Welcome to CSCB36

Introduction To The Theory Of Computation

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IC220	*Wednesdays*	*15:10-17:00*

Course Topics

Review: Induction Proofs

Recurrences

Logic

Program Correctness

Formal Languages

The Numbers

Assignments

3 each worth 15%

Term Test

Week 7 or 8 worth 15%

Final Exam

Worth 40%

Resources

Course Slides and Handouts

Posted each week.

You are *required to bring them to each class.*

Website

www.utsc.utoronto.ca/~bretscher/cscb36

e-Textbook

http://www.cs.toronto.edu/~vassos/b36-notes/

Office Hours

Mondays 11.10-12

Wednesdays 1.10-2

Fridays 12.10-1

Tutorials

4

Course Expectations

Expectations of the lecturer

- Give clear, organized lectures
- Assign fair, challenging assignments that ensure that you, the student, understand the material
- Be available for help in office hours
- Help every student achieve their goals in the course (this requires your help!)

Course Expectations

Expectations of the student

Attend lectures and participate

Bring course notes to class

Review lecture notes after each class, not just before the exam

Complete homework fully, neatly and independently

Have respect for your classmates, lecturer and teaching assistants

Course Expectations

What does neatly mean?

Staple sheets

Write legibly (if you are incapable of this skill, please type)

Your work should be of a quality that you would feel comfortable giving to your boss in a work environment.

Proof By Induction

Q. What are the 3 steps of an inductive proof?

A.

1. *Base Case: Prove the statement for the smallest cases.*
2. *Induction Hypothesis: Consider any case k, bigger than or equal to the base case.*
3. *Inductive Step: Prove that if the k^{th} case is true, it must be true that the $k+1^{st}$ case is true.*

8

Intuition

Imagine a set of dominos.

Suppose we want to prove that all the dominos fall down using induction.

Q. What is the Base Case?
A. That the first domino falls down.

Q. What is the Induction Hypothesis?
A. Suppose that some domino falls down. Let's say it's the k^{th} domino.

Q. What is the Inductive Step?
A. Prove that *if the* k^{th} domino falls down *then* it knocks over the $k+1^{st}$ domino.

9

Prove for every natural number n that

$$n(n+5) \text{ is divisible by } 6.$$

Let $S(n)$ represent "$n(n+5)$ is divisible by 6".

Q. What is the *base case?*

$S(0)$: $0(0 + 5) = 0 = 0(6)$ which is divisible by 6.

OR

$S(1)$: $1(1+5) = 6 = 1(6)$ which is divisible by 6.

10

Prove for every natural number n that

$$n(n\ +5) \text{ is divisible by } 6.$$

Let $S(n)$ represent "$n(n\ +5)$ *is divisible by* 6".

Q. What is the *induction hypothesis?*

A. Pick an arbitrary numbe k bigger than or equal to our base case, so $k \geq 0$.

Q. What is the *inductive step?*

A. Show that if $S(k)$ *is true*, then $S(k+1)$ *has to be true.*

11

Prove for every natural number n that

$$n(n\ +5) \text{ is divisible by } 6.$$

Let $S(n)$ represent "$n(n\ +5)$ *is divisible by* 6".

Suppose that $S(k)$ *is true*, then what do we know?

$$k(k^2+5) \text{ is divisible by } 6$$

Mathematically, we express this as:

There exists a natural number c such that

$$k(k^2+5) = 6c$$

12

Prove for every natural number n that

$$n(n^2+5) \text{ is divisible by } 6.$$

Let $S(n)$ represent "$n(n^2+5)$ is divisible by 6".

We are trying to prove that $S(k+1)$ *is true*, ie, that:

$$(k+1)((k+1)^2+5) = 6b$$

for some b in the natural numbers.

Now what?

13

We are trying to prove that there exists a natural number b such that:

$$(k+1)((k+1)^2+5) = 6b$$

Using the fact that:

$$k(k^2+5) = 6c$$

So…

$$(k+1)((k+1)^2+5) = (k+1)(k^2+2k+1+5)$$

$$=k(k^2+2k+5 + 1) + 1(k^2+2k+6)$$

$$=k(k^2+5) +2k^2+k+k^2+2k+6$$

$$=6c +3k^2+3k+6 \qquad \text{Why?}$$

14

So far we have…

$$(k+1)((k+1)^2+5) = 6c + 3k^2+3k+6$$

Notice 6 is divisible by 6. What about

$$3k^2+3k\,?$$

$$= 3k(k+1)$$

Q. Is $3k(k+1)$ divisible by 6?

A. Yes. Why?

Because k(k+1) must be even (think about this!) so divisible by 2.

