The Pumping Lemma–Showing L is *Not* Regular

Proof Structure.

- Use a proof by *contradiction*.
- By the *Pigeon Hole Principle*, any sufficiently long string in L must *repeat* some *state* in the FSA, thus creating a *cycle*.
- **“Pumping”** the *cycle* leads to a string *not* in L.
- **Contradiction.**
- Conclude that L is *not regular*.

Q: What is the *Pigeon Hole Principle*?

If there are n pigeon holes and $n + 1$ pigeons, then there exists at least one hole with two pigeons.

Formal Definition–Pumping Lemma

The Pumping Lemma. *Let $L \subseteq \Sigma^*$ be a regular language. Then there is some $p \in \mathbb{N}$ such that every $s \in L$ that has length $|s| \geq p$, can be divided into 3 pieces, $s = xyz$ such that:*

- $|y| > 0$, i.e., $y \neq \epsilon$,
- $|xy| \leq p$,
- $\forall i \in \mathbb{N}, xy^i z \in L$

Q: Which part of xyz represents the *cycling* part of s? y is the cycling part.

Showing L is *not* regular by building a *contradiction*:

1. **Assume** L is regular, so some FSA accepts it.
2. **Let** p be the number of states of the FSA.
3. **Choose** a string $s \in L$ such that $|s| \geq p$
4. **Consider** $s = xyz$ such that
   - $|xy| \leq p$ and
   - $y \neq \epsilon$
5. **Choose** an i such that $xy^i z \notin L$.

Example.

Prove that $L = a^m b^m : m \geq 0$ *is not regular.*

1. **Assume** L is *regular*, so there exists an *FSA* on p states that accepts L.

2. **Choose** a string $s = a^i b^i$ where $i > p$, so $|s| \geq p$ and in particular, any *prefix* of *length* i consists entirely of a's.

3. **Consider** $s = a^i b^i = xyz$.
   - Since $|xy| \leq i$,
   - xy must consist entirely of a's.
   - y cannot be empty.

4. **Choose** $k = 0$. This has the effect of dropping $|y| a$'s out of the string, *without* affecting the number of *b*'s.

5. The resultant string has fewer a's than b's, hence does not belong to L. Therefore L is not *regular*.