Combinatorics Worksheet

When solving a combinatorics problem, what are the two questions you need to ask yourself?

**Q1.** Does order matter? yes...then it’s a permutation, no...then it’s a combination.

**Q2.** Is there repetition?

**Example.** Creating a *secure* password.

To create a Google account, you are required to give 8 or more *alphanumeric* characters.

**Self test:** Ask Q1 and Q2. What type of problem is this?

A. permutation with repetition

Why does google require 8 characters? Why not just 4?

Let’s count how many *possible passwords* there are of lengths 4 and 8 and compare.

**Q.** How many *alphanumeric* characters are there?

A. a-z, A-Z, 0-9 = 62

**Q.** How many *passwords* are there of length 4?

A. \(62^4 > 1.4 \times 10^7\)

**Q.** Why do we multiply instead of add?

A. Think of it as: we have 62 choices for the first character then 62 choices for the second. So there are 62 choices for the second character for each of the 62 choices for the first character or \(62 \times 62\). Continue this thinking for the 3\(^{rd}\) and 4\(^{th}\) characters.

**Q.** How many *passwords* are there of length 8?

A. \(62^8 > 2.1 \times 10^{14}\)

So there are more than *14 million times* as many passwords of length 8 than of length 4.
Take home. The formula to count \( r \) items taken from \( n \) objects with \textit{repetition} when \textit{order matters} is:

\[ n^r \]

**Example.** Unique Neighbourhood Design.

A builder has the following requirements:

- 5 different model homes.
- Each home can come in 3 different colours.
- A street will have 10 homes on it.

The builder wants to make sure each home is unique (ie, no two houses are the same on a street).

Q. How many possible different streets can the builder design if no two streets are the same?

**Self test:** Ask Q1 and Q2. What type of problem is this?

A. order matters, repetition not allowed \( \Rightarrow \) permutation without repetition.

**Now solve the problem:**

5 different models, 3 colours so 15 possible homes.

Choices for street layout are:

\[
P(n, r) = P(15, 10) = \frac{15!}{5!}
\]

Note: we can also think of this as order does not matter - in this case no two streets can have the same set of houses and the answer is \( C(15, 10) \).

Take Home. The formula to arrange \( r \) items taken from \( n \) objects \textit{without repetition} when \textit{order matters} is:

\[
\frac{n!}{(n - r)!}
\]

We denote this formula by \( P(n, r) \) or \( nP_r \).
Arrangements – Order Does Not Matter

These are combinations. At first we will look at when there is no repetition.

Example. Tennis tryouts! 30 athletes are trying out for the U of T Blues Tennis team. The coach plans to have a team of 6 players. How many different possible teams are there?

A.
The coach has a choice of 30 athletes for her first choice, 29 for the second choice, 28 for the third choice, 27 for the 4th choice and 26 for the 5th choice.

We can think of this first as how to select 6 players. Then we can account for the fact that order doesn’t matter. If order matters, then we have $P(30, 6) = \frac{30!}{24!}$.

Order doesn’t matter so that means we can rearrange these 6 players in many different ways. How many?

The first choice has 6 options, the second 5 options, the third 4 options etc. for a total number of $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ ways.

$\frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6!} = \frac{30!}{6!24!}$

Take Home. The formula for arrangement of $r$ objects chosen from $n$ objects when order does not matter and there is no repetition is

$$\frac{n!}{(n-r)!r!}$$

We denote this combination as $C(n, r)$ or $\binom{n}{r}$.

Example. In Poker each player has 5 cards. A standard deck has 52 cards. How many 5-card hands are possible?

$$C(52, 5) = \binom{52}{5} = \frac{52!}{47!5!} = 2,598,960$$

A flush is when all 5 cards are of the same suit. If there are 4 suits, i.e., 13 cards per suit, how many ways are there to obtain a flush?

$$4 \times C(13, 5) = 4 \times \frac{13!}{8!5!} = 4 \times 1287 = 5148$$

Q. Should flushes happen very often? no, about 2% of the time

Example. How many different 8-digit binary sequences are there with six 1s and two 0s?

Hint: Can you think of way to rephrase this into a question you know how to answer?