15

This means there exists a d such that

$$(k+1)((k+1)^2+5) = 6c + 6d.$$

Therefore, there exists a $b = c + d$ such that

$$(k+1)((k+1)^2+5) = 6b$$

and S(k+1) is true. Therefore S(n) is true for all n in the natural numbers.

16

Strong Induction

Q. What is the difference between simple and strong induction?

A.

1. *Base Case: Can have more than one base case.*
2. *Induction Hypothesis: We assume all cases less than some case k are true.*
3. *Inductive Step: Prove that if every case smaller than the k^{th} case is true, then the k^{th} case must be true.*

17

Strong Induction

Two typical formats.

Base Case Format:

1. *Multiple base cases.*
2. *Induction Hypothesis: We assume all cases less than some case k but at least as large as the largest base case are true.*
3. *Inductive Step: Prove that if every case smaller than the k^{th} case is true, then the k^{th} case must be true.*

18

Strong Induction

Two typical formats.

No Base Case Format:

1. *Induction Hypothesis: We assume all cases less than some case k are true.*
2. *Inductive Step: Consider all possible options for k. This will include the base cases. Prove that if every case smaller than the k^{th} case is true, then the k^{th} case must be true.*

19

Strong Induction Example

The Game of Nim

Rules

1. A positive number of *stones* are thrown onto the ground.
2. Two players take turns, where each turn a player *removes* 1, 2 or 3 *stones*.
3. The one to remove the *last stone* loses.

Let's give our players names: Red and Gold.
Assume that Red always starts the game.

20

Nim

Q. If there are *n stones* to start, which player, Red or Gold can *guarantee* that they *win*?

Lemma: Player Gold (2nd player) has a *winning* strategy *iff*

$$n = 4j+1 \text{ for } j \text{ in } \mathbf{N}$$

21

Nim

Lemma: Player Gold has a *winning* strategy *iff*

$$n = 4j+1 \text{ for } j \text{ in } \mathbf{N}$$

Proof by induction (using 2 format).

Define P(n):

"Player #2 (Gold) can always win iff $n = 4j+1$ for j *in* $\mathbf{N}$*."*

Induction Hypothesis: *Assume that for arbitrary m in N, m>1, P(k) holds for all 1≤ k<m.*

22

Nim

P(n): *"Gold can always win iff $n = 4j+1$* for *j in* ***N**.*"

Induction Step: Prove P(m).

Case 1. m=1; then Red loses, or Gold wins.

Q. What are the other cases?

A. m>1 and m=4j+1 or m>1 and m≠4j+1

23

Nim

P(n): *"Gold can always win iff $n = 4j+1$* for *j in* ***N**.*"

Induction Step: Prove P(m).

Case 2a. m>1, m=4j+1; then if Red selects x stones, Gold will take 4-x stones.

Now we have m=4(k-1)+1 and Gold can win.
Why?

24

Nim

P(n): *"Gold can always win iff $n = 4j+1$* for *j in **N**."*

Case 2b. m>1 and m=4j + c, c in {0,2,3}.

Q. What are we trying to prove here?

A. That there is a strategy for Red that ensures Gold cannot win.

25

Nim

P(n): : *"Gold can always win iff n = 4j+1* for *j in **N**."*

Case 2b. m>1 and m=4j + c, c in {0,2,3}.

Q. What is Red's strategy?

A. To select enough stones so that *4j'+1* stones are left for Gold.

c = 0	Red takes 3 stones
c = 2	Red takes 1 stone
c = 3	Red takes 2 stones

26

Nim

P(n): *"Gold can always win iff $n = 4j+1$* for *j in **N**."*

Q.How does this prove that Gold loses?

A. It is Gold's turn and there are < m stones so the *Induction Hypothesis* holds.

I.e., the *2nd player* has a winning strategy iff there are *4j'+1* stones to start. The 2nd player is now Red.

This is a new game where there are *4j'+1* stones and Gold is the *first* player. Therefore Gold loses.

27