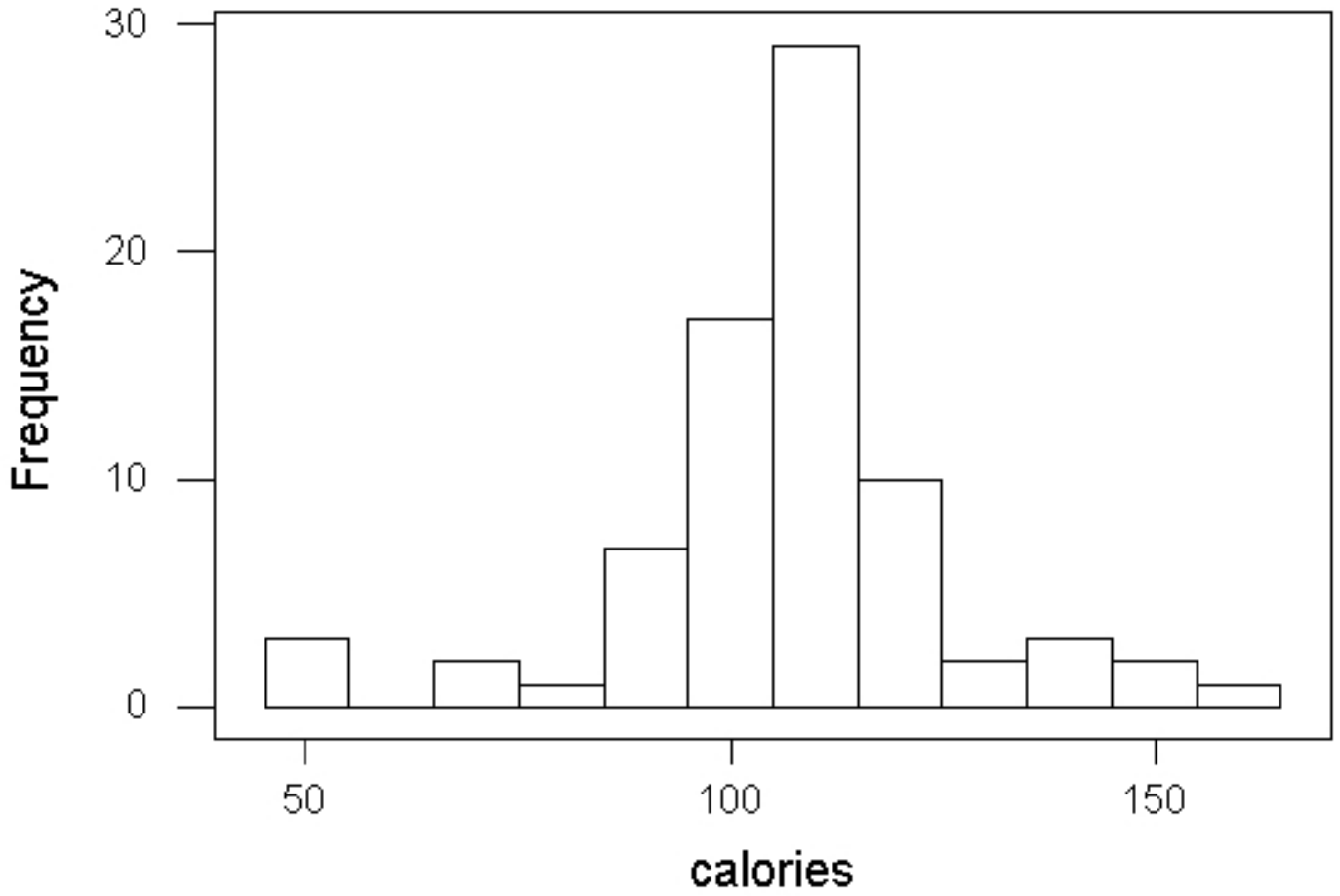
- If too many stems, choose next digit to left (eg. 100s digit) to be new stem, next digit to be leaf, and *trim* one remaining digits from right of values.
- If too few stems, *split* in half.
- Idea: want to show *shape* of distribution.

Shape, centre and spread: the cereal data

Different data: information on 77 kinds of breakfast cereal (individuals), lots of variables such as:

- calories per serving
- protein per serving
- fat per serving
- sodium per serving
- fibre per serving
- potassium per serving
- shelf on which found at supermarket (1, 2, 3)
- serving size (cups)

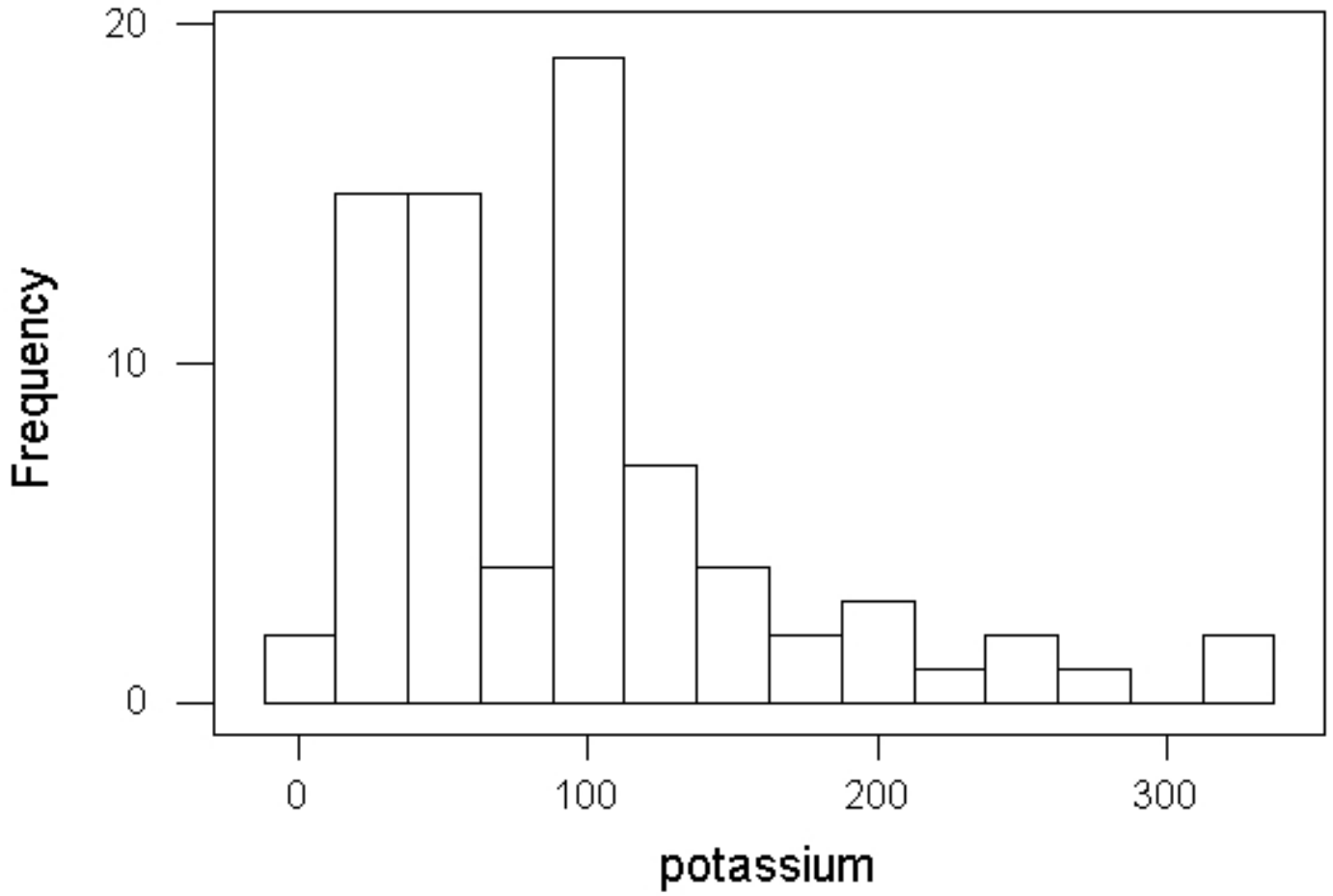
Histogram of calories per serving



What histogram shows

Most cereals have around 110 calories/serving. Only a very few cereals have a lot more or fewer calories.
That is, **centre** of distribution of values around 110, **spread** fairly small. Also, **shape** fairly symmetric: picture goes “up” same way as “down”.

Compare potassium per serving



Discussion

Most cereals have 100 mg or less of potassium (difficult to say where exact centre is), but there is a lot of spread – from no potassium to over 300 mg.
Shape not symmetric: more cereals with unusually high potassium than with unusually low. Up quickly, down slowly.
Called **skewed to right**.

Shape, centre and spread with stemplots

Calorie content (rounded to nearest 10 in data)

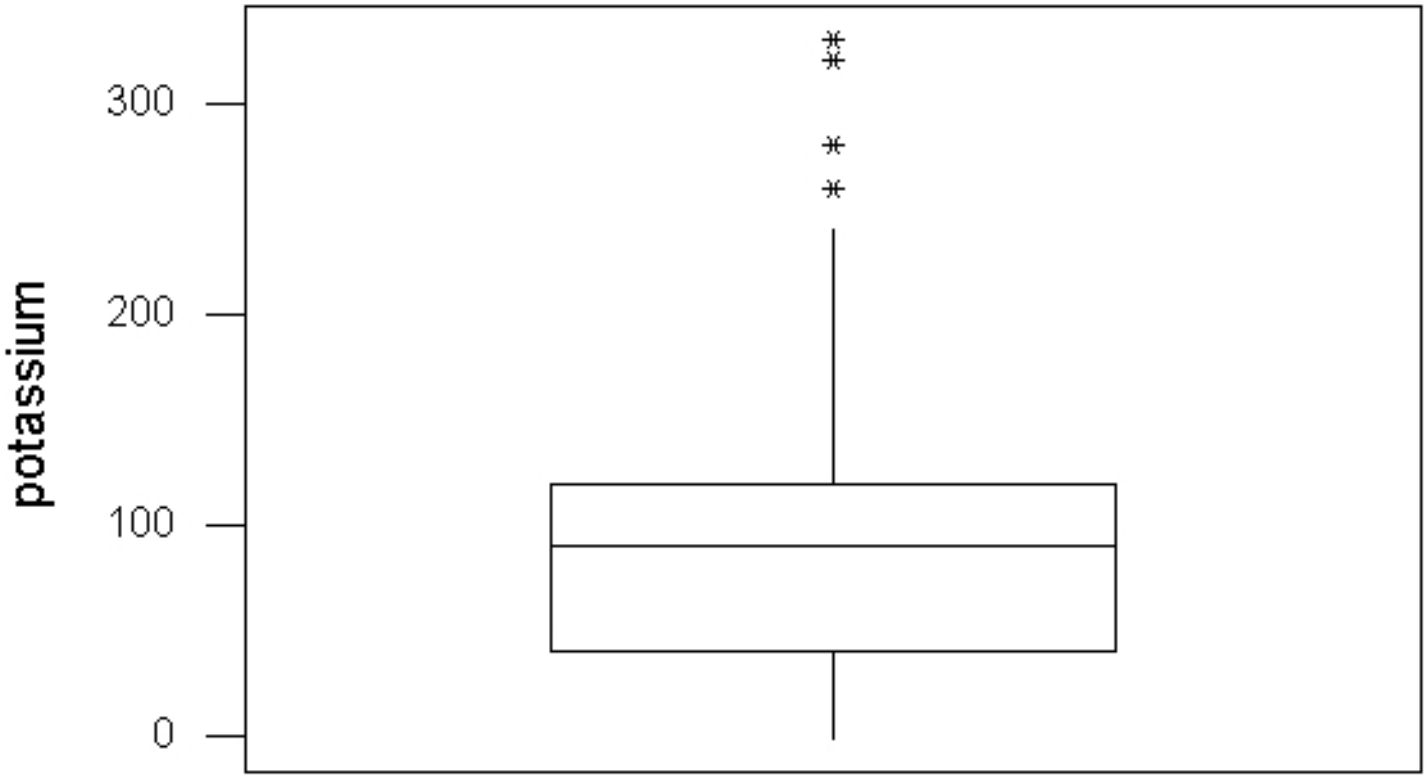
```
Stem-and-leaf of calories  N  = 77
Leaf Unit = 1.0

    3    5 000
    3    6
    5    7 00
    6    8 0
   13    9 0000000
   30   10 00000000000000000
  (29)  11 00000000000000000000000000000
   18   12 0000000000
    8   13 00
    6   14 000
    3   15 00
    1   16 0
```

Stemplot of potassium

```
Stem-and-leaf of potassiu  N  = 77; Leaf Unit = 10
    3    0 001
   17    0 22222333333333
   29    0 444444445555
   34    0 66667
  (11)   0 88999999999
   32    1 00000111111
   21    1 222233
   15    1 44
   13    1 6677
    9    1 99
    7    2 0
    6    2 3
    5    2 4
    4    2 6
    3    2 8
    2    3
    2    3 23
    0    3
```

Boxplots (from §1.2)

Boxplots designed to show shape, centre, spread (when "centre" makes sense). For potassium per serving:

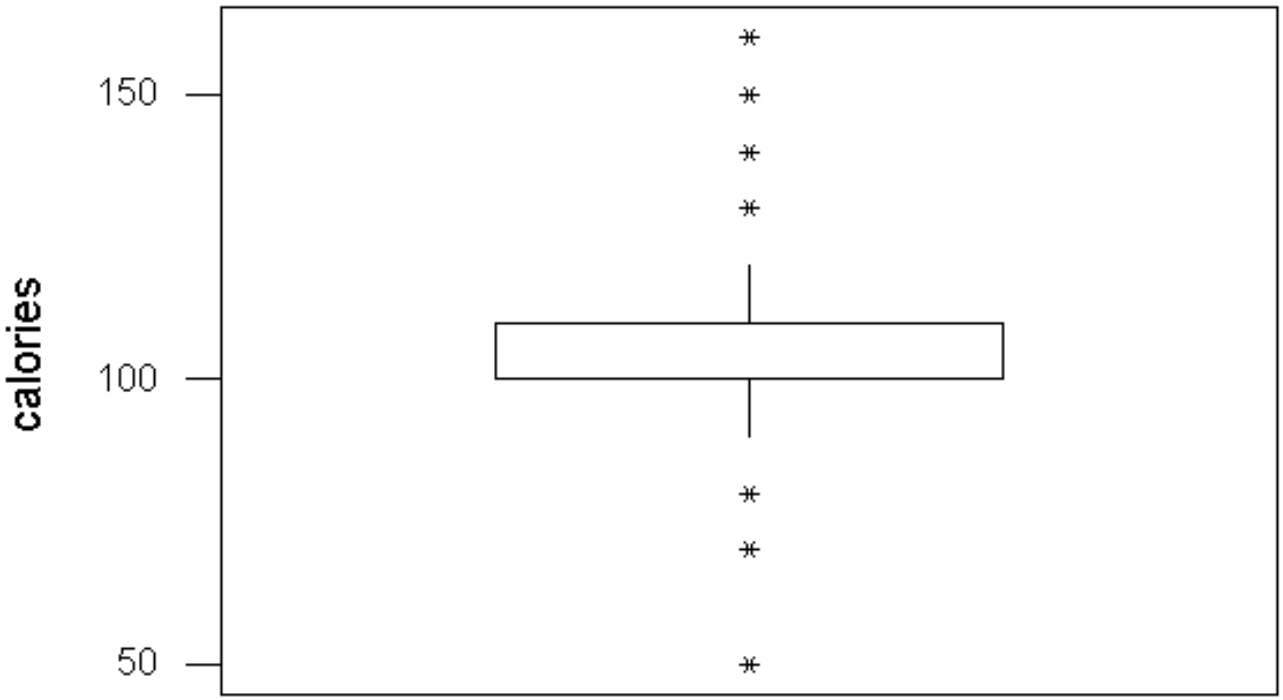


Parts of boxplot

- rectangle with line across it. Line marks “centre”, top/bottom of rectangle show middle 50% of data. Typical potassium level just under 100, middle 50% between 40 and 120.
- vertical lines (“whiskers”) above/below rectangle show extent of “plausible” values (0 to 240).
- * marks individual “unusual” values (4, above 260).

Heights of rectangle and vertical lines show spread.
Comparative lengths of whiskers above & below show symmetry/skewness (long upper whisker – skewed to right).

Boxplot for calorie content



Centre around 100, compressed box & whiskers show little spread.
Whiskers same length – symmetric shape. Many unusual values.

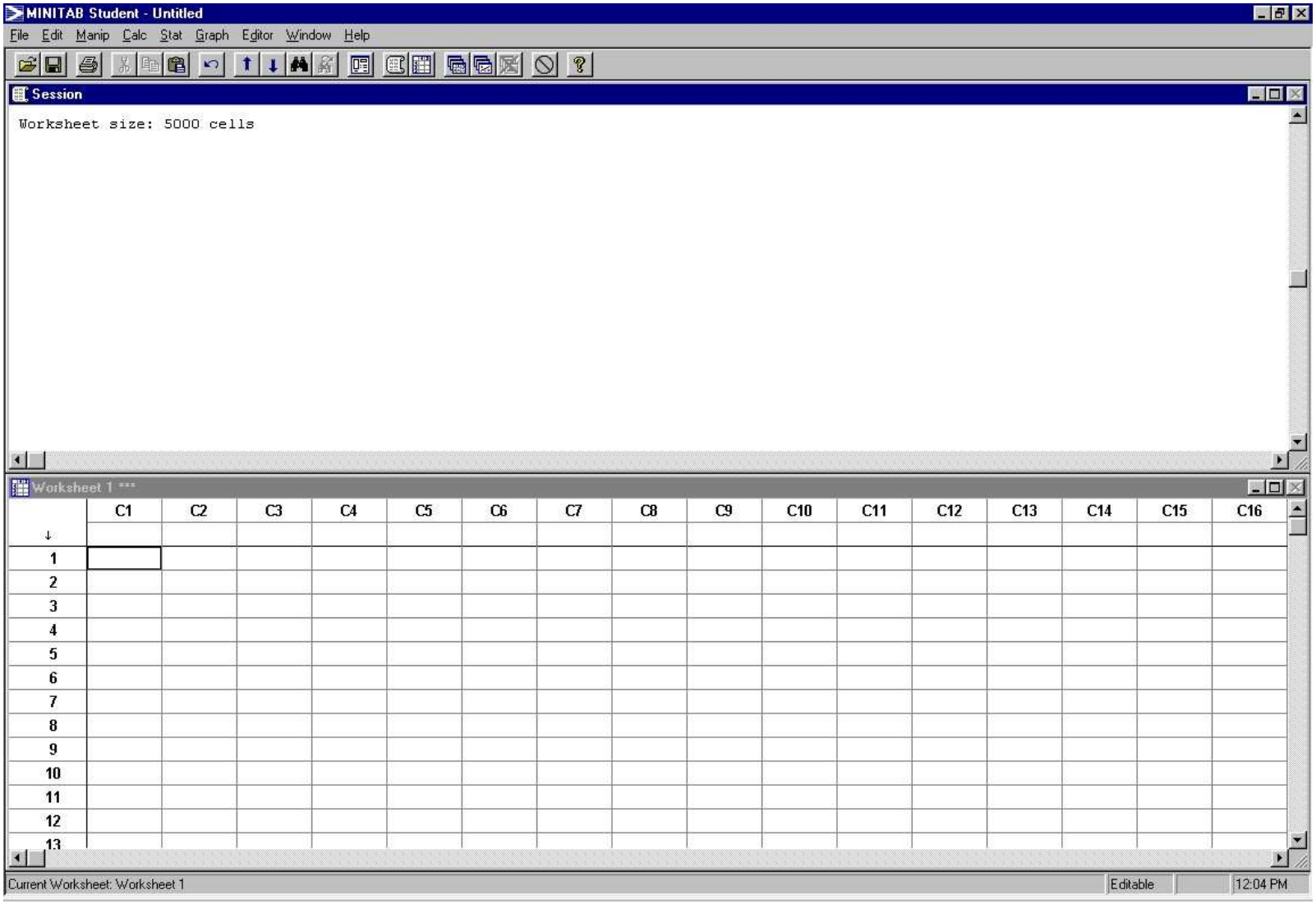
Introduction to Minitab

Minitab is software designed especially for doing Statistics.
Comes bundled with textbook (with manual), also can be accessed in Windows labs on campus.
We use Minitab in this course because:

- it is not difficult to learn (in our opinion)
- it has been well tested over the years of its existence (you can trust its calculations)
- it does all the statistical analyses you are likely to need.

Spreadsheet software (like Excel) is not to be trusted for Statistics!
Startup: Start button, Programs, Minitab.

Startup


MINITAB Student - Untitled
File Edit Manip Calc Stat Graph Editor Window Help
Session
Worksheet size: 5000 cells
Worksheet 1 ***
C1 C2 C3 C4 C5 C6 C7 C8 C9 C10 C11 C12 C13 C14 C15 C16
1 2 3 4 5 6 7 8 9 10 11 12 13
Current Worksheet: Worksheet 1
Editable
12:04 PM

The Minitab display

- Bottom window for data (variables in columns and individuals in rows). Each variable can have a name (use boxes directly below C1, C2), above 1st row.
- Results of calculations appear in the top window (called the Session window).
- The menus at the top let you reorganize your data set (Manip), do calculations with it (Calc and Stat) and draw pictures (Graph).

Getting data in

- File, Open Worksheet:
  - Make sure you're looking in the right place!
  - Textbook data sets: find on disk in textbook (Data Sets, PC Data Sets, Minitab). Search for "Minitab Portable" files (`.mtp`).
  - Class data sets: save from webpage to your desktop, *then* open in Minitab.
- Type in yourself: Worksheet window works like spreadsheet. Space above rows used for column name.

Minitab allows multiple worksheets. Click on the one to use, or select from Window menu.

Drawing graphs

Use Graph menu (duh!), select type of graph. (Examples below for car data.)

- Bar graph: Graph, Bar chart. Click OK. Select country (double-click in left window, appears in right). Click OK.
- Histogram: Graph, Histogram. Choose Simple, click OK. Select MPG (double-click). Click OK.
- Stemplot: Graph, Stem and Leaf, select MPG. Appears in Session window.
- Boxplot: Graph, Boxplot. Choose Simple, click OK. Select variable, click OK.

Describing distributions with numbers (§1.2)

Picture often says much about distribution, but to convince others, may need *numbers* to describe.
Three major features of distribution:

- shape (picture)
- centre (number: mean, median)
- spread (number: IQR, SD)

Centre: the mean

Centre says what “typical”, “average” value is.
Mean: add up values, divide by how many there are.
Example: running back on football team carries ball 5, 3, 11, 1 and 4 yards. Mean distance carried is

$$\frac{5+3+11+1+4}{5} = 4.8 \text{ yards.}$$

Cereal calorie content: from histogram, centre around 110.
The mean for all 77 cereals is 106.88.

Centre: the median

Median: arrange values in order, pick out middle one.

- Football running back: in order, 1, 3, 4, 5, 11. Median is 4 yards (compare mean 4.8 yards).
- Cereal calorie content: median is 110 calories (compare mean 106.88).

If even number of data values, no single middle one. Use mean of middle 2. Example: median of 10, 11, 14, 20 is $(11 + 14)/2 = 12.5$.

Mean vs. median

Both measuring centre: why have both?
When distribution shape *symmetric*, mean and median close together. But if distribution *skewed*, or if unusually high or low values, can be quite different.
Running back: mean (4.8) bigger than median (4) because of one unusually long run (11 yards). Affects mean (adds to total) but only affects median by being bigger than 4.
So use:

- *mean* if shape close to symmetric,
- *median* if shape skewed.

Spread: quartiles and interquartile range

Median splits data values in half (half above median, half below).

Quartiles: split data values $\frac{1}{4} - \frac{3}{4}$. (Quarter below or quarter above).

Tiny example: 10, 11, 13, 15, 18, 19, 21.

- 7 values, median is 15. Split values into lower half and upper half; if odd number of values, don't use median.
- So: lower half 10, 11, 13; upper half 18, 19, 21.
- Median of lower half is 1st quartile: here $Q1 = 11$. Median of upper half is 3rd quartile, $Q3 = 19$.

Warning: other textbooks (and Minitab) have different ways of getting quartiles, but difference is not important.

Interquartile range

- Cereal potassium content: 1st quartile is 40 (quarter of cereals have less potassium than this, three-quarters more). 3rd quartile is 120 (three-quarters less, one quarter more).
- **Interquartile range (IQR)**: difference between 3rd quartile and 1st quartile.
- For potassium content, is $120 - 40 = 80$.
- Larger IQR means data more spread out. (If all values same, IQR would be 0.)
- Compare calorie content of cereals. 1st quartile 100, 3rd quartile 110, IQR $110 - 100 = 10$.
- IQR for calorie content much smaller than for potassium, because values much less spread out.

Numerical descriptions in Minitab

Use Stat menu: Stat, Basic Statistics, Display Descriptive Statistics. Select (double-click) variables to display, eg. calorie and potassium content of cereals (can be more than one). Result:

```
Descriptive Statistics: calories, potassium

Variable    N  N*    Mean  SE Mean  StDev  Minimum
calories   77   0  106.88     2.22  19.48    50.00
potassium  77   0   96.08     8.12  71.29    -1.00

Variable        Q1  Median      Q3   Maximum
calories    100.00  110.00  110.00    160.00
potassium    40.00   90.00  120.00    330.00
```

Calculate IQR by hand as $Q3 - Q1$.

Boxplot revisited: outliers

- IQR measures spread of middle 50% of data. Also yardstick to decide whether value unusual compared to rest: an **outlier**.
- Criterion: *more than* $1.5\times$ *IQR above 3rd quartile, or more than* $1.5\times$ *IQR below 1st quartile* considered outlier. (Example below.)
- Boxplot (eg. for potassium): centre line at *median*, top and bottom of box at 1st and 3rd quartiles. Whiskers from box to most distant non-outlier; outliers plotted individually.

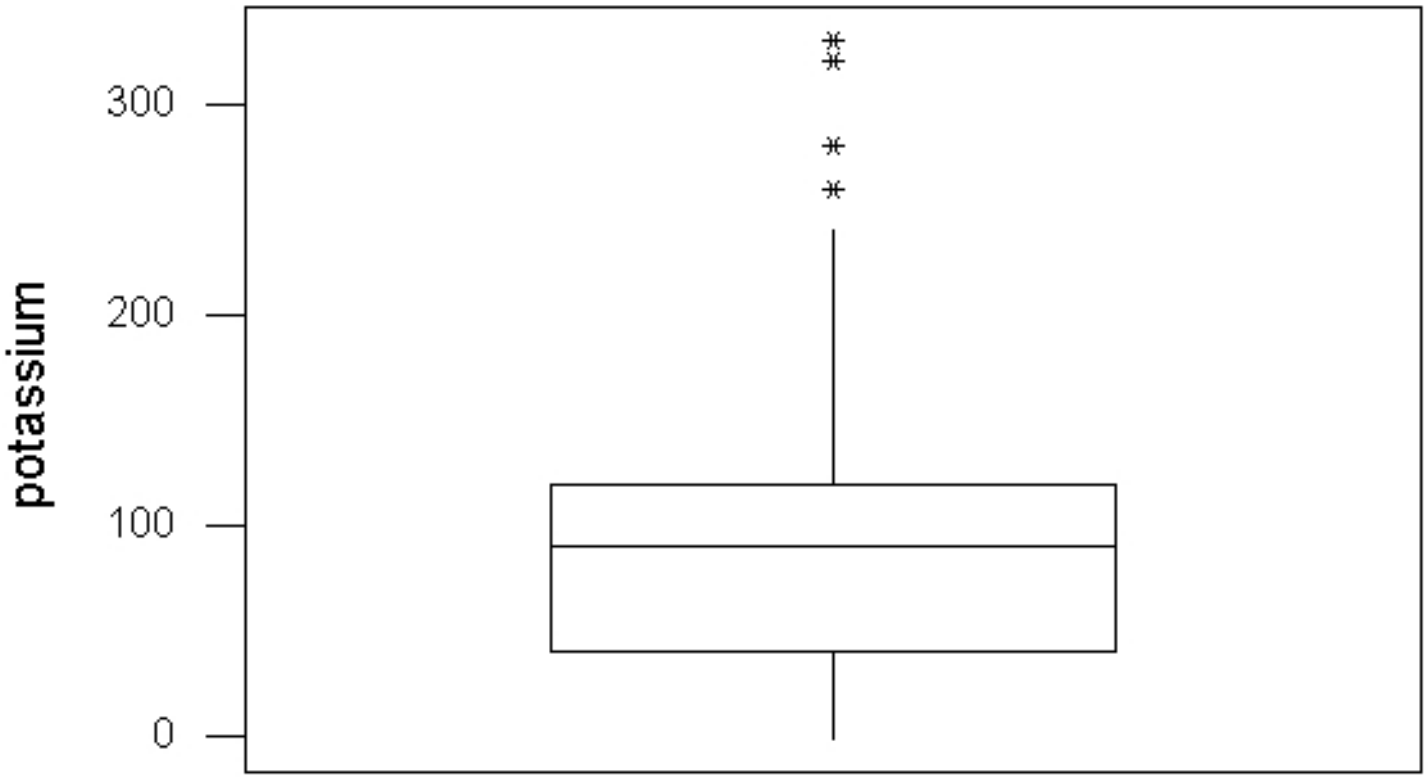
Example

Check calculation for potassium: median 90, 1st quartile (Q1) 40, 3rd quartile (Q3) 120.

$$IQR = 120 - 40 = 80; \quad 1.5 \times IQR = (1.5)(80) = 120.$$

Therefore anything below $40 - 120 = -80$ is an outlier, also anything above $120 + 120 = 240$.
In data: smallest value 0 (no outliers at low end). 4 values above 240, one exactly equal. 240 is not outlier; is highest non-outlier, so extend whisker to 240, plot higher values individually.

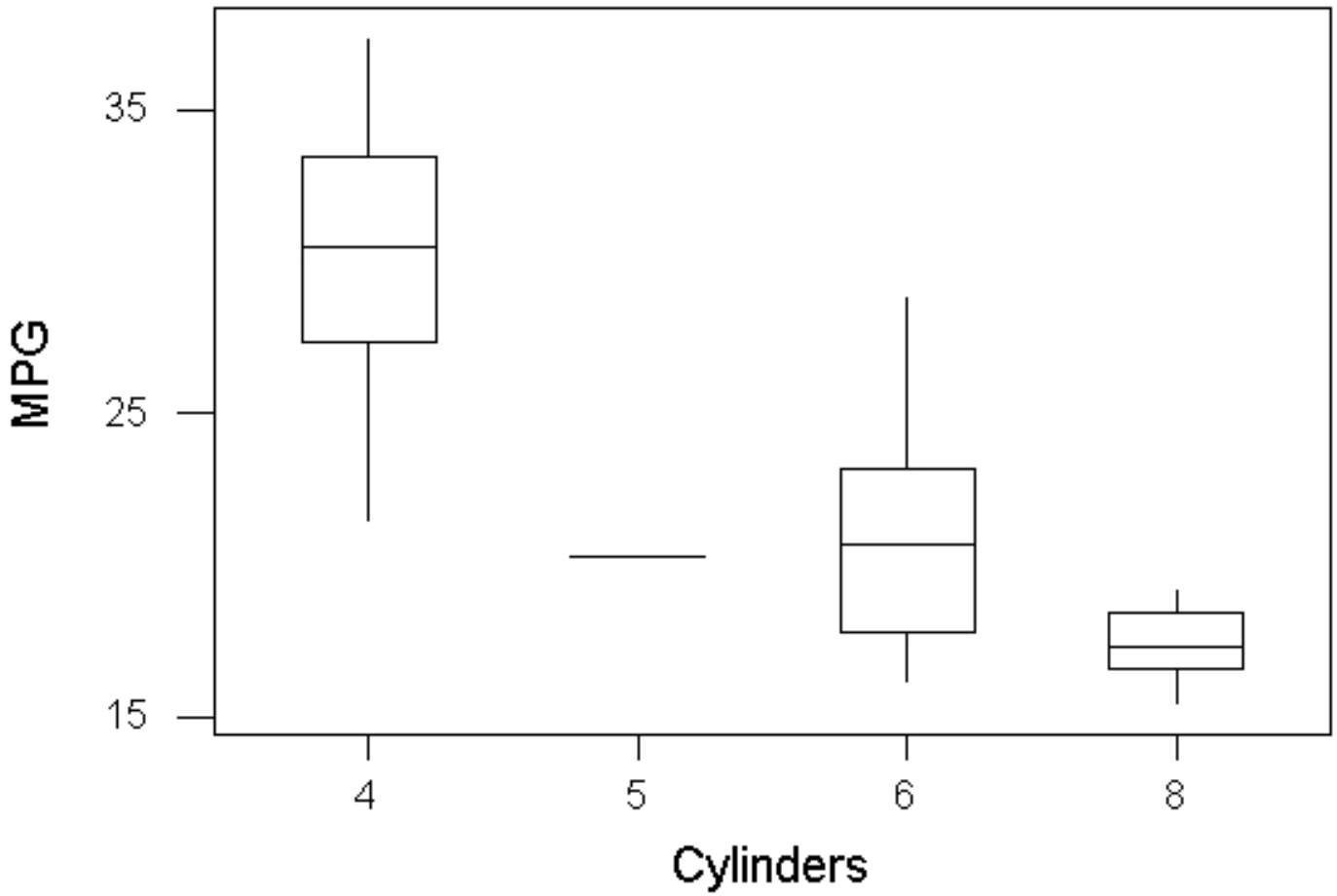
Review boxplot



Boxplots for comparison

- Boxplots useful for comparison – plot side by side.
- Cars: More cylinders means worse gas mileage? Boxplot for cars with 4, 6, . . . cylinders.
- Minitab: choose With Groups instead of Simple. Select MPG as Graph Variable, click on Categorical Variable box and select Cylinders there.

MPG by cylinders boxplot



Standard deviation

Median, IQR use only middle of distribution. Mean uses all values; spread that does same?