At first one may think that we are trying to count specific permutations, but really we are considering which of the 8 places we put 1s. i.e., the number of ways to select 6 of the 8 slots in the binary number (equivalently 2 of the 8 for the 0s) and don’t care about the order we select them in. Therefore $C(8, 6) = 28$

Exercise. Given $n$ non-parallel lines such that no three intersect in a point, determine how many triangles are formed.
Q. How many triangles are formed by 3 lines?
A. 1 = \( \binom{3}{3} \)

Q. How many triangles are formed by 4 lines?
A. 4 = \( \binom{4}{3} \)

Q. How can we phrase this question in terms we understand...answer questions Q1 and Q2.

A. Notice that any three lines create one triangle. Therefore the problem is really, in how many ways can we choose 3 lines from \( n \) lines in which we don’t care about the order we choose the lines and repetition is not allowed. Therefore the number of triangles is \( \binom{n}{3} \).

**Permutations with Repeated Items**

Recall the example that asks for the number of permutations of the set of characters \{a, a, a, b, n, n\}. How did we determine the number of permutations of these letters?

- There are 6 characters. If all the letters were unique we would have 6! possible permutations.
- There are 3 a's which means 6! over counts by... 3!
- Similarly, there are 2 n's so 6! over counts by 2!.
- This means there are \( \frac{6!}{3!2!} \) permutations of \{a, a, a, b, n, n\}.

Let’s practice.

Q. How many nine-digit numbers can be formed with the numbers \{2, 2, 2, 3, 3, 3, 4, 4\}?  
A. \( n = 9, a = 3, b = 4, c = 2, a + b + c = 9 \)  
The order of the elements does matter.

The elements are repeated.

\[ P(9, 4, 3, 2) = \frac{9!}{4!3!2!} \]

Q. The signal mast of a ship can raise 10 flags at one time (4 red, 2 blue and 4 green). How many different signals can be communicated by the placement (ie, ordering) of these nine flags?

A. \( n = 10, r = 4, b = 2, g = 4, r + b + g = 10 \)  
The order of the elements does matter.
The elements are repeated.

\[ P(10, 4, 4, 2) = \frac{10!}{4!4!2!} \]

**Q.** A gym teacher must make 4 volleyball teams from 36 students in her class. In how many ways can she select these four teams.

Let’s call the teams \( A, B, C \) and \( D \).

**A.** Let’s solve this two ways:

**Method 1.** Use combinations to solve the problem. Of the 36 students, we can choose team \( A \) first \( C(36, 9) \) ways. Now to choose team \( B \), we have 27 students to pick from, so have \( C(27, 9) \) ways. For team \( C \) there are \( C(18, 9) \) ways to make the team and for team \( D \) there is 1 way of \( C(9, 9) \). The final answer is:

\[
\begin{align*}
\left( \begin{array}{c} 36 \\ 9 \end{array} \right) & \times \left( \begin{array}{c} 27 \\ 9 \end{array} \right) \times \left( \begin{array}{c} 18 \\ 9 \end{array} \right) \times \left( \begin{array}{c} 9 \\ 9 \end{array} \right) \\
&= 36! \quad 9! \quad 27! \quad 9! \quad 18! \quad 9! \quad 9! \\
&= 2.145 \times 10^{19}
\end{align*}
\]

**Method 2.** Use permutations with repeated objects to solve the problem. Hint: Line up the 36 students and then determine how many ways you can give each student a team name. Think of this as having 9 \( A \)s, 9 \( B \)s, 9 \( C \)s and 9 \( D \)s. Consider lining up the 36 students and then labelling \( A, B, C \) or \( D \). This is permutations or arrangements with repeated items. So,

\[
\begin{align*}
P(36; 9, 9, 9, 9) &= \frac{36!}{9!9!9!9!} \\
&= 2.145 \times 10^{19}
\end{align*}
\]

**Order matters, repeated items.** Given \( n \) objects, with \( r_1 \) of type 1, \( r_2 \) of type 2, \ldots, \( r_m \) of type \( m \) where

\[ r_1 + r_2 + \ldots + r_m = n \]

then the number of arrangements of the \( n \) objects, denoted by \( P(n; r_1, r_2, \ldots, r_m) \) is:

\[
\left( \begin{array}{c} n \\ r_1 \\ r_2 \\ \vdots \\ r_m \\ \end{array} \right) = \frac{n!}{r_1!r_2!\ldots r_m!}
\]

or equivalently

\[
\frac{n!}{r_1!r_2!\ldots r_m!}
\]

### Combinations with Repetition

When order doesn’t matter and we are “choosing” items that may be repeated, we have combinations with repetition. The bagel example illustrates this situation. Let’s try another:

**Q.** On your way home from UTSC with your 6 buddies (7 of you total) you stop for dinner. The menu offers a choice of hamburger, beyond the meat burger, chicken tenders or a fish sandwich. Let \( h, b, c, f \) denote the 4 meal choices.

