## 2 Rabbits!

The reproductive rate of *rabbits* is rather remarkable. In fact, if you start with a pair of rabbits, once they are *two months old*, they can produce *another pair each month*.

Assuming no rabbits die, how many *pairs of rabbits* does one have after $n$ *months*?

| Months | Number of Pairs or Rabbits |
|---|---|
| 1 | 1 |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| ⋮ | ⋮ |
| n | |

We can express this as a formula:

$$
\begin{aligned}
F(0) &= \\
F(1) &= \\
F(n) &=
\end{aligned}
$$

**Q:** What is this sequence?

## 3 Some Fun Properties

**Example 1.** The sum of the Fibonacci numbers up to and including the $n^{th}$ number is the $(n+2)^{nd}$ number -1, *i.e.*,

$$f_0 + f_1 + f_2 + f_3 + \cdots + f_{n-1} + f_n = f_{n+2} - 1$$

Prove this property for all natural numbers $n$.

**Solution.**

**Example 2.** Prove that consecutive Fibonacci numbers are relatively prime.

**Solution.**

Try this application to plot sunflower seeds so that they are packed tightly.

http://www.mathsisfun.com/numbers/nature-golden-ratio-fibonacci.html

Sceptical? For every advocate for the golden ratio, there is a sceptic:

http://www.lhup.edu/~dsimanek/pseudo/fibonacc.htm

***Practice your proofs! Try to prove these properties of the Fibonacci numbers.***

Prove that for all $n \in \mathbb{N}_{>0}$ that

$$\sum_{j=1}^{n} F_{2j-1} = F_{2n}.$$

Prove that for all $n \in \mathbb{N}_{>0}$ that

$$\sum_{j=1}^{n} F_j = F_{n+2} - 1.$$

# 4 Other Recurrence Relations

Let's look at building recurrence relations.

**Example.** Express the number of ways to arrange $n$ distinct objects in a row as a *recurrence relation*. Suppose $n = 3$ then we have

$$(1, 2, 3), (1, 3, 2, )(2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$$

**Solution.**

**Example.** Draw 4 straight non-parallel lines in the plane such that no three intersect in one point. How many regions do the lines split the plane into?

**Q**. How many regions do we add if we draw a $5^{\text{th}}$ line?

**A**.

**Q**. How about if we have $n-1$ lines and we add an $n^{\text{th}}$ line? Explain.

**A**.

Write a recurrence relation for $a_n$ the number of regions in the plane formed by $n$ non-parallel, straight lines such that no three intersect at a single point.

**Example.** Express the number of strings of length $n$ formed using the alphabet $\{0, 1, 2, 3\}$ that have an even number of 0s as a recurrence relation.

**Solution.**

**Example.** Draw a binary tree height 3. What is the upper bound on the number of nodes in this tree in terms of $h$.

Express an upper bound on the number of nodes in a *binary* tree of *height* $h$ as a *recurrence relation*.

**Solution.**

**Question.** How can we *solve* these recurrences?

# 5 Backward's Substitution.

$$T(h) =$$

$$T(h) =$$

$$T(h) =$$

$$= \vdots$$

$$T(h) =$$

**Question.** What is $i$?

Therefore,

$$T(h) =$$

**Example.** Consider the following algorithm to compute $n!$.

```
Algorithm Factorial(n)
   if n=1
       return 1;
   else
       return (n*Factorial(n-1));
```

What is the *complexity* $T(n)$ for `Factorial(n)`? Write $T(n)$ as a *recurrence relation*.

**Solution.**

**Exercise.** Find a closed form formula for $T(n)$.

**Example.** What if we had a recurrence relation like:

$$T(1) = a; \quad T(n) = 2T(\frac{n}{2}) + g(n) \text{ for } n > 1$$

where $g(n)$ is a function of $n$.

Can we use *repeated substitution* to find the ***closed form*** of $T(n)$?

**Solution.**

**Q:** For what value of $i$ is $\frac{n}{2^i} = 1$?

Therefore,