Predicate Calculus

Review

- A *predicate* is a *boolean function*., eg. $E(x)$: x is even. $PR(x, y)$: course x is a prerequisite for course y.

- A *predicate* P of *arity* n maps from the same *domain of discourse* D, formally

$$D \times D \times \ldots \times D \rightarrow \{0, 1\}$$

- Combine *predicates* using the *connectives* from *propositional logic*: $\{\wedge, \vee, \neg, \rightarrow, \leftrightarrow\}$.

- Two *quantifiers*:

  1. UNIVERSAL: ($\forall$) If $D = \{x_1, x_2, \ldots,\}$ then $\forall x, E(x)$ is true if $E(x_1) \wedge E(x_2) \wedge \ldots$ is true.

  2. EXISTENTIAL: ($\exists$) If $D = \{x_1, x_2, \ldots,\}$ then $\exists x, E(x)$ is true if $E(x_1) \vee E(x_2) \vee \ldots$ is true.

Example:

Let $D = \mathbb{N}$ and $E(x)$: x is even.

Is $\forall x, E(x)$ $\mathbf{true}$?

How about $\exists x, E(x)$?

Let

- D ={The set of CS courses at U of T}.
- $PR(x, y)$: course x is a *prerequisite* for course y.

Q: Is $\forall x, \exists y, PR(x, y)$ $\mathbf{true}$? no

Is every course a prerequisite for some course.

Q: Is $\exists x, \forall y, PR(x, y)$ $\mathbf{true}$?

Q: Do $\exists x, \forall y, PR(x, y)$ and $\forall y, \exists x, PR(x, y)$ mean the same thing? I.e., are they *logically equivalent*?

First-Order Language

We can define the set of *predicate formulas* called a FIRST-ORDER LANGUAGE $\mathcal{L}$ using *structural induction.*

Terminology

- **Terms of** $\mathcal{L}$: A *term* is a *constant* or *variable* in $\mathcal{L}$.
- **Atomic formula of** $\mathcal{L}$: An *atomic* formula is a predicate $P(t_1, t_2, \ldots, t_k)$ of $arity$ k where each t_i is a *term in* $\mathcal{L}$.

The set of *First-Order Formulas* of $\mathcal{L}$ is the smallest set such that:

Basis: *Any atomic formula in* $\mathcal{L}$ *is in the set.*

Induction Step: If $\mathcal{F}_1$ and $\mathcal{F}_2$ are in the *set*, x is a *variable* in $\mathcal{L}$, then each of $F_1 \vee F_2$, $F_1 \wedge F_2$, $F_1 \rightarrow F_2$, $\neg F_1$, $\forall x F_1$, $\exists x F_1$ are in the *set*.

Example: The *language of arithmetic, $\mathcal{LA}$*.

- An infinite set of *variables*, $\{x, y, z, u, v, \ldots\}$
- *Constant* symbols $\{0, 1\}$
- Predicates:
  - $L(x, y) : x < y$,
  - $S(x, y, z) : x + y = z$,
  - $P(x, y, z) : x \cdot y = z$,
  - $E(x, y) : x = y$.

Q: Let our *domain* be $\mathbb{N}$. What does

$$\forall x(\exists y S(y, y, x) \vee \exists y \exists z(S(y, y, z) \wedge S(z, 1, x)))$$

represent?

every natural number is even or odd

Q: Do the two y's on either side of the $\vee$ *represent* the *same* thing?

The Scope of Quantifiers

- A variable is *free* if it is *not* referred to by any *quantifier*.
- A variable is *bound* if it *is* referred to by a *quantifier*.
- A *sentence* is a formula that has *no free variables*.

Q: In the following predicate formula, which *variables* are *free* and which are *bound*?

$$\forall x[\exists y(R(x,y) \wedge T(z,x)) \to \forall z(N(z,x,y))]$$

free **bound**

free

Q: What problem may occur if the *same symbol* is used to represent *more than one variable* in a formula?

Q: Soln? Replace variable names so that they are all unique.

Evaluating Predicate Formulas

First suppose that there are *no free variables*.

