Question 1. [10 MARKS]

Consider the following sequence of natural numbers:

$$e_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 3 & n = 2 \\ e_{n-1} + e_{n-2} + e_{n-3} & n \geq 3 \end{cases}$$

Prove that $e_n \leq 3^n$ for all $n \geq 0$.

Solution

Let $S(n)$ be $e_n \leq 3^n$. RTP $S(n)$ for all $n \geq 0$.

Basis. $n = 0$, $3^0 = 1$✓
$n = 1$, $3^1 \geq 2$✓
$n = 2$, $3^2 \geq 3$✓.

I.H. Assume for arbitrary n that $S(k)$ is true for all $2 \leq k < n$, $n, k \in \mathbb{N}$.

I.S Prove $S(n)$.

$$\begin{aligned} e_n &= e_{n-1} + e_{n-2} + e_{n-3} \text{ by defn} \\ &\leq 3^{n-1} + 3^{n-2} + 3^{n-3} \text{ by IH and the fact that } n > 2, \text{ so } n-3, n-2, n-1 \geq 0 \\ &= 3^2 3^{n-3} + 3^1 3^{n-3} + 3^{n-3} \\ &= (9+3+1)3^{n-3} = \frac{13}{27}3^n \\ &< 3^n \end{aligned}$$

Question 2. [10 MARKS]

Use the *Well Ordering Principle* to prove that:

$$6 \text{ divides } n^3 - n.$$

Do ***not*** use induction directly.

Solution.

We will build a contradiction. Assume that there exists at least one natural number k such that 6 does not divide $k^3 - k$. Let A be the set of all such natural numbers. By the WOP, there exists a smallest number a in A. Notice that if $k = 0$, then 6 divides 0 (0 times), so $a \geq 1$. Since a is the smallest number in A, it must be the case that $0 \leq a - 1 \notin A$.

Notice, that if $a - 1 \notin A$, then there exists i such that $(a-1)^3 - (a-1) = 6i$ and furthermore, notice that $(a-1)^3-(a-1) = (a-1)((a-1)^2-1) = (a-1)(a^2-2a+1-1) = (a-1)(a^2-2a) = (a^3-2a^2-a^2+2a)$ $a^3 - a - 3(a^2 - a) = a^3 - a + 3a(a-1) = 6i$ and since $a - 1$ is divisible by 6, so too must $3a(a-1)$ and $a^3 - a$. Therefore, $a \notin A$, contradiction.

Question 3. [10 MARKS]

Consider the recursive definition of the set B of binary strings:

Basis. $0 \in B$ and $11 \in B$.

Induction. If $b \in B$ then $b + b0 \in B$.

For example, if $11 \in B$ then $11 + 110 = 1001 \in B$.

Prove that for all $b \in B$, the decimal number represented by b is divisible by 3.

Proof Idea. Note this is not properly written up, just gives the idea of the solution.

Consider an arbitrary element $b \in B$. Then $b + b0 \in B$. By structural induction b is divisible by 3 and $b0$ is equal to 2 times b, so it too must be divisible by 3. Therefore, $b + b0$ is divisible by 3.

Question 4. [10 MARKS]

Prove that for every $n \geq 1, n \in \mathbb{N}$,

$$\sum_{i=2}^{2^n} \frac{1}{i} \geq \frac{n}{2}$$

Solution
Let $S(n)$ be: $\sum_{i=2}^{2^n} \frac{1}{i} \geq \frac{n}{2}$. Prove $S(n)$ true for all $n \geq 1, n \in \mathbb{N}$.
Basis. $n = 1$, then $\sum_{i=2}^{2} \frac{1}{i} = \frac{1}{2}$ ✓.

IH. Assume that for arbitrary $k \in \mathbb{N}$, $S(k)$ holds.

IS. We will prove that $S(k+1)$ holds.

$$\begin{aligned}
\sum_{i=2}^{2^{k+1}} \frac{1}{i} &= \sum_{i=2}^{2^k} \frac{1}{i} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \text{ there are } 2^k \text{ terms here} \\
&\geq \sum_{i=2}^{2^k} \frac{1}{i} + 2^k \cdot \frac{1}{2^{k+1}} \\
&= \sum_{i=2}^{2^k} \frac{1}{i} + \frac{1}{2} \\
&\geq \frac{k}{2} + \frac{1}{2} = \frac{k+1}{2} \text{ by induction hypothesis.}
\end{aligned}$$

Question 5. [10 MARKS]

Use repeated substitution to find an upper bound $\mathcal{O}()$ for $T(n)$. You may assume that $n = 5^k$ for some k.

$$T(n) = \begin{cases} a & n = 1 \\ 4T(\frac{n}{5}) + 2n^3 & n \geq 2 \end{cases}$$

$$\begin{aligned}
T(n) &= 4T(\frac{n}{5}) + 2n^3 \\
&= 4^2T(\frac{n}{5^2}) + 2n^3 + 4 \cdot 2(\frac{n}{2})^3 \\
&= \ldots \\
&= 4^iT(\frac{n}{5^i}) + \sum_{j=0}^{i-1} 2 \cdot 4^j(\frac{n}{2^j})^3 \\
&= 4^{\log_5 n}a + 2\sum_{j=0}^{\log_5 n-1} 4^j(\frac{n}{2^j})^3 \\
&= 4^{\log_5 n}a + 2\sum_{j=0}^{\log_5 n-1} \mathcal{O}(4^j(\frac{n}{2^j})^3)
\end{aligned}$$

there are a couple of different ways to simplify this, here is one way:

$$\begin{aligned}
&= a5^{\log_5 4^{\log_5 n}} + 2n^3 \sum_{j=0}^{\log_5 n-1} \frac{1}{2^j} \\
&= a5^{\log_5 n^{\log_5 4}} + 2n^3 \frac{\frac{1}{2}^{\log_5 n} - 1}{\frac{1}{2} - 1} \\
&= an^{\log_5 4} + 2n^3 \frac{\frac{1}{n^{\log_5 2}} - 1}{-.5} \\
&= an^{\log_5 4} + 2n^3(-2)(\frac{1}{n^{\log_5 2}} - 1) \\
&= an^{\log_5 4} + 2n^3(2)(1 - \frac{1}{n^{\log_5 2}}) \\
&\leq an^{\log_5 4} + 2n^3(2) \in \mathcal{O}(n^3)
\end{aligned}$$