1 Proof by Induction

Lemma 1. *Prove that for all natural numbers n,*

$$12^n - 1 \text{ is an integer multiple of } 11.$$

Proof.

1. **Define** $P(n)$:

   $P(n)$: $12^n - 1$ is an integer multiple of 11.

   RTP: $P(n)$ is true $\forall n \in \mathbb{N}$.

2. **Base Case:**

   $n = 0$. Then $12^n - 1 = 0 = 0 \cdot 11$.

3. **Inductive Hypothesis:**

   Let $k \in \mathbb{N}$. Suppose $P(k)$.

4. **Inductive Step:**

   Since we assume $P(k)$, $12^k - 1 = 11c$ for some $c \in \mathbb{Z}$. Then, $12^{k+1} - 1 = 12 \cdot (12^k) - 1$ and $12^k = 11c + 1$.

   Therefore, $12^{k+1} - 1 = 12 \cdot (11c + 1) - 1 = 12c \cdot 11 + 12 - 1 = (12c + 1) \cdot 11$.

   Since $c \in \mathbb{Z}$, $12c + 1 \in \mathbb{Z}$.

□

Splitting Piles Consider this "*trick*". Start with a pile of n stones. Ask your friend to split the piles into two smaller piles of any size of at least 1. Multiply the sizes of the two piles and add to a sum that we will call $total$. Repeat until all piles are of size 1.

Example.

Let $n = 6$. Try splitting the pile in different ways and seeing what your total is. Here is one way:

$6 \rightarrow \frac{4}{2}$ makes a sum of 8. $total = 8$.
$4 \rightarrow \frac{3}{1}$ makes a sum of 3. $total = 11$.
$2 \rightarrow \frac{1}{1}$ makes a sum of 1. $total = 12$.
$3 \rightarrow \frac{1}{2}$ makes a sum of 2. $total = 14$.
$2 \rightarrow \frac{1}{1}$ makes a sum of 1. $total = 15$.

Try another way of splitting the piles with $n = 6$.

Should add up to 15 always = $\frac{(n(n-1)}{2}$.
Try with $n = 8$ and $n = 9$. What do you notice about the $total$?
$total = \frac{(n(n-1)}{2}$.

Let's prove the claim.

$P(n) : total = \frac{(n(n-1)}{2}$.
Base Case. $n = 2$. $total = 1 = 2(1)/2$.
Induction Hypothesis. Consider an arbitrary number of stones, say $n > 2$. Assume that $P(k)$ holds for $k < n$.
Induction Step. Consider splitting n into k and j piles. Then the total for splitting each of these piles is $total_k = \frac{k(k-1)}{2}$ and $total_j = \frac{j(j-1)}{2}$. The total for n is then

$$\begin{aligned}
total_n &= k \cdot j + \frac{k(k-1)}{2} + \frac{j(j-1)}{2} && (1)\\
&= \frac{2kj + k^2 - k + j^2 - j}{2} && (2)\\
&= \frac{k^2 + kj - k + j^2 + kj - j}{2} && (3)\\
&= \frac{k(k+j-1) + j(j+k-1)}{2} && (4)\\
&= (k+j)(k+j-1)/2 && (5)\\
&= n(n-1)/2 && (6)\\
& && (7)
\end{aligned}$$