Determining *satisfiability* or *unsatisfiability* of such a *predicate formula* is a *3-step* process:

1. Give meaning to our predicates.
2. Define the domain.
3. Define which elements of the domain the constants represent.

This defines a *structure* $\mathcal{S}$.

Interpretations and Truth

Examples: Give a *structure* $\mathcal{S}_1$ that *satisfies* the following predicate and another *structure* $\mathcal{S}_2$ that *falsifies* the following *predicate formula*:

1. $\forall x L(0, x)$
   (a) Let $L(a,b): a \leq b$. $D = \mathbb{N}$; $0^{S_1} = 0$
   (b) Let $L(a,b): a \leq b$; $D = \mathbb{Z}$, $0^{S_2} = 0$.
2. $\forall x \forall y \neg A(x, y, x)$
   (a) Pick $D = \mathbb{N}^{>0}$, $A: x+y=x$
   (b) Pick $D = \mathbb{N}$, $A: x+y=x$

If there exists a *free variable* then we need to define a *valuation* of $\mathcal{S}$.

Valuations

- A *valuation* of $\mathcal{S}$ is a *function* σ that *maps* each variable in $\mathcal{L}$ to an *element* of the domain $\mathcal{D}$.

  e.g., If $\mathcal{S}$ is defined as:

  – *Domain*: $\{$Simpsons Characters$\}$,

  – *predicate* $P(x, y)$: x is a parent of y,

  – $\sigma(x, y, z) = (Homer, Bart, Lisa)$

  then $P(x, y)$ expresses *Homer is a parent of Bart.*.

- Given σ,

  σ^x_a means that σ^x_a is *identical* to σ with the exception that x is mapped to a.

  e.g., If $\mathcal{S}$ is as above, then $\sigma^x_{Marge} = (Marge, Bart, Lisa)$.

- Together, a *structure* $\mathcal{S}$ and a *valuation* σ for a language $\mathcal{L}$, define an *interpretation* denoted $\mathcal{I} = (\mathcal{S}, \sigma)$.

- We say $\mathcal{I}$ *satisfies* a formula F if F is **true** in interpretation $\mathcal{I}$.

- Similary, $\mathcal{I}$ *falsifies* F if F is **false** in interpretation $\mathcal{I}$.

Logical Equivalences for Predicates

We can define *valid, satisfiable* and *unsatisfiable* in the same manner as with propositional logic.

Let F be a formula of a first order language L. We say F is:

1. *valid* or a *tautology* if it is satisfied by *every interpretation* of L

2. *satisfiable* if *some interpretation* of L satisfies it.

3. *unsatisfiable* if it is *not satisfied* by *any interpretation* of L

Examples Which of the following is *valid*, *satisfiable* or *unsatisfiable*?

- $\forall x A(x) \rightarrow A(1)$

  Valid.

  Why? if left is false, statement is true

  - if the left is true then every value in the domain makes A() true including A(1).

- $\exists x A(x) \rightarrow A(1)$

  SATISFIABLE

  Why?

  TRUE: $D = \{1\}$, define $A(1) = $ TRUE

  FALSE: $D = \{1, 2\}$, define $A(1) = $ False
  $A(2) = $ True

$A(1)=T$ $B(1)=F$

- $F \ : \ \forall x(A(x) \rightarrow B(x)) \wedge A(1) \wedge \neg B(1)$

  UNSATISFIABLE

  Why?

  For *some interpretation* to *satisfy* F, what must be true?

  1. $A(1)=T$

  2. $B(1)=F$
  3. $A(1) \rightarrow B(1)$ to be true. But it is $T \rightarrow F$ which is false.

  Conclusion?

  This leads to a contradiction, so no interpretation can satisfy F

- F: $\forall x \exists y P(x, y)$

Q: How do we *know* if F is *valid*? perhaps we haven't thought of an *interpretation* that doesn't satisfy F.

Theorem *Let F and H be formulas of a first-order language, then:*

1. $F \Rightarrow H$ *iff* $F \rightarrow H$ is *valid*

2. $F \Leftrightarrow H$ *iff* $F \leftrightarrow H$ is *valid*

Logical Equivalences – Predicate Logic

For any formulas F and E and variables x and y.

