FUNCTIONS - PART 1

Introduction

This handout is a summary of the basic concepts you should understand and be comfortable working with for the first math review module on functions. This is intended as a summary and should be used together with the references given below.


If you are not familiar with any of the material below you need to spend time studying these concepts and doing some exercises.

Definition of a Function

A function $f$ is a rule that assigns to each element in a set $A$ exactly one element, called $f(x)$ in a set $B$.

The set $A$ is called the domain of $f$, and the set of all values $f(x)$ for all $x$ in $A$ is called the range of $f$. That is, Range = \{f(x)|x \in A\}. Sometimes a function is written as an equation $y = f(x)$.

The symbol that represents an arbitrary number in the domain of a function $f$ is called the independent variable. In the definition of a function above, the independent variable is $x$.

The symbol that represents a number in the range of a function $f$ is called the dependent variable. In the definition of a function above, the dependent variable is $y$.

Example: Find the domain and range of $f(x) = x^2$.

The domain is the set of real numbers for which the definition of the function makes sense. Since we can calculate the square of any real number then the domain of $f$ is
\[ R. \] Since \( x^2 \geq 0 \) for all real numbers, and every positive number \( y \) is the square of \( \sqrt{y} \), then the range of \( f \) is \( \{ y \in R | y \geq 0 \} \).

**Exercise:** Find the domain of \( g(x) = \frac{1}{x^2 - x} \) and write it in interval notation.

**Answer:** The domain is \((-\infty, 0) \cup (0, 1) \cup (1, \infty)\).

**Exercise:** Find the domain of \( h(x) = \sqrt{9 - x^2} \) and write it in interval notation.

**Answer:** The domain is \([-3, 3]\).

### Operations with Functions

Given two functions we can add, subtract, multiply and divide them. Let \( f \) and \( g \) be functions with domains \( A \) and \( B \) respectively. Then the functions \( f + g \), \( f - g \), \( fg \) and \( f/g \) are defined as follows:

\[
(f + g)(x) = f(x) + g(x) \quad \text{Domain: } A \cap B \\
(f - g)(x) = f(x) - g(x) \quad \text{Domain: } A \cap B \\
(fg)(x) = f(x)g(x) \quad \text{Domain: } A \cap B \\
\frac{f}{g}(x) = \frac{f(x)}{g(x)} \quad \text{Domain: } \{ x \in A \cap B | g(x) \neq 0 \}
\]

**Exercise:** If \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt{9 - x^2} \), find the functions \( f + g \), \( f - g \), \( fg \), and \( \frac{f}{g} \) and their domains.

**Answer:**
\[
(f + g)(x) = \sqrt{x} + \sqrt{9 - x^2} \quad \text{Domain: } [0, 3] \\
(f - g)(x) = \sqrt{x} - \sqrt{9 - x^2} \quad \text{Domain: } [0, 3] \\
(fg)(x) = \sqrt{x} \cdot \sqrt{9 - x^2} = \sqrt{9x - x^3} \quad \text{Domain: } [0, 3] \\
\frac{f}{g}(x) = \frac{\sqrt{x}}{\sqrt{9 - x^2}} \quad \text{Domain: } [0, 3]
\]

### Transformations of Functions

We will consider how certain transformations of a function will affect its graph.
Vertical Shifting

Suppose we know the graph of \( y = f(x) \). A vertical shift of \( f(x) \) by \( c \) units is given by the graph of \( y = f(x) + c \).

\[
\begin{align*}
  y &= f(x) + c \quad (c > 0) \quad \text{Shifts the graph of } y = f(x) \text{ upwards } c \text{ units.} \\
  y &= f(x) + c \quad (c < 0) \quad \text{Shifts the graph of } y = f(x) \text{ downwards } c \text{ units.}
\end{align*}
\]

Horizontal Shifting

Suppose we know the graph of \( y = f(x) \). A horizontal shift of \( f(x) \) by \( c \) is given by the graph of \( y = f(x + c) \).

\[
\begin{align*}
  y &= f(x + c) \quad (c > 0) \quad \text{Shifts the graph of } y = f(x) \text{ to the left } c \text{ units.} \\
  y &= f(x + c) \quad (c < 0) \quad \text{Shifts the graph of } y = f(x) \text{ to the right } c \text{ units.}
\end{align*}
\]

Reflecting Graphs

Given the graph of \( y = f(x) \), the reflection of \( f(x) \) in the \( x \)-axis or \( y \)-axis is given by the following transformations.

\[
\begin{align*}
  y &= -f(x) \quad \text{Reflects the graph of } y = f(x) \text{ in the } x\text{-axis.} \\
  y &= f(-x) \quad \text{Reflects the graph of } y = f(x) \text{ in the } y\text{-axis.}
\end{align*}
\]

Vertical Stretching and Shrinking Graphs

Given the graph of \( y = f(x) \), a vertical stretching or shrinking of the graph is given by the following transformations.

\[
\begin{align*}
  y &= af(x) \quad (a > 1) \quad \text{Stretches the graph of } y = f(x) \text{ vertically by a factor of } a \\
  y &= af(x) \quad (0 < a < 1) \quad \text{Shrinks the graph of } y = f(x) \text{ vertically by a factor of } a.
\end{align*}
\]

Horizontal Stretching and Shrinking of Graphs

Given the graph of \( y = f(x) \), a horizontal stretching or shrinking of the graph is given by the following transformations.

\[
\begin{align*}
  y &= f(ax) \quad (a > 1) \quad \text{Shrinks the graph of } y = f(x) \text{ horizontally by a factor of } 1/a \\
  y &= f(ax) \quad (0 < a < 1) \quad \text{Stretches the graph of } y = f(x) \text{ horizontally by a factor of } 1/a.
\end{align*}
\]