How many different purchase combinations are possible if each person buys one option.

**A.** We care about how many of each item is ordered, not the order. Therefore combination with repetition. Draw a table to illustrate how to solve this problem (if you forget how to do this, go back and look at the lecture 2 readings).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( b )</th>
<th>( c )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>xx</td>
<td>xxx</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>xxx</td>
<td>xxx</td>
</tr>
<tr>
<td></td>
<td>x</td>
<td>x</td>
<td>xx</td>
</tr>
</tbody>
</table>

Now solve the problem:

Notice that we have 4 choices. We have 7 people. Think of it as an \( x \) for each person and there are 4-1=3 possible \( | \) to put in. Therefore \( C(7 + 4 - 1, 7) = \frac{10!}{7!3!} \).
Q. How many ways are there to distribute 20 identical chocolate bars and 15 identical sticks of gum to 5 children?

A. Think of this as selecting combinations with repetition.

First the chocolate bars. We ask, how many ways are there to select 20 children’s names from a set of 5 different names (types). This is just \( \binom{20+5-1}{20} \).

Same for the gum. Pick 15 children’s names from the 5 different names: \( \binom{15+5-1}{15} \).

Now multiply to get the total number of combinations. \( \frac{24!19!}{20!4!15!4!} \).

Q. How many ways can you distribute 12 dog treats to 4 dogs?

A. \( C(12+3, 12) = C(15, 12) = 455 \).

Q. Repeat now with the requirement that every dog must get at least one treat.

A. Give each dog one treat. Now there are 8 left over and we have the same problem but with 8. \( C(8+3, 8) = C(11, 8) \)

**Example.** Consider the following problem. How many integer solutions are there to the equation \( x_1 + x_2 + x_3 + x_4 = 12 \) with \( x_i \geq 0 \) and \( 1 \leq i \leq 4 \)?

Q. How is this related to the first dog question?

A. Think of each \( x_i \) as being one of the dogs and there being 12 treats to distribute amongst the \( x_i \) or dogs.

Solve the problem.

A. Exactly the same as above \( C(15, 12) \)

Q. What if we require that each \( x_i \geq 1 \)?

A. Same as the second dog example. Then this means putting at least one object in each box so imagine placing subtracting one value for each box from 12 leaving 8. Now we are considering the same problem but with 8 objects and 4 boxes \( \rightarrow C(8+4−1, 8) \).

**Order does not matter, repeated objects.** Given \( r \) objects and \( n \) types of objects to choose from, the number of selections with repetitions is:

\[
\binom{r + (n - 1)}{r}
\]

**Example.** How many ways can 20 different diplomats be assigned to 5 different continents?

Q. How is this different than the other examples we have seen?

A. Here the diplomats are unique and so are the continents so there is no repetition.

**Solve** the problem by rephrasing as an arrangement we already know how to solve.

**Solution.** Think of this as lining up the 20 diplomats and labelling each with one of 5 continents - this is really just \( 5^{20} \). Or think of this as simply permutation with repetition (not repeated items), 20 spaces, 5 options each.

Q. What if each continent needs to have 4 diplomats each?
A, then we have 5 countries each repeated 4 times to make a total of 20. This is permutation with repeated objects or \( P(20, 4, 4, 4, 4) = \frac{20!}{(4!)^5} \).

Understanding Permutations and Combinations

Definition. An \( r \)-permutation of \( n \) distinct objects is an ordered arrangement of \( r \) of the \( n \) objects. We use the notation \( P(n, r) \).

We can derive the formula as follows:

\[
P(n, r) = \frac{n!}{(n-r)!}
\]

Q. In terms of factorials, how can we rewrite this formula?

A. \[
\frac{n!}{(n-r)!}
\]

Given \( n \) objects, how many ways can we rearrange or permute them?

\[
P(n, n) = n(n-1)(n-2)\ldots(2)(1) = n!
\]

Combinations

Definition. An \( r \)-combination of \( n \) distinct objects is an unordered selection, or subset of \( r \) of the \( n \) objects.

We can think of combinations in terms of permutations.

Q. Given \( P(n, r) \), the number of \( r \)-permutations of \( n \) objects, how can we derive the number \( C(n, r) \) of \( r \)-combinations of \( n \) objects?

A. All \( r \)-permutations of the \( n \) objects can be generated by first taking an \( r \)-combination and then creating all permutations of this combination. In other words, \( P(n, r) = C(n, r) \times P(r, r) \). Therefore, \( C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!r!} \).