1. All the **propositional logical equivalences**.

2. The $\forall, \exists$ version of *DeMorgan's* called the **Quantifier Duality**

$$\neg\forall x F \Leftrightarrow \exists x \neg F \quad \text{and} \quad \neg\exists x F \Leftrightarrow \forall x \neg F$$

3. **Rename Quantified Variables** Why might we want to *rename variables*?

   Consider, $\forall x(P(x) \vee \exists x Q(x))$.

   $\Leftrightarrow \forall x (P(x) \vee \exists y Q(y))$

4. **Substitute Equivalent Formulas**: $P \leftrightarrow \neg Q$

   For example, $((P \wedge Q) \vee (\neg Q \wedge \neg P)) \rightarrow R$ is *logically equivalent* to $(P \leftrightarrow Q) \rightarrow R$.

5. **Factorizing Quantifiers over $\vee$ and $\wedge$**: Suppose that x is *not free* in E, then what can we say about:

$$(E \wedge \forall x F) \Leftrightarrow \forall x (E \wedge F)$$

$$(E \wedge \exists x F) \Leftrightarrow \exists x (E \wedge F)$$

$$(E \vee \forall x F) \Leftrightarrow \forall x (E \vee F)$$

$$(E \vee \exists x F) \Leftrightarrow \exists x (E \vee F)$$

   This equivalence illustrates how we can *factor quantifiers*.

6. **Factorizing Quantifiers over Implications**

   Assuming x is *not free in F*, is

$$\forall x E \rightarrow F \overset{??}{\Leftrightarrow} \exists x (E \rightarrow F)?$$

Let's prove

$$\forall x E \to F \overset{??}{\Leftrightarrow} \exists x(E \to F)?$$

$$\begin{aligned}
(\forall x E) \to F &\Leftrightarrow \neg(\forall x\, E) \vee F \\
&\Leftrightarrow (\exists x\, \neg E) \vee F \\
&\Leftrightarrow \exists x\, (\neg E \vee F) \\
&\Leftrightarrow \exists x\, (E \to F)
\end{aligned}$$

Similar proofs show that:

$$\begin{aligned}
\exists x E \to F &\Leftrightarrow \forall x(E \to F) \\
E \to \forall x F &\Leftrightarrow \forall x(E \to F) \\
E \to \exists x F &\Leftrightarrow \exists x(E \to F)
\end{aligned}$$

An Example Prove that

$$(\forall x P(x)) \rightarrow \forall x(Q(x) \rightarrow A(x) \vee B(x))$$

is *logically equivalent* to

$$\exists x \forall y(\neg P(x) \vee \neg Q(y) \vee A(y) \vee B(y)).$$

Proof.

$$(\forall x P(x)) \rightarrow \forall x(Q(x) \rightarrow A(x) \vee B(x)) \quad \text{subst.}$$

$$\Leftrightarrow (\forall x P(x)) \rightarrow \forall y(Q(y) \rightarrow A(y) \vee B(y))$$

$$\Leftrightarrow \neg(\forall x P(x)) \vee \forall y(Q(y) \rightarrow A(y) \vee B(y)) \quad (\overset{\rightarrow}{\text{law}})$$

$$\Leftrightarrow \exists x \neg P(x) \vee \forall y(\quad\underline{\qquad\qquad\qquad}\quad)$$

$$\Leftrightarrow \exists x \forall y(\neg P(x) \vee [Q(y) \rightarrow A(y) \vee B(y)])$$

$$\exists x \forall y(\neg P(x) \vee [\neg Q(y) \vee A(y) \vee B(y)]) \qquad \square$$

What do we notice about this second formula...

Prenex Normal Form or PNF

A formula is in *Prenex Normal Form (PNF)* if all quantifiers *precede* a *quantifier-free* sub-formula.

Q: Is the following formula *valid*?

$$\forall x \exists y (L(x,y)) \leftrightarrow \exists y \forall x (L(x,y))$$

This suggests that formulas in *PNF* are *very sensitive* to the *order* of the *quantifiers*.

Q: Is the following formula *valid*?

$$\forall x \exists y (M(x) \wedge F(y)) \leftrightarrow \exists y \forall x (M(x) \wedge F(y))$$

Proof. no. Left as exercise.

$\square$