Standard deviation (SD) based on how far away from mean each value is. If many values far from mean (or some very far), SD large. If all values close, SD small.

Value minus mean $+$ if value above mean, $-$ if value below. So calculation squares each value first (makes $+$), then takes square root at end. (Details in section 1.2 of text.)

SD and IQR

Examples from cereal data:

Variable	IQR	SD
Calories	10	19.48
Potassium	80	71.29

IQR, SD numbers not directly comparable, but pattern same: calories less spread-out than potassium content.

SD vs. IQR

- SD, like mean, uses all data values. So can be badly affected by outliers or skewed shape.
- So use SD when would use mean: shape symmetric, no outliers. Otherwise use median and IQR.

Changing units of measurement

Could measure eg. heights in feet or metres, temperatures in Celsius or Fahrenheit. Conclusions shouldn't be affected by choice.

Common kinds of change:

- multiplying each value by the same number b
- adding the same number a to each value

Changes numbers on plots, but *no effect* on shape.

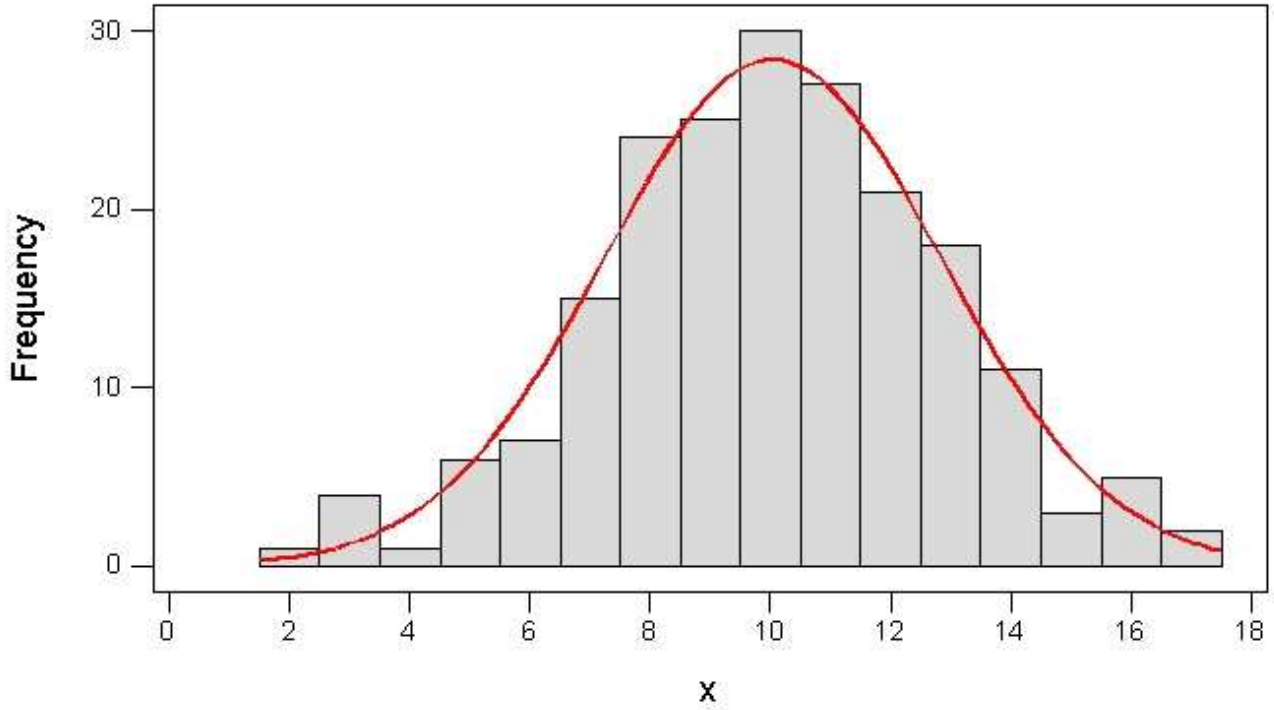
Effect of changes of unit

Effects of these changes on quantitative descriptions predictable:

- multiplying by b multiplies mean, median, SD and IQR by b
- adding a adds a to mean and median, but leaves SD and IQR unchanged.

Makes sense: adding a number changes centre, but “shifts distribution along” without changing its spread.

The normal distributions (§1.3)



Histogram has fairly symmetric shape. Smooth curve superimposed passes (more or less) through top of histogram bars.

Normal density function

Curve called a **normal density function**. Might guess that histogram looks jagged because of randomness of data; "underlying" picture is smooth.

Normal density function is **mathematical model** of process producing data.

If histogram with bars matching normal density function, data said to have **normal distribution**.

Normal — mean and SD

Normal distribution described completely by *mean* and *standard deviation*.
Notation with Greek letters: mean μ (mu), SD σ (sigma).
Picture: $\mu = 10, \sigma = 3$.
Mean at peak of curve (obvious “centre”).
σ distance from peak to “shoulder” (same both sides).
If σ smaller, curve is more peaked; if σ larger, curve lower and flatter.

Why normal distribution is useful

- distribution of *real data* sometimes
- distribution of *chance outcomes*
- central to *statistical inference* (drawing conclusions from data)

68–95–99.7

On *any* normal distribution (whatever mean and SD):

- 68% of values fall within σ of the mean (between $\mu - \sigma$ and $\mu + \sigma$)
- 95% of values fall within 2σ of the mean (between $\mu - 2\sigma$ and $\mu + 2\sigma$)
- 99.7% of values fall within 3σ of the mean (between $\mu - 3\sigma$ and $\mu + 3\sigma$)

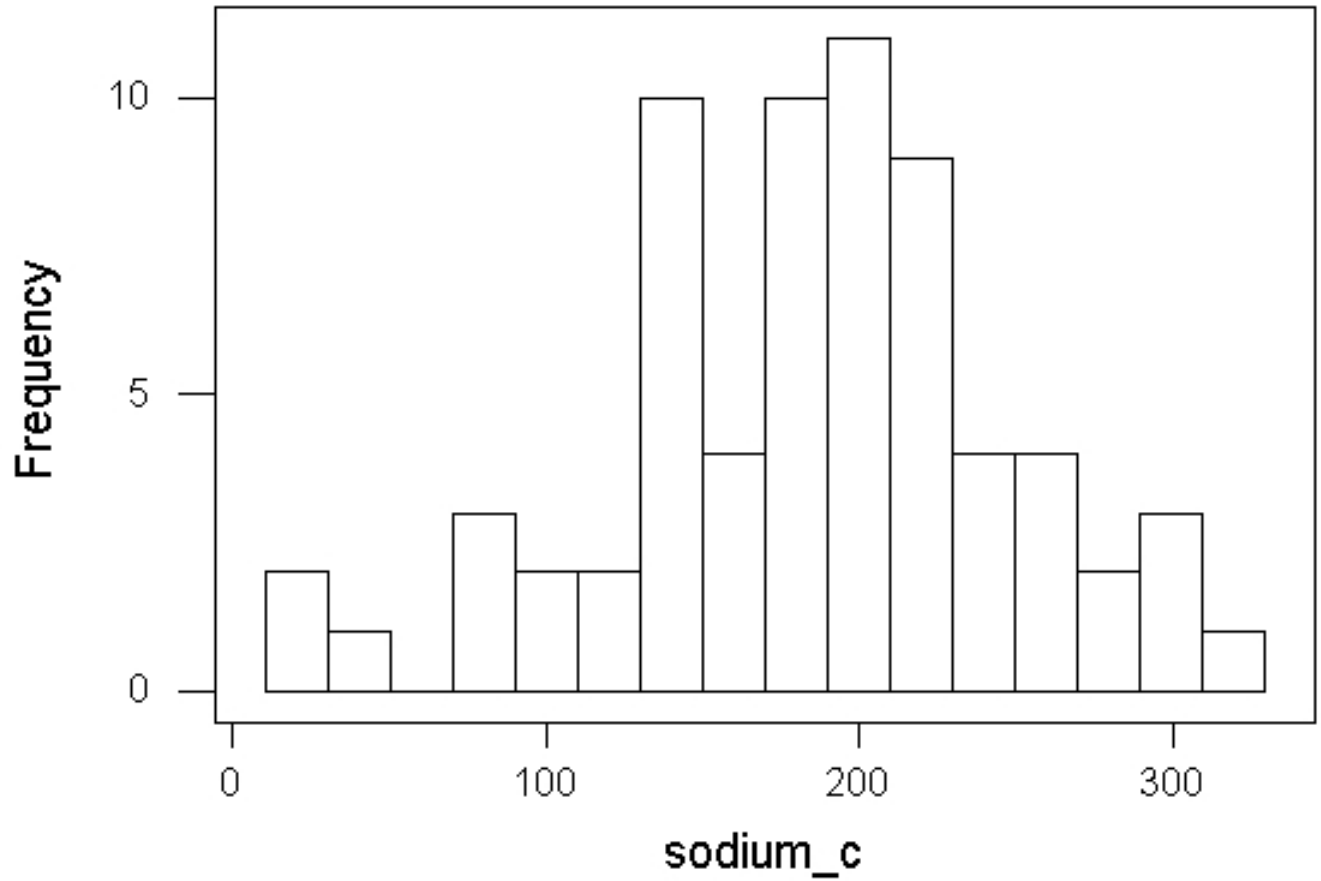
Example 1

In a normal distribution with mean 10 and SD 3:

- 68% of values between $10 - 3 = 7$ and $10 + 3 = 13$.
- 95% of values between $10 - (2)(3) = 4$ and $10 + (2)(3) = 16$.
- What % of values between 1 and 19? $1 = 10 - (3)(3)$ and $19 = 10 + (3)(3)$, so 99.7% of them.
- What % of values between 10 and 13? 68% of values between 7 and 13, normal curve symmetric, so half of that: 34%. (Draw picture.)
- What % of values below 4? 95% between 4 and 16, so 5% either below 4 or above 16. Symmetric, so half each end: 2.5%.

Example 2: Cereal sodium levels

Cereal data: sodium content levels look normal-distribution-ish:



Rule for cereal sodium levels

Try rule. Mean is 180.8, SD 64.0, 68 values. (To find out how many observations between two values, go to data set and count them.)

Low	High	Count	Actual %	Rule
116.8	244.8	50	74%	68%
52.8	308.8	64	94%	95%
–11.2	372.8	68	100%	99.7%

Pretty close!

Standardizing and z-scores

- Can find out how unusual a value from a normal distribution is.
- Example: Cheerios have 290 mg sodium/serving. Subtract mean, divide SD to get

$$z = \frac{290 - 180.8}{64.0} = 1.71.$$

  Higher than average sodium; in fact, notably higher relative to SD.
- Process called **standardizing**; result called **z-score**.
- (Fact: if normal distribution correct, only 4% of cereals higher in sodium.)

Normal distribution calculations

- No nice formula, but can be done with Table A in text.
- Example 1.27 in text: SAT scores have normal distribution, mean 1026, SD 209. What proportion of students have scores 820 or bigger?
- Steps: value to z-score to proportion.
- Value here 820. Then

$$z = \frac{820 - 1026}{209} = -0.99.$$

- Look up -0.99 in Table A: 0.1611. *Table gives you "less"*, so proportion 0.1611 of students have SAT scores less than 820, and proportion $1 - 0.1611 = 0.8389$ have scores greater than 820.

Proportion between

To get the proportion between two values, turn *both* values into z-scores, look them *both* up in Table A, then take difference.

- In previous example, proportion of students scoring between 720 and 820?
- 820 corresponds to $z = -0.99$ and proportion 0.1611. For 720,

$$z = \frac{720 - 1026}{209} = -1.46.$$

  In Table A, $z = -1.46$ goes with proportion 0.0721.
- So proportion of students scoring between 720 and 820 is $0.1611 - 0.0721 = 0.0890$.

Getting value from proportion

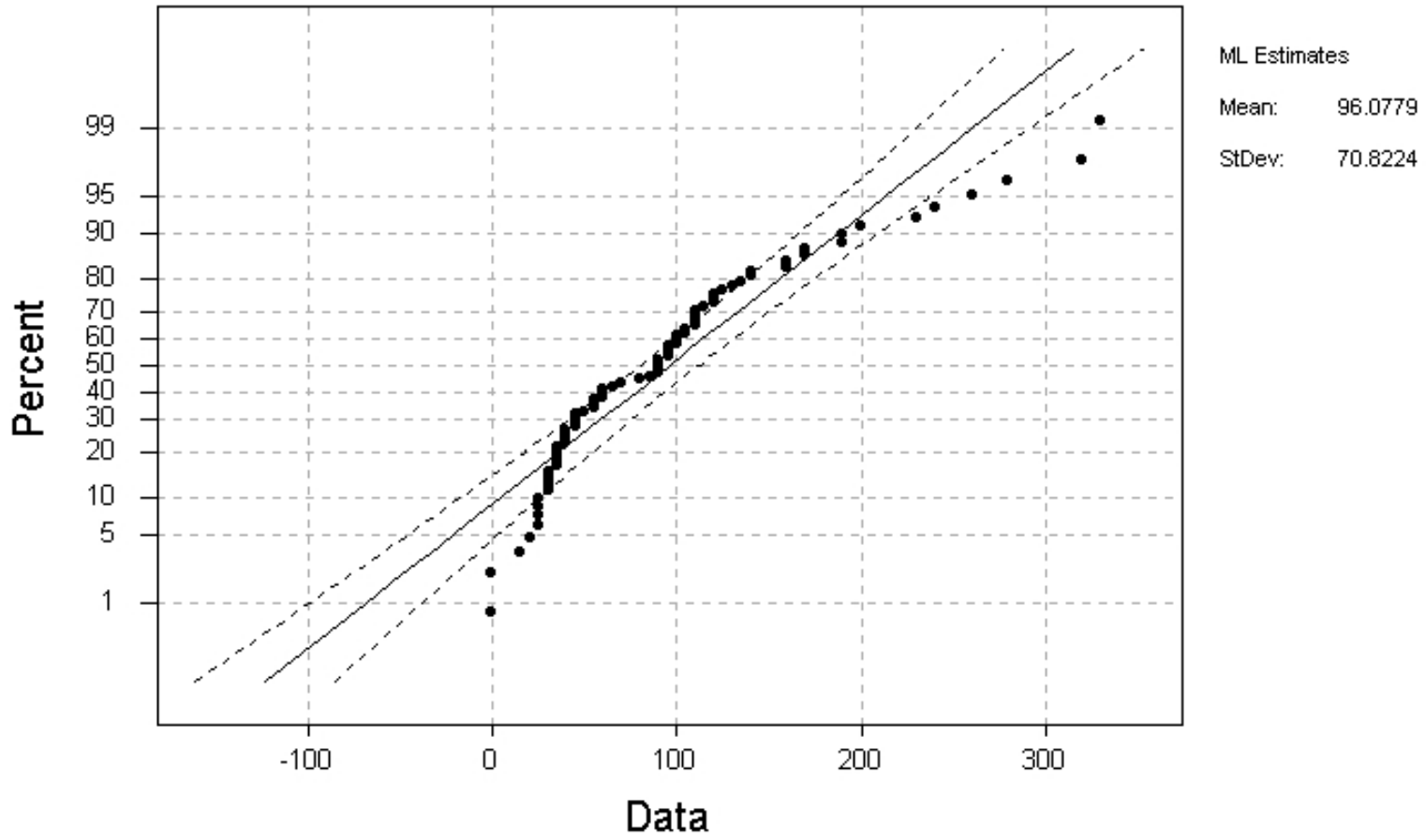
If you have a proportion and you want to get a value, reverse above procedure.

- What SAT score do 10% (0.1000) of students score less than?
- Use Table A backwards first: look up 0.1000 in *body* of table to get $z = -1.28$ (approx).
- Then "unstandardize" to get SAT score of $\mu + z\sigma = 1026 + (-1.28)(209) = 758$ (rounding).
- 10% of students will score less than 758, and the other 90% will score more.

Normal quantile plot

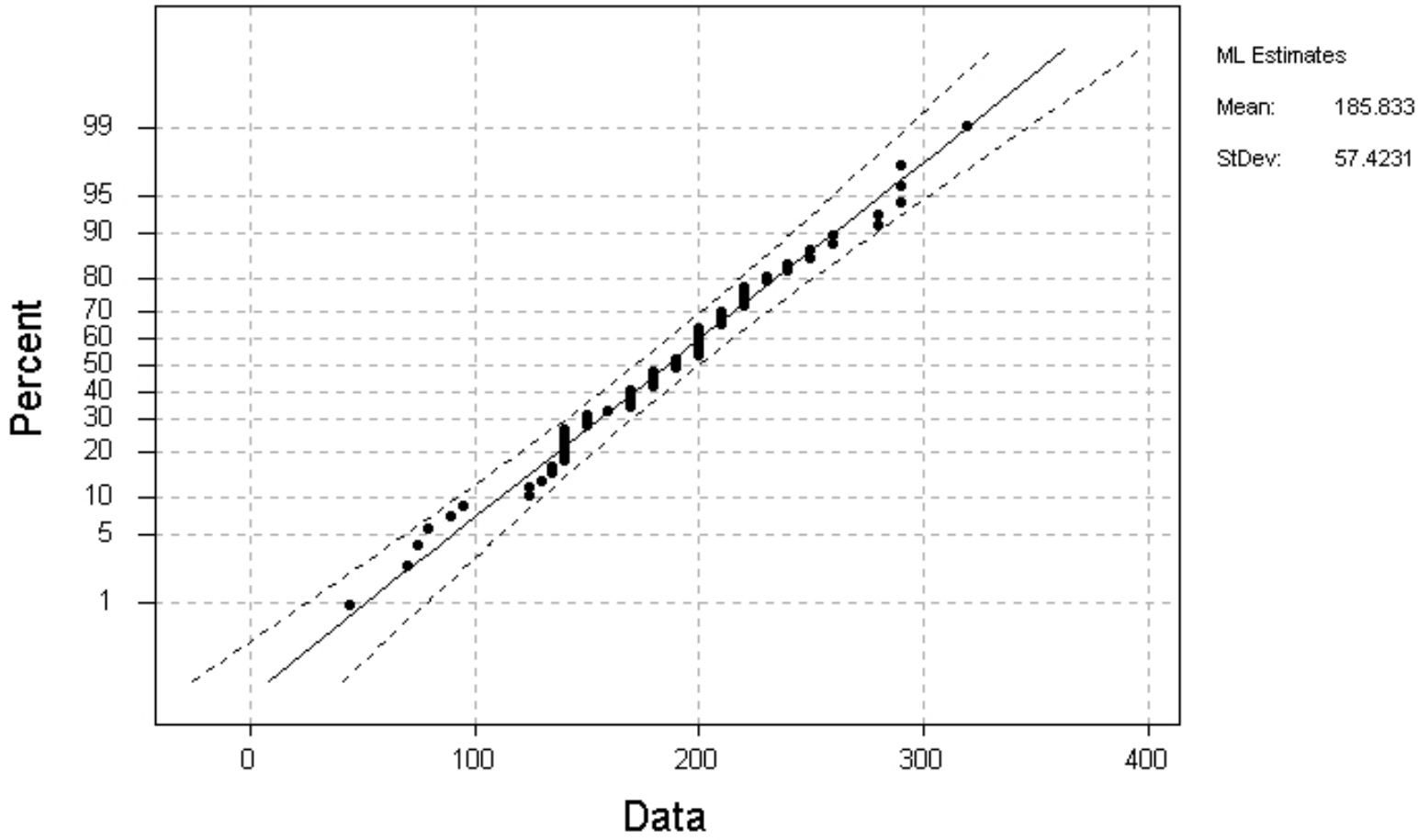
- To do any calculations with a normal distribution, need to know that shape is correct.
- Example: cereal data, potassium content values had skewed distribution (so normal no good).
- If normal dist. correct, normal quantile plot should show straight line. If it doesn't, normal dist. no good:

Normal quantile plot for potassium



Curve indicates a skewed distribution.

Normal quantile plot for sodium



Pattern of dots wiggles, but follows the central line well.

Chapter 2: Looking at Data – Relationships

Introduction

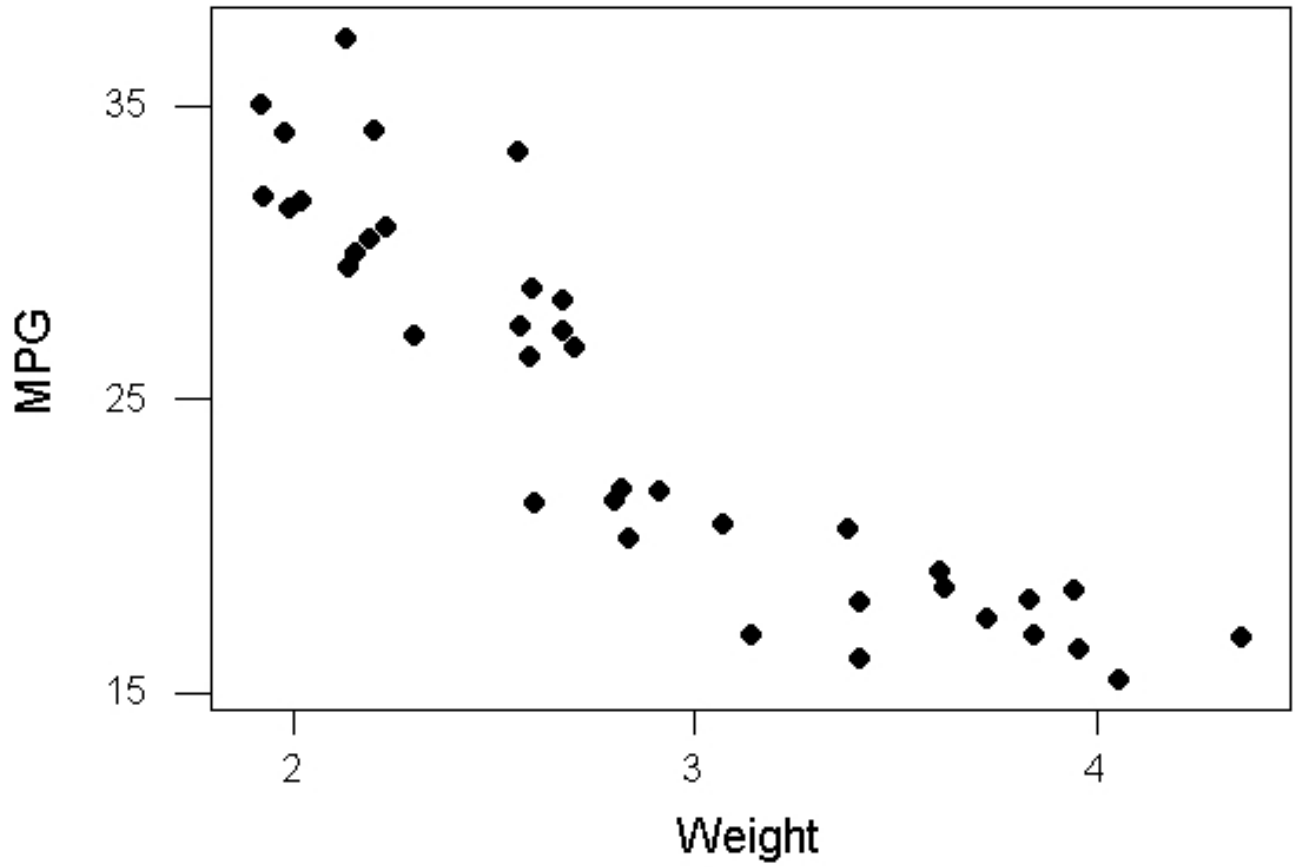
Returning to car data: cars usually cheap to run or powerful but not both. So might expect eg. heavier car to be more powerful, so less gas-efficient – variables “weight” and “MPG” **associated**.

- New idea: so far, consider variables one at a time, but now have to consider 2 variables *together*.
- Relationships between 2 variables might also be affected by other variables – potential for confusion.
- Start with quantitative variables. (Look at categorical variables in §2.5.)

Scatterplots (§2.1)

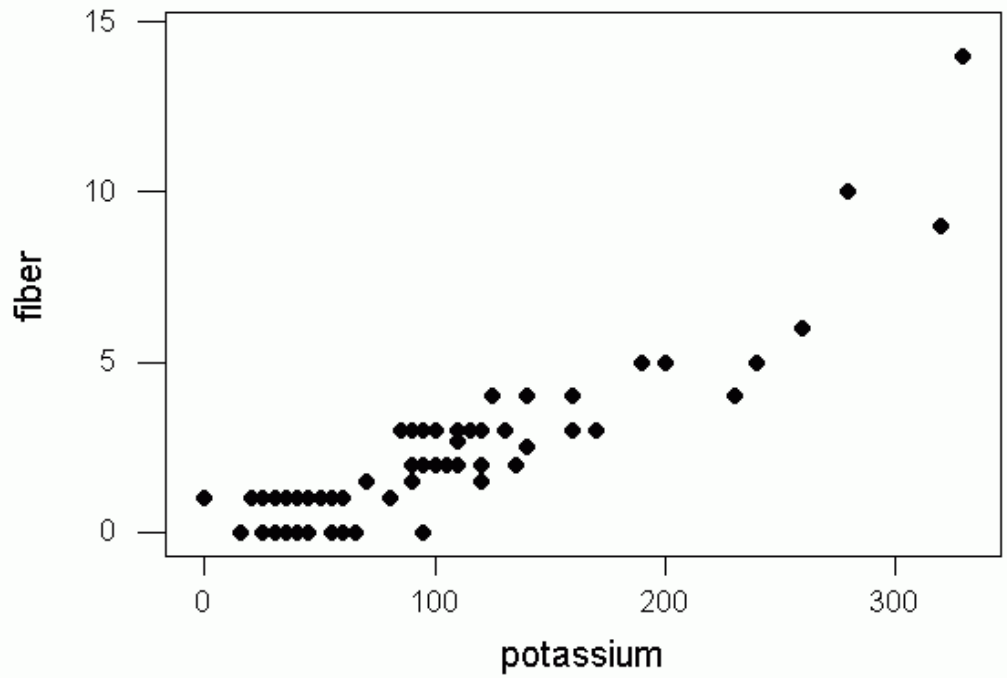
- Good graphical display of association for quantitative variables is **scatterplot**. For each individual, plot values for the two variables on an x-y graph.
- Car data: MPG and weight.
- Get a scatterplot in Minitab from Graph, Plot then selecting MPG, weight by double-clicking.

Cars: MPG vs. weight scatterplot



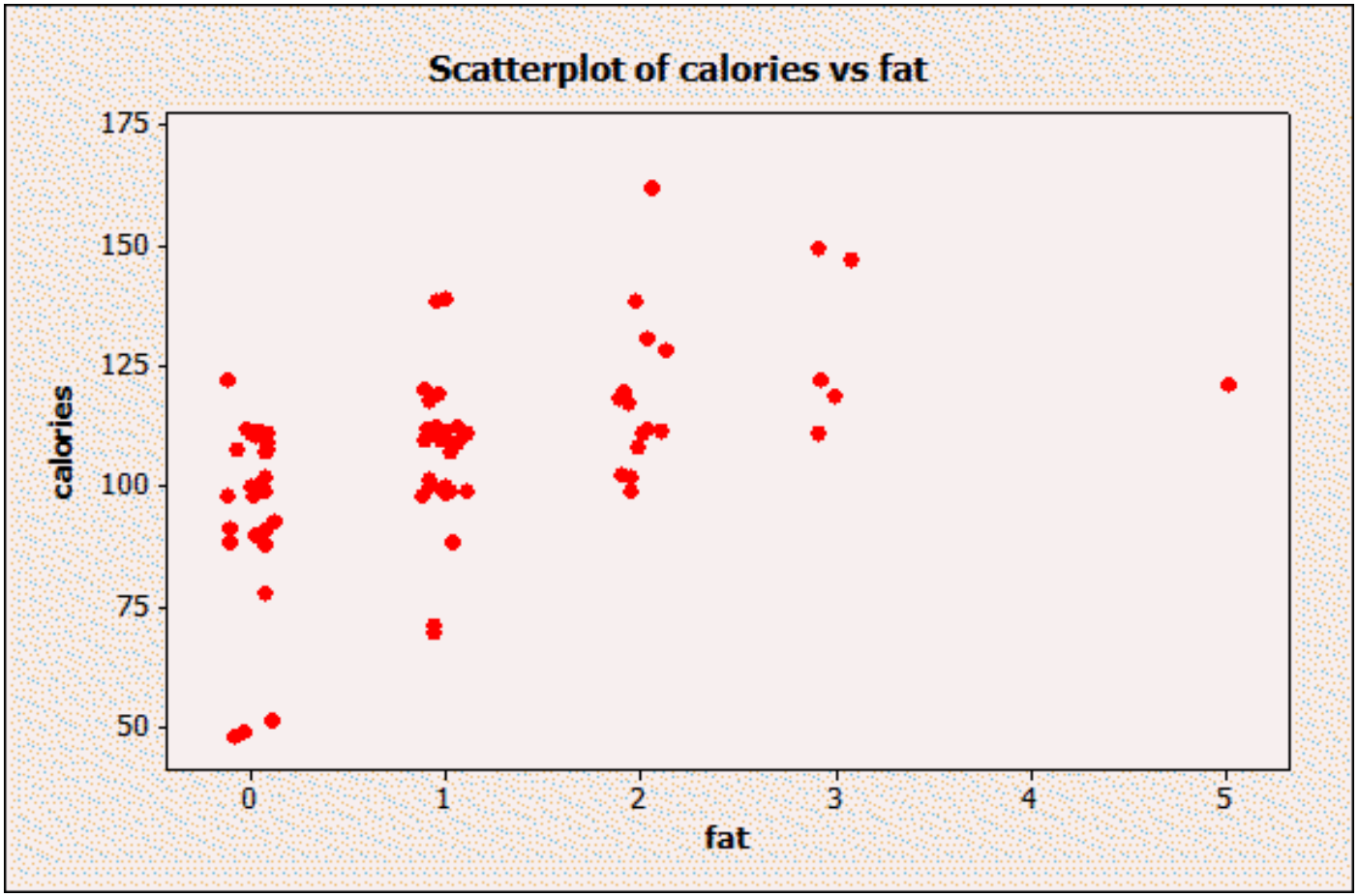
Heavier cars generally have worse MPG (though exceptions). Called a **negative association**.

Cereals: potassium vs. fibre



Cereals with more potassium generally have more fibre.
Positive association.

Cereals: calories vs. fat



As fat increases, calories increase for a while, but then levels off. Non-linear association.

Other points

- Statistical associations are *general tendencies*, not ironclad rules. Almost always exceptions to trend. (What works "on average".)
- Many studies have one variable that is *outcome* (final result). Eg. MPG is result of design, construction and driving of car. Called **response variable.** Other variables called **explanatory variables**: "explain" how response changes.
- In a scatterplot, put response variable on vertical (y) axis.

Summary

- With two related variables, need to look at both at once.
- Scatterplot is good display for quantitative variables.
- In scatterplot, can have:
  - positive association (big on one variable means big on other)
  - negative association (big on one is small on other)
  - non-linear (curved) association
  - no apparent association
- Statistical associations usually general trend with individual exceptions.

Correlation (§2.2)

- First step in assessing 2-variable relationship is scatterplot.
- Can see if *is* association, and what kind (straight line, curve).
- If association looks like straight line, describe strength of relationship using number called **correlation**.

Properties of correlation

- Number between -1 and 1. 1 means perfectly straight *upward* trend; -1 means perfectly straight *downward* trend, 0 means no trend at all.
- Measures how “predictable” one variable is if you know other.
- Only works for straight-line associations (misleading otherwise).
- Has no units of measurement (pure number), so same for any measurement units of variables (km per litre instead of MPG).
- Based on mean and SD, so can be badly affected by outliers.

Examples of correlation

In Minitab, Stat, Basic Statistics, Correlation. Select 2 (or more) variables, click OK.
For fibre and potassium in cereals, high potassium usually meant high fibre:

```
Correlation of fiber and potassium = 0.903
```

Correlation high and positive. Fibre predictable from potassium. (Compare scatterplot.)

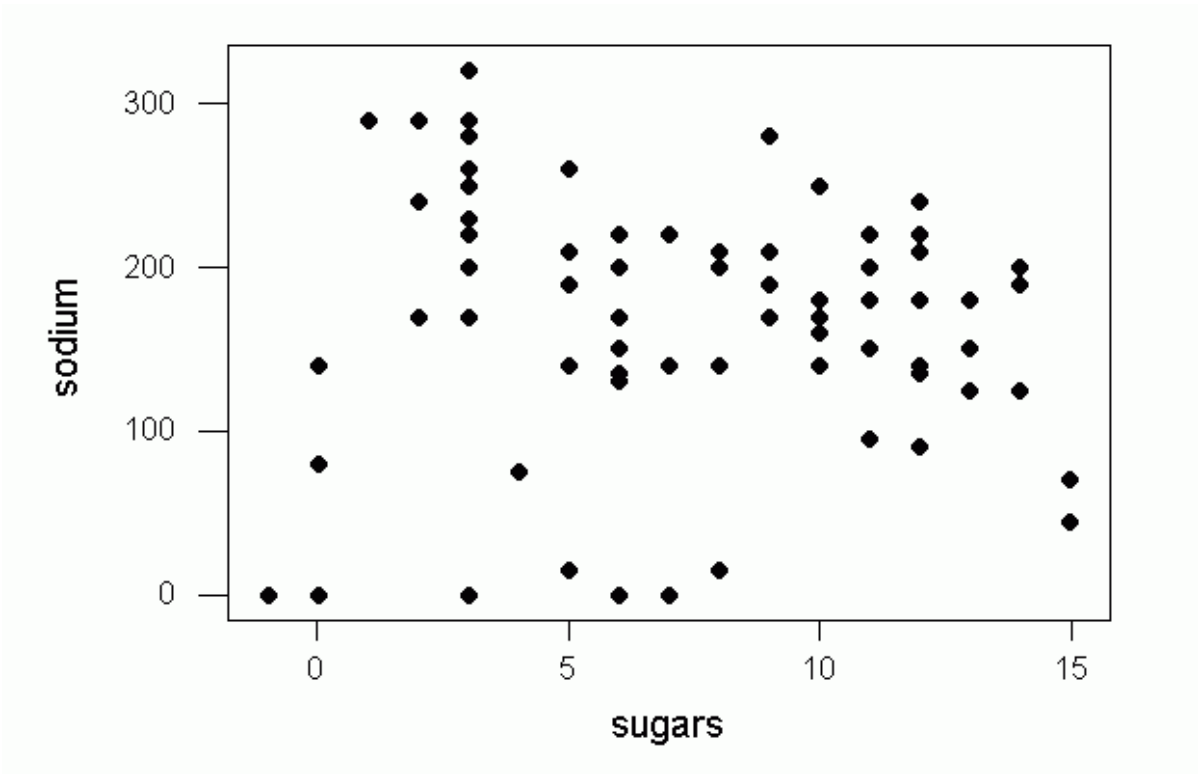
Cars: weight and MPG again

Cars: high weight meant low MPG. Correlation shows this too:

```
Correlation of Weight and MPG = -0.903
```

Identical numerical values of 0.903 coincidence!

