Proofs

- Direct Proofs
- Indirect Proofs – Contradiction More practice today
- Indirect Proofs – Contrapositive More practice today
- Proof by Induction (Simple) NEW TODAY
- Proof by Induction (Strong) Next Week
Proof by Induction

Proof by *induction* is a powerful tool when used properly.

**Visual:** The Domino Argument

Think of a *row of dominos*.

If set up properly, when the $i^{th}$ falls, so too will the $(i + 1)^{st}$ domino for all values of $i \geq 0$ or $i \geq 1$. 
Proof by Induction

We often use induction to prove statements of the form:

$$\forall n \in \mathbb{N}, S(n)$$

Note that we include 0 in the natural numbers.

There are three steps to an inductive proof.
The Steps

To prove:

\[ \forall n \in \mathbb{N}, S(n) \]

We have three main steps:

1. **Base Case.** Prove that \( S(0) \) holds.

2. **Inductive Hypothesis.** Assume that \( S(k) \) holds for an arbitrary value \( k \geq 0 \).

3. **Inductive Step.** Prove that

   \[ \forall k \in \mathbb{N}, S(k) \rightarrow S(k + 1) \].

   I.e., that if \( S(k) \) is true then \( S(k + 1) \) is true.
Dominos!

How do our three main steps relate to the domino analogy?

1. **Base Case.** Someone knocks over the $1^{st}$ domino.

2. **Inductive Hypothesis.** Assume for arbitrary $k \geq 0$, that the $k^{th}$ domino falls.

   Q. What does *arbitrary* mean?

3. **Inductive Step.** If the dominos were set up properly, then for all $k$, if the $k^{th}$ domino falls, it will hit the $(k + 1)^{st}$ domino and make it fall.
A First Example

Prove:

\[ \forall n \in \mathbb{N}, n(n^2 + 5) \text{ is divisible by } 6 \]

1. Base Case. \( n = 0, 0(0^2 + 5) = 0 \) which is divisible by 6.

2. Inductive Hypothesis.
   Assume that \( S(k) \) holds for arbitrary \( k \geq 0 \). So \( k(k^2 + 5) \) is divisible by 6.

3. Inductive Step. We need to show that \( S(k) \rightarrow S(k + 1) \).
A First Example

Prove: \( \forall n \in \mathbb{N}, n(n^2 + 5) \) is divisible by 6

Inductive Step. We need to show that \( S(k) \rightarrow S(k+1) \).

\( S(k + 1) \) says that \( (k + 1)((k + 1)^2 + 5) \) is divisible by 6.

By I.H., we know \( S(k) \), that \( k(k^2 + 5) \) is divisible by 6.

Let’s work with \( S(k + 1) \):

\[
(k + 1)(k^2 + 2k + 1 + 5) = k(k^2 + 2k + 5 + 1) + (k^2 + 2k + 5 + 1)
\]
A First Example

For all natural numbers $n$, $n(n^2+5)$ is divisible by 6

**Inductive Step.** We need to show that $S(k) \Rightarrow S(k+1)$.

$$((k+1)(k^2+2k + 1 + 5) = k(k^2 + 2k + 5 + 1) + (k^2 + 2k + 5 + 1)$$

$$= k(k^2 + 5) + 2k^2 + k + k^2 + 2k + 5 + 1$$

$$= k(k^2 + 5) + 3k^2 + 3k + 6$$

$$= k(k^2 + 5) + 3k(k+1) + 6$$

Q. Why are each of these terms divisible by 6?
A. $S(k)$ by I.H., 3 times an even number, $6 | 6$. 
A First Example

For all natural numbers $n$, $n(n^2+5)$ is divisible by 6

**Inductive Step.** We need to show that $S(k) \rightarrow S(k+1)$.

$\begin{align*}
((k+1)(k^2+2k + 1 + 5) &= k(k^2 + 2k + 5 + 1) + (k^2 + 2k + 5 + 1) \\
&= k(k^2 + 5) + 2k^2 + k + k^2 + 2k + 5 + 1 \\
&= k(k^2 + 5) + 3k^2 + 3k + 6 \\
&= k(k^2 + 5) + 3k(k+1) + 6
\end{align*}$

Therefore, we have shown that $S(k) \rightarrow S(k+1)$ completing the proof.
Q. What if we want to prove $S(n)$ for all natural numbers $n \geq b$?

A. Two options:

1. Prove $\forall n \in \mathbb{N}, n \geq b \rightarrow S(n)$

2. Prove $\forall n \in \mathbb{N}_{\geq b}, S(n)$

There are other options, but these are the simplest.