Week 6 Worksheet
Pigeon Hole Principle Examples

Example. Explain why in a class of 36 students, at least 6 were born on the same day of the week.

Solution.

Make each day of the week a container, then there are 7. That means that by the _generalized PHP_ there are at least 6 in one of the 7 boxes \(7 \times 5 = 35 < 36\).

This problem illustrates the **Generalized Pigeon Hole Principle**.

\[
\text{GENERALIZED PIGEON HOLE PRINCIPLE.}
\]

\[
\text{If } n \text{ items are put into } m \text{ containers, with } n > m(r - 1), \text{ then at least one container must contain at least } r \text{ items.}
\]

In the previous example then, we had \(n = 36, m = 7\) and since \(n = 36 > m(r - 1) = 7(6 - 1) = 35\) so \(r=6\).

Example. A website displays an image each day from an image bank of 30 images. In any given 100-day period, show that some image must be displayed four times.

Solution.

**Holes:** The 30 images.

**Pigeons:** the 100 days.

Then notice that \(30(4 - 1) = 30(3) = 90 < n\) so by the **GPHP** there are at least 4 days with the same image.

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**Definition: Graph.**

A graph \(G = (V, E)\) consists of a set \(V\) of vertices of size \(n\) (think of vertices as objects) and a set \(E\) of edges of size \(m\) joining pairs of vertices.

For example, the graph \(G\) with \(V = \{1, 2, 3, 4\}\) and \(E = \{(1, 2), (1, 4), (2, 3), (2, 4)\}\) can be drawn as follows where points represent vertices and lines represent edges.
**Example.** Let $G$ be a graph with 6 vertices such that every pair of vertices has an edge between them. Let $V = \{a, b, c, d, e, f\}$ be the vertices. Draw the graph.

![Graph](image)

**Q.** How many edges should your graph have?

**A.** 15

**Q.** Suppose that some edges are coloured red and the rest blue. Show that there must be some triangular (three edges forming a triangle) of the same colour.

**A.** Pick a vertex (doesn’t matter which one, say vertex 1). There are 5 edges, so by pigeon hole, three or more of the edges have to have the same colour, suppose that colour is red and that the three (or more vertices) are vertex numbers 4, 5, 6. Consider these three vertices, if they form an all blue triangle then we are done. If not, then one of their edges is red and so that edge along with vertex 1 forms a red triangle.

![Coloured Graph](image)
Propositional Logic

Example. Use truth tables to evaluate each of the following statements. Show that the following pairs of statements are equivalent:

\[ \neg(s \land t) \text{ and } \neg s \lor \neg t \]

\[ \neg(s \lor t) \text{ and } \neg s \land \neg t \]

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<th>\neg(s \land t)</th>
<th>\neg s \lor \neg t</th>
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We denote equivalent formulas with the following symbol \( \iff \). You can imagine this symbol to be analogous to an “=” sign. These two equivalent pairs of statements have a special name: De Morgan’s Law.

Example. Write English sentences that illustrate De Morgan’s Law (the above two equivalence laws). You may use these definitions of \( s \) and \( t \).

Let \( s \) be: \( s \) is an odd integer. Let \( t \) be: \( t \) is an odd integer.

- \( \neg(s \land t) \) and \( \neg s \lor \neg t \):

  It is not the case that \( s \) and \( t \) are both odd integers \( \iff \) \( s \) is even or \( t \) is even or both are even.

- \( \neg(s \lor t) \) and \( \neg s \land \neg t \):

  It is not the case that either of \( s \) is odd or \( t \) is odd \( \iff \) \( s \) and \( t \) are even.
Implies

Fill in the truth table for $t \rightarrow s$. You may find it helpful to draw a Venn diagram to help visualize the meaning of $t \rightarrow s$.

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<th>$\neg t \lor s$</th>
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Now fill in the column for $\neg t \lor s$.

**Q.** What do you notice about the truth table?

Fill in the column for $\neg s \rightarrow \neg t$.

**Q.** Now what do you notice?

**A.** All of $t \rightarrow s$, $\neg t \lor s$ and $\neg s \rightarrow \neg t$ evaluate to the same set of truth values.

We say that $(t \rightarrow s) \iff (\neg t \lor s) \iff (\neg s \rightarrow \neg t)$ or that they are equivalent.

**Example.** Which of the following are equivalent to $t \rightarrow s$? Use your truth table to help you figure out “sufficient” and “necessary”.

- $\neg s \rightarrow \neg t$ ✓
- $\neg t \rightarrow \neg s$
- $s \rightarrow t$
- if $s$ then $t$
- if $t$ then $s$ ✓
- $t$, if $s$
- $\neg s$, if $\neg t$
- $\neg s \lor t$
- $s \lor \neg t$ ✓
- $\neg t \lor s$ ✓
- $s$, if $t$ ✓
- $s$ is sufficient for $t$
- $t$ is sufficient for $s$ ✓
- $s$ is necessary for $t$ ✓
- $s$ is necessary for $t$ ✓
- $t$ is necessary for $s$
- $\neg (t \land \neg s)$ ✓
**Example.** Let's use real mathematical statements. Let's consider the integers $\mathbb{Z}$. Let $x \in \mathbb{Z}$ ($x$ belongs to the set of integers).

If $x > 0$ then $x^2 > 0$.

Rewrite this implication using *sufficient, necessary, if, and negations*.

- $x > 0$ is sufficient for $x^2 > 0$.
- $x^2 > 0$ is necessary for $x > 0$.
- $x^2 > 0$, if $x > 0$.
- If $x^2 \neq 0$ then $x \neq 0$, we can also say this as: $x^2 \leq 0 \rightarrow x \leq 0$.

**Q.** What do you notice when $x > 0$ is *false*? Is it still possible for $x^2 > 0$ to be true?

**A.** Yes. When $x < 0$.

We have learned some *logical equivalences*. We say that two statements are *logically equivalent* when they *evaluate* to the same truth value given an assignment of truth values to their variables.

So far we have seen:

- **De Morgan’s Law** $\neg (p \lor q) \iff (\neg p \land \neg q)$ and $\neg (p \land q) \iff (\neg p \lor \neg q)$
- **Implication law** $p \rightarrow q \iff \neg p \lor q$
- **Contrapositive** $p \rightarrow q \iff \neg q \rightarrow \neg p$