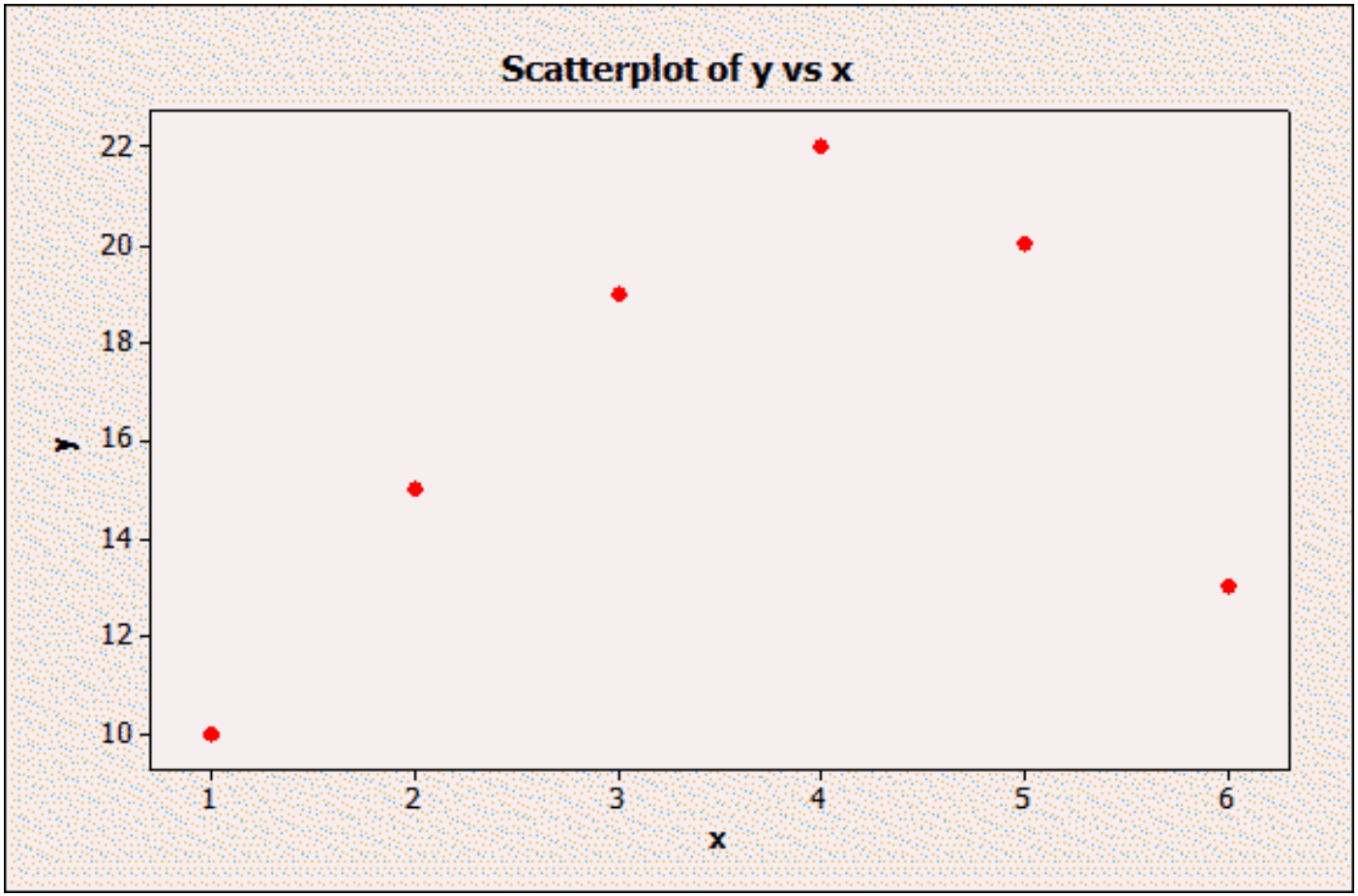
Cereals: Sodium and sugars



`Correlation of sodium and sugars = 0.101`

Almost no association here at all.

A curved association



`Pearson correlation of y and x = 0.384`

Association is non-linear, but more up than down, so correlation is small but positive. (What if curve is less curvy?)

Summary

- Correlation is number summarizing extent and kind of relationship.
- Correlation $+1$ shows perfect positive association.
- Correlation -1 shows perfect negative association.
- Correlation 0 shows no (straight-line) association.
- Doesn’t depend on units of measurement.
- Only intended for straight-line relationships (misleading for curves).

Regression (§2.3)

Correlation: "*is there* a straight-line association?"
Regression: "*what is* that straight line?"
When line known, can use it to predict value of response variable from a value of explanatory variable – given a car's weight, can predict its MPG.

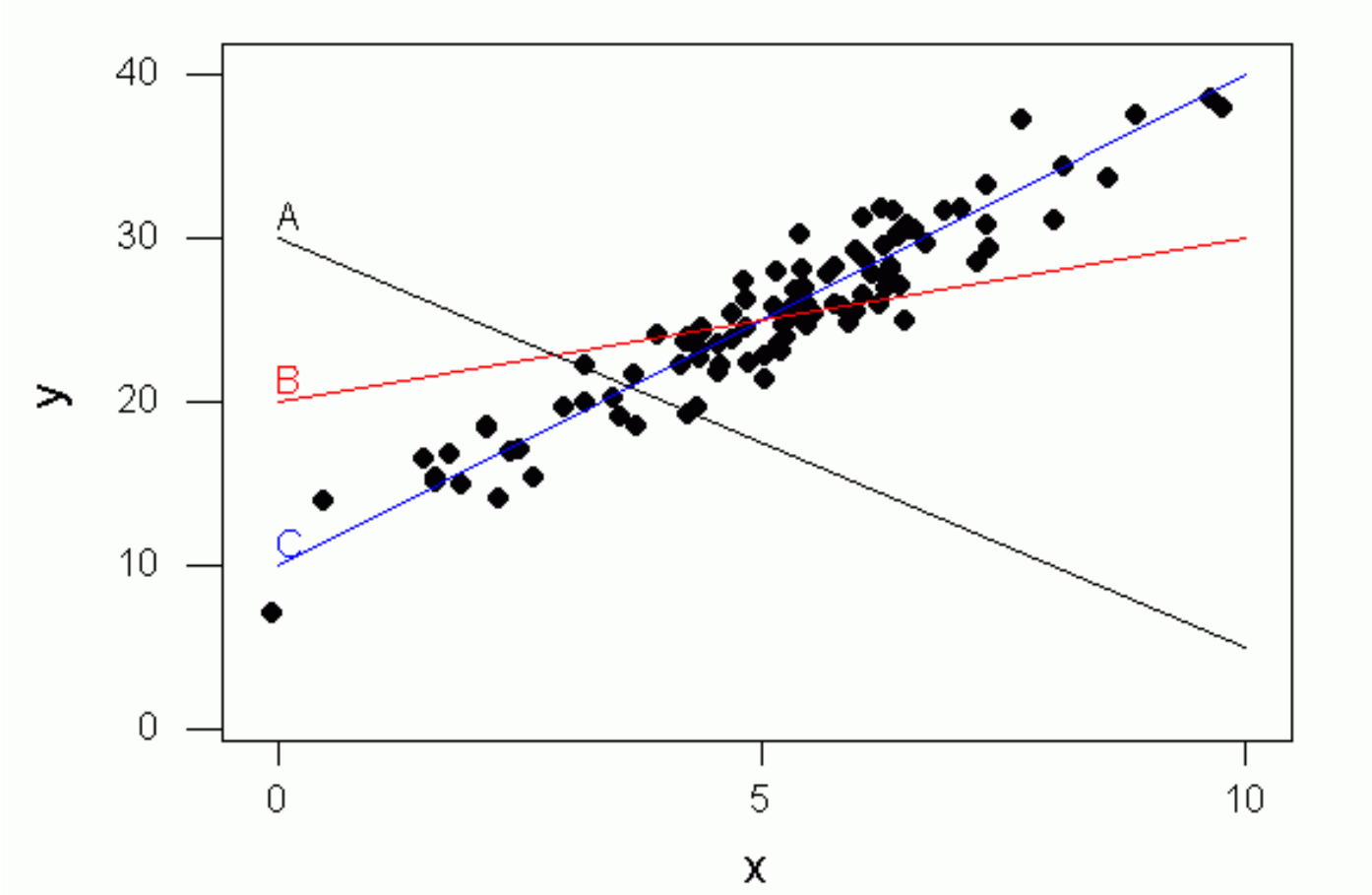
Straight lines in mathematics

Mathematical equation for straight line association between x and y is:

$$y = a + bx$$

a is value of y when $x = 0$, called **intercept**.
b says how much y changes when x increases by 1, called **slope**.
So choosing a line means choosing its intercept and slope.

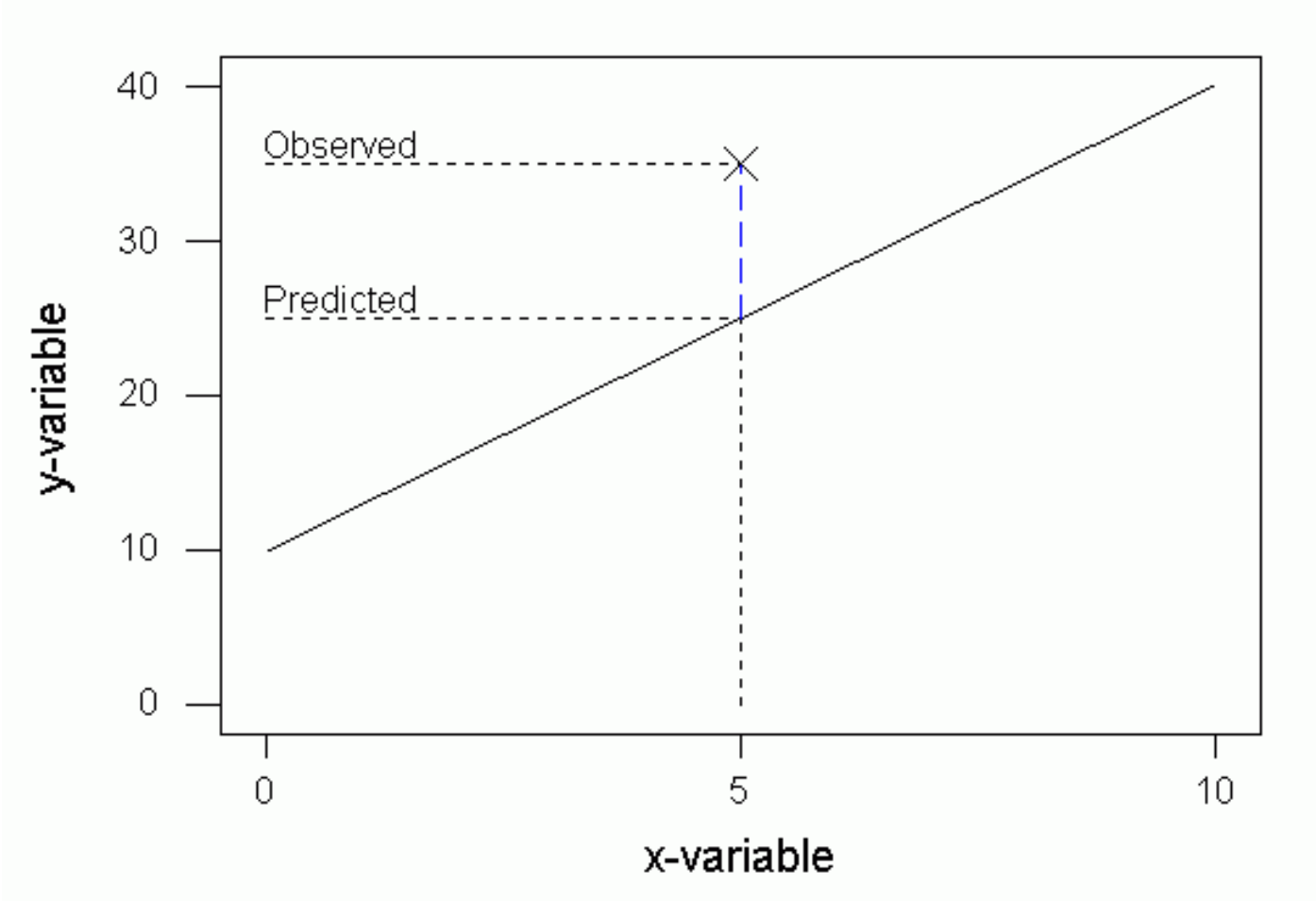
Choosing a line “through” the data



Line A all wrong; B not steep enough; C about right.

Least squares

Want line going “closest” to data with smallest “error”.



Observed: $x = 5, y = 35$. But line predicts $\hat{y} = 25$ when $x = 5$, so error in using line is $e = y - \hat{y} = 35 - 25 = 10$ (height of dashed line)

Residuals and least squares

- Error $e = y - \hat{y}$ called **residual**.
- Each individual has x, y, predicted $\hat{y}$; for any particular line, work out residuals. Residual $+$ if observed $>$ predicted, $-$ if $<$.
- Can't combine residuals by adding up (always get 0), but can *square first* to make positive and then add up.
- For a good line, all residuals small, sum of squared residuals small. For a bad line, some residuals large, sum of squared residuals large.
- Choose line with *smallest sum of squared residuals*. Called **least squares regression line**.

Regression line for MPG and weight

Return to MPG and weight for cars. Correlation was negative: larger weight usually means smaller MPG.

- Regression line in Minitab: Stat, Regression, Regression. Response is MPG (trying to predict), Predictor is weight (predicting MPG from it). Click OK:

  ```
  The regression equation is
  MPG = 48.7 - 8.36 Weight
  ```

  plus other stuff.

- Each increase in weight by 1 ton associated with *decrease* of 8 MPG.

- Predicted MPG for car with weight 2.5 tons is

$$\hat{y} = 48.7 - 8.36(2.5) = 27.8.$$

Finding regression equation

To calculate, need: means of x and y data $(\bar{x}, \bar{y})$, SDs of x and y (s_x, s_y), correlation (r). Then regression line has equation

$$\hat{y} = b_0 + b_1 x$$

with

$$\begin{aligned} b_1 &= r\frac{s_y}{s_x} \\ b_0 &= \bar{y} - b_1\bar{x} \end{aligned}$$

Calculations for weight and MPG

For weight (x), MPG (y),

$$\begin{aligned} \bar{x} &= 2.863, \\ \bar{y} &= 24.76, \\ s_x &= 0.707, \\ s_y &= 6.55, \\ r &= -0.903, \end{aligned}$$

so

$$\text{slope } b_1 = (-0.903)(6.55/0.707) = -8.365;$$

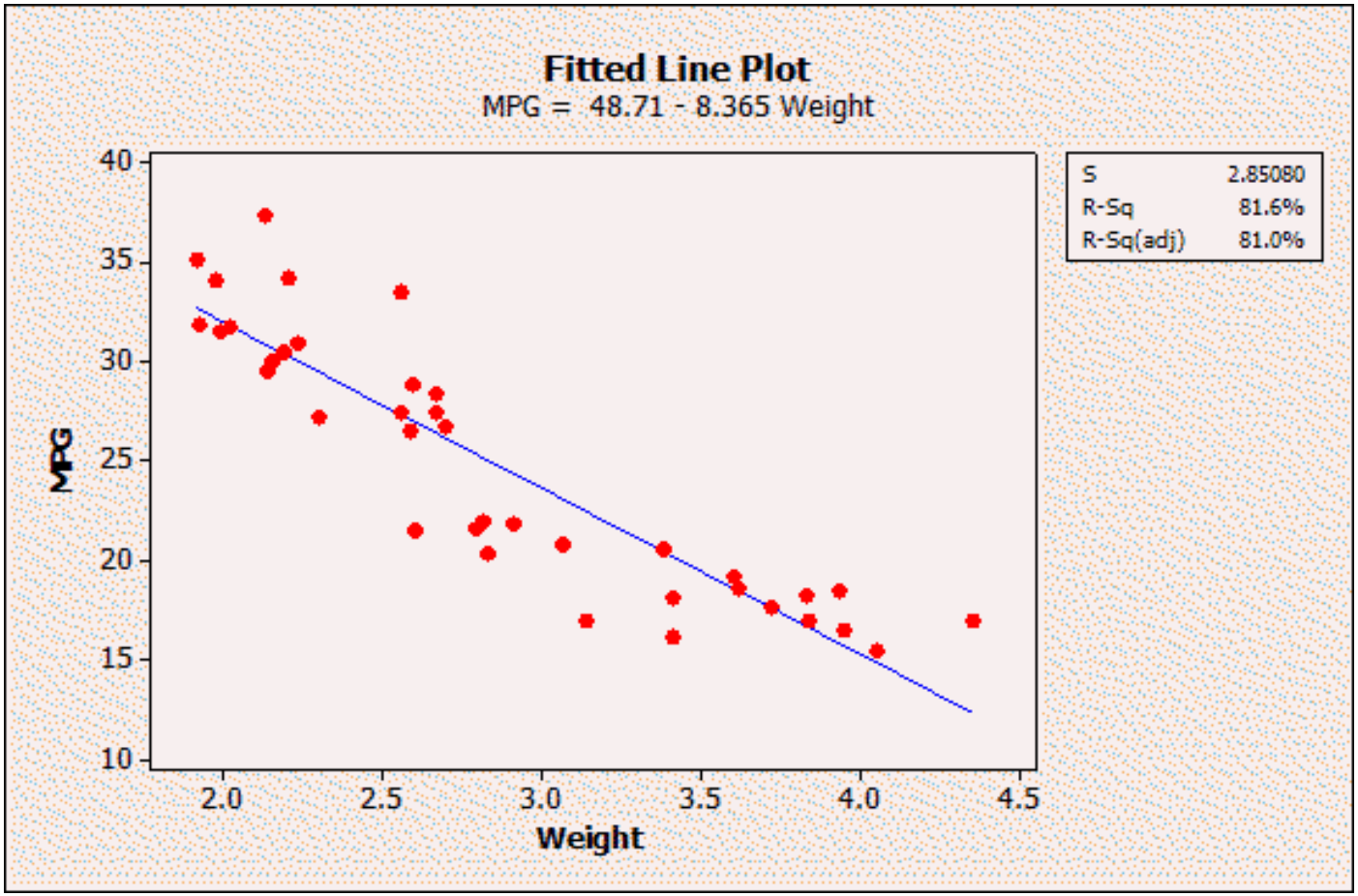
$$\text{Intercept } b_0 = 24.76 - (-8.365)(2.863) = 48.71.$$

Agrees with Minitab to within rounding.

R-squared

- Square of correlation, r^2, is *fraction of variation in y explained by regression of y on x*.
- Above, $r = -0.903$, so $r^2 = (-0.903)^2 = 0.815 = 81.5\%$.
- $r^2 = 0$ means x tells you nothing about y.
- $r^2 = 100\%$ means that y only varies because it depends on x.
- Higher r^2 better.

Scatterplot with regression line



Extrapolation

- What happens if we predict MPG for car weight 6 tons?

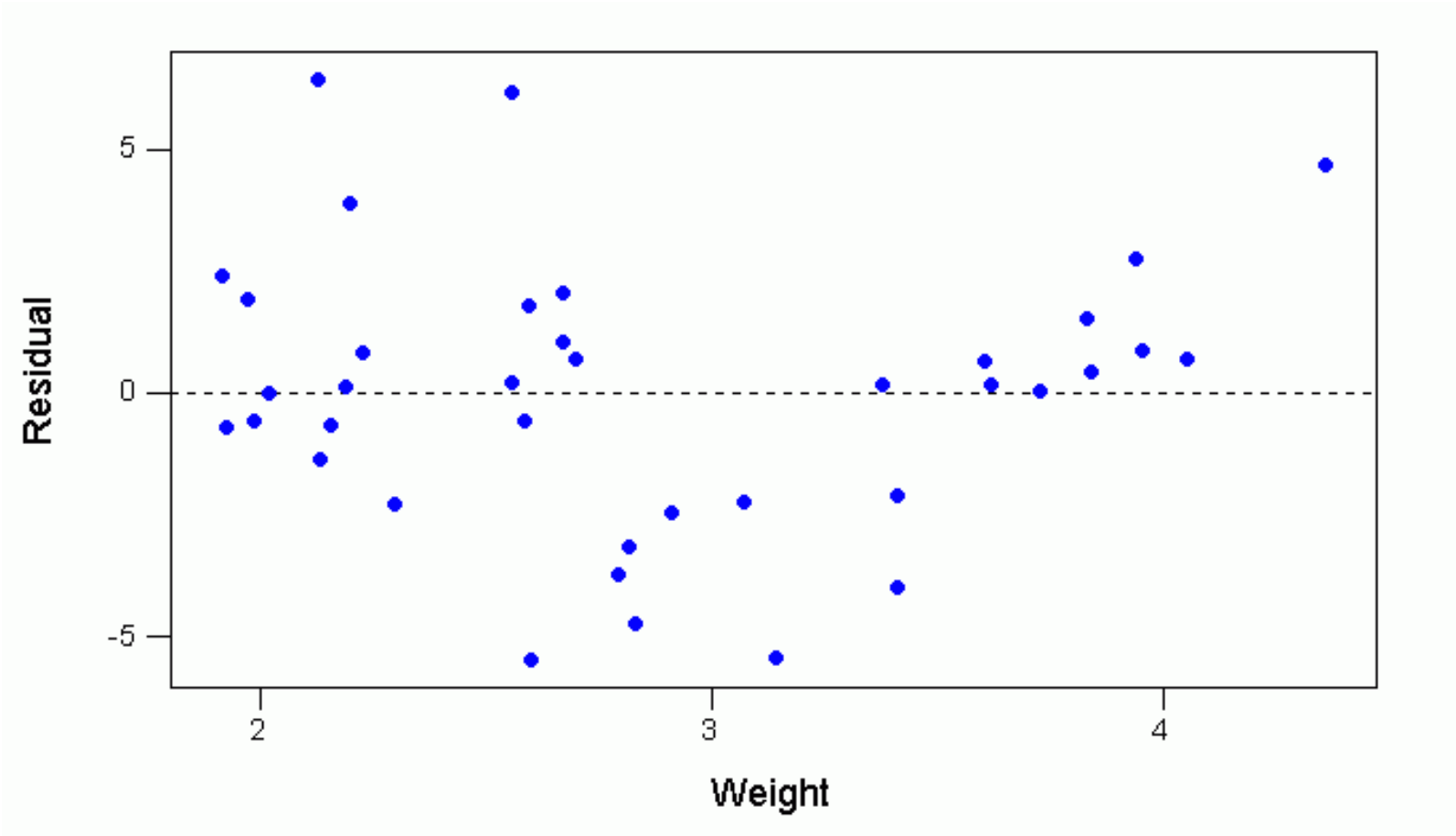
$$\hat{y} = 48.7 - 8.36(6) = -1.46$$

  but MPG cannot be negative!
- In data, have no cars as heavy as 6 tons, so are taking on faith that regression line continues to apply – bad idea!
- For these data, MPG seems to decrease as weight increases, but not so fast for bigger weight.
- Using regression line to predict beyond data should not be done.
- Sometimes can see that prediction is nonsense, but often not.

A second look at fitted line plot

- Line provides simple description of data.
- But maybe *too* simple for these data. Cars weight around 3 tons below line (worse MPG than predicted by line), lightest and heaviest cars above line (better MPG than predicted).
- Maybe a curve would be better, but is there a good way to tell?
- Think about *residuals* $y - \hat{y}$ for each observation.
- Plot residuals against explanatory variable.

Residual plot (§2.4) for MPG-weight

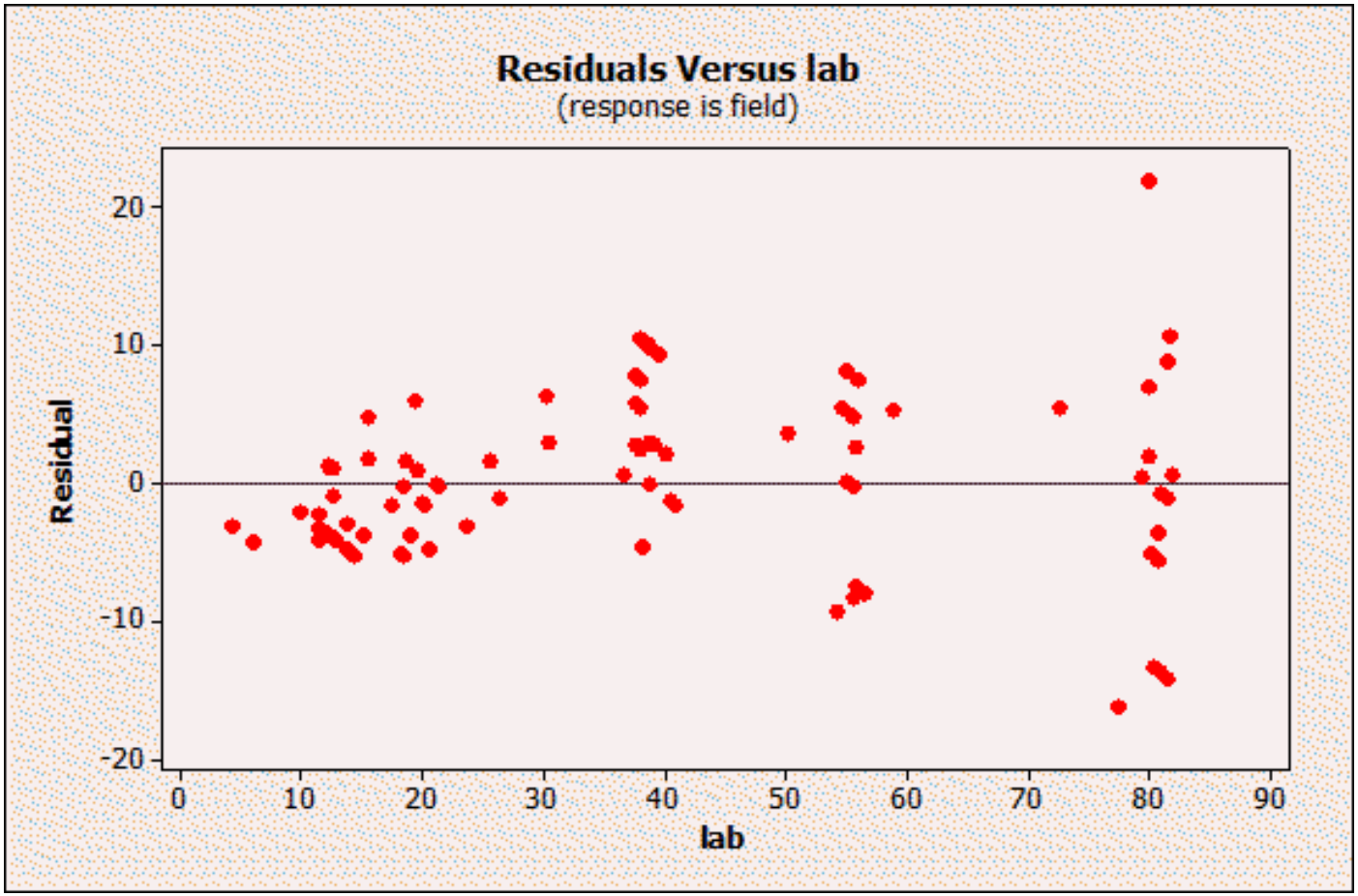


Clear down-and-up pattern.

Looking for lack of pattern

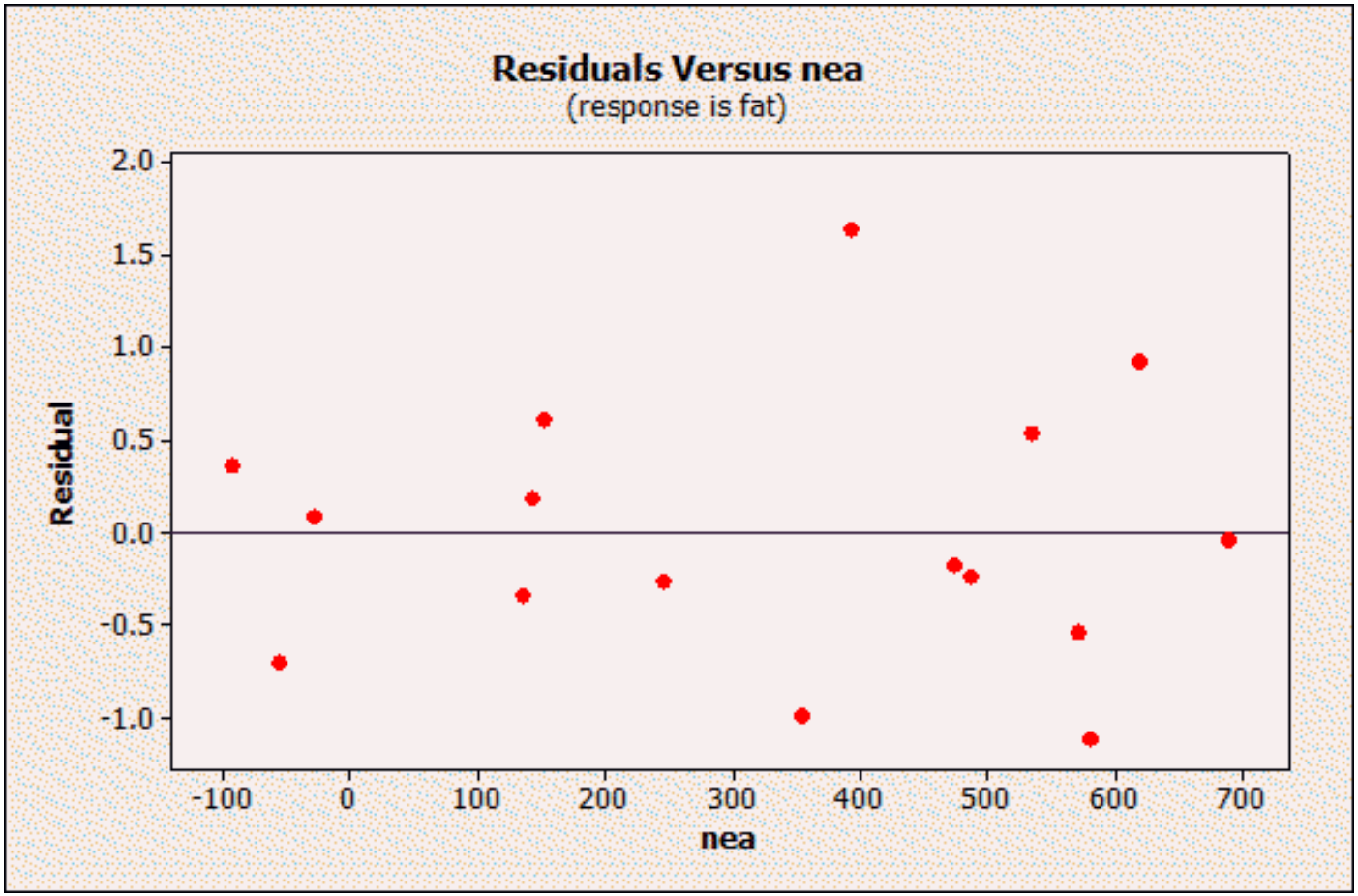
- Minitab: in Regression dialogue, click Graphs, in “Residuals vs. variables” box select explanatory (weight).
- Aim: all association between variables summarized in *regression*. Should be nothing left over, so residual plot should have *no pattern*.
- If pattern, something wrong:
  - curved pattern (above) shows curved not linear association
  - fan-out pattern (next page) shows increasing variation about line as x increases – predictions more precise for smaller x.

Fan-out pattern (Example 2.20)



Predictions less accurate when x larger. Bad – regression should be equally good all along.

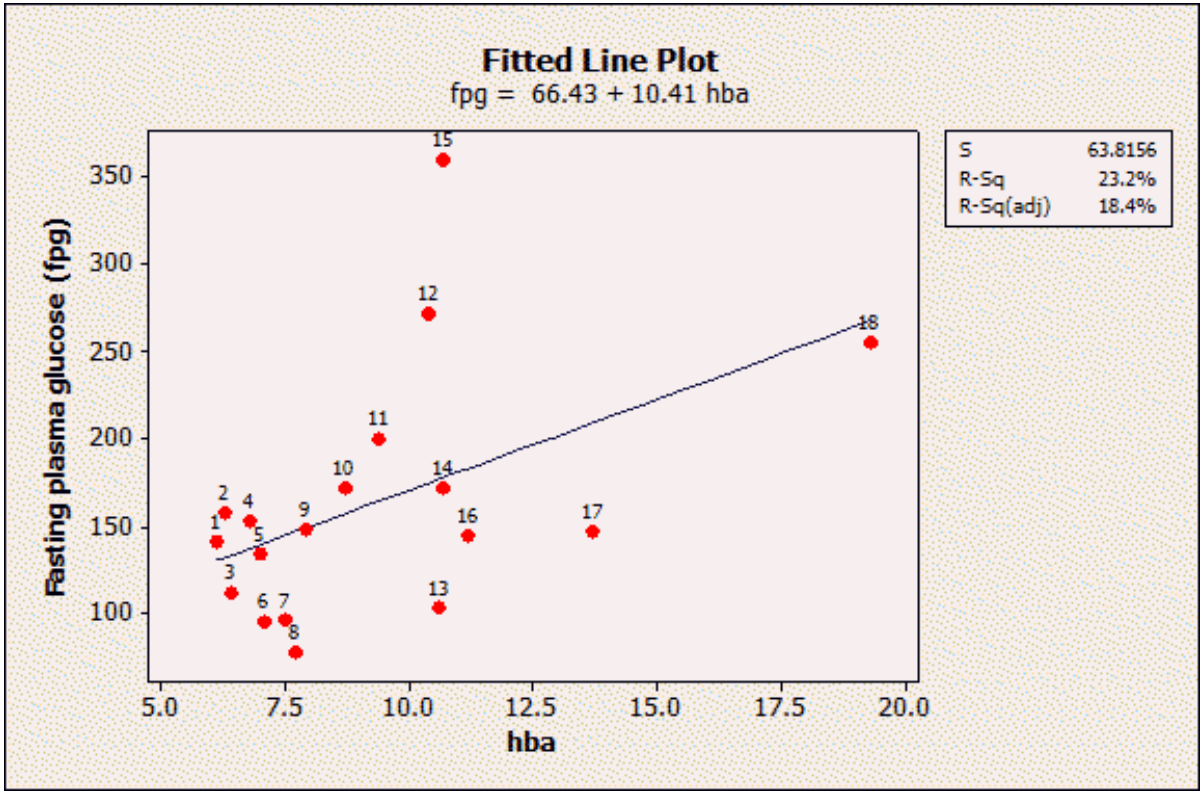
A good residual plot (Example 2.19)



No pattern. Ask: can I predict residual from x? (Here: no.)
Beware of looking too hard for pattern.

Outliers and influential points (example 2.21)

Diabetics manage blood sugar: measure FPG themselves, HbA measured in doctor's office.



Observations 15 and 18 outliers. $r^2 = 23.2\%$.

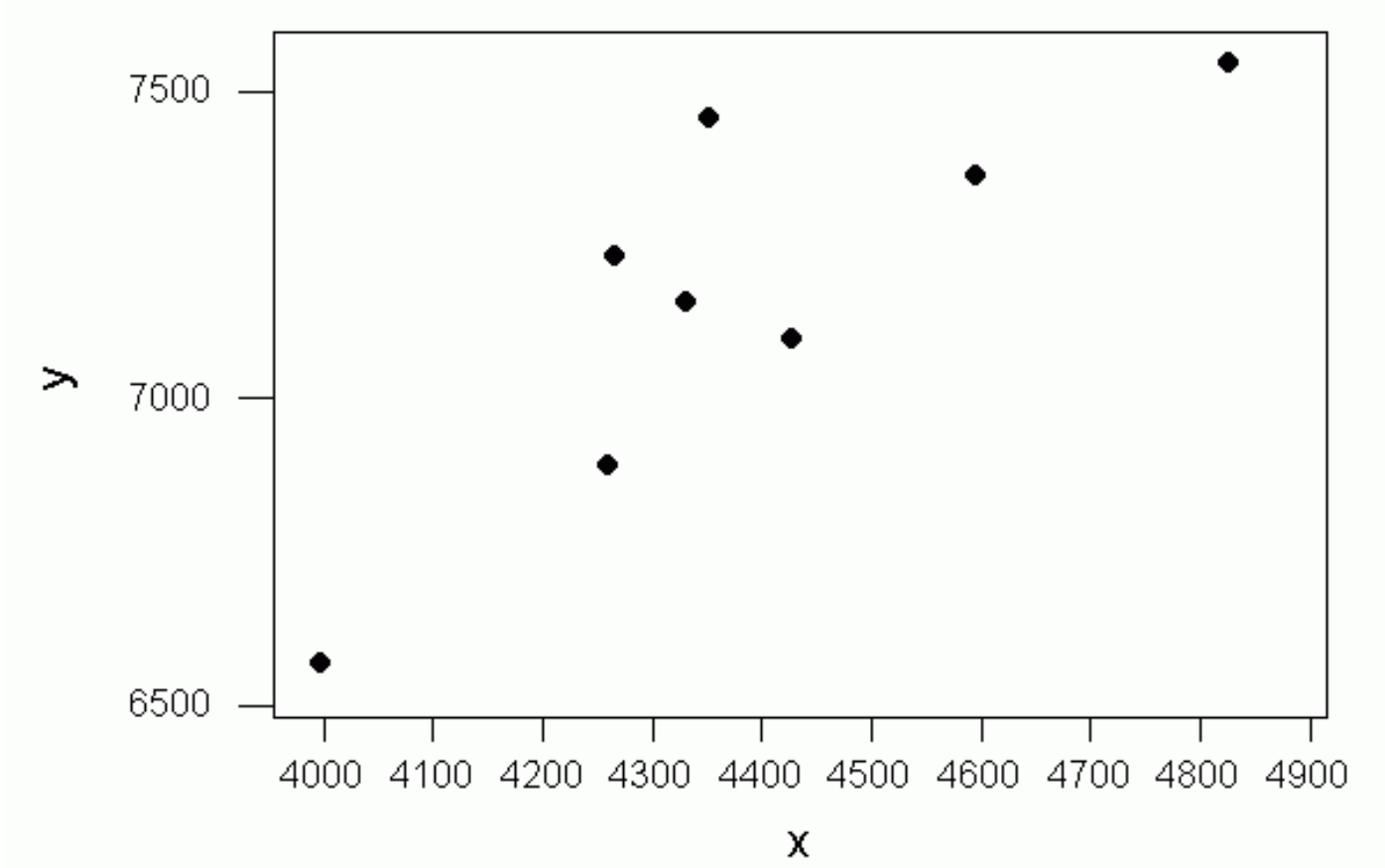
Effects of removing outliers

Remove	outlier	r^2	regression
		23.2%	$\hat{y} = 66.4 + 10.4x$
15	y	32.3%	$\hat{y} = 69.5 + 8.52x$
18	x	14.7%	$\hat{y} = 52.3 + 12.1x$

- Regression line and r^2 change in both cases.
- Obs. #15 off trend, so removing it improves correlation.
- Obs. #18 on trend, but far away from other data. Removing it worsens correlation.
- Obs. #18 "drags regression towards itself" by having unusual x, so when removed, regression free to change.

Lurking variables

If *important variable missed* from regression, can be misled. Predicting number of students in elementary university math courses (y) from total number of 1st years (x).



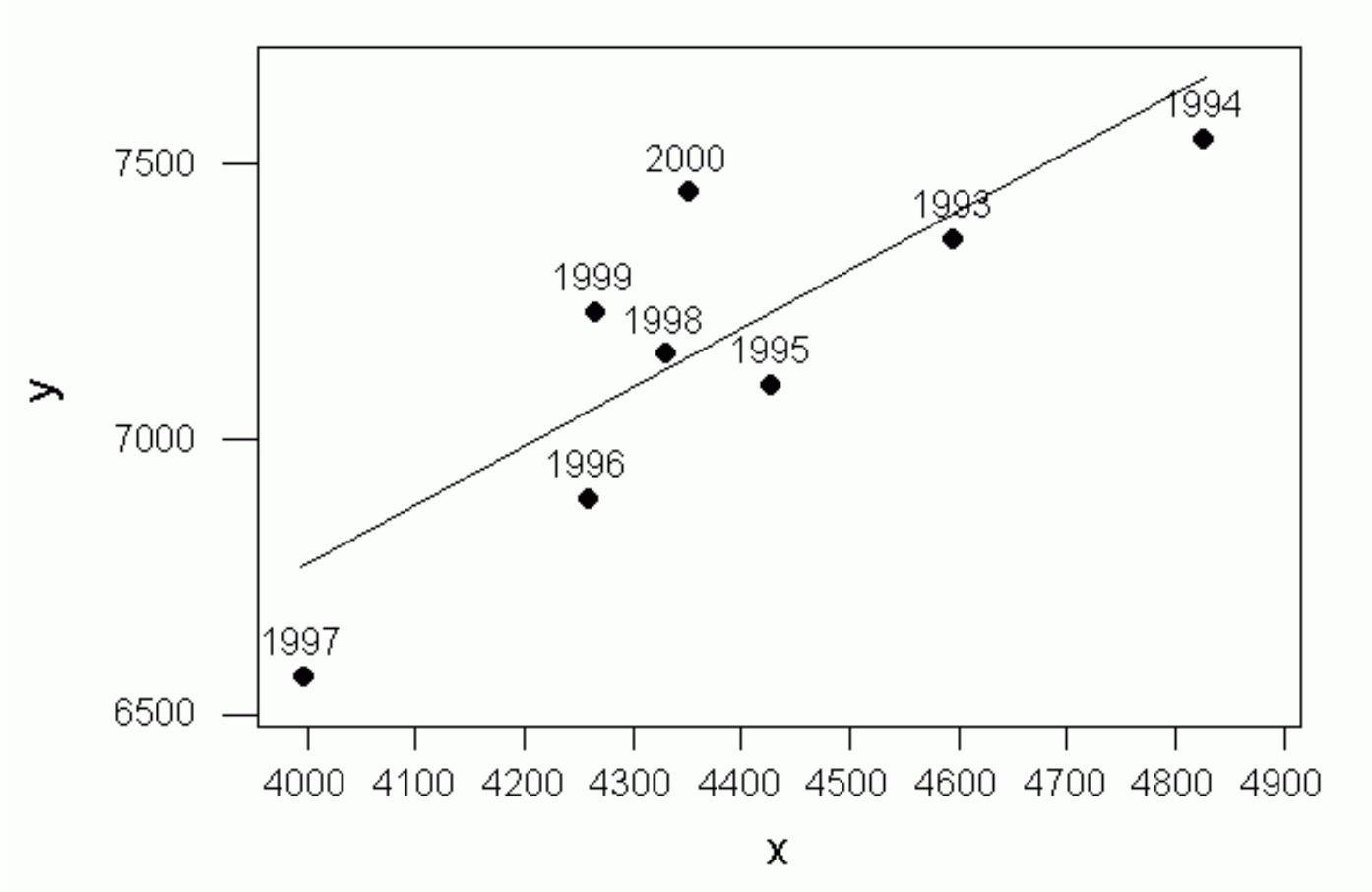
Predicting math students from 1st years

Upward trend (correlation 0.831).
Regression line:

```
The regression equation is
y = 2493 + 1.07 x
```

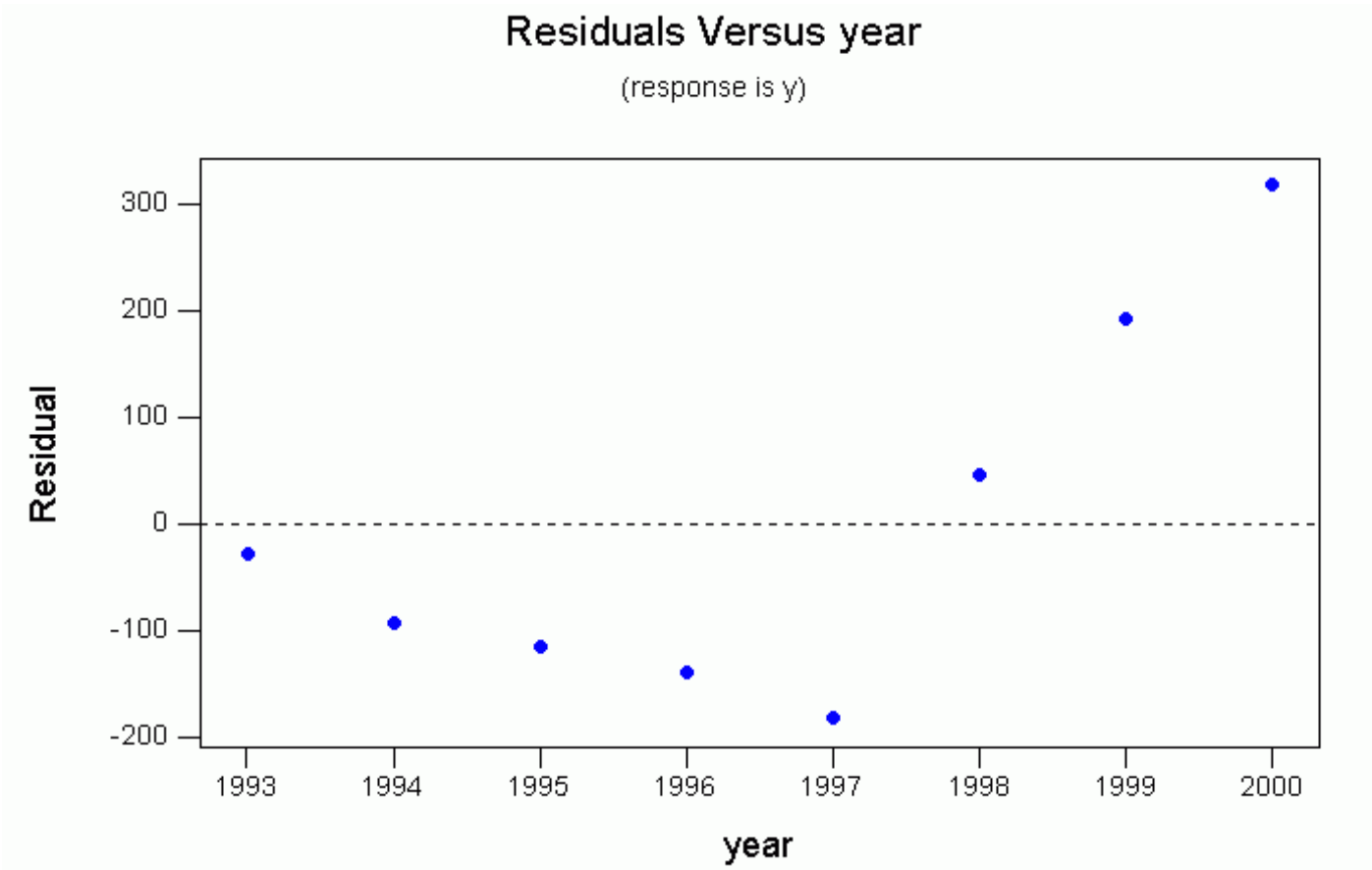
Fit reasonably good, but plot line with years:

Lurking variable “year”



1st 5 years all below line, last 3 all above. Plot residuals against year to show more clearly. (Can plot residuals against anything.)

Plot of residuals against year



Change in 1998: a dept required another math course.
Should not use pre-1998 data to estimate future enrollment.

Summary

- Correlation: does a straight line describe data?
- Regression: which straight line describes data?
- Choose line through data by least squares idea.
- Intercept is y when $x = 0$; slope is increase in y going with x increase of 1.
- Look at residuals to decide whether regression line useful.
- Beware of lurking variables that can distort apparent relationship.

Two-way tables (§2.5)

- Suppose you have two *categorical* variables. How do you assess association then?
- Example: students, age and status (full/part time). Summarize in **two-way table**:

	Status	
Age	Full-time	Part-time
up to 24	8626	1553
25 and over	2465	3744

- 8626 students are aged 24 or less *and* are full-time.
- Association between age and status? Eg. are older students more likely to be part-time? Hard to answer because more younger students overall.
- Key to understanding: calculating proportions, because not affected by how many (students in each category).

Joint distribution

- Total of $8626 + 1553 + 2465 + 3744 = 16388$ students.
- Divide each entry by this grand total, eg. $8626/16388 = 0.526$. Called the **joint distribution**, gives proportions in each category combination:

	Status	
Age	Full-time	Part-time
up to 24	0.526	0.095
25 and over	0.150	0.228

- Proportions add up to 1 (to within rounding).

Marginal distributions

- What proportion of students fall into each age category? Add them up.
  - Up to 24: $0.526 + 0.095 = 0.621$.
  - 25 and over: $0.150 + 0.228 = 0.378$.
- What proportion of students fall into each status category? Add them up.
  - Full-time: $0.526 + 0.150 = 0.676$
  - Part-time: $0.095 + 0.228 = 0.323$.
- These called **marginal distributions**. Overall, majority of students are younger (rather than older) and majority are full-time (rather than part-time). Add marginal proportions to table:

	Status		
Age	Full-time	Part-time	Total
up to 24	0.526	0.095	0.621
25 and over	0.150	0.228	0.378
Total	0.676	0.323	1

Conditional distributions

- Out of younger students, what proportion full-time or part-time? Divide by marginal total (0.621) to get
  - full-time: $0.526/0.621 = 0.847$
  - part-time: $0.095/0.621 = 0.153$
  - younger students very likely to be full-time.
- Out of older students, what proportion full-time or part-time? Divide by their marginal total (0.378):
  - full-time: $0.150/0.378 = 0.397$
  - part-time: $0.228/0.378 = 0.603$
  - older students more likely to be part-time.
- These called **conditional distributions** because have condition attached: "*if* student is younger, how likely is that student to be full-time?"
- If we know age of student, can make guess at their status, because conditional distributions different. Thus age and status associated: tell by looking at conditional distributions.

Goodman and Kruskal's lambda (not in text)

Age	Status		Total
	Full-time	Part-time	
up to 24	0.526	0.095	0.621
25 and over	0.150	0.228	0.378
Total	0.676	0.323	1

Age	Guessed status	Proportion of errors
unknown	full-time	0.323
up to 24	full-time	0.095
25 and over	part-time	0.150

- **Proportional reduction in error** from knowing age is:

$$\lambda = \frac{0.323 - 0.245}{0.323} = 0.241.$$

- Between 0 and 1. 0 if knowing x (age) of no help in predicting y (status); 1 means if knowing x means no mistakes in predicting y.

Simpson's paradox

- (Example *like* Example 2.36 in text)
- Airline business competitive: airlines compete on price, service, punctuality.
- Counts of flights on time, delayed at 5 different airports for 2 different airlines (June 1991):

	Alaska Airlines		America West	
	On time	delayed	On time	delayed
Los Angeles	497	62	694	117
Phoenix	221	12	4840	415
San Diego	212	20	383	65
San Francisco	503	102	320	129
Seattle	1841	305	201	61
Total	3274	501	6438	787

- Hard to make sense.

Percentages

- Calculate percentages. Overall, Alaska had $3274 + 501 = 3775$ flights, 501 delayed, 13.3%. America West: $6438 + 787 = 7225$ flights, 787 delayed, 10.9%. America West more punctual overall.
- Now calculate %'s delayed by airport and airline:

	Alaska Airlines	America West
Los Angeles	11.1	14.4
Phoenix	5.2	7.9
San Diego	8.6	14.5
San Francisco	16.9	28.7
Seattle	14.2	23.2
Total	13.3	10.9

- America West better overall but *worse at every single airport*!

Simpson's paradox

- Seems impossible, but numbers all correct. Called **Simpson's paradox**.
- Airports vary in % delays. Phoenix very low, San Francisco, Seattle high. Go back to original numbers: America West flies mostly into Phoenix, where easy to be on time; Alaska flies mostly into Seattle, where hard to be on time.
- General principle: *beware of comparing percentages for summarized data*. Only fair comparison here is airport by airport.
- Another way to say it: punctuality depends on airline *and* airport, so to summarize by airline only is misleading.

Correlation does not imply causation (§2.6)

In science, find out how world works: want to know that changes in one variable *cause* changes in another.
But correlation and regression only show *association*: that the two variables change together. *Does not show whether either one causes the other.*
High correlation can be caused by lurking variable. Example: for countries, measure TV sets per person and average life expectancy.
High positive correlation, but TV ownership not *cause* of long life. Cause for both variables is standard of living.

Establishing causation

Science: do *experiment* in which other lurking variables controlled (in lab). Gives most convincing evidence of cause and effect.

But in real world, impossible to experiment. To see whether smoking causes lung cancer, would have to choose 2 similar groups of people, make 1 group smoke, measure lung cancer rates. Not ethical!

How to convince others of cause and effect without experiment?

Evidence for cause and effect

- **Strong association** (smokers suffer more from lung cancer than others)
- **Consistent association** (many different studies get same result)
- **Larger effect** with larger exposure (heavier smokers get more lung cancer)
- **Cause before effect** (lung cancer comes after starting smoking)
- **Cause is plausible** (experiments on animals show cigarette smoke causes cancer).

Summary

Correlation cannot prove that one variable is *cause* of another, but if:

- association strong and consistent
- larger effect associated with larger value of explanatory variable
- supposed cause comes before effect and is scientifically plausible

then have convincing evidence of cause.

Chapter 3: Producing data — Finding out what we want to know

What can we learn?

- So far: have data, asked *What do we see?*
- Use graphs, numbers to *describe*.
- But this only a start. Ask *what can we learn?* Needs more work with:
  - calculations and reasoning
  - collecting data to make calculations/reasoning work
- Aim here: collecting data well.

Finding available data (Chap. 3 intro)

- Question of interest, eg.:
  - how are Canadians' eating habits changing?
  - social/economic backgrounds of college students
  - relative safety of flying vs. driving
- **Anecdotal evidence** comes from own/others' experiences. But may not be typical, or remember untypical cases (plane crash killing 200). May not represent whole phenomenon.

The whole story

- Need to get data telling *whole story*. Visit library or Internet: much data (not gathered especially for us but helpful eg. for first question).
- From Statistics Canada website:
  - Canadians eating more pasta/bakery/cereals, less red meat than in past years
  - Milk consumption steady (after falling in previous years), but now more low-fat milk.
  - Consumption of cream has increased (because consumption of coffee has increased?)
- Reliable? Probably yes. Detailed surveys of representative samples of Canadians gathered for information (not by group with axe to grind).

Producing data

- Producing new data expensive, but often only good route to clear answers.
- Ideal: **census**. Measure *every* individual of interest, whole **population**. Slow, impossibly expensive.
- In practice: **sample**. Choose small collection of individuals to "represent" population, draw conclusions about population based on sample.
- Samples are kind of *observational study*. Aim to collect data without changing anything, and conclude whatever possible.

Statistical experiment

- Compare statistical **experiment**: conditions deliberately changed to see what happens. Better than observational study, because can observe effect on response by changing right thing (get cause/effect).
- Example: to find what Canadians eat, observational study (sample) fine.
- But to find whether one diet healthier than another, need experiment. Get 2 groups with same mix of age/sex/lifestyle; one group gets diet 1, other gets diet 2. Groups started similar, so differences at end caused by diets.

Summary

- Ask: what can we learn?
- Using available data
- Sample from large population
- Observational study: observe only
- Statistical experiment: deliberately change conditions
- In experiment: difference at end can mean factor modified *made* difference.

Design of experiments (§3.1)

- Jargon:
  - Individuals on which experiment done: **experimental units**, or **subjects** if people.
  - Specific experimental condition applied to units: **treatment**.
- Example: comparing success rates of surgical procedure at 2 hospitals. Could just observe, but success differences might be result of patient differences.
- Experiment: control which patient (subject) goes to which hospital (treatment). Share out more/less critical cases among hospitals; difference in success rates then evidence of difference between hospitals.
- Success rate here response; hospital is (categorical) explanatory variable, called **factor**. 2 specific hospitals called **levels** of factor.

Placebo effect and control groups

- Many subjects in an experiment respond favorably even if treatment is ineffective (eg. patients getting pill with no active ingredient). Called **placebo effect**, and **placebo** is treatment designed to do nothing except look like real treatment.
- Group of subjects getting placebo called **control group**.
- Important to have control group because *controls* for psychological effect of receiving "treatment" at all. Better to assess *one* treatment by comparing *two* groups, treatment and control.

Randomization

- How to assign experimental units to treatments? Need to make treatment groups “similar”.
- Can try *matching* on all relevant variables. Helpful, but if variables forgotten, can give dissimilar groups.
- Easier: use **random** assignment. Ensures that group assignment not related to anything else. Groups usually similar in terms of anything important.

Using Minitab to do randomization

- Suppose we have 30 subjects, want 15 in treatment group and 15 in control group. Effectively “draw names from hat”.
- Give each subject a number, 1–30. Calc, Make Patterned Data, Simple Set of Numbers. Store in: type C1, First Value 1, Last Value 30. Click OK.
- Now select 15 subjects for treatment group: Calc, Random Data, Sample from Columns. In box, sample 15 rows, click From box, select C1, type C2 in other box. I got:

26	25	8	27	18
2	22	19	9	20
14	4	11	24	23

- These in Treatment group, rest in Control. (Different if repeated.)
- For more than two groups, easier to “shuffle” whole list of 30 (sample all 30 rows). Then pick eg. first 10 for Treatment 1, next 10 for Treatment 2, last 10 for Control.

Conclusions from randomized experiments

- Groups in randomized experiment not *exactly* same, though will be very similar. So any differences afterwards could be:
  - chance: the groups weren't quite the same to begin with
  - treatment effect: one of the treatments really is better.
- Later, learn how big chance differences can be. Observed difference bigger than chance: must be treatment effect. Such difference called **statistically significant**.

Randomization at work

- In many diseases, success of treatment depends on age of patient. So want treatment, control groups to have similar mean age.
- Worksheet `trtcontrol` has ages for 30 people. Mean age is 54.
- Select 15 people for treatment group, calculate mean age. In Sample from Columns, selecting 2 input and 2 output columns carries age info along. My results:

Simulated treatment groups

```
Mean of C5 = 50.733
Mean of C5 = 54.400
Mean of C5 = 56.667
Mean of C5 = 56.200
Mean of C5 = 47.600
Mean of C5 = 55.267
Mean of C5 = 55.600
Mean of C5 = 53.067
```

Mean age of treatment group generally very close to 54. Taking more people and larger groups would give even better results.

Cautions about experimentation

- Experiments always conducted under artificial conditions. Setting of experiment may not reflect real world – **lack of realism**.
- Example: TV commercials tried out in “focus groups” where people invited to watch and vote on several alternative commercials for same product. Very unlike TV-watching, so best focus-group commercial may not be best TV commercial.

Double-blind

- Also, people administering and receiving treatment may have prejudices. (As patient, what if you knew you had placebo?)
- Important to design experiment **double-blind** so no-one in running of experiment can be biased. (Each subject gets something identical with code number attached.)

Matching and blocking

- Often measure each subject *before* experiment as well as after – gives baseline for comparison (eg. learning or exercise tasks). Example of **matched pairs** – 2 measurements per subject.
- "Case-control study" for cause of disease: for each "case" with disease, select similar person without disease ("control"), see whether supposed cause more common in cases than controls.
- Idea: compare *within* pairs.

Blocking example

- Another example: comparing effectiveness of pesticides in farming. But soil type, fertility differ by location. So apparent effectiveness depends on pesticide *and* location.
- Idea: divide locations into **blocks** – in each block, soil type, fertility same. Then randomize pesticides into locations *within* each block. Then each block has fair comparison of pesticides A, B, C, D:

	Location			
	1	2	3	4
Low fertility	C	A	B	D
Mod fertility	C	B	D	A
High fertility	B	C	A	D

Summary

- Individuals in experiment called experimental units or subjects.
- Condition being modified called treatment.
- May be effect of “nothing” called placebo effect; use control group to provide comparison with “nothing”.
- Assign units (subjects) to control/treatment groups by randomizing: groups “similar on average”.
- Big enough observed difference can be effect of treatment.
- Experiments can lack realism or lack blinding.
- Before-after studies can isolate effect of treatment.
- Units very different: test each treatment on mixture (blocking).

Sampling (§3.2)

- When can't experiment, can observe, but impractical to observe everyone. Jargon:
  - Entire group of individuals of interest called **population**.
  - Part of population actually examined called **sample**.
- How to choose sample?
  - Bad: *voluntary response*, eg. polls on radio stations. People choose themselves to be in sample, usually because of interest in issue discussed. Results biased towards strong opinions.
  - Good: use *randomization*. No connection between being in sample and issue being addressed.

Simple random sample

- Simplest: put names (population) in hat, draw out some (sample), say n.
- This is **simple random sample**. Each set of n individuals equally likely to be sample actually selected.
- In Minitab: same idea as for randomizing treatment group. Make list of population, use Calc, Random Data, Sample from Columns to select individuals for sample.

Stratified sampling

- A population often has dissimilar groups in it. Eg. opinions of Ontario people different from those in Québec, West, Maritimes.
- Simple random sample might under-represent some groups by chance. Idea: take simple random sample *from each group*. Combined **stratified random sample** represents all groups.

Multistage sampling

- Drawing simple random sample requires list of population – big if sampling all of Canada!
- Also, SRS of Canada would have small numbers of people in widely scattered places – impractical.
  - Divide Canada into areas (eg. electoral ridings), select random sample of ridings
  - divide selected ridings into smaller areas (city blocks etc), select random sample
  - select random sample of people in smaller areas
- Easier process in stages; final sample less scattered.

Sample survey problems

- Requires list of population/subpopulations. If list incomplete, sample biased because of **undercoverage**.
- Individual chosen for sample may refuse to take part (**nonresponse**). Nonresponders may be different from responders; if so, causes bias in results. (Eg. calling homes in working hours.)
- Questions on sample survey need careful, neutral **wording**:
  - 44% think America spends too much on *welfare*
  - 13% think America spends too much on *assistance to the poor*.
- Good statistical design only part of successful sampling.

Towards statistical inference (§3.3)

- So what is it possible to learn about a population when all we have is a sample?
- Answer: quite a lot, as long as it's a *random* sample.
- Really the purpose of statistics: from a sample, making **statistical inference** about population.
- Because of randomness in sampling, won't get exactly right answer, but hope to be close (and to know how close).

Parameter and statistic

- Imagine mean height of all adult women in Canada. Some number of inches, but can only know by measuring everyone. But draw random sample of 100 Canadian women; mean height of those women can be calculated.
- **Parameter** is number describing *population*. Fixed but *unknown*.
- **Statistic** is number describing *sample*. Can be *calculated*, but different in different samples.
- Mean height of all Canadian women is *parameter*, mean height of women in sample is *statistic*. Take different sample, get different statistic.

Sampling variability

- Random sampling eliminates bias/favoritism (no control over which individuals end up in sample).
- But different samples have different sample statistics – **sampling variability**. Hence sample gives imperfect answer. But maybe answer close enough to be useful.
- Going from (one) sample to population difficult. But going from population to sample(s) can be easy.

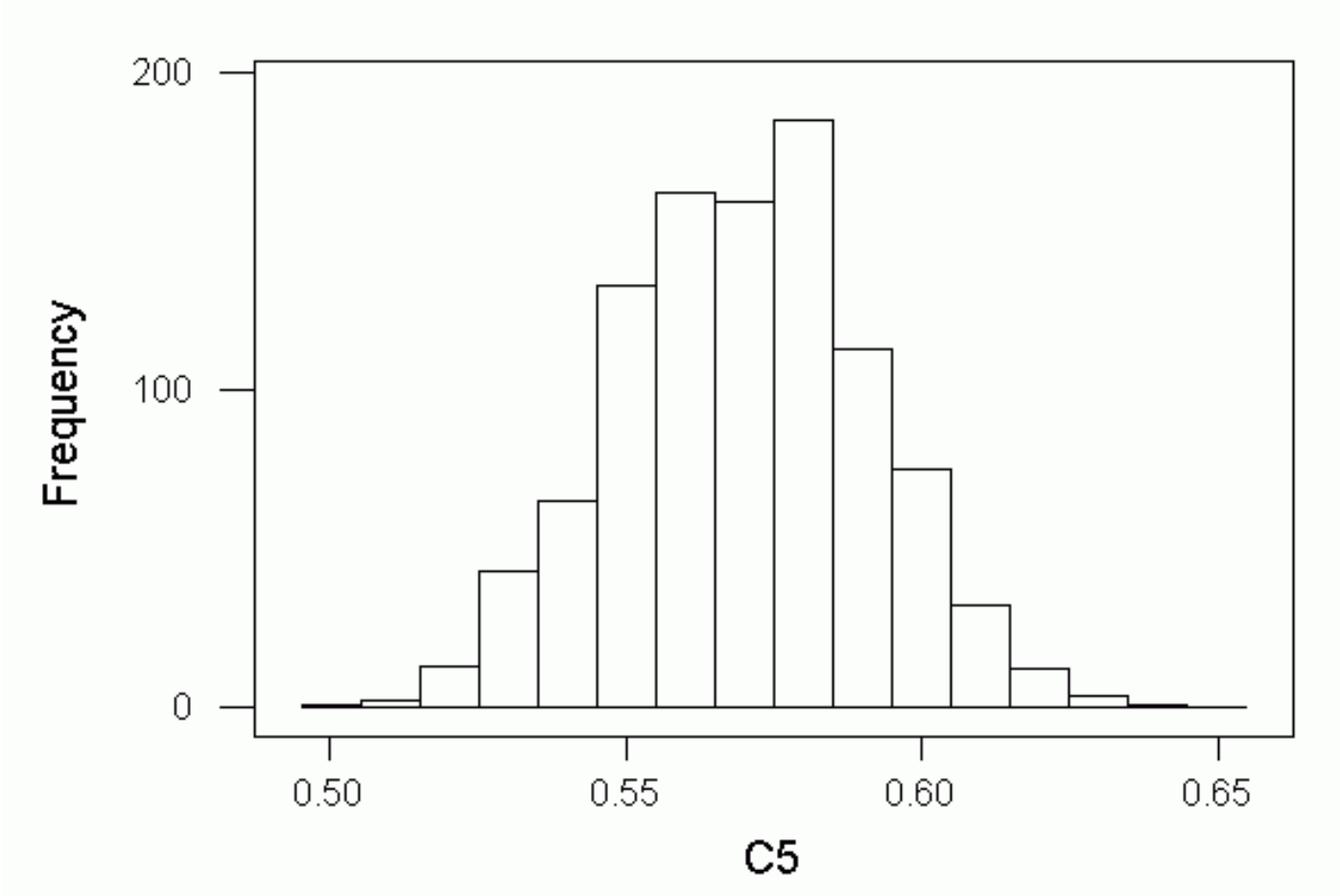
Simulation

Many possible samples from most populations (number of possible samples of 10 people from 100 is number with 14 digits). But looking at a lot of samples gives good enough idea. Steps:

- pick a population to sample from
- generate a random sample from the population, calculate sample statistic
- repeat previous step many times
- make histogram of results

Simulation example

Example: 57% of students at college female. In samples of 500 students, what proportion of females might we get?



Discussion of simulation results

- Centre near 0.57.
- Spread *small* (50–65% women, values close to 57% commoner than extreme values).
- Shape close to *normal*.
- Collection of possible sample proportions called **sampling distribution** – shows what sample statistic might be, if population parameter known.

Bias and variability

- Statistic usually used to estimate population parameter (sample proportion to estimate population proportion). In this role, called **estimator**.
- Sampling distribution of estimator desirably:
  - centre around the population parameter (**unbiased**)
  - has small spread (low **variability**).
- Fact: provided sample small part of population, sampling distribution *does not depend on size of population*.

Benefits of randomization

- Possible to work out sampling distributions by mathematics as well as simulation. Only true for random samples – take sample any other way, have no idea of sample-population relationship.
- With random samples, can reduce variability by taking larger sample. With very large random sample, sample statistic very close to population parameter.
- Sampling distribution shows how close sample, population quantities might be by going from population to sample. Later, use same idea to go from (one) sample to population.

Summary

- Choose sample by randomization (no connection between sample and variable studied)
- Simple random sample (each individual has same chance to be in sample, independent of others)
- Stratified, multistage sampling when simple random sample inconvenient
- Conclusions only as good as sampling method: beware of undercoverage, nonresponse, biased wording.

Summary part 2

- Want to learn about population parameter, only have sample statistic. But with random sampling, can know how close we may be to “right answer”, even though different samples will give different results.
- Reasoning from sample to population hard, but if we know population, can figure out what kind of samples might come from it.
- Can understand effect of looking at many different samples by simulation.
- If sample is not drawn randomly, we are stuck!

Probability: The Study of Randomness

Randomness (§4.1)

- Toss a coin – can't predict whether outcome will be Heads or Tails.
- Roll a die – can't predict whether you'll roll a 1 or something else.
- Play roulette (or other casino game) – can't predict whether you'll win or lose.
- In each case, individual outcome cannot be predicted, but pattern emerges:
- After many coin tosses, you will observe about 50% heads and 50% tails.
- After many die rolls, about $\frac{1}{6}$ of the rolls will be 1.

Randomness and probability

- When individual outcomes are uncertain, but a pattern emerges when many outcomes are observed, phenomenon called **random**.
- Proportion of times an outcome is observed over the long run called **probability** of that outcome ($\frac{1}{2}$ for Heads, $\frac{1}{6}$ for 1 on die). Write as $P(H) = \frac{1}{2}$, $P(1) = \frac{1}{6}$.
- Probability is a number between 0 and 1. 0 means outcome *impossible*, 1 means outcome *certain*.

Probability models: some jargon (§4.2)

- To describe a random phenomenon, need two things:
  - List of all possible outcomes (**sample space**, written S).
  - Probability for each outcome.
- These two things together called **probability model**.
- Examples of sample spaces:
  - flip a coin once. $S = \{H, T\}$.
  - flip 4 coins, count number of heads. $S = \{0, 1, 2, 3, 4\}$.
  - choose a Canadian at random, note down province of residence.
    $S = \{\text{BC}, \text{Alberta}, \ldots, \text{Ontario}, \ldots, \text{Newfoundland}\}$.

Events

- An **event** is a collection of outcomes.
- Example: toss 2 coins. Sample space is $S = \{HH, HT, TH, TT\}$. "Exactly one head" is an event, consisting of outcomes HT and TH.
- An event has a probability, which you get by *adding* the probabilities of its constituent outcomes.
- In this example, if the coin is fair (unbiased), any outcome has probability $\frac{1}{4}$, so prob. of "exactly one head" is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
- The whole sample space is an event. Prob. of S here is

$$P(S) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1.$$

  This true in general: $P(S) = 1$.

Equally likely outcomes

- $P(S) = 1$, so probs for all the outcomes must add up to 1.
- If can assume that each outcome *equally likely*, then probs are all same (and must add up to 1). In 2-coin example above, were 4 equally likely outcomes, so prob. of any one is $\frac{1}{4}$.
- In general, for n equally likely outcomes, prob. is $1/n$.
- Even if *outcomes* equally likely, *events* may not be; depends on how many outcomes they contain. Example (2 coins):
  - $P(\text{exactly 0 heads}) = P(\{TT\}) = \frac{1}{4}$.
  - $P(\text{exactly 1 head}) = P(\{HT, TH\}) = \frac{2}{4}$.
  - $P(\text{exactly 2 heads}) = P(\{HH\}) = \frac{1}{4}$).
- If n equally likely outcomes, and event A contains r of them, $P(A) = r/n$.
- Most outcomes not equally likely. Eg. weather on a June day likely to be sunny, and very unlikely to be snow!

The event "not A"

- Returning to two-fair-coin example, suppose $A = \{HH\}$. A contains 1 of 4 outcomes (equally likely), so $P(A) = \frac{1}{4}$.
- The event "not A", written A^c, contains all the other outcomes. So here $A^c = \{HT, TH, TT\}$. Then $P(A^c) = \frac{3}{4} = 1 - \frac{1}{4}$.
- General rule: for any event A, $P(A^c) = 1 - P(A)$.

Independent and disjoint events

- Imagine that you toss 2 coins. Let A be event "get H on 1st coin", B be event "get H on 2nd coin". Knowing that A happens doesn't make B any more or less likely: A and B are *independent* events.
- Again imagine tossing 2 coins. Let A be event HH, let B be event "head on 1st coin", so $B = \{HH, HT\}$. If A happens, B is certain to happen, so A and B *not* independent events. A and B have outcome HH in common, so are *overlapping* events.
- Now let C be event TT. If A happens, C cannot happen (coin cannot show both heads and tails), so A and C not independent. But have no outcomes in common; called *disjoint* events.

Finding probabilities from independent and disjoint events

- For general events A and B,
  - if A and B are independent, **multiplication rule** says

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

  - if A and B are disjoint, **addition rule** says

$$P(\text{either } A \text{ or } B) = P(A) + P(B)$$

- To use these rules:
  - First, are events independent? If they are, can use multiplication rule.
  - If not, are they are disjoint? If they are, can use addition rule.
  - Otherwise, cannot use either rule.

Examples

- In a certain lottery, you can win either prize A, prize B, or no prize. $P(A) = 0.01$, $P(0) = 0.90$.
- To find $P(B)$, 3 probs have to add up to 1, so $P(B) = 1 - 0.01 - 0.90 = 0.09$.
- $P(\text{some prize}) = P(A \text{ or } B)$. A and B not independent, but are disjoint (can't win more than 1 prize from 1 ticket), so can use addition rule.
  $P(\text{some prize}) = P(A) + P(B) = 0.01 + 0.09 = 0.10$. (Or: "some prize" means "not no prize", which gives answer as $1 - 0.90 = 0.10$.)

Lottery example continued

- Suppose two people play this lottery. Prob. that both win some prize? Neither wins a prize?
- If they pick numbers separately, events independent, so can use multiplication rule.

$$P(\text{both win some prize}) = 0.10 \times 0.10 = 0.01$$

$$P(\text{neither wins}) = 0.90 \times 0.90 = 0.81.$$

These two probs don't add up to 1 because one person could win and the other not.

Random variables (§4.3)

- Suppose you:
  - toss 4 coins and count the number of heads
  - roll 2 dice and count the total number of spots
- In each case, each outcome gives you a number. (In 1st case, don't care *which* came up heads, just *how many*).
- This number called a **random variable**, labelled X.
- Each value of random variable has probability. $P(X = 3)$ is sum of probs of all outcomes leading to X being 3.

Example

Value of X	1	2	3
Probability	0.3	0.6	0.1

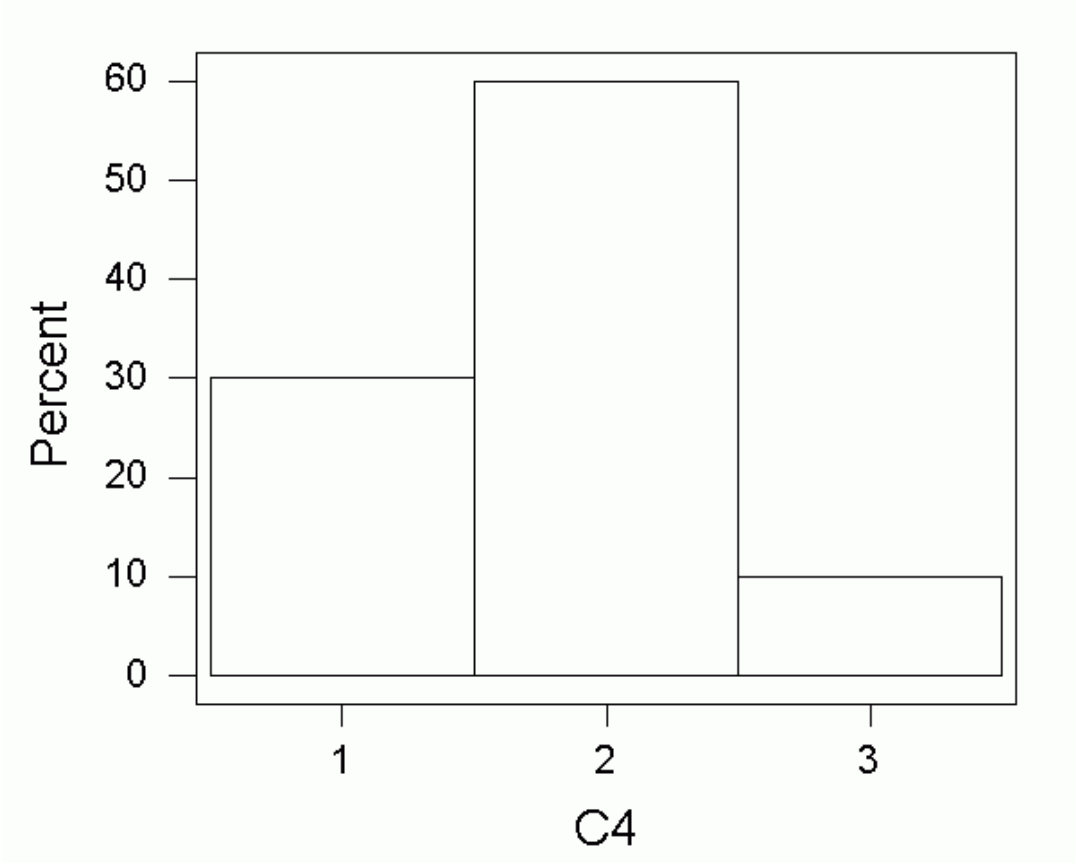
- Is example of *probability distribution* (list of values and probs).
- $P(X \geq 2) = P(X = 2) + P(X = 3) = 0.6 + 0.1 = 0.7.$
- Note also that

$$P(X = 1, 2 \text{ or } 3) = 0.3 + 0.6 + 0.1 = 1,$$

that is, all the probs together add up to 1.

Probability histogram

Can make a plot: above each value of X, draw a bar whose height is the probability. Looks like a histogram. For above distribution:



Highest bar above 2, showing that $X = 2$ has highest prob.

Continuous random variables

- So far, can list possible values and their probs: **discrete random variable**. But what if *all* values in an interval (say between 0 and 1) are possible?
- Cannot talk about prob. of individual value in a continuous distribution (there are infinitely many values). So talk about prob. of *interval*: $P(3 \leq X \leq 4)$ or $P(X > 7)$.
- Probability histogram becomes **density curve** (compare §1.3). Probability of getting value in any interval is *area* under density curve (recall normal distribution: Table A gives area).
- Extra notes: example of density function.

Mean of a random variable (§4.4)

- Data distribution has mean and SD you can calculate. Likewise, probability distribution (and thus random variable) has a mean and SD. But the way to calculate them is different.
- For the mean of a random variable, multiply each value by its probability, and add them up. To recycle our example:

Value of X	1	2	3
Probability	0.3	0.6	0.1

- Mean of X is $(1)(0.3) + (2)(0.6) + (3)(0.1) = 1.8$. Often use symbol μ (μ_X) to represent mean (of X).
- Here, $\mu_X < 2$ because $P(X = 1) > P(X = 3)$.

Estimation and law of large numbers

- Suppose we want to know mean height of all adults in Canada. Idea from Chapter 3: take sample, look at sample mean. But how close is that to mean height of *all* adults?
- Height of randomly chosen Canadian adult is random variable H. H has a mean μ. Hope that sample mean $\bar{x}$ will be close to μ, especially with large sample.
- **Law of large numbers** says this more carefully. Decide how close you want to be to μ (say within 1 inch), and how likely you want to be that close (say 90%). Then, with large enough sample, will be as close as you want with specified probability.
- Supports idea of using sample mean as estimate of population mean.

Variance and SD of a random variable

- SD of a random variable measures its spread: that is, whether values far from mean are likely or not.
- To calculate SD, first find variance, using recipe:
  1. Take each value of the random variable and subtract the mean.
  2. Square each result.
  3. Multiply by the probability and add up.
- SD is then square root of variance.

Example

- To use our example again:

Value of X	1	2	3
Probability	0.3	0.6	0.1

Mean was $\mu = 1.8$.

- Summarize calculations in table:

x	$x - \mu$	$(x - \mu)^2$	$\times$ prob.
1	-0.8	0.64	0.192
2	0.2	0.04	0.024
3	1.2	1.44	0.144
Total			0.360

- SD is square root of this, so SD of X is $\sqrt{0.36} = 0.6$.
- SD small, so values far from mean unlikely.

Rules for means, SDs and variances

- Multiplying X by a constant multiplies the mean by same constant.
- Adding a constant to X adds the same constant to the mean.
- Multiplying X by a constant multiplies the SD by the same constant.
- Adding a constant to X doesn't change the SD.
- For two random variables X and Y, mean of $X + Y$ is sum of means of X and Y.
- For two *independent* random variables X, Y, *variance* of $X + Y$ is sum of variances of X and Y.

Examples

- Suppose the temperature in a city has mean 7 and SD 10 (degrees C). What are the mean and SD in degrees F?
- Get degrees F by multiplying by 1.8 and adding 32.
- So mean in degrees F is $(7)(1.8) + 32 = 44.6$. But for SD, adding makes no difference, so SD in degrees F is $(10)(1.8) = 18$.
- Now let X be sales of cars at a dealership: X has mean 20 and SD 5. Y is sales of trucks, with mean 8 and SD 4.
- Total sales have mean $20 + 8 = 28$. *If* X and Y independent, variance of total sales is $5^2 + 4^2 = 41$, so SD is $\sqrt{41} = 6.4$. But X and Y may rise and fall together (eg. with economic conditions).

Chapter 5: Sampling distributions

Introduction

- Statistical inference: from sample to population.
- Simulation (mathematics): from population to possible samples (sampling distribution).
- Look at some more sampling distributions. First simulate, then use exact mathematical results. Along way, learn why operating big casino is money-maker even though casino games based on chance and randomness.

Sampling distributions for counts and proportions (§5.1)

- Sample surveys often ask questions with “yes/no” type answers, like “Do you consider that this university is well run?”.
- Industrial quality control: samples of TVs are taken from production line and tested. Each TV “satisfactory” or not.
- In each case, “successes” and “failures” in the sample tell something about the populations.

Real and hypothetical populations

- Population of “all students at UTSC” actually exists (registrar’s records). Population of “all TVs produced at factory” harder to grasp. But often justified in treating sampled TVs as random sample from hypothetical population.
- Many possible sampling distributions (corresponding to different kinds of statistic we might measure). Nature of sampling distribution depends on kind of population and sampling procedure.

Sample counts and proportions

- Suppose we sample 20 TVs from our production line, and find that 17 of them work satisfactorily. Just want to know what % of *all* TVs would work, so doesn't matter *which ones* in sample are OK.
- 17 here is **sample count** (x) of satisfactory TVs out of 20 sampled.
- **Sample proportion** ($\hat{p}$) is sample count as fraction of whole: here $17/20 = 0.85$.

Binomial distribution

Sampling distribution of sample count depends on sampling method, but common scenario is:

- *Fixed number* n of observations.
- Observations fall into one of two categories (“success” and “failure”, alternatively “heads” and “tails”)
- Observations all *independent* (knowing one observation to be success doesn’t affect chance of others being success)
- Chance of success for each observation *always same* (p).

Simple example: tossing coin 10 times; # heads has binomial distribution.

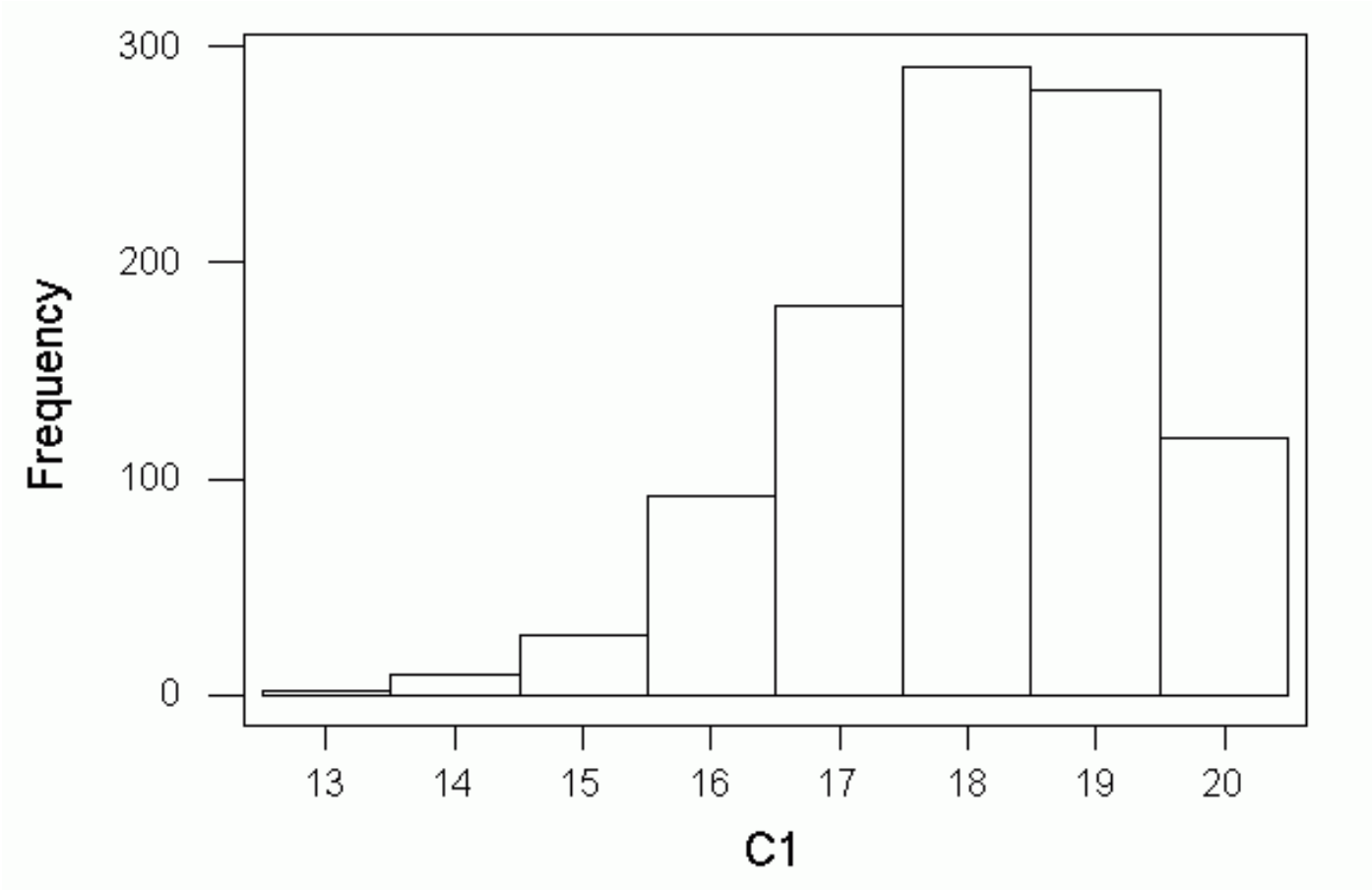
Binomial distribution in sampling

- Counts of successes in sample often described by binomial distribution. True provided sample is small fraction of population.
- (If sample most of population, might get almost all population successes before finished sampling, then know we have failures to come.)

Simulating binomial

- TVs: suppose 90% of all TVs produced in past satisfactory. Treat as population proportion $p = 0.90$. Sample $n = 20$ TVs; how many might be satisfactory?
- Simulate. Minitab: Calc, Random Data, Binomial. Generate 1000 rows of data ("many samples"), store in C1, Number of Trials 20, Probability of Success 0.90.

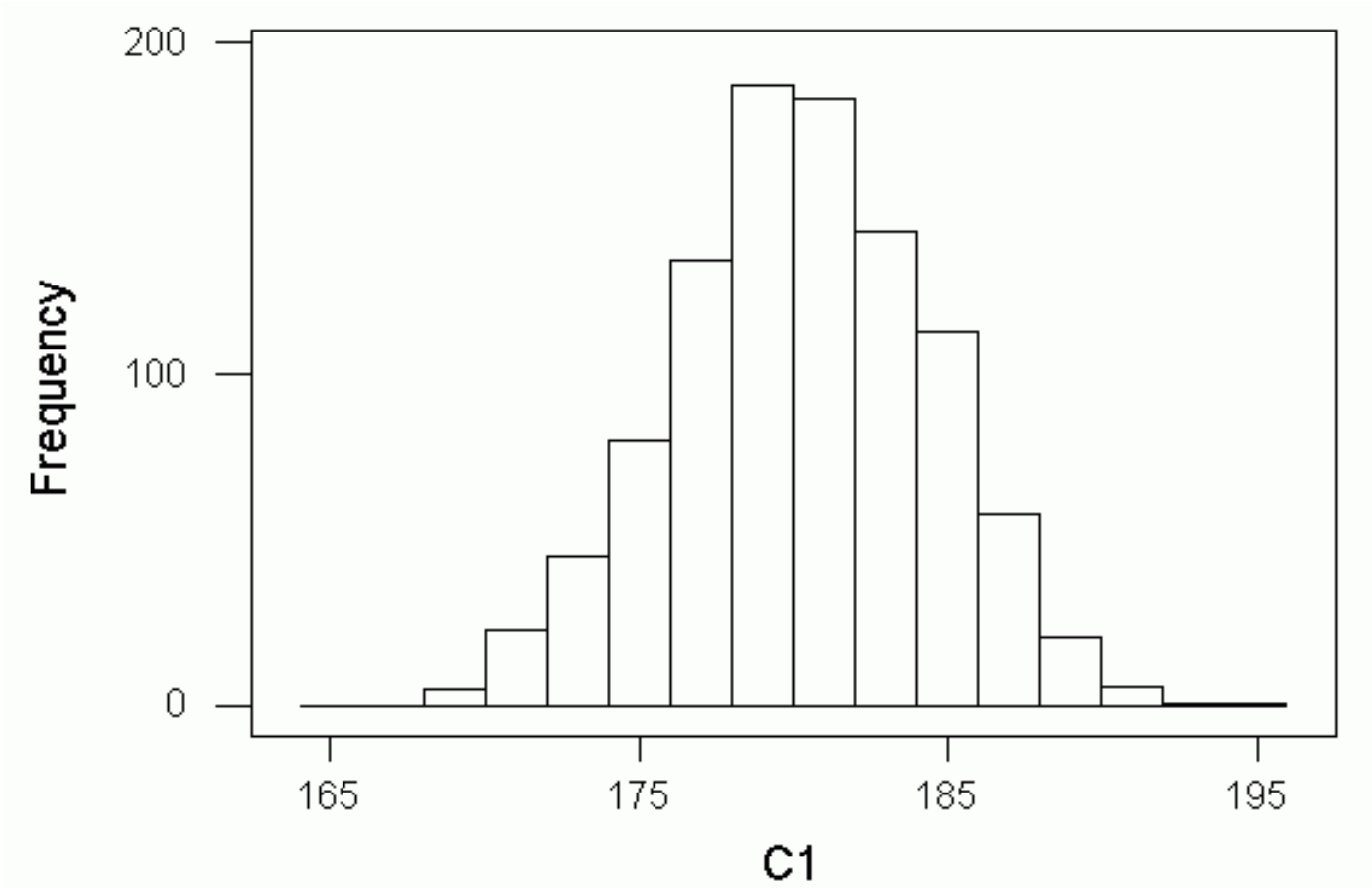
Simulated binomial: $n = 20, p = 0.9$



Discussion of histogram

- Shape skewed to left.
- Centre around 18 (90% of 20).
- One sample had only 13 satisfactory TVs.
- Now try with samples of 200 TVs ($n = 200, p = 0.90$):

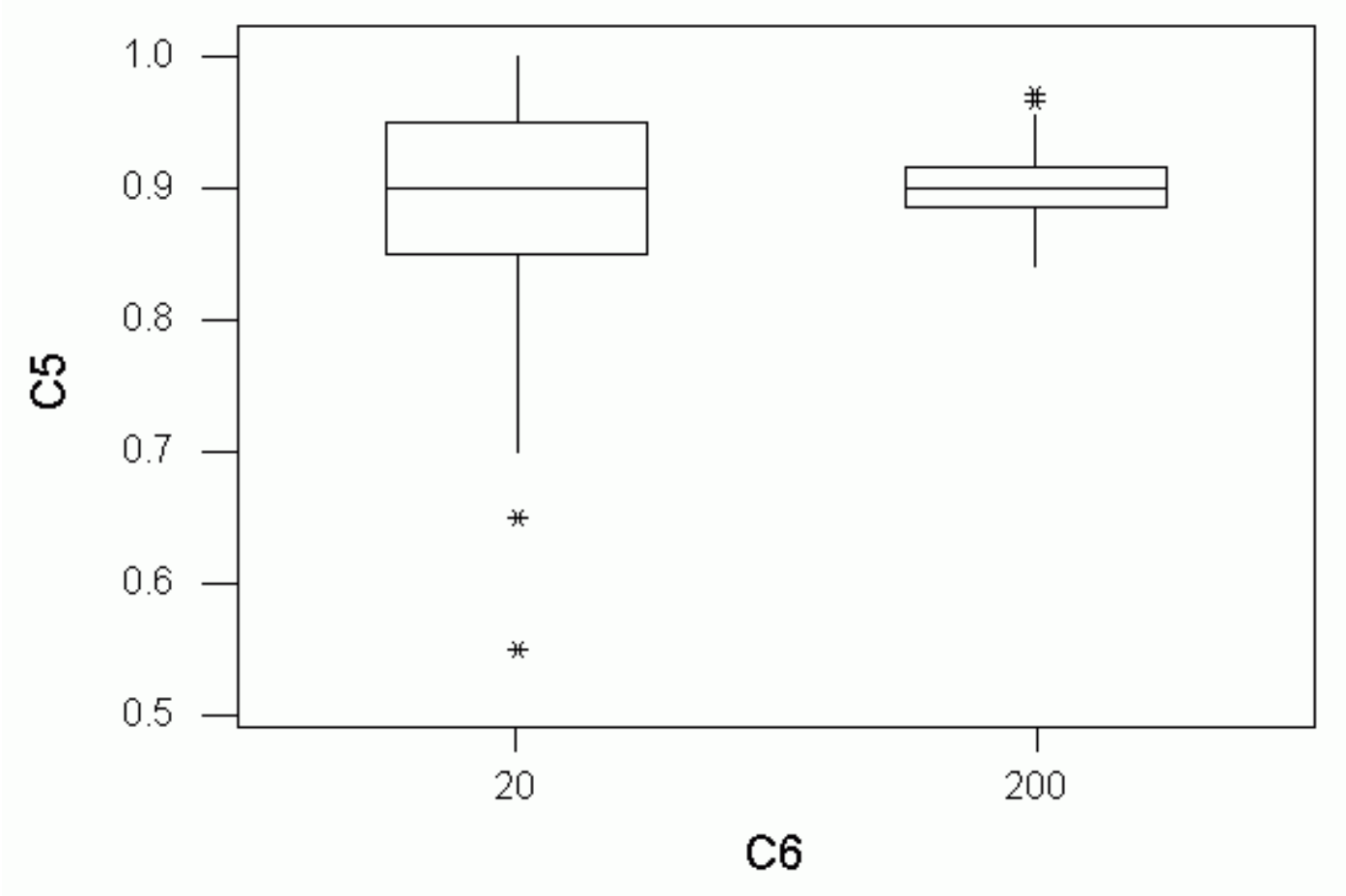
Simulated binomial: $n = 200, p = 0.9$



Centre around 180 (90% of 200). Shape like *normal distribution*.

Comparing proportions

Counts difficult to compare (out of 20 vs. 200). Calculate *sample proportions* $\hat{p}$ (divide by 20 or 200), show boxplots of proportions.



Discussion

- For both sample sizes, centre of sampling distribution close to 0.90. But sampling distribution for $n = 200$ much less spread out, and more symmetrically shaped.
- General facts for sample proportions based on large sample:
  - sample proportion almost certainly very close to the population proportion. **Law of Large Numbers.**
  - sampling distribution shape very close to *normal*.

The point of all this

- Sampling distribution of count has binomial distribution; for large samples, sampling distribution of count or proportion close to normal.
- That is, if we know population, we know what samples from it will look like.
- In particular, know how close sample proportion likely to be to population proportion.
- Later, provides key to reversing logic: going from sample to population. *This* is useful thing in practice – eg. tells factory manager whether enough of *all* TVs off production line work satisfactorily.

Summary

- May have a population divided into two parts: “success” and “failure” say.
- If:
  - sample size of n trials fixed
  - successes equally likely on each trial
  - trials independent

  then number of successes described by binomial distribution.
- Often describes sampling from this population (unless sample is most of population)

Summary continued

- Can simulate from a population, to see what samples from it may look like.
- In a large sample:
  - sample proportion will be close to population proportion
  - distribution of possible sample proportions will look normal
- Later, reverse logic: from sample, draw conclusion about population.

Sampling distribution of sample mean (§5.2)

- Counts and proportions describe *categorical* data. *Quantitative* data: describe using sample mean, median, SD etc.
- Here, concentrate on *sample mean* $\bar{x}$ and how close it might be to population mean μ.
- Same attitude as proportions: start from population, see what samples from it look like.

Roulette

Gambling game popular in casinos. Based on wheel with numbers (from gambling-hall-online.com):

American roulette wheel diagram

In American Roulette the wheel consist of 38 identical slots, numbered from 0, 00, 1 through 36.

On the standard roulette wheel the numbers are not distributed in increasing series or randomly.

On the contrary, the numbers are ordered to achieve a certain mathematical balance between high and low, red and black and even and odd:

- Red and black numbers alternate,
- Usually two odd numbers alternate with two even numbers.
- All red numbers are opposite the black numbers.
- Every odd number is opposite the next higher even number.
- Most numbers are part of a pair, with one number between them. The sums of these pairs are either 37 or 39.

Zero and double zero are both green pockets, while the remaining 36 are split evenly between red and black.

Numbers face the outside of the wheel.

How roulette works

- Ball spun around wheel, eventually falls into “pocket” next to a number. Thus outcome is a single number.
- Bet on single number or combination. If outcome matches *any* of numbers in a combination, you win.
- Wheel has numbers 1–36 plus 0 and 00. Bets paid as if 0 and 00 absent (gives casino small edge).
  - “High”: win on 19–36, lose otherwise. Win $1, lose $1.
  - “4 numbers”: win on 7, 8, 9 or 10 (say), lose otherwise. Win $8, lose $1.

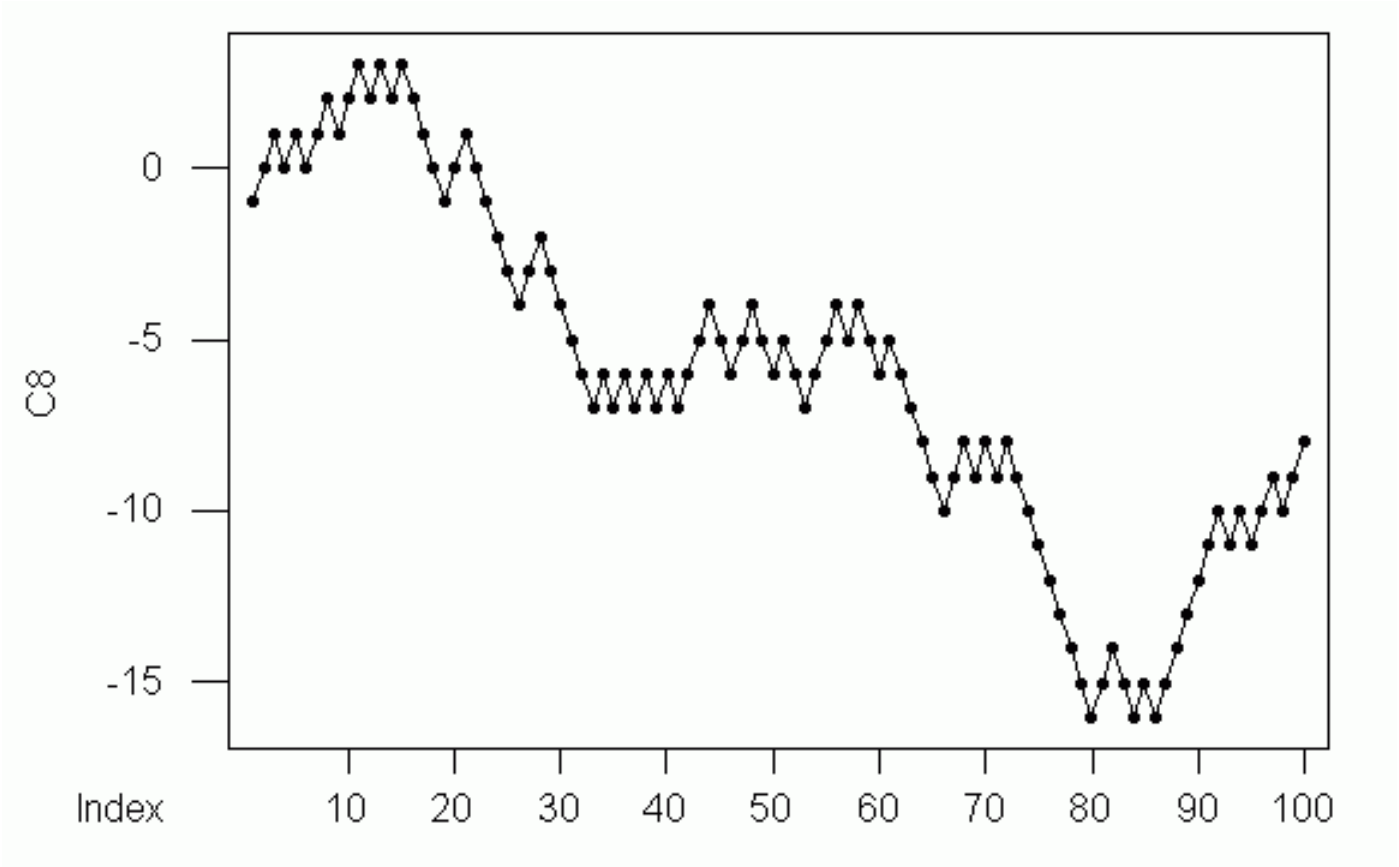
Simulated roulette in Minitab

- Column with numbers 1–36, 0, 00. For each bet, create column with winnings for each number (worksheet `roulette`).
- Session window: turn on command language (Editor, Enable Command Language).
- Simulate 100 plays of high-low bet. Calc, Random Data, Sample from Columns. Sample from C1 and C2 (to get winning number and winnings). Put results in empty columns, say C6, C7. Check “sample with replacement” box.

Winnings over time

- See evolution of winnings over time: Calc, Calculator. Store Result in C8. Scan through list of functions to Partial Sums, double-click. In Formula box, see `PARS(number)`. Double-click C7, OK. C8 contains total winnings so far.
- Plot this over time: Graph, Time Series Plot. Double-click C8, OK.

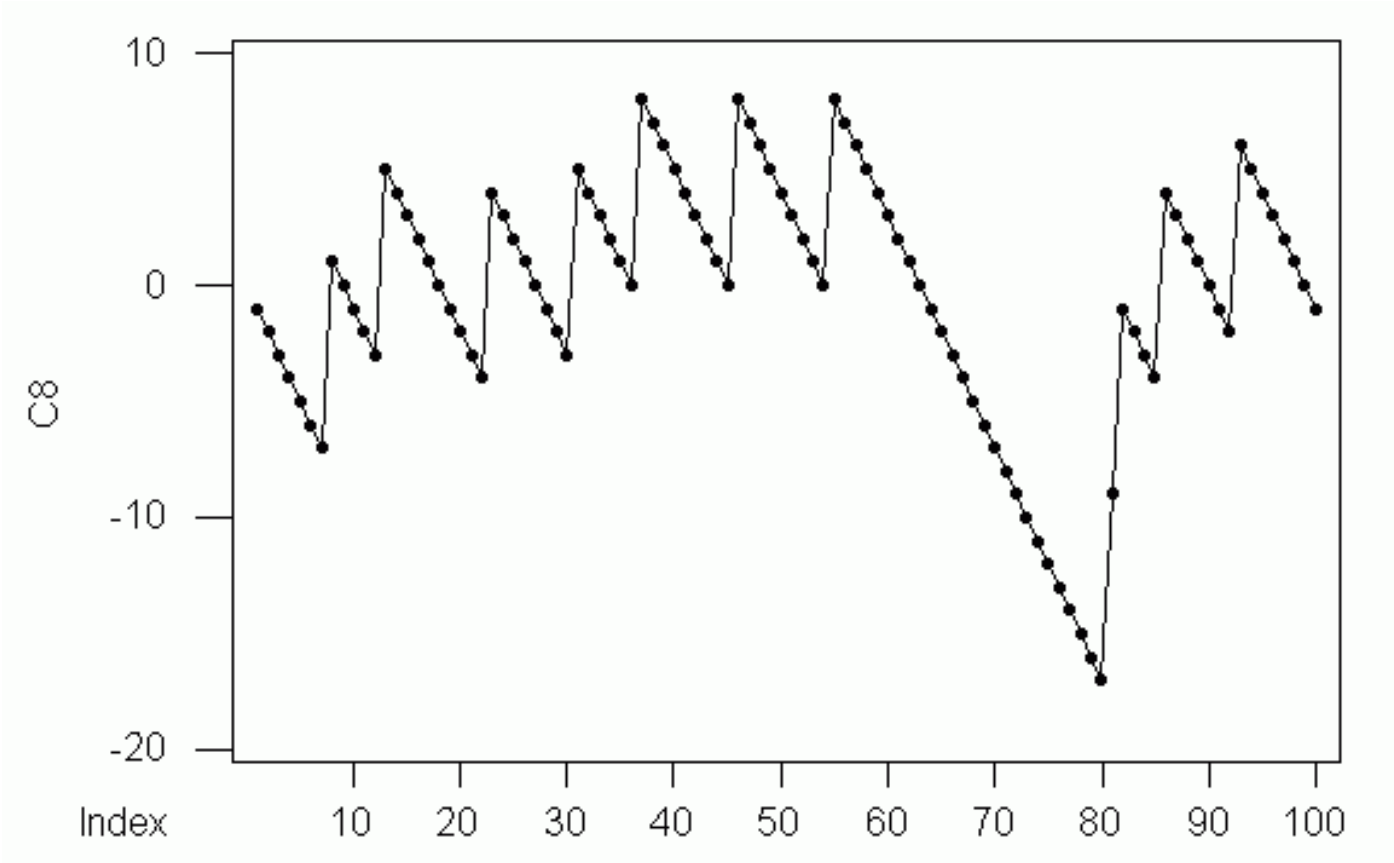
Example run of high-low bet



More runs

- Select commands in Session window back to Sample, Edit-Copy. Go to end of Session window (last `MTB >`), Edit-Paste, Enter. Get another graph.
- Try with 4-number bet. Calc, Random Data, Sample from Columns. Sample from C1 and *C3*, put results in C6, C7 (as before), check With Replacement. Total winnings, plot as before.
- Typical picture more “jagged”: lose for some time, occasionally win big.

4-number bet



Here, down $1 at end, but fluctuates wildly.

Comparison of high-low and 4-number

- Repeated simulations suggest 4-number-bet results “wilder” than for high-low – more likely to come out ahead, but also more likely to lose big.
- Mathematical calculations support this. For single plays, work out mean and SD of winnings per play for 2 bets (using methods of §4.4):

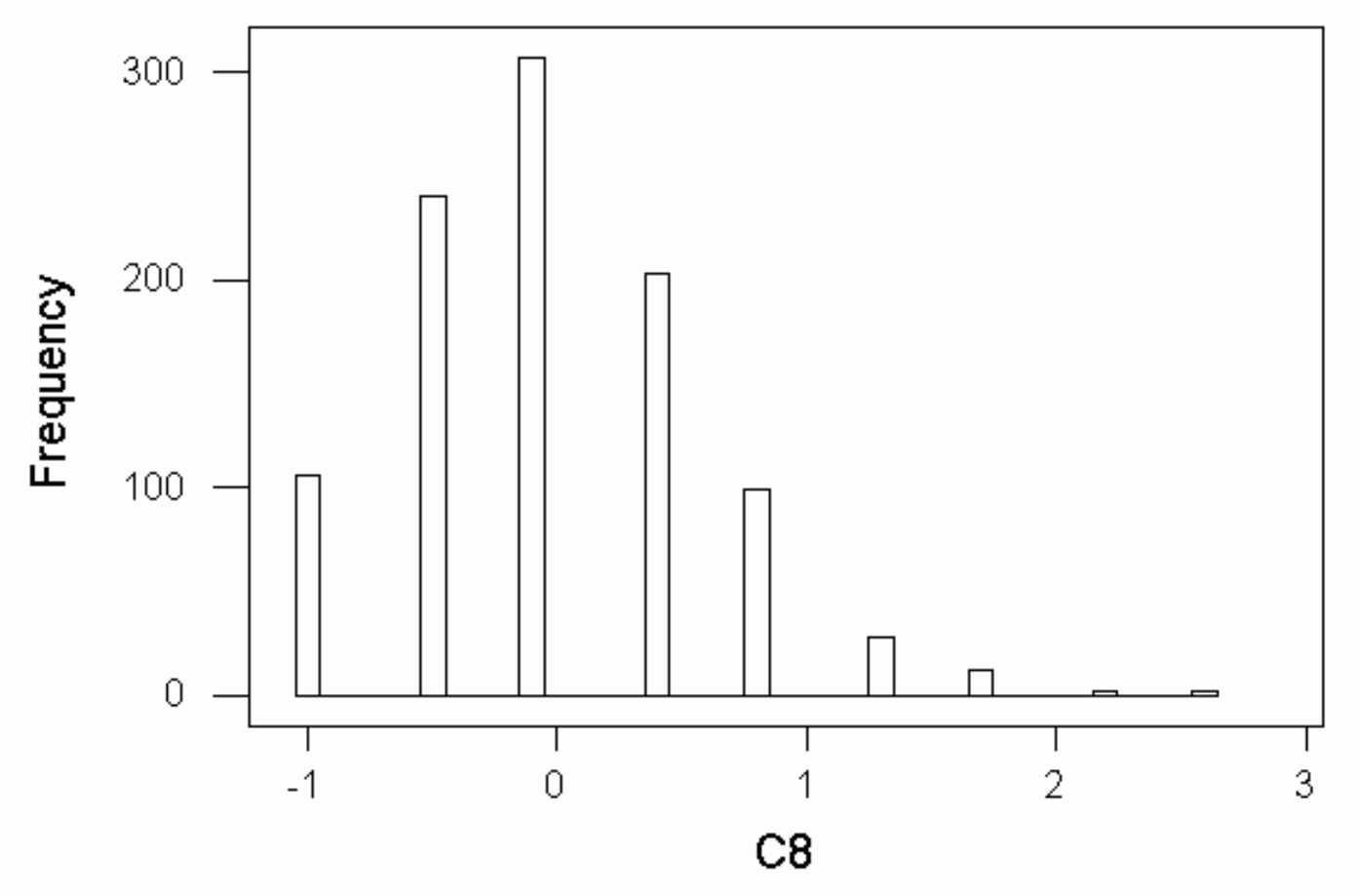
Bet	Mean	SD
High-low	–0.0526	0.9986
4-number	–0.0526	2.7620

Sampling distribution of sample mean

- Can think of 100 plays of roulette as *sample* from *population* of "all possible plays".
- Mathematical results. Write population mean as μ ("mu") and population SD as σ ("sigma").
- Then sampling distribution of sample mean $\bar{x}$ has:
  - mean μ (same as population mean)
  - SD $\sigma / \sqrt{n}$ (*smaller* than population SD).

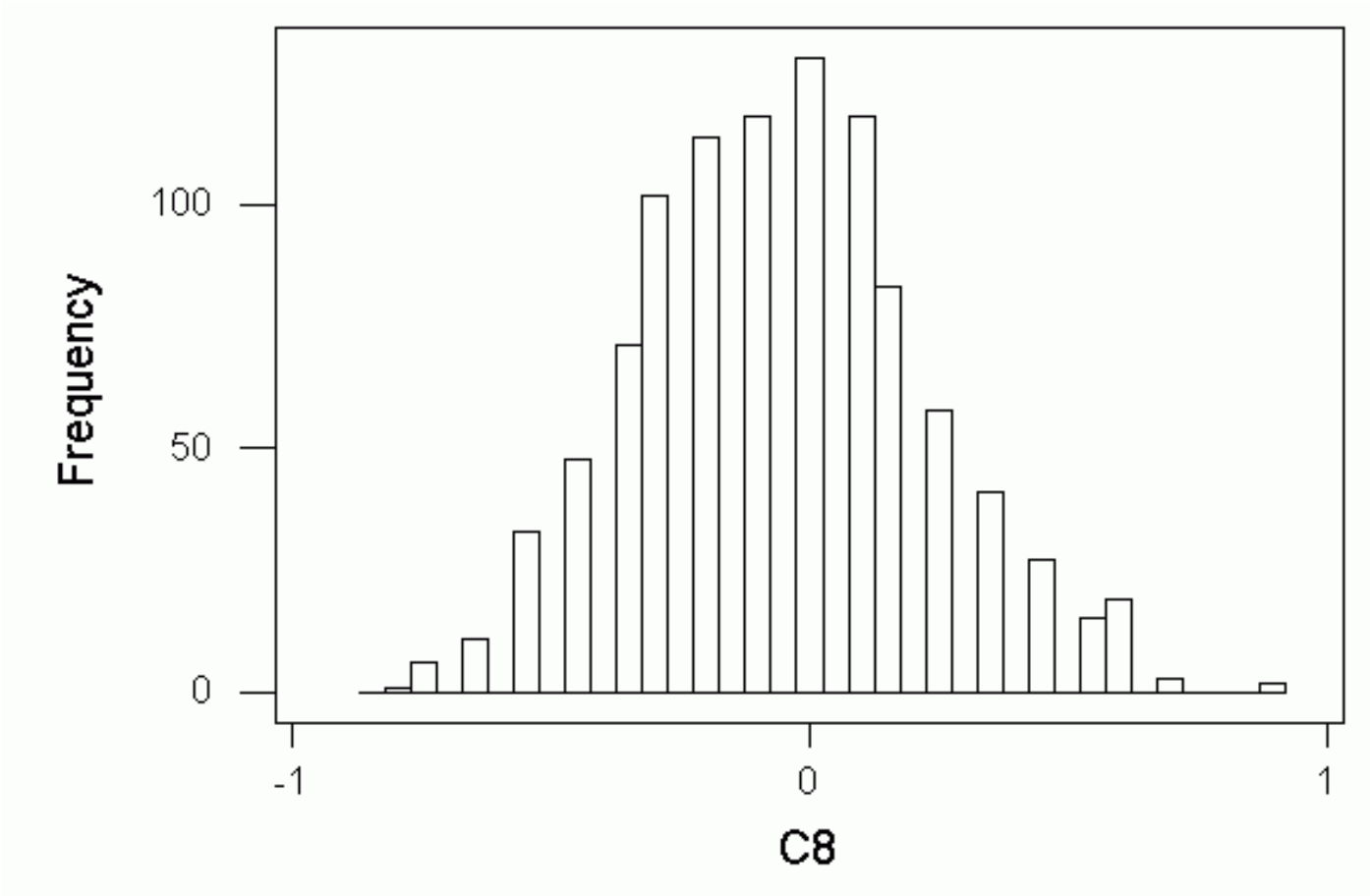
Shape of sampling distribution

Investigate *shape* by simulation. Depends on sample size – for roulette, times each bet made. 4-number, 20 bets:



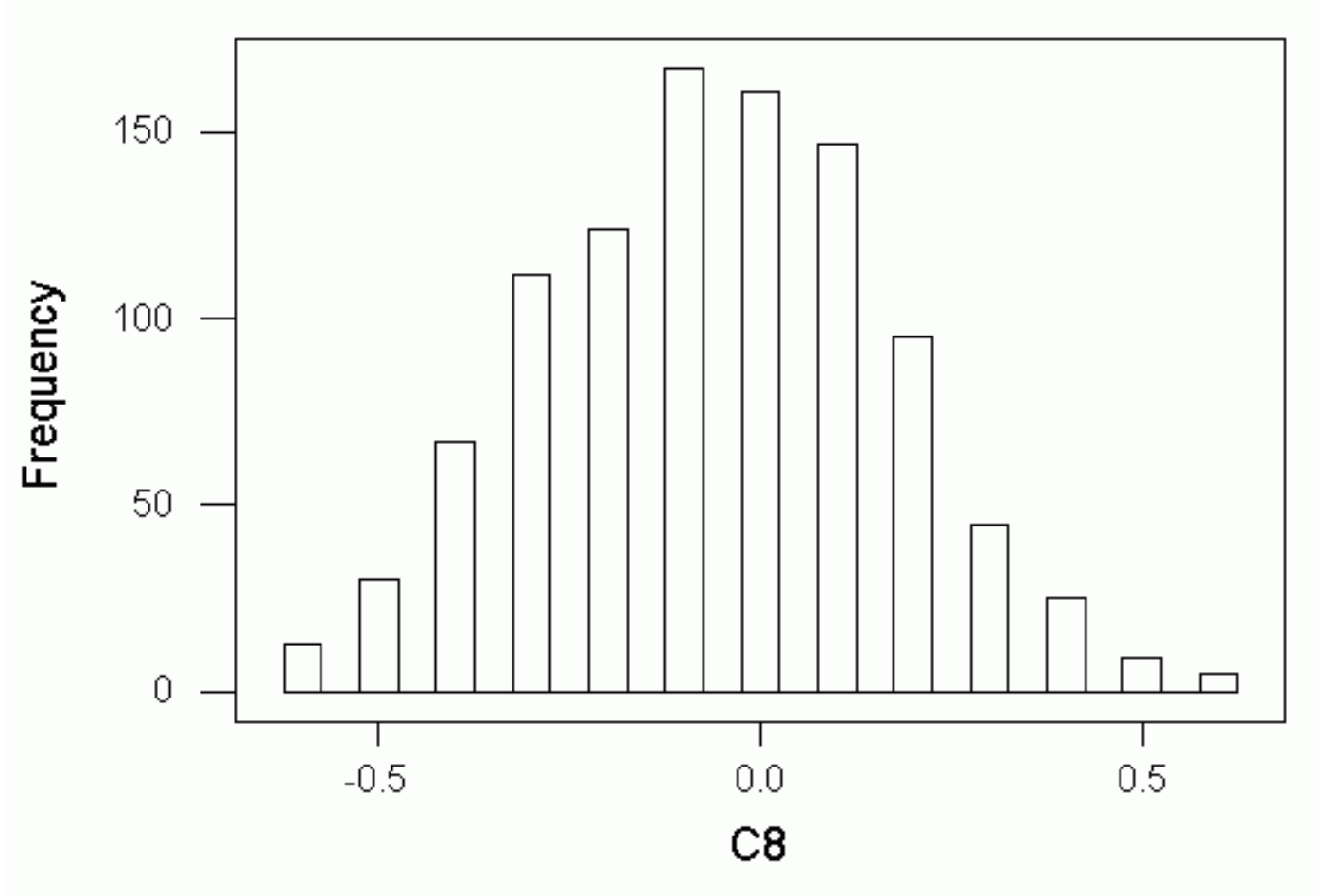
Skewed right.

4-number, $n = 100$ bets



More bets: less spread, shape like normal.

High-low, $n = 20$



High-low, 20 bets, also shape like normal.

Central Limit Theorem

- So where does normal distribution come from?
- Fact: draw simple random sample size n from *any* population. If n large, sampling distribution of $\bar{x}$ has approx. **normal** shape.
- Remarkable fact, called **central limit theorem**.
- Required largeness of n depends on population – must be large enough to "iron out" skewness. Roulette: high-low population near symmetric to start, so $n = 20$ large enough. But 4-number population skewed (small chance to win big), so needed $n = 100$.

Calculations for sample means

- Central Limit Theorem allows use of normal distribution to calculate chances, provided sample size n big enough.
- First: get mean μ, SD σ of population. (Also mean, SD of *single observation* randomly drawn from population.)
- Next: calculate mean, SD of *sampling distribution*. Respectively μ and $\sigma / \sqrt{n}$.
- Finally: complete calculation using Table A.

Chances of profit and of losing at least $0.10 per play

- Roulette: high-low bet, 100 plays.
- Mean of population -0.0526, SD of population 0.9986. Sampling distribution has mean -0.0526 (same), SD $0.9986/\sqrt{100} = 0.09986$.
- Chance of profit? Mean winnings per play > 0. $z = (0 - (-0.0526))/0.09986 = 0.53$.
- Table gives 0.7019, chance of mean winnings *less than 0* (loss). Chance of profit $1 - 0.7019 = 0.2981$.
- Chance of losing $0.10 per play or worse? Replace 0 by -0.10, repeat. $z = (-0.10 - (-0.0526))/0.09986 = -0.47$, answer 0.3192.

4-number bet

- Compare with 4-number bet, 100 plays. Population mean -0.0526, SD 2.7620. Sampling distribution mean -0.0526, SD $2.7620/\sqrt{100} = 0.2762$.
- Chance of profit? As before: $z = (0 - (-0.0526))/0.2762 = 0.19$, chance of profit $1 - 0.5753 = 0.4247$.
- Chance of losing \$0.10 per play or worse? $z = (-0.10 - (-0.0526))/0.2762 = -0.17$; answer 0.4325.
- 4-number bet offers bigger chance of profit (0.42 vs. 0.30), but bigger chance of large loss (0.43 vs. 0.32).
- Because 4-number bet more variable, result you get could be further from mean, hence likelier profit or large loss. But "average average" same for both bets.

Summary

- Understand (sampling) distribution of sample mean from population with mean μ and SD σ.
- Can do by simulation or mathematics.
- Mathematics: sampling distribution of sample mean $\bar{x}$ has mean μ and SD $\sigma/\sqrt{n}$.
- For large sample, sampling distribution has normal shape (central limit theorem).
- Enables (approx.) calculations about values of sample mean when population mean, SD known.

Chapter 6: Introduction to Inference

Estimating with confidence (§6.1)

Previous simulations, calculations say that

> if I know about the population, I can say what kind of samples I might get from that population. In particular: sample mean and sample proportion have sampling distributions with mean, SD I can calculate, and a normal shape if n large.

But in practice, *don't* know about population; just have one sample from it.

Note logic of this example

SAT-M scores have SD $\sigma = 100$ points. Take random sample of 500 California high-school seniors, give test to all. Sample mean is 461. What about mean score of *all* California seniors?

- Sample size large. So sampling distribution of sample mean approx. normal, mean μ, SD $100/\sqrt{500} = 4.5$.
- 68-95-99.7 rule: in about 95% of all samples, $\bar{x}$ within 2 times right SD (ie. $2 \times 4.5 = 9$ points) of μ.
- same as saying μ within 9 points of $\bar{x}$.
- So interval $\bar{x} - 9$ to $\bar{x} + 9$ has μ in it somewhere for 95% of all possible samples.

Confidence interval process

- Example: $\bar{x} = 461$, so interval for our data from $461 - 9 = 452$ to $461 + 9 = 470$. Called **95% confidence interval for μ.**
- Understand process:
  - Randomness from "all possible samples". So confidence in *procedure*, not any one answer.
  - Cannot know whether our one interval from 95% "good" ones that contain μ, or 5% "bad" ones that don't.
- See figure 6.3 in text (next page).

Figure 6.3 (text)

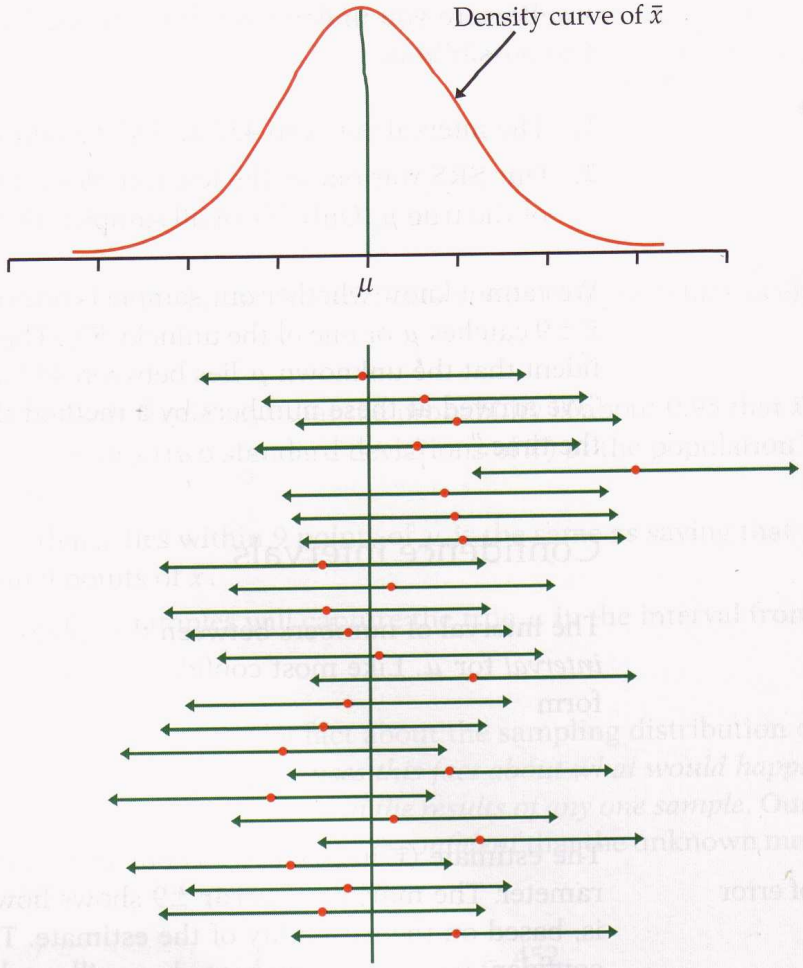


FIGURE 6.2 Twenty-five samples from the same population gave these 95% confidence intervals. In the long run, 95% of all samples give an interval that covers μ.

Confidence intervals

- 95% confidence interval for μ above was $\bar{x} \pm 9$. Typical interval is

  estimate $\pm$ margin of error.

- Estimate calculated from sample; margin of error expresses accuracy of estimate.
- Many different kinds of confidence interval, depending on parameter being estimated, sampling methods etc., but all have:
  - two numbers giving lower and upper limits for interval
  - “confidence level” (95% above) giving chance that procedure gives interval containing parameter

Confidence interval for population mean

- By going up and down 2 times the right SD, got 95% confidence interval (more or less) because of 68-95-99.7.
- Can make confidence interval for any level of confidence. Lower confidence level (like 90%) gives shorter interval, but greater chance of interval *not* containing μ.
- General formula for confidence interval:

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

where choose z^* to get confidence level right:

Level	90%	95%	99%
z^*	1.645	1.960	2.576

Getting z^*

- 95% confidence interval based on *middle 95%* of normal distribution.
- Leaves $100\% - 95\% = 5\%$ for two ends.
- That is, $2.5\% = 0.025$ for each end.
- Look up 0.0250 in body of Table A: get $z = -1.96$. Ignore minus sign to get $z^* = 1.96$ for 95% CI.
- Same idea for 80% CI: 20% in the ends, $10\% = 0.1000$ in each end, $z = -1.28$, so $z^* = 1.28$.
- Can verify 90% and 99% z^* values also.

Example of confidence interval

- A large hospital wants to estimate the average length of time patients stay in the hospital. Sample 90 records, find mean stay is 4.63 days; SD of length of stay known from previous data to be 3.7 days.
- 95% confidence interval uses $z^* = 1.96$:

$$4.63 \pm 1.96 \times 3.7/\sqrt{90} = 4.63 \pm 0.76,$$

  from 3.87 to 5.39 days.
- 99% confidence interval uses $z^* = 2.576$:

$$4.63 \pm 2.576 \times 3.7/\sqrt{90} = 4.63 \pm 1.00,$$

  from 3.63 to 5.63 days.
- 99% confidence interval bigger than 95%, because have to be more confident in answer (only wrong in 1% of all possible samples).

Confidence interval for mean in Minitab

- Can use software for calculations. Data in worksheet `hospital`.
- Stat, Basic Statistics, 1-sample Z. in Variables, select column ("stay"), ensure Confidence Interval selected, put 95 in Level, put in 3.7 for Sigma:

```
The assumed sigma = 3.70

Variable  N  Mean StDev SE Mean    95.0 % CI
stay     90 4.633 3.785   0.390 ( 3.869, 5.398)
```

basically as before.

- Repeat for 99%: change Level to 99:

```
The assumed sigma = 3.70

Variable  N  Mean StDev SE Mean     99.0 % CI
stay     90 4.633 3.785   0.390  ( 3.629, 5.638)
```

Summary

- Know how far apart sample and population means might be, so can make guess at possible values for population mean based on sample.
- Meaning of eg. 90% confidence interval for pop. mean: in 90% of all possible samples, procedure will give interval containing population mean. (Confidence is in *procedure*).
- General formula for confidence interval for pop. mean, when σ known: $\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$ where choose z^* to get confidence level right.
- 95% CI more likely to contain pop. mean than 90% interval, but interval will be bigger (less precise).

How confidence intervals behave

- Size of confidence interval depends on *margin of error* $m = z^* \sigma / \sqrt{n}$. Want *small* margin of error – then pinned down parameter precisely.
- If margin of error too large:
  - use lower confidence level
  - reduce population SD σ
  - increase sample size n.
- Sometimes σ can be reduced by measuring more carefully, but usually only good way to reduce margin of error is larger sample.

Choosing sample size in advance

- Previously, used available data and accepted margin of error. But can plan study more carefully: decide on required margin of error, then figure out sample size needed.
- Take margin of error formula equation above:

$$m = \frac{z^* \sigma}{\sqrt{n}};$$

  rewrite in terms of sample size n.
- This gives

$$n = \left(\frac{z^* \sigma}{m} \right)^2$$

Sample size for hospital example

- Hospital example: in 95% interval, margin of error 0.76 for $n = 90$, using $\sigma = 3.7$. How many patient records needed to reduce margin of error to 0.5?
- Formula:

$$n = \left(\frac{1.96 \times 3.7}{0.5}\right)^2 = 210.4.$$

- To be safe, round up: need to sample 211 patient records to get this small a confidence interval.
- Required margin of error only a little smaller, but sample is over twice as big as original 90, so will cost much more.

Cautions with confidence intervals

- Works for *simple random sample*, not for stratified/multistage samples.
- Sample mean can be affected by *outliers*.
- Sample size large enough to “iron out” skewness etc.
- *Must know population SD* σ. (This dealt with in Chapter 7.)

Summary

- Margin of error can be made smaller by taking larger sample.
- Can choose sample size required for desired margin of error: $n = (z^* \sigma / m)^2$.
- This CI calculation only works for simple random sample with population SD σ known.

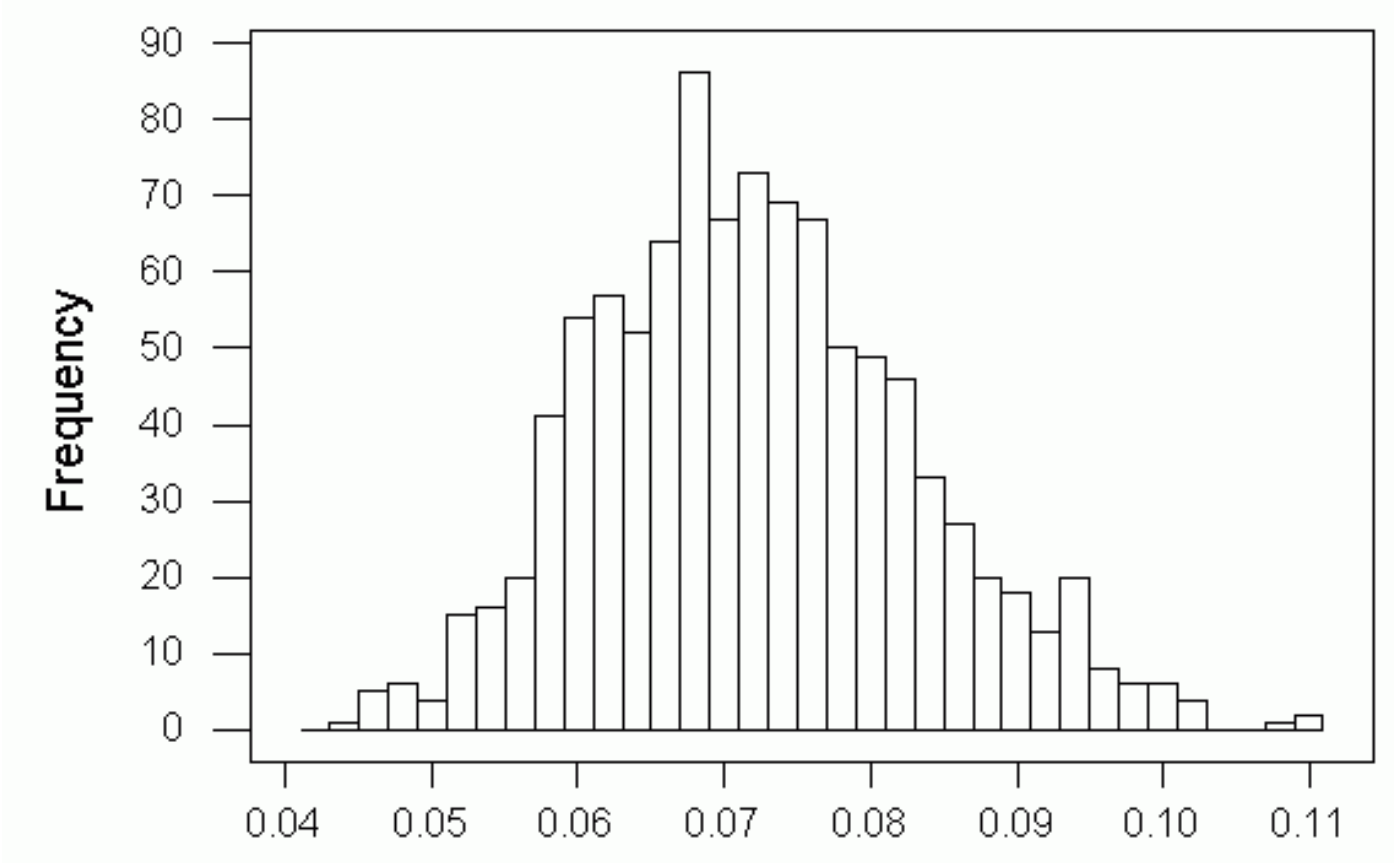
Confidence interval answering question “What values of population parameter plausible?”. Another question: “I have a value; is this value plausible?” Answer this by *test of significance*.

Tests of Significance (§6.2)

- Logic:
  - suppose unemployment rate is 7.2% (as in 2003)
  - take random sample of 500 people, get 23 unemployed (4.6%).
  - examine kinds of random samples with $n = 500$ and rate 7.2%
  - find that 4.6% unusually low in comparison
  - conclude unemployment rate changed (such sample rare if population rate 7.2%)
- Using sample to *draw conclusion* about population: *particular* population proportion, 7.2%, not plausible.
- This logic called **test of significance**.
- Might be wrong: is *possible* to get sample proportion 4.6% from population proportion 7.2%, but very unlikely. Always have to accept risk of being wrong.

Sampling distribution

Sampling distribution of sample proportion if rate 7.2% (simulation), for sample size $n = 500$:



Chance of observing 4.6% *or more extreme* very low.

Test procedure for population mean μ

- assume that a given value for μ correct, eg. $\mu = 10$. Called **null hypothesis**. Opposite, $\mu \neq 10$, called **alternative hypothesis**.
- Take random sample from population, calculate $\bar{x}$.
- Find chance of $\bar{x}$ value as extreme or more extreme, *if null hypothesis true*. Called **P-value**.
- If P-value small, **reject** null hypothesis in favour of alternative. If not, **do not reject** null hypothesis.

Analogy: court of law

Court of law	Test of significance
Accused innocent until proven guilty	Null hypothesis assumed true until proven wrong
Found guilty only with strong evidence	Reject only if P-value small
Innocent	Null hypothesis true
Guilty	Null hypothesis false
Found not guilty	Do not reject null hypothesis
Found guilty	Reject null hypothesis

Legal and statistical decision process

	Legal	Decision	
		Find not guilty	Find guilty
Truth	Innocent	Correct	Serious error
	Guilty	Error	Correct

	Testing	Decision	
		Not reject null	Reject null
Truth	Null hyp. true	Correct	Type I error
	Null hyp. false	Type II error	Correct

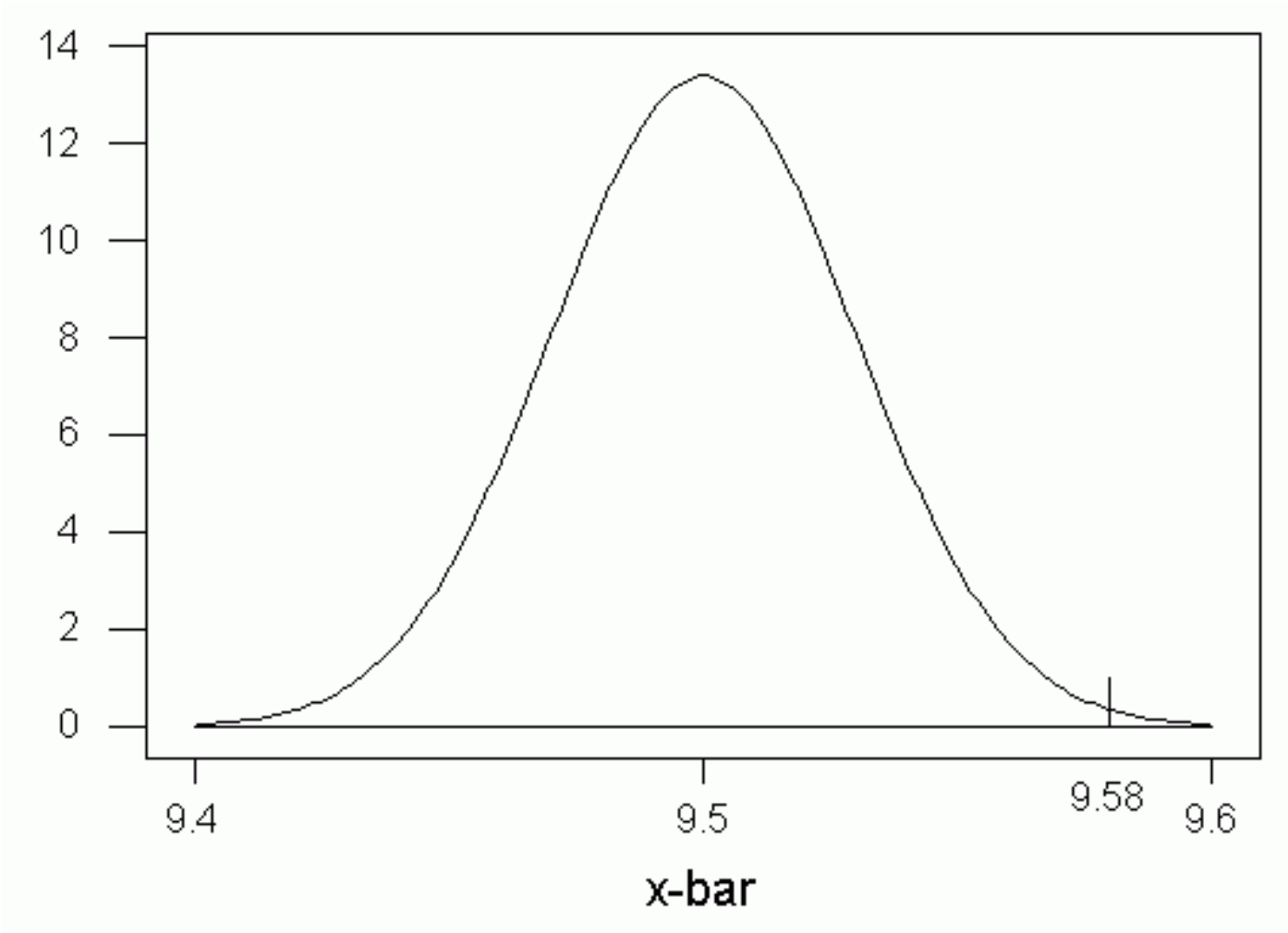
Stating hypotheses: example

- Calcium levels in blood of healthy young adults vary with mean $\mu = 9.5$ mg/dl and SD $\sigma = 0.4$ mg/dl. Clinic in rural Guatemala measured blood Ca levels of 180 pregnant women; sample mean $\bar{x} = 9.58$.
- Trying to prove, based on sample, that blood calcium levels higher for these women. Thing "trying to prove" is *alternative* hypothesis: in symbols, $H_a : \mu > 9.5$ for these women.
- This alternative hypothesis **one-sided**. $\mu \neq 9.5$ would have been **two-sided**.
- Other conclusion is that these women same as healthy young adults generally. "Same" is *null* hypothesis, $H_0 : \mu = 9.5$.

Sampling distribution

- Ask: if null hypothesis true, is sample mean of 9.58 typical or untypical?
- To answer, look at *sampling distribution* of sample mean. If null hypothesis true, sampling distribution has mean $\mu = 9.5$, SD $\sigma/\sqrt{n} = 0.4/\sqrt{180} = 0.0298$ and approx. normal shape (because of large sample).

Sampling distribution of sample mean if H_0 true



9.58 is a long way out.

P-values

- How far out is “far out”?
- Idea: find chance of being *as extreme or more extreme* in sampling distribution from null hypothesis. “Extreme” means “in the direction of the alternative hypothesis”.
- For a one-sided test, two steps:
  - check whether sample mean is correct side of H_0 mean (that is, if H_a has $>$, want $\bar{x}$ bigger; if H_a has $<$, want $\bar{x}$ smaller). If not, “wrong side”, don’t reject H_0.
  - If sample mean is correct side, find prob. of whichever direction H_a says.
- In our case, $H_a : \mu > 9.5$, and $\bar{x} = 9.58 > 9.5$, so we are on “correct side”. Find $P(\bar{x} > 9.58)$. Then $\sigma = 0.4, n = 180$ so $z = (9.58 - 9.5)/(0.4/\sqrt{180}) = 2.68$. Prob of $>$ is $1 - 0.9963 = 0.0037$, small.

P-values for two-sided tests

- Now suppose H_a had been $H_a : \mu \neq 9.5$. This *two-sided*: looking for *any* difference from null.
- For two-sided test, P-value found this way:
  - Compare $\bar{x}$ to H_0 value of mean.
  - If $>$, find prob. of $>$; if $<$, find prob. of $<$.
  - P-value is 2 $\times$ prob. just found ("$\times$ by number of sides").
- In example, $H_0 : \mu = 9.5$ and $\bar{x} = 9.58$, so $9.58 > 9.5$. Find z as before (2.68); find prob. of greater (0.0037); P-value is $2 \times 0.0037 = 0.0074$.

z here called **test statistic**: stepping stone to getting P-value.

Statistical significance

- Reject null hypothesis if P-value "small". But how small?
- Compare P-value to number α (alpha) chosen in advance. $\alpha = 0.05$ common choice. If P-value smaller, result called **statistically significant at level α**.
- "Significant" here means only "evidence against null hypothesis reaches stated level" – not "significant" in sense of "important".
- Statement of conclusion in example: Result statistically significant (P=0.0074). Or, choosing $\alpha = 0.05$ in advance, "the P-value is 0.0074 and H_0 can be rejected."
- Whichever way conclusion expressed, *state your P-value*. Enables reader to make own decision (if they disagree with your choice of α).

Choice of α

- Reflects strength of evidence against null hypothesis regarded as convincing. Case-by-case.
- Small α like 0.01 requires strong evidence. Hard to reject null hypothesis when true (good), but also hard to reject when false (bad).
- Large α like 0.10: Often reject null hypothesis when true (bad), but also often reject when false (good).
- No easy route to correct decision!

Another example

- The General Health Questionnaire (GHQ) measures mental health (low score better). In general population, SD is $\sigma = 5$. Researcher wants to show that mean GHQ for all unemployed men exceeds 10. Sample of 49 unemployed men, sample mean 10.94.
- Researcher wants $\mu > 10$, so is alternative hypothesis. Null hypothesis $\mu = 10$ (always "="). Choose $\alpha = 0.05$.
- One-sided test. Sample mean on correct side of 10: $10.94 > 10$ so alternative could be true. So work out P-value as prob. of $>$.
- Test statistic $z = (10.94 - 10)/(5/\sqrt{49}) = 1.32$.
- P-value is $1 - 0.9066 = 0.0934$.
- P-value not smaller than α, so cannot reject H_0. Result not statistically significant.

Interpreting result

- Sample mean this high in about 10% of samples if population mean really 10.
- Have *not* proved that population mean GHQ is 10 – have only shown that it could be 10 (but could be many other values as well).
- (Confidence interval for μ here: 9.54 to 12.34. Anything in here "plausible" value for μ.)

One and two-sided P-values (summary)

- If alternative has $\neq$ (2-sided):
  - if $z > 0$, P-value is twice prob. of greater
  - if $z < 0$, P-value is twice prob. of less
- If alternative has $<$ (1-sided):
  - if $z > 0$, P-value is large: do not reject null. “Wrong side”.
  - if $z < 0$, P-value is prob. of less.
- If alternative has $>$ (1-sided):
  - if $z > 0$, P-value is prob. of greater
  - if $z < 0$, P-value is large: do not reject null. “Wrong side”.

Tests of significance in Minitab

When have data, can get Minitab to do all the calculations. In worksheet ghq, 49 scores in column C1.
Stat, Basic Statistics, 1-sample z (again). Select C1, click Test Mean. Change 0.0 to 10, change Alternative to Greater Than. Put in value 5 for sigma. Click OK:

```
Test of mu = 10.000 vs mu > 10.000
The assumed sigma = 5.00

Variable     N      Mean    StDev   SE Mean        Z           P
ghq         49    10.939    5.113     0.714     1.31       0.095
```

Minitab's P-value 0.095, conclusion as before.

Summary

- Is given population parameter value plausible?
- What we try to prove called alternative hypothesis (accused is guilty).
- Status quo called null hypothesis (always contains $=$). (accused innocent)
- Calculate P-value: chance of result as or more extreme, if null hypothesis true. Small P-value, reject null hypothesis, otherwise don't reject.
- To decide "small", choose α before doing test. Reject null if P-value smaller than α. Often choose $\alpha = 0.05$.

CIs and tests of significance

- Confidence interval: *what values are plausible?*
- Significance test: *is this value plausible?*
- Related ideas. Suggests: pick null hypothesis value for μ. If this null hypothesis not rejected ("plausible"), value inside confidence interval. If null hypothesis rejected, value outside.
- True, provided: (a) doing 2-sided test, (b) match up α and confidence level. (Eg. for 95% interval, use $\alpha = 0.05$.)
- If test 1-sided, have to check for "correct side" *and* adjust α (double it).

Example

Weights (kg) of 24 male runners. In worksheet `runner_weight`. Use $\sigma = 4.5$ kg, $\alpha = 0.05$.

- Suppose null hypothesis is $\mu = 59$ (against $\mu \neq 59$):

```
Test of mu = 59.000 vs mu not = 59.000
The assumed sigma = 4.50

Variable     N      Mean    StDev   SE Mean         Z           P
weight      24    61.792    4.808     0.919      3.04      0.0024
```

  Null hypothesis rejected. 59 outside confidence interval.

- Now try null hyp. $\mu = 62$:

```
Test of mu = 62.000 vs mu not = 62.000
The assumed sigma = 4.50

Variable     N      Mean    StDev   SE Mean         Z           P
weight      24    61.792    4.808     0.919     -0.23        0.82
```

  Null hyp. not rejected. 62 inside confidence interval.

Example continued

- Finally, null hyp. $\mu = 64$:

```
Test of mu = 64.000 vs mu not = 64.000
The assumed sigma = 4.50

Variable     N      Mean    StDev   SE Mean         Z           P
weight      24    61.792    4.808     0.919     -2.40       0.016
```

  Null hyp. rejected, 64 outside interval.

- To check this, calculate 95% confidence interval (to go with $\alpha = 0.05$):

```
The assumed sigma = 4.50

Variable     N      Mean    StDev  SE Mean       95.0 % CI
weight      24    61.792    4.808    0.919  (  59.991,  63.592)
```

  As predicted, 62 inside, 59 and 64 outside.

- P-value for testing $\mu = 64$ vs. $\mu \neq 64$ was 0.016. If used $\alpha = 0.01$, would not reject. 64 should be *inside* 99% confidence interval, as it is (not shown).

Use and abuse of tests (§6.3)

Calculations for tests simple (with software), but wise use of tests difficult. Each test valid in certain circumstances with certain assumptions.

In testing a mean, as we have done:

- Works for *simple random sample*, not for stratified/multistage samples.
- Sample mean can be affected by *outliers*.
- Sample size large enough to “iron out” skewness etc.
- *Must know population SD* σ*.*

same as for confidence intervals.

Strength of evidence vs. decision-making

- Best way to give result of significance test is to give *P-value*. Shows strength of evidence against null hyp. Enables reader to judge whether evidence "strong enough".
- To use test to make decision, must choose α *before looking at data*. Choice depends on consequences of wrong decision. If rejecting null in favour of alternative expensive, need strong evidence to reject null (small α). Subjective, extra-statistical.
- $\alpha = 0.05$ often used, reasons mainly historical. No real difference between P-values 0.049 and 0.051.

Statistical and practical significance

- Example: test $H_0 : \mu = 20$ vs. $H_a : \mu \neq 20$ using sample $n = 10,000$, $\sigma = 0.2$. $\bar{x} = 20.005$: P-value 0.012. If $\alpha = 0.05$, *reject null hyp.* even though sample mean very close to 20.
- Difference between 20 and 20.005 very small, maybe of no practical relevance. P-value says only that with this big a sample, sample mean of 20.005 unusually high.
- *Statistical significance not same as practical significance.*
- Other way around too: with small sample, can get sample mean far from null hyp. but not statistically significant.

Don’t ignore lack of significance

- Tests with large P-values also important, even though not statistically significant.
- Typical science: researcher has new theory about some effect (alternative hyp.) tested against existing theory (null). Tries to gather evidence against null.
- Suppose now P-value not small. Could indicate theory wrong, or flaw in experiment. If theory plausible, result worth knowing about.
- However, tendency in scientific work only to publish statistically significant results.

Statistical inference not always valid

- Badly designed surveys/experiments give invalid results – cannot rescue by clever analysis.
- Need to use proper experimental design and appropriate analysis, with right kind of randomization.
- But also face data not from experiments. Eg. diameters of holes bored in engine blocks in car-making. Check whether reasonable to treat as independent observations from normal distribution using descriptives (graphs, numbers).
- In general, *learn how data produced*, assess whether test/interval meaningful.

Searching for significance

- Right way: decide on effect sought, design experiment to assess effect, significance test on results.
- But tempting: collect a bunch of data, do a bunch of tests, see what is significant.
- Example: industrial process with desired mean 50, SD 10. Take samples size 50 each day; test $\mu = 50$ vs. $\mu \neq 50$.

Results

```
Test of mu = 50.00 vs mu not = 50.00
The assumed sigma = 10.0

Variable     N      Mean    StDev   SE Mean         Z           P
C1          50     49.69    11.35      1.41     -0.22        0.83
C2          50     52.21     8.39      1.41      1.56        0.12
C3          50     51.39     8.71      1.41      0.99        0.32
C4          50     50.49     9.61      1.41      0.35        0.73
C5          50     49.43     9.50      1.41     -0.40        0.69
C6          50     49.02    10.91      1.41     -0.69        0.49
C7          50     52.55     8.64      1.41      1.80       0.072
C8          50     52.88     8.30      1.41      2.04       0.042
C9          50     48.25    10.20      1.41     -1.24        0.22
C10         50     49.55     7.81      1.41     -0.32        0.75
```

- Result on 8th day (C8) significant at $\alpha = 0.05$. What happened?
- Answer: nothing. Data randomly generated from normal distribution mean 50, SD 10. Result only of doing many tests.

Summary

- Value for parameter inside confidence interval — value plausible — null hyp. of that value wouldn't be rejected. True as long as test 2-sided and α, confidence level match up.
- Tests come with assumptions. Here: simple random sample, mean appropriate for "centre", sample size large enough, must know σ.
- P-value measures strength of evidence (smaller $=$ stronger). Use for decision-making by choosing α first and rejecting null if P-value $< \alpha$.
- Statistical significance not same as practical significance.
- Lack of (statistical) significance may mean plausible theory incorrect.
- Statistical inference depends on correct design, randomization, analysis. Can sometimes treat data "as if" random sample.
- Doing many tests will probably produce some significant results by chance.

Power (§6.4)

- Two types of error in significance testing:
  - Type I: rejecting null hyp. when true
  - Type II: not rejecting null hyp. when false
- Concentrated mostly on Type I (by choosing α), but type II also important. Want to design experiment that has some chance of detecting desired effect – low chance of type II error, or high **power** — a high chance to *correctly* reject the null.
- Can sometimes figure power by calculation, but usually easier to simulate or use software (later).

Power example

- Consider text example *** check *** 6.28 (p. 431). Researchers testing whether exercise program increases total bone mineral content (TBBMC) in young women over 6-month period. Based on previous data, prepared to assume population SD is 2. Intend to use $n = 25$ subjects.
- How likely are they to detect change of 1 unit in TBBMC? That is, if actually $\mu = 1$ and go through test procedure, how often would they reject null with P-value < 0.05?

Power by simulation

- Simulate data from *true* population (mean 1, SD 2). Carry out test for each simulated sample, see how many times null rejected.
- Minitab: do 50 simulated tests: Calc, Random Data, Normal. Generate 25 rows of data, store in `C1-C50` (type into box). Mean 1, SD 2. Click OK. Then Stat, Basic Statistics, 1-sample Z. Variables C1–C50, test mean 0, alternative "greater than", sigma 2.

Power example results

```
Test of mu = 0.000 vs mu > 0.000
The assumed sigma = 2.00

Variable     N      Mean    StDev   SE Mean         Z           P
C1          25     1.236    1.671     0.400      3.09      0.0010
C2          25     0.675    2.057     0.400      1.69       0.046
C3          25     0.319    2.449     0.400      0.80        0.21
C4          25     0.965    2.469     0.400      2.41      0.0080
C5          25     1.123    1.982     0.400      2.81      0.0025
C6          25     0.743    1.749     0.400      1.86       0.032
C7          25     0.781    1.825     0.400      1.95       0.026
C8          25     0.311    1.747     0.400      0.78        0.22
...
```

6 rejections in 8 tests shown (didn't reject for C3 and C8); got 39 rejections in 50 tests, estimate power $39/50 = 0.78$.

Power by Minitab

Minitab does power directly: Stat, Power and Sample Size, 1-sample Z. Select “Calculate power for each sample size”, fill in sample size 25. “Difference” is difference between null hyp. and true means, here 1 (put in). Put in 2 for sigma (bottom). Click Options, select Greater Than for alternative. Click OK:

```
1-Sample Z Test

Testing mean = null (versus > null)
Calculating power for mean = null + 1
Alpha = 0.05  Sigma = 2

Sample
  Size   Power
    25  0.8038
```

Difference detectable in 80% of all possible samples.

(Simulation close.) Researchers have good chance of success.

Determining sample size

- At beginning of study, need to know how many subjects/experimental units to use. Common way of deciding: choose power for alternative of interest, then use software to find sample size making this work.
- Suppose researchers of previous example actually wanted power 0.90 to detect an increase of 1 unit in TBBMC. Would need bigger sample than 25, but how big? Minitab: Stat, Power and Sample Size, 1-sample Z as before. Select “Sample size for each power”, fill in 0.90 for power; 1 for difference, 2 for sigma (as before). In Options, make sure Greater Than selected for alternative.

Sample size results

Results:

```
1-Sample Z Test

Testing mean = null (versus > null)
Calculating power for mean = null + 1
Alpha = 0.05  Sigma = 2

Sample   Target  Actual
  Size    Power   Power
    35   0.9000  0.9054
```

35 subjects are needed. Actually, 35 subjects gives power > 0.90, but 34 subjects gives power < 0.90, so 35 safe.

Summary

- Power is chance of rejecting null hypothesis when it is false. Want power to be high.
- Study power by deciding what change is important to detect, and then finding chance of detecting that change.
- Can simulate: assume important change has happened (change in μ), simulate from that population, do original test, see how often original null rejected.
- Can calculate power directly with Minitab.
- Before study, can determine sample size needed to obtain a specified power against a specified alternative.

Chapter 7: Inference for Distributions

Inference for Population Mean (§7.1)

- In Chapter 6, did tests and confidence intervals for population mean μ, but needed to know population SD σ – not realistic!
- Can always calculate *sample* SD. Use symbol s.
- Obvious remedy: use sample SD in place of population SD.
- Will z procedures still work?
- Investigate confidence intervals by simulation.

Simulation in Minitab

- Minitab: generate 100 random samples of size $n = 5$ *in rows*. Calc, Random Data, Normal. Generate 100 rows, store in C1-C5, mean 10, SD 3.
- Calculate sample means: Calc, Row Statistics. Click Mean under Statistic, select columns C1-C5, store results in C7. Do sample SDs same way, selecting Standard Deviation and storing in C8.

Simulation continued

- Now calculate confidence intervals “by hand”. First get “margin of error” for 95% interval as $1.96 \times s/\sqrt{5}$: Calc, Calculator. Store result in C9, in Expression box type `1.96 * C8 / sqrt(5)` (or find “square root” in the list of functions). Click OK.
- Ends of confidence interval for each sample are mean minus margin of error and mean plus margin of error, ie. `C7-C9` and `C7+C9`. Use Calculator again, store in C11, C12.
- My results (selected):

Simulation results

Row	x-bar	s	+/-	C10	C11	C12
1	10.1868	2.23416	1.95833		8.2285	12.1451
2	10.1366	1.47570	1.29351		8.8430	11.4301
3	8.9175	1.49579	1.31112		7.6064	10.2286
12	7.1230	2.77318	2.43080		4.6922	9.5538
13	10.1301	2.28851	2.00597		8.1242	12.1361
31	12.7054	2.02251	1.77281		10.9326	14.4782
32	11.0099	2.66603	2.33688		8.6730	13.3468

- Population mean 10, so confidence intervals should contain 10. Those from rows 1,2,3,13,32 do, those from 12 and 31 don't.
- In my simulation, only 89 of 100 simulated intervals contain 10. Supposed to be 95% interval, but only about 89%.

Fixing it up

- Simulations showed that using Chapter 6 procedure with s not σ gave intervals that are *too short*.
- Previously got margin of error using 1.96 for 95% interval, 1.645 for 90% interval.
- Idea: make these numbers bigger to account for lack of knowledge of σ.
- Large sample: sample SD s probably close to σ. Loss of information small, adjustment small.
- Small sample: sample SD s might be far from σ. Loss of information large, adjustment large.

The t-distributions

- So for 95% confidence interval, margin of error calculated with something other than 1.96, but exact number depends on sample size.
- Correct number comes from **t-distribution**. Distribution different for each sample size; traditionally labelled by **degrees of freedom (df)**, which is $n - 1$.
- Get number from *table*, Table D:

t-table

TABLE D *t* distribution critical values

	Upper tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
z^*	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291

Using t-table to get t^*

- For sample size $n = 5$, df $5 - 1 = 4$. Find 95% along bottom of table, look up to 4 df, get 2.776. Much bigger than 1.96.
- Now, try $n = 51$. 50 df row, 95% column, 2.009. Much nearer 1.96.
- With a *very* large sample, t^* is very close to 1.96, eg. with 1000 df.
- Do other CIs same way: eg. 99% for $n = 10$, look in 99% column, $10 - 1 = 9$ df: $t^* = 3.250$.

Example

- When people buy bicycles, they often buy other accessories too (helmet, water bottle, etc.) A store took a random sample of 12 bike-buying customers, and found accessory purchases to be (in $) 38, 65, 82, 114, 77, 19, 142, 93, 63, 107, 58, 76.
- Sample mean $\bar{x} = 77.83$, sample SD $s = 33.51$. $n = 12$, so 11 df. For 95% confidence interval, $t^* = 2.201$. So interval is

$$77.83 \pm 2.201 \frac{33.51}{\sqrt{12}},$$

from $56.50 to $99.12.

t confidence intervals in Minitab

With data, can use Minitab. I entered the 12 numbers into column C1. Then: Stat, Basic Statistics, 1-sample t. Select column C1, ensure Confidence Interval checked and 95.0 in box. Click OK. (Don't need to give "sigma".) Results:

```
T Confidence Intervals

Variable     N      Mean    StDev  SE Mean        95.0 % CI
C1          12     77.83    33.51     9.67  (    56.54,    99.13)
```

a more accurate version of our calculation.

Summary

- When σ not known, using procedure of Chapter 6 with s in place of σ gives confidence intervals that are too short.
- Adjustment depends on sample size. Use t distribution with df $n-1$.
- Confidence interval for μ now

$$\bar{x} \pm t^* \frac{s}{\sqrt{n}},$$

where t^* from table.

Significance tests based on t distribution

- To continue bike-accessories example: at similar bike shop, mean accessory sales are \$90 per bike sold. Is there evidence that accessory sales at this shop less?
- Test of significance. Population mean sales μ. Alternative $H_a : \mu < 90$, null $H_0 : \mu = 90$.
- Cannot do using previous test procedure because *don't know* σ. So calculate test statistic t using sample SD, and get P-value from Table D.

Bike-shop example

- $H_0 : \mu = 90$, $H_a : \mu < 90$.
- Here, $\bar{x} = 77.83$ (correct side), $s = 33.51$. So:

$$t = \frac{77.83 - 90}{(33.51/\sqrt{12})} = -1.26.$$

  P-value is prob. of being below -1.26. But table only gives *positive* values of t. t distribution is symmetric, like normal, so P-value also prob. of being *above* 1.26.
- Go to table with $12 - 1 = 11$ df. Look along 11 df row, and find where 1.26 is – between 1.088 and 1.363. Look to very top row: P-value between 0.10 and 0.15.
- Approximate, but accurate enough: not small, so do not reject null. The population mean could be 90.

1-sided and 2-sided P-values

As for z, but slightly different (because Table D different):

- If alternative has $\neq$ (2-sided):
  - if $t < 0$, ignore minus sign
  - then P-value is twice what you get from Table D.
- If alternative has $<$ (1-sided):
  - if $t > 0$, P-value is large: do not reject null. “Wrong side”.
  - if $t < 0$, ignore minus sign, get P-value from Table D.
- If alternative has $>$ (1-sided):
  - if $t > 0$, get P-value from Table D.
  - if $t < 0$, P-value is large: do not reject null. “Wrong side”.

t-test in Minitab

Minitab: data in column C1. Stat, Basic Statistics, 1-sample t (as for interval). Click Test Mean, fill in 90. Change Alternative to “less than”. Click OK:

```
T-Test of the Mean

Test of mu = 90.00 vs mu < 90.00

Variable     N      Mean    StDev   SE Mean        T          P
acc_sold    12     77.83    33.51      9.67    -1.26       0.12
```

P-value 0.12 not small, cannot reject null hypothesis. No evidence that mean sales less than $90. (Note that 0.12 consistent with “between 0.10 and 0.15” from before.)

Matched pairs

- When designing experiment, *comparative* usually better (treatment vs. control etc.) – protects against other variables affecting result.
- Two ways to compare:
  - **Matching**: pair each subject with another similar subject. One subject gets treatment, other control. Alternative: get "before", "after" measurements from each subject.
  - **Randomizing**: divide subjects into 2 large groups at random. One group gets treatment, other control.
- Think about matched pairs next.
- Think about two separate groups in §7.2.

Matched pairs example

- Measurements come in pairs: each “before” with particular “after”, each “treatment” with particular “control”.
- Example: “Apnea” is brief stoppage of breathing during sleep. 13 premature infants given drug; measure “apneic episodes per hour” before and after drug given. Data like this:

Row	before	after
1	1.71	0.13
2	1.25	0.88
10	0.67	0.75
13	1.96	1.13

- All infants except #10 had fewer episodes after.

Example continued

- Idea: *differences*, after minus before. Write μ for population mean difference. Trying to prove drug *reduces* apnea, so alternative hyp. $\mu < 0$, null $\mu = 0$.
- Test just 1-sample t-test on differences. Get 90% confidence interval too.
- Minitab: Stat, Basic Statistics, Paired t. Want after minus before, so `after` is 1st sample, `before` 2nd (illogical!) Click Options, change Confidence Level to 90, Alternative to "less than".

Example continued

```
Paired T for after - before

                 N      Mean     StDev   SE Mean
after           13     0.984     0.833     0.231
before          13     1.751     0.855     0.237
Difference      13    -0.767     0.524     0.145

90% CI for mean difference: (-1.026, -0.508)
T-Test of mean difference = 0 (vs < 0): T-Value = -5.28
P-Value = 0.000
```

- P-value “0.000” very small, so reject null hypothesis; real decrease in apnea. Same infants before and after, so decrease due to drug.
- With 90% confidence, drug reduces apnea by between 0.5 and 1 episodes per hour. (Gives idea of *size* of drug effect.)

Robustness of t procedures

- Mathematics of t procedures assumes that population has normal-distribution shape (and therefore that mean, SD measure centre, spread).
- But in practice, shape must be badly non-normal to cause problems, particularly if sample large. More important to have simple random sample. Guidelines:
  - $n < 15$: don't use t if clearly non-normal shape or outliers.
  - $15 \leq n < 40$: beware only of outliers or strong skewness.
  - $n > 40$: unlikely to be any trouble.

Assessing appropriateness of t

- Assess using pictures of data (histogram, normal quantile plot).
- For matched pairs, only *differences* matter; original variables can have any shape.
- If an inference procedure doesn't depend much on its assumptions (as here), called **robust**.

What to do if you can't use the t procedures?

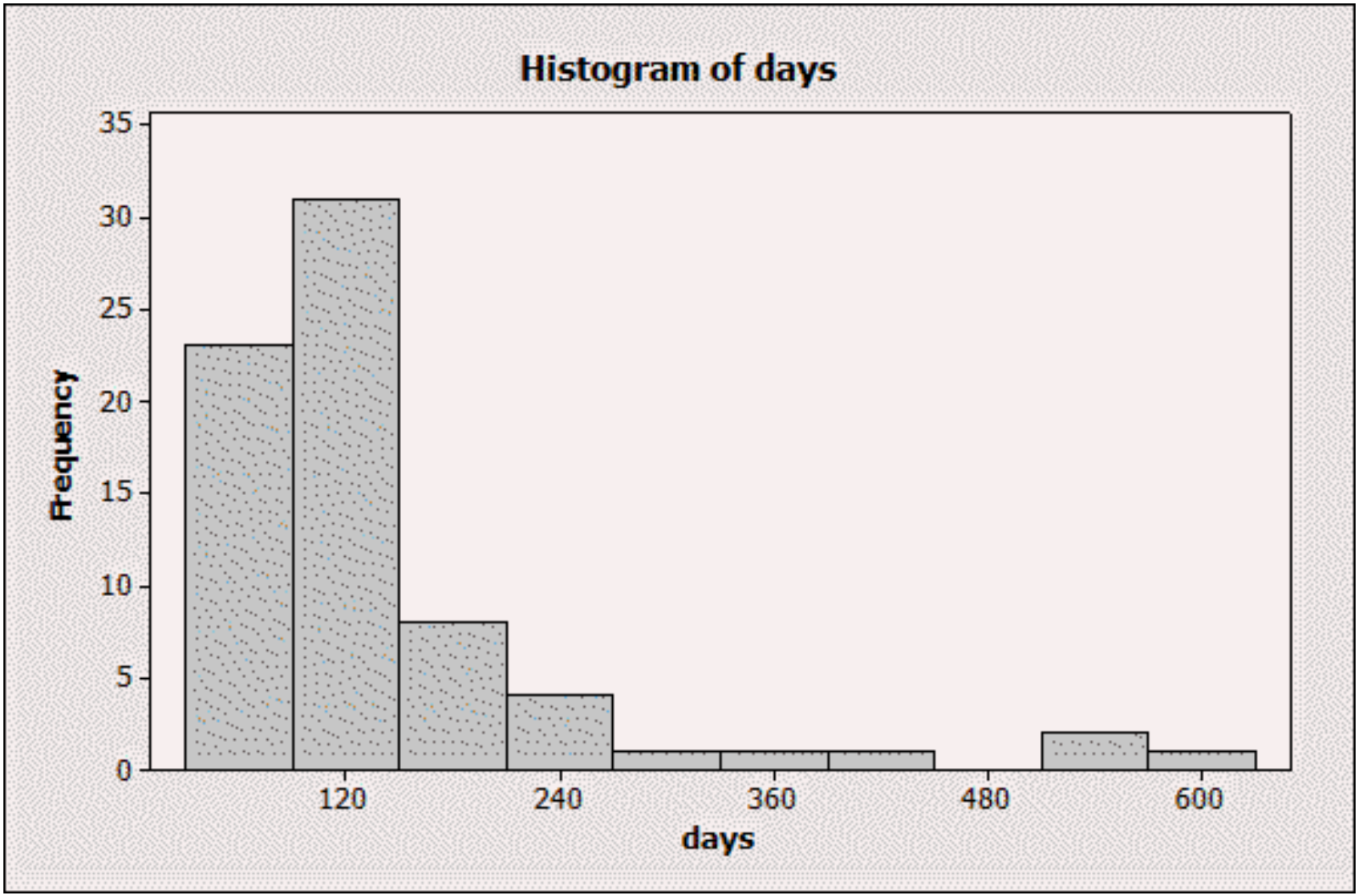
Two main approaches to choose from:

- *transform* the data so that the distribution is more nearly normal. In Example 7.10 in text (p. 436), analysis is based on logarithms of data values because data very right-skewed.
- *use a different test*, such as those in Chapter 15, which do not use the distribution shape at all (but not such good tests if the distribution shape is close to normal).

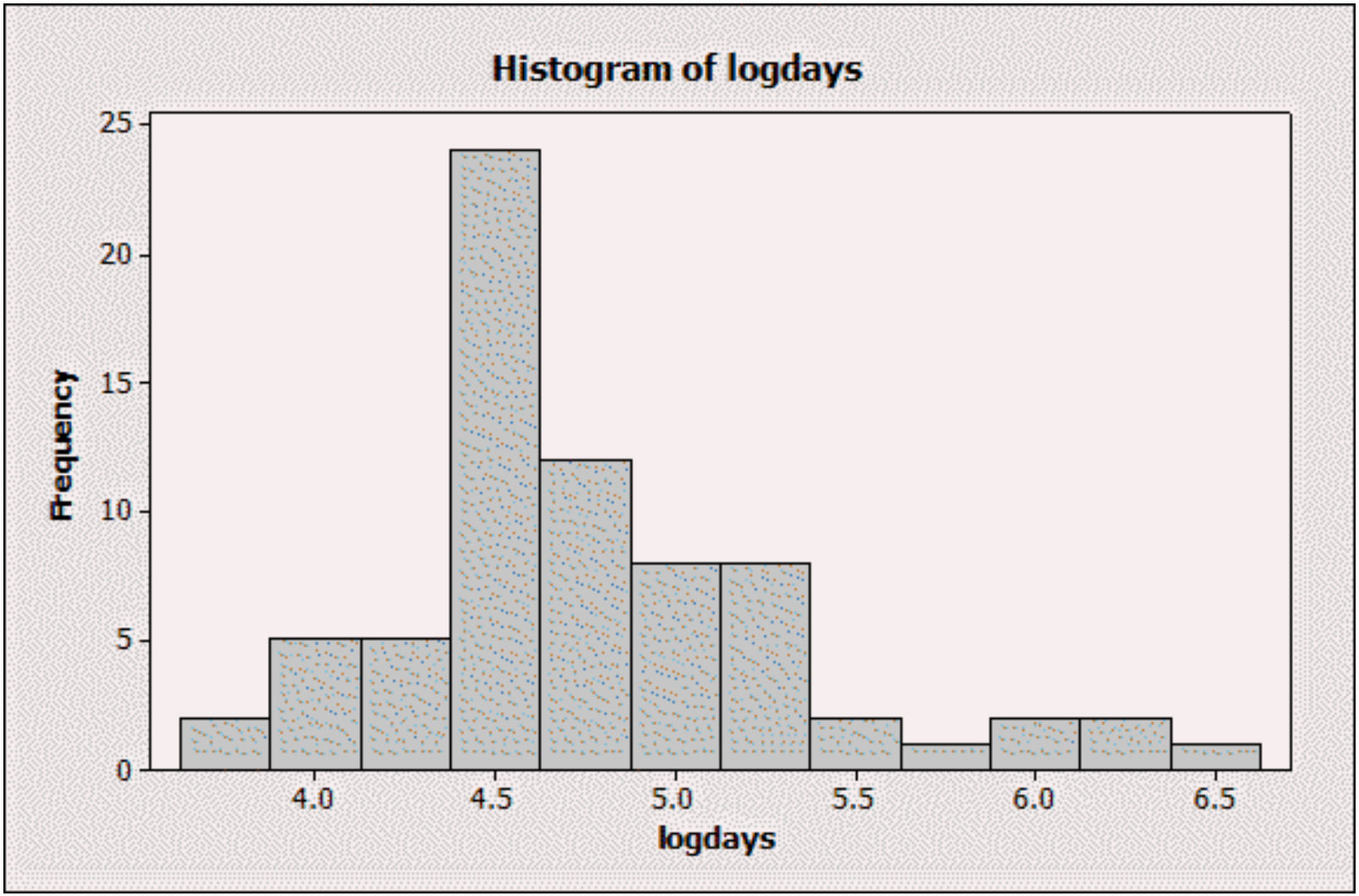
Guinea pig survival example

- Text table 1.8 has survival times for guinea pigs in medical experiment.
- Very right-skewed (some very big values), so have doubts about t procedures.
- Take logarithms: Calc, Calculator, type name (`logdays`) in box for result, calculation is `log(c1)`.
- Still right-skewed, but not nearly as bad. Sample size $n = 72$ should overcome this skewness. See pictures next 2 pages.

Guinea-pig survival times



Guinea-pig survival times, logs



Guinea pigs continued

- Get Minitab to find confidence interval for mean of log data. Stat, Basic Statistics, 1-sample t. Click Samples in Columns, select `logdays`. Output like this:

```
One-Sample T: logdays

Variable   N     Mean    StDev  SE Mean         95% CI
logdays   72  4.77107  0.55956  0.06595  (4.63958, 4.90256)
```

- This is confidence interval for log-mean, so undo log. Use e^x button on calculator, or type into empty column and use `exp` in Calculator to get 95% CI of 103.5 to 134.6 days.
- Compare less reliable CI of 116.1 to 167.5 from original data (sample mean affected by outliers).

Power and sample size for t tests

- How likely is t-test to reject null hyp. when it is false?
- Mathematics complicated (needs to account for replacing σ by s), but can use simulation, or Minitab.
- Return to apnea example. With sample of 13 infants, how likely is detection of decrease of 0.25 episodes/hour?
- Minitab: Power and sample size, Calculate power for each sample size. Fill in sample size 13, difference -0.25 (lower better). Need value for σ; use our sample SD 0.524. Click Options, select alternative “less than”.

Power example

Results:

```
1-Sample t Test

Testing mean = null (versus < null)
Calculating power for mean = null - 0.25
Alpha = 0.05  Sigma = 0.524

Sample
  Size   Power
    13  0.4910
```

Only about 50-50 chance to detect this size difference.

Sample size

- How big a sample size would be needed to make this power 0.8?
- Power & sample size, 1-sample t as before. Click Calculate Sample Size. Fill in power 0.80, difference -0.25. Check sigma correct, check alternative (via Options):

```
Testing mean = null (versus < null)
Calculating power for mean = null - 0.25
Alpha = 0.05  Sigma = 0.524

Sample   Target  Actual
  Size    Power   Power
    29   0.8000  0.8055
```

  Need 29 infants to have this chance of detecting a 0.25 decrease.

Summary

- Significance test when σ unknown: replace σ by s, use t distribution to get P-value. In practice, use software.
- With before-after measurements on same individual (measurements on paired individuals), matched-pair analysis: t procedures on differences (null: mean difference is 0).
- In practice, with reasonably large sample(s), can use t procedures with most data (even though mathematics assumes data normal). Otherwise: transformation, sign test.
- When null hypothesis known to be false, can find chance of (correctly) rejecting it: power. Can also find sample size needed to obtain desired power against given alternative.

Comparing two means (§7.2)

- Earlier: best experiments *comparative*. Compare by matching or by randomizing.
- Example: teaching reading. Compare new method to standard method. Can't teach a child by both methods, so: take random sample of 20 children, choose at random 8 children for new method (other 12 get standard method).
- Children taught by qualified people for 6 months, given reading test at end. Record test score.
- Two *separate* samples, no pairing. Needs different analysis.

Two means: notation

- Two populations. (Reading example: “all children taught by new method”, “all children taught by standard method”.)
- Population 1: mean μ_1, SD σ_1. Population 2: mean μ_2, SD σ_2.
- Don’t know any of these. Don’t care about actual values of μ_1, μ_2 – care about how they *compare*. So think about $\mu_1 - \mu_2$.
- Reading example: 1 is “new”, 2 is “standard”. $\mu_1 = \mu_2$ or $\mu_1 - \mu_2 = 0$ means two methods equally good, $\mu_1 > \mu_2$ or $\mu_1 - \mu_2 > 0$ means new better.

Notation continued

- Collect samples from each population. Population 1: sample size n_1, sample mean $\bar{x}_1$, sample SD s_1. Population 2: sample size n_2, sample mean $\bar{x}_2$, sample SD s_2.
  item Sample sizes can be same, but don't have to be (8 and 12 in reading example).
- Don't know population SDs σ_1, σ_2, so use sample SDs s_1, s_2 to estimate them, use t-distribution.

The formulas

- To see why: p. 450–451 of the text. Actually t-distribution only approx. correct, but usually good enough.
- Calculate

$$d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}.$$

- Base df on smaller sample size.
- Test statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{d}$$

and confidence interval is

$$\bar{x}_1 - \bar{x}_2 \pm t^* d.$$

Reading example revisited

- Want to detect any differences between methods (new method might be worse), so $H_a : \mu_1 \neq \mu_2$. Null $H_0 : \mu_1 = \mu_2$.
- Data, labelling 1 as "new": $n_1 = 8, \bar{x}_1 = 77.13, s_1 = 4.85$. $n_2 = 12, \bar{x}_2 = 72.33, s_2 = 6.34$.
- Thus $d = \sqrt{4.85^2/8 + 6.34^2/12} = 2.508$. Test statistic $t = (77.13 - 72.33)/2.508 = 1.91$. Using $8 - 1 = 7$ df, P-value between 0.05 and 0.10 (2-sided; double values in Table D). Cannot quite reject null: not quite evidence that new reading method makes difference.
- 95% confidence interval for difference in means, using $t^* = 2.365$:

$$77.13 - 72.33 \pm (2.365)(2.508),$$

from -1.13 to 10.73.

Reading example: Minitab

- Minitab: enter “new” scores in one column, “standard” in another (worksheet `reading`).
- Stat, Basic Statistics, 2-sample t. Click “Samples in different columns”. Select “new” as 1st column, “standard” as 2nd. Ensure alternative “not equal”.

```
Two sample T for new vs standard

            N      Mean     StDev   SE Mean
new         8     77.13      4.85       1.7
standard   12     72.33      6.34       1.8

95% CI for mu new - mu standard: ( -0.5,  10.1)
T-Test mu new = mu standard (vs not =):
T = 1.91  P = 0.073  DF = 17
```

- Note different df, from formula page 460. Hence smaller P-value, shorter CI than hand calculation.

Conclusions

- P-value 0.073, no evidence of difference as before.
- Confidence interval from –0.5 to 10.1, includes 0. “No difference” plausible, though suggests positive effect of new method.
- Minitab gives both test and interval; choose as necessary.
- Sample sizes small in example (8 and 12); might not have much *power* to detect useful difference. (Sample means 77.13 and 72.33 about 5 marks apart – raising reading scores on average by 5 marks might be important.)

Robustness of two-sample procedures

- Generally as for 1-sample t, only replacing "sample size" with sum $n_1 + n_2$.
- Better to use equal sample sizes when possible.
- Overall conclusion: non-normality/outliers not a problem in large sample sizes; worry mainly about outliers in moderate-sized samples.
- Difficult to assess normality anyway in small samples.

Power and sample size for two-sample t tests

- Will we be able to prove that new reading method has effect? In general, how likely is rejection of null hyp. when it is false? This is *power*.
- Minitab's power calculation assumes equal sample sizes and equal population SDs. To be safe, use *smaller* sample size, *larger* SD.
- Reading example: sample sizes 8 and 12 (use 8), sample SDs 4.85, 6.34 (so use 6.34 as guess of population SD). Suppose that increase of 5 marks meaningful.

Power in Minitab

- Stat, Power and Sample Size, 2-sample t. Select “Power for each sample size”, type in sample size 8. Enter difference 5 (marks). Enter 6.34 for “sigma”. Click Options, ensure Alternative is “not equal”, alpha 0.05.

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + 5
Alpha = 0.05  Sigma = 6.34

Sample
  Size   Power
     8  0.3123
```

- Little chance of detecting this difference.

Sample size

- How many children should be in each sample to get this power up to 70% (0.70)?
- Minitab: Stat, Power and Sample Size, 2-sample t. Select “Calculate sample size for each power value”, fill in 0.70 for power, 5 for difference. Ensure “sigma” still 6.34, Options correct:

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + 5
Alpha = 0.05  Sigma = 6.34

Sample   Target  Actual
  Size    Power   Power
    21   0.7000  0.7033
```

- Need 21 children *in each group* to get this much power.

Summary

- When data not paired, use two-sample t-procedures to compare two means.
- Hypotheses based on difference between population means.
- Sample sizes don't have to be the same.
- These procedures generally robust (provided $n_1 + n_2$ large enough).
- For power, use smaller n, larger s. For sample size, use larger s. Answer gives number of individuals in each group.

Chapter 8: Inference for Proportions

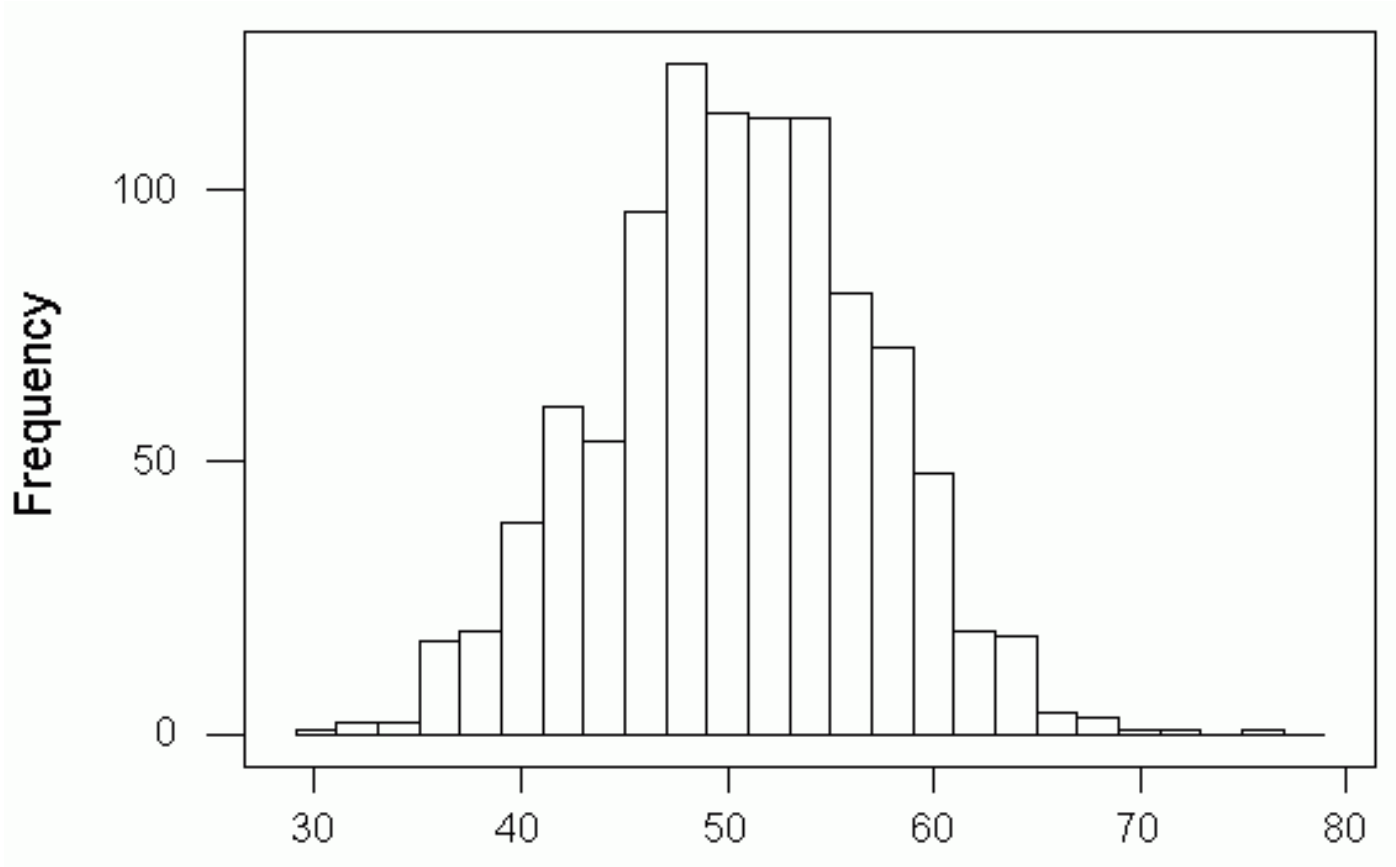
Inference for a single proportion (§8.1)

- Suppose that a new snack food is being tried out. 500 students sampled at random, and 71 of them said they would definitely buy it.
- Idea: “success” (definitely buy), “failure” (wouldn’t). Count successes in sample (71) or proportion ($71/500 = 0.142$).
- What can we say about proportion of *all* students who would buy the new snack? (Confidence interval.)
- Company estimates new snack profitable if more than 10% of all students buy it – evidence for this? (Significance test.)

Significance tests for proportions

- Back in chapter 5, studied sampling distribution of proportions: if we know the population proportion, can work out what kinds of sample proportions we might get.
- Under usual assumptions for sampling, **binomial distribution** describes number of successes, with success prob. or population proportion p.
- Example: is there evidence that more than 10% of all students would buy? Trying to prove this: alternative hypothesis $H_a : p > 0.10$. Null hyp. $H_0 : p = 0.10$.
- Now, *if null hyp. true*, know population proportion p: 0.10. So number of people in sample who will buy has binomial distribution with $n = 500$ and $p = 0.10$. Is observed value 71 (of 500) consistent?
- Simulate: Calc, Random Data, Binomial. Generate (say) 1000 rows of data, store in C1. Number of Trials 500, Probability of Success 0.10.

Simulation for $n = 500, p = 0.10$



- If p really 0.10, could observe 71 successes, but very unlikely.
- So conclude that we observed 71 successes because $p > 0.10$: reject null hypothesis in favour of alternative.

Test of significance in Minitab

- Mathematics: can calculate P-values exactly (binomial distribution). Used by Minitab.
- Stat, Basic Statistics, 1 proportion. Click Summarized Data, fill in number of trials (500), successes (71). Click Options. Test Proportion 0.10, Alternative “greater than”:

```
Test of p = 0.1 vs p > 0.1

                                                          Exact
Sample   X   N  Sample p         95.0 % CI            P-Value
1       71 500  0.142000  (0.112597, 0.175711)          0.002
```

- P-value is very small (0.002), so reject H_0 without question. Can be almost certain that more than 10% of students will buy.

Confidence interval for proportion

- Continuing example: Minitab gave 95% confidence interval of 0.1126 to 0.1757. So, from information in sample, believe that between 11.3% and 17.6% of all students will buy this snack food.
- Where did this come from?
- Recall connection between confidence interval and 2-sided test. Value *inside* interval "plausible" – would not reject that null hyp. But value *outside* not plausible, would reject.
- If we test $H_0 : p = 0.113$ vs. $H_a : p \neq 0.113$ on same data, get P-value 0.056, so do not reject H_0; 0.113 *inside* CI.
- Test $H_0 : p = 0.176$ vs. $H_a : p \neq 0.176$: P-value 0.046. Reject $H_0 : p = 0.176$. 0.176 *outside* CI.
- Consistent with Minitab's CI.
- Minitab finds 95% CI by finding two values of p with P-value exactly 0.05, but hard to do by hand!
- On next page, see how to do by hand.

Test and CI without Minitab

Recall some stuff from §5.1:

- *number of successes* in simple random sample has binomial distribution.
- *sample proportion of successes* has mean p and SD $\sqrt{p(1-p)/n}$.
- If sample large, number and proportion of successes have approx. normal distribution.

Use normal approximation to make our test and CI.

Test of significance

- Sampling distribution of sample proportion $\hat{p}$ has mean p and SD $\sqrt{p(1-p)/n}$.
- As before, testing null $p = 0.10$ against alternative $p > 0.10$ with data of 71 successes in 500 trials, so $\hat{p} = 71/500 = 0.142$.
- Use *null hypothesis* value of p to calculate test statistic:

$$z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} = \frac{0.142 - 0.1}{\sqrt{(0.10)(0.90)/500}} = 3.13$$

  with 1-sided P-value (Table A) of $1 - 0.9991 = 0.0009$. Compare approx. 0.0009 and exact 0.002 from Minitab.
- Conclusion (either way): reject null and conclude that $p > 0.10$.

Confidence interval

- Problem: in $\sqrt{p(1-p)/n}$, have no value for p. So use best value we have, namely $\hat{p}$. Hence formula:

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

- In example, 95% CI is

$$0.142 \pm 1.96 \sqrt{\frac{(0.142)(0.858)}{500}} = 0.142 \pm 0.031,$$

  or from 0.111 to 0.173. This is approx.; compare exact 0.113 to 0.176.
- Usually have a large sample, so usually, as here, approximation very good.

Power and sample size for proportions

- As always with test of significance, want to know whether interesting alternative has some chance of being detected with current sample size (power), or what sample size needed to be able to detect alternative of interest.
- To recycle snack food example: if in fact 12% of students would buy product, how likely are we to reject null hyp. $p = 0.10$ in favour of alternative $p > 0.10$ with sample of 500 students?

Calculating power for test of proportion

Minitab: Stat, Power and Sample Size, 1 proportion. Fill in sample size 500, alternative p 0.12 (this is the "true" 12%). Leave Power blank (this will be calculated). Put in null hypothesis p, 0.10, at bottom. Click Options, select Greater Than.

```
Testing proportion = 0.1 (versus > 0.1)
Calculating power for proportion = 0.12
Alpha = 0.05  Difference = 0.02

Sample
  Size   Power
   500  0.4434
```

Only sometimes detect this difference.

Sample size for test of proportion

- How many students should we sample to have power 0.60 when in fact 12% of students will buy the product?
- Minitab: Stat, Power and Sample Size, 1 proportion. Fill in power 0.60, alternative value $p = 0.12$, ensure Hypothesized Value still 0.10. Leave Sample Size blank. Click Options, ensure still Greater Than:

```
Testing proportion = 0.1 (versus > 0.1)
Calculating power for proportion = 0.12
Alpha = 0.05  Difference = 0.02

Sample   Target  Actual
  Size    Power   Power
   829   0.6000  0.6001
```

- Need to sample 829 students to have 60% chance of rejecting null hyp. when $p = 0.12$.
- Each sample value doesn't give much information ("success" / "failure"), so need large sample to get much power.

Summary

- Do significance test for proportion by asking: if null hypothesis p correct, is observed #successes plausible or not?
- Get P-value using binomial distribution, or normal as approx.
- Confidence interval for p: “undo” test by asking what values of p would make you fail to reject null.
- Calculate power by giving null hypothesis value of p and correct value of p, along with sample size.
- Given null hypothesis p, correct p and desired power, can calculate sample size needed to achieve power.

Inference for two proportions (§8.2)

- Often do comparative studies: have two groups (eg. treatment and control) and want to compare groups (eg. test null hypothesis that groups are same).
- If data are measured: calculate means, SDs, use methods of §7.2.
- If data are success/failure: have two binomial counts of successes, one for each group. What to do?

Example

- Example: In a study of syntax texts, are references to females more or less likely to refer to juveniles (“girl” vs. “woman”) than references to males (“boy” vs. “man”)? Data considered to be a random sample from all syntax texts.
- Data: for females, $n_1 = 60$ of which $X_1 = 48$ were juvenile; for males, $n_2 = 132$ of which $X_2 = 52$ were juvenile.
- Thinking of “success” as a “juvenile” reference, number of successes has a binomial distribution. Sample proportions are $\hat{p}_1 = 48/60 = 0.80$ and $\hat{p}_2 = 52/132 = 0.394$.

Getting confidence interval for $p_1 - p_2$

- Suppose X_1 has a binomial distribution with n_1 trials and success probability p_1, and X_2 has also with n_2 trials and prob. p_2.
- Then: X_1 has mean $n_1 p_1$, variance $n_1 p_1 (1 - p_1)$. Also $\hat{p}_1 = X_1 / n_1$ has mean p_1, variance $p_1(1 - p_1)/n_1$.
- Same applies for X_2 and $\hat{p}_2$.
- Also, if n_1 and n_2 are large (only case we consider), $\hat{p}_1$ and $\hat{p}_2$ approx. normally distributed.

Getting CI continued

- Can estimate population proportion difference $p_1 - p_2$ by sample proportion difference $D = \hat{p}_1 - \hat{p}_2$.
- D is difference of normals, so its variance is *sum* of individual variances:

$$\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}.$$

- Don't know p_1 and p_2, so replace by sample estimates to get SE (standard error) of D as

$$SE_D = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}.$$

- Then a confidence interval for $p_1 - p_2$ is

$$\hat{p}_1 - \hat{p}_2 \pm z^* SE_D.$$

CI for example

- Example data: $\hat{p}_1 = 0.80, n_1 = 60$; $\hat{p}_2 = 0.394, n_2 = 132$.
- Hence

$$SE_D = \sqrt{\frac{(0.80)(0.20)}{60} + \frac{(0.394)(0.606)}{132}} = 0.067,$$

and for a 95% confidence interval, $z^* = 1.96$, giving

$$0.80 - 0.394 \pm (1.96)(0.067) = 0.406 \pm 0.131,$$

from 0.275 to 0.537.

- We think the proportion of words for females that are juvenile is bigger than the proportion of words for males that are juvenile by between 0.275 and 0.537.
- This confidence interval contains only positive values, so we believe that the proportion of juvenile words for females is bigger.

Confidence interval in Minitab

- In Minitab, select Stat, Basic Statistics, 2 Proportions. Select Summarized Data (the bottom one). Fill in the number of trials and the number of successes for each group. For our data, there are 60 trials and 48 successes, then 132 trials and 52 successes.
- If you want a confidence level other than 95%, click Options and change it.
- This command will also give you a test of significance, which you can ignore for now (we will learn it later). Output (same result as by hand):

```
Sample      X      N  Sample p
1          48     60  0.800000
2          52    132  0.393939

Estimate for p(1) - p(2):  0.406061
95% CI for p(1) - p(2):  (0.274942, 0.537179)
```

Test of significance for two proportions

- Confidence interval says “how far apart could proportions be?”
- To see whether proportions believably the same, need a test of significance, with null hypothesis $H_0 : p_1 = p_2$. Base test on

$$D = \hat{p}_1 - \hat{p}_2,$$

and reject null if too far from (above, below) zero.

The mathematics

Now,

$$SE_D = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

but if null is true, then $p_1 = p_2 = p$, say, so that SE_D becomes

$$SE_{Dp} = \sqrt{\frac{p(1-p)}{n_1} + \frac{p(1-p)}{n_2}} = \sqrt{p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}.$$

Of course, don't know p either, but estimate it using overall proportion of successes:

$$\hat{p} = \frac{X_1 + X_2}{n_1 + n_2}.$$

Test procedure

1. Calculate $\hat{p}$ from data.
2. Calculate SE_{Dp} using $\hat{p}$ in place of p.
3. Calculate test statistic $z = (\hat{p}_1 - \hat{p}_2)/SE_{Dp}$
4. Get P-value from normal distribution.

Example

- Data $X_1 = 48, n_1 = 60, X_2 = 52, n_2 = 132$ and alternative $p_1 \neq p_2$:

$$\hat{p} = (48 + 52)/(60 + 132) = 100/192 = 0.521;$$

$$SE_{Dp} = \sqrt{(0.521)(0.479)\left(\frac{1}{60} + \frac{1}{132}\right)} = 0.0778;$$

$$z = \frac{0.800 - 0.394}{0.0778} = 5.22.$$

- Off the end of the table, so the P-value is close to $2 \times 0 = 0$. We can conclude that the two proportions are different.

Minitab / discussion

- Minitab produces the same answer for our data. Select Stat, Basic Statistics, 2 Proportions; click Summarized Data and fill in the sample values. Click Options; make sure that Alternative is Not Equal, and select Use Pooled Estimate. Click OK. Get the output from before plus this:

```
Test for p(1) - p(2) = 0 (vs not = 0):  Z = 5.22  P-Value = 0.000
```

- Same results as by hand.
- In this example, the sample proportions are very different, so (not surprisingly) we could conclude that the population proportions are different.
- In general, proving that population proportions are different requires either
  - sample proportions that are very different, or
  - sample sizes that are very large.

Summary of §8.2

- To compare population proportions p_1 and p_2, use difference in sample proportions $D = \hat{p}_1 - \hat{p}_2$.
- Standard error $SE_D = \sqrt{p_1(1-p_1)/n_1 + p_2(1-p_2)/n_2}$.
- For CI, substitute sample proportions in SE_D; interval is $\hat{p}_1 - \hat{p}_2 \pm z^* SE_D$.
- For test, null is $p_1 = p_2$. Use SE_D with p replacing p_1 and p_2.
- Estimate p using overall sample proportion of successes.
- Test statistic $z = (\hat{p}_1 - \hat{p}_2)/SE_{Dp}$, get P-value from normal table.

Chapter 9: Analysis of two-way tables

Comparing multiple proportions (§9.1)

- We have been stressing the need to do *comparative experiments* where possible. For instance: in type A people who have survived heart attacks, is it helpful to offer behavioral training as well as medical care?
- 290 patients randomized into 2 groups. Treatment group got behavioral training plus medical care, while control group got medical care only. Each person either suffered a 2nd heart attack or not.
- For two proportions, can use methods of §8.2; for more, need something new. (Compare this example using two methods.)

Contingency table

- Summarize results this way:

	2nd attack	No 2nd attack	Total
Treatment	17	123	140
Control	29	121	150
Total	46	244	290

- For instance, 123 patients were in treatment group and did not get a 2nd heart attack. Layout called **contingency table**.
- Research question: does treatment affect rate of 2nd heart attacks?

Percentages

- Starting point: percentages. Depending on data, row/column/overall %'s may be best. Here, want % of 2nd attacks for treatment & control groups.
- Treatment: $17/140 \times 100\% = 12\%$, control: $29/150 \times 100\% = 19\%$. Fewer 2nd attacks in treatment group – but could be just chance.

Inference for two or more proportions (§9.2)

- Alternative hypothesis: treatment has some effect (treatment proportion of 2nd attacks different from control proportion). Null hyp.: proportions same.
- In example, expect more 2nd attacks in control group because 150 patients vs. 140. This true even if treatment has no effect. But how many?
- Idea: 46 of 290 patients (0.1586) had 2nd attack overall. So, if null hyp. true, expect $0.1586 \times 140 = 22.21$ in treatment group, $0.1586 \times 150 = 23.79$ in control group.

Expected counts

Work out expected numbers of patients without 2nd attacks in same way, get this:

```
Expected counts are printed below observed counts

             C1       C2    Total
    1        17      123      140
          22.21   117.79

    2        29      121      150
          23.79   126.21

Total        46      244      290
```

Chisquare test

Now want one number summarizing this: small if observed and expected all close, large otherwise. Right mathematics p. 610; use Minitab calculation:

```
Chi-Sq =  1.221 +  0.230 +
          1.139 +  0.215 = 2.805
DF = 1, P-Value = 0.094
```

At $\alpha = 0.05$, can't reject null hyp. Evidence in data set not strong enough to conclude that treatment has effect.

Procedure called a **chisquare test**.

Doing it all in Minitab

- Calculations above were actually Minitab output. First step is to get table into Minitab. Just enter counts as laid out in table (without totals): 17 and 29 in column C1, 123 and 121 in column C2.
- Then: Stat, Tables, Chisquare Test. Select C1 and C2 as columns containing table; get output as above.

Another example

Altruism defined as “interest in welfare of others”. Questionnaire developed to measure altruism – results low, medium, high. Students from different majors studied:

Major	Low	Medium	High
Agriculture	5	27	35
Family studies	1	32	34
Engineering	12	129	94
Education	7	77	129
Management	3	44	28
Science	7	29	24
Technology	2	62	64

Comparing many proportions. Analysis from Minitab (tidied):

Expected counts

Expected counts are printed below observed counts

	Low	Medium	High	Total
Agric	5	27	35	67
	2.87	30.98	33.15	
Family	1	32	54	87
	3.72	40.23	43.05	
Engin	12	129	94	235
	10.05	108.67	116.28	
Educat	7	77	129	213
	9.11	98.50	105.39	
Manage	3	44	28	75
	3.21	34.68	37.11	
Science	7	29	24	60
	2.57	27.75	29.69	
Techno	2	62	64	128
	5.48	59.19	63.33	
Total	37	400	428	865

Chisquare test

```
Chi-Sq =  1.589 +  0.512 +  0.103 +
          1.990 +  1.684 +  2.787 +
          0.377 +  3.803 +  4.268 +
          0.489 +  4.692 +  5.288 +
          0.013 +  2.503 +  2.236 +
          7.659 +  0.057 +  1.090 +
          2.206 +  0.133 +  0.007 = 43.487
DF = 12, P-Value = 0.000
4 cells with expected counts less than 5.0
```

- P-value is small; reject null hyp. Definitely difference in proportions of low, medium, high among majors.
- How do majors differ? Compare observed and expected; something interesting happening where very different:

Why was H_0 rejected?

- Science (row 6): more lows than expected.
- Education (row 4): more highs, fewer medium/low than expected.
- Engineering (row 3): fewer highs, more medium/low than expected.

According to questionnaire, education students more altruistic than average, science and engineering students less so.

Summary

- Often do comparative studies of two or more groups, yes/no type answers.
- Contingency table to display results; percents to summarize.
- Calculate expected frequencies; leads to chi-square test: null of all proportions same, alternative not all same. Reject: conclude some differences